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SUBNORMALITY FOR THE ADJOINT OF A COMPOSITION OPERATOR ON L^2

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1. INTRODUCTION

Consider a composition operator C on $L^2(X, \Sigma, m) : Cf = f \circ T$ where $T : X \to \to X$. In the recent past, special operator properties of C have been characterized by measure theoretic properties. In [18] Whitley shows that C is quasinormal exactly when $h = h \circ T$, where h is the Radon-Nikodym derivative $\mathrm{d}m \circ T^{-1}/\mathrm{d}m$, and that C is normal exactly when $h \circ T$ and $T^{-1}\Sigma = \Sigma$. Similar results are found in [12], [16] and [17]. Lambert shows in [11] that C is hyponormal exactly when $h > 0$ a.e. and $E(l/h) \leqslant l/h \circ T$ a.e., where E is the conditional expectation with respect to the σ-algebra $T^{-1}\Sigma$, and in [12] that C is subnormal exactly when for almost all x, $\{h_n(x)\}$ is a moment sequence, where h_n is the Radon-Nikodym derivative $\mathrm{d}m \circ T^{-n}/\mathrm{d}m$. In [3] Dibrell and Campbell investigate hyponormal powers of C. Finally in [8] the authors show that C is centered exactly when h is measurable with respect to the σ-algebra $\bigcap_{n=1}^{\infty} T^{-n}\Sigma$.

Less attention has been focused on the special properties of the adjoint C^* of C. In [9] Harrington and Whitley show that C^* is hyponormal if and only if $H \cap \Sigma \subset T^{-1}\Sigma$ and $h \circ T \geqslant h$ a.e. and C^* is quasinormal if and only if $H \cap \Sigma \subset T^{-1}\Sigma$ and $h \circ T = h$ a.e., where H is the support of h. In this paper we shall continue the study of special properties of C^*. In particular we show that C^* is both centered and power hyponormal when C^* is hyponormal. Additionally we show that C^* is subnormal exactly when C is centered and $\{h_n \circ T^n\}$ is a moment sequence a.e. $\mathrm{d}m$. A straightforward application of this last result yields Berger's characterization of subnormal weighted shifts.

2. NOTATION AND TERMINOLOGY

Let (X, Σ, m) be a complete σ-finite measure space and let T be a measurable transformation from X to X such that $m \circ T^{-1} \ll m$. Then composition with T defines a linear transformation on the space of all measurable function. Under the assumption that $\frac{\mathrm{d}m \circ T^{-1}}{\mathrm{d}m} \in L^\infty$, this linear transformation acts on $L^2(X, \Sigma, m)$. We shall refer to this operator as C, the *composition operator* induced by T. The following notational conventions will be used throughout this article:

- $h_n = \frac{\mathrm{d}m \circ T^{-n}}{\mathrm{d}m}$.
- E_n is the *conditional expectation operator* with respect to $T^{-n}\Sigma$: $E_n(f) = E(f|T^{-n}\Sigma)$.
- All set and function statements are to be interpreted as holding up to sets of measure 0.
- H_n is the support of h_n.
- $\Sigma_\infty = \bigcap_{n=1}^{\infty} T^{-n}\Sigma$.

We shall make use of the following general properties of measurable transformations:

- Each $T^{-1}\Sigma$-measurable function F has the form $f \circ T$ for some measurable function f. Further if $f \circ T = g \circ T$, then $f = g$ on H. Therefore, even when T fails to be invertible, $h \cdot (Ef) \circ T^{-1}$ is well-defined. In fact, $C^*f = h \cdot (Ef) \circ T^{-1}$. (See [1] and [12].)
- $C^n C^{*n} f = (h_n \circ T^n) E_n f$.
- $\{H_n\}$ is a decreasing sequence of sets ([8]).
- $T^{-1}H = X$ ([8]).
- $h_{n+1} = h \cdot (Eh_n) \circ T^{-1} = h_n \cdot (E_n h) \circ T^{-n}$ ([11]).

In this paper the term *moment sequence* will always refer to a Hausdorff moment sequence. Recall that a numerical sequence $\{\alpha_n\}$ is a Hausdorff moment sequence if there is an interval $I = [0, a]$ and a Borel probability measure μ on I such that for each integer $n \geqslant 0$, $\alpha_n = \int_I t^n \mathrm{d}\mu$. Similary a sequence $\{A_n\}$ of operators on a Hilbert space is called a Hausdorff moment sequence if there is an interval $I = [0, a]$ and an operator-valued measure μ such that for each integer $n \geqslant 0$, $A_n = \int_I t^n \mathrm{d}\mu$.

We shall use the following standard terminology for special Hilbert space operators. An operator A is *positive* if $(Af, f) \geqslant 0$ for each f; A is *hyponormal* if $A^*A - AA^*$ is positive; A is *normal* if $A^*A - AA^* = 0$. Also A is *quasinormal* if A commutes with A^*A and *subnormal* if A is the restriction of a normal operator to an invariant subspace. An indepth discussion of these classes is found in [2]. The hierarchical relationship between the classes is as follows:

$$A \text{ normal} \Rightarrow A \text{ quasinormal} \Rightarrow A \text{ subnormal} \Rightarrow A \text{ hyponormal}.$$

3. HYPONORMALITY AND SUBORMALITY OF C^*

We begin our investigation by establishing a series of results when C^* is hyponormal. We shall refer frequently to Harrington and Whitley's characterization in [9]:

(1) C^* is hyponormal if and only if $\Sigma \cap H \subset T^{-1}\Sigma$ and $h \circ T \geqslant h$.

LEMMA 1. *If C^* is hyponormal, then $T^{-1}\Sigma = \Sigma_\infty$.*

Proof. If $A \subset X - H$, then $T^{-1}A = \emptyset$. Assume that $A \subset H$. By (1) there is a set B with $A = T^{-1}B$. Therefore, $T^{-1}A \in T^{-2}\Sigma$. Thus $T^{-1}\Sigma = T^{-2}\Sigma$, which yields the asserted equality. ■

In [14] B. Morrell and P. Muhly introduced the concept of a *centered operator*. A is centered if the family $\{A^{*n}A^n, A^mA^{*m} : n, m \geqslant 0\}$ is commutative. Note that A is centered if and only if A^* is centered. In general the composition operator C need not be centered. We shall make use several times of the following characterization in [8]:

(2) C is centered if and only if h is Σ_∞-measurable.

LEMMA 2. *If C^* is hyponormal, then C is centered.*

Proof. By (1) and Lemma 1 every measurable subset of the support H of h is Σ_∞-measurable, and so h is Σ_∞-measurable. Thus by (2) C is centered. ■

One useful conseqence of C being centered is that for each positive integer n we have

$$h_n \circ T^n = \prod_{k=1}^{n} h \circ T^k.$$

(This is easily verified using the recursion for h_n mentioned earlier in the paper and the Σ_∞-measurability of h.)

We note that there are many examples of hyponormal operators A for which some higher power of A fails to be hyponormal. (Indeed, examples of such composition operators are known; see [3] and [11].) The next result shows that if C^* is hyponormal, then all its powers are hyponormal, i.e., it is *power hyponormal*.

THEOREM. 3. *If C^* is hyponormal, then C^* is power hyponormal.*

Proof. Let $g_n = h_n \circ T^n$. Then $g_n = \prod_{k=1}^{n} h \circ T^k$ since C is centered. Suppose that C^{*n} is hyponormal for some $n \geqslant 1$. Then $g_n \geqslant h_n$ by (1), so that for every measurable set A,

$$\int_A g_n \mathrm{d}m \geqslant \int_A h_n \mathrm{d}m.$$

Let A be in Σ. Then

$$\int_A g_{n+1}\mathrm{d}m = \int_A \prod_{k=1}^{n+1} h \circ T^k \mathrm{d}m = \int_A h \circ T \prod_{k=2}^{n+1} h \circ T^k \mathrm{d}m \geqslant$$

$$\geqslant \int_A h \cdot \prod_{k=1}^{n} h \circ T^k \mathrm{d}m = \int_{T^{-1}A} \prod_{k=1}^{n} h \circ T^{k+1} \mathrm{d}m \geqslant$$

$$\geqslant \int_{T^{-1}A} \prod_{k=1}^{n} h \circ T^k \mathrm{d}m = m(T^{-n}T^{-1}A) =$$

$$= \int_A h_{n+1} \mathrm{d}m.$$

Thus, $h_{n+1} \circ T^{n+1} \geqslant h_{n+1}$. Since C is centered, h_{n+1} is Σ_∞-measurable. Since also $\Sigma_\infty = T^{-(n+1)}\Sigma$, then h_{n+1} is $T^{-(n+1)}\Sigma$-measurable. It follows from (1) that C^{*n+1} is hyponormal. Thus by induction we see that all powers of C^* are hyponormal if C^* is hyponormal. ■

THEOREM 4. *If C^* is hyponormal and irreductible, then either X consists of a single atom (equivalently, L^2 is one dimensional) or $T^{-1}\Sigma = \Sigma$.*

Proof. By Lemmas 1 and 2, C is centered and $T^{-1}\Sigma = \Sigma_\infty$. It follows from [8] that $L^2(\Sigma_\infty)$ reduces C. Thus, Σ_∞ is either trivial or all of Σ. Suppose $T^{-1}\Sigma = \{\emptyset, X\}$. Since C is centered, h is Σ_∞-measurable, so that $H \in T^{-1}\Sigma$ and either $H = \emptyset$ or $H = X$. The first case, $H = \emptyset$, leads to X having measure zero. Assume now that $H = X$, or $h > 0$ a.e. dm. In this case $m(T^{-1}A) = 0$ only when $m(A) = 0$. Assume that $X = A_1 \cup A_2$ where A_1 and A_2 are disjoint sets. Then $T^{-1}X = T^{-1}A_1 \cup T^{-1}A_2$ and $T^{-1}A_1 \cap T^{-1}A_2 = \emptyset$. Consequently, $T^{-1}A_1 = \emptyset$ or $T^{-1}A_2 = \emptyset$, resulting in $A_1 = \emptyset$ or $A_2 = \emptyset$. This argument shows that X is an atom. ■

We are now able to completely characterize those transformation T for which C^* is subnormal.

THEOREM 5. *C^* is subnormal if and only if C is centered, $T^{-1}\Sigma = \Sigma_\infty$, and $\{(h_n \circ T^n)(x)\}$ is a moment sequence for almost all x in X.*

Proof. Suppose that C^* is subnormal. By [10] we know that $\{\|C^{*n}f\|^2\}$ is a moment sequence for each f in $L^2(X, \Sigma, m)$. Also C^* is centered by Lemma 2 and $E_n = E$, $n \geqslant 1$, by Lemma 1. A straightforward computation now shows that for $n \geqslant 1$.

$$\|C^{*n}f\|^2 = \int_X h_n \circ T^n |Ef|^2 \mathrm{d}m.$$

We note that Ef is an arbitrary $L^2(X, \Sigma_\infty, m)$ function and conclude that

$$\int_X h_n \circ T^n |g|^2 \mathrm{d}m$$

is a moment sequence for each g in $L^2(X, \Sigma_\infty, m)$. In particular let $G = \chi_A$, where A is a subset of X of finite measure. Then

$$\int_A h_n \circ T^n \mathrm{d}m = \int_I t^n \mathrm{d}\mu,$$

where $\int_I t^n \mathrm{d}\mu$ is the moment sequence $\|C^{*n}\chi_A\|^2$. Observe now that if $\sum a_n t^n$ is a polynomial which is nonnegative on I, then $\int_A \sum (h_n \circ T^n) a_n \mathrm{d}m \geqslant 0$. It follows now for each nonnegative polynomial $\sum a_n t^n$ that $\sum a_n (h_n \circ T^n)(x) \geqslant 0$ a.e. $\mathrm{d}m$. Therefore by the classical result in [19, Chapter III], $\sum (h_n \circ T^n)(x)$ is a moment seqence a.e. $\mathrm{d}m$.

Now suppose the stated conditions hold. We shall construct a quasinormal extension of C^*. Since this extension is itself subnormal and the restriction of a subnormal operator to an invariant subspace is subnormal, we will have completed the proof. We write $h_n \circ T^n(x) = \int_I t^n d\mu_x(t)$.

ASSERTION. For each x in X, $\mu_{Tx} \ll \mu_x$ and $\frac{\mathrm{d}\mu_{Tx}}{\mathrm{d}\mu_x}(t) = \frac{t}{h \circ T(x)}$.

Indeed, since h is Σ_∞-measurable, we have $h_{n+1} \circ T^{n+1} = [h \circ T]\,[h_n \circ T^n] \circ T$ and

$$\int_I t^n t \mathrm{d}\mu_x(t) = \int_I t^n h \circ T(x) \mathrm{d}\mu_{Tx}(t) \quad \text{a.e.d}m(x), \qquad n = 0, 1 \ldots .$$

Thus $t\mathrm{d}\mu_x = h \circ T(x)\mathrm{d}\mu_{Tx}$. Since $h \circ T > 0$ a.e. $\mathrm{d}m$, $\mu_{Tx} \ll \mu_x$ and the desired formula holds. Let B be the collection of Borel subsets of I, let $\Gamma = \Sigma \times B$ and define

$$\nu(A \times J) = \int_A \mu_x(J)\mathrm{d}m(x).$$

Essentially the same argument as in [13] shows that ν extends to a σ-finite measure on Γ. Define the weighted composition operator W on $L^2(X \times I, \Gamma, \nu)$ by

$$(WF)(x,t) = \frac{t}{h \circ T(x)} F(Tx, t).$$

Then W is bounded since

$$\|WF\|^2 = \int_X \int_I \left(\frac{t}{h \circ T(x)}\right)^2 |F(Tx,t)|^2 \mathrm{d}\mu_x(t)\mathrm{d}m(x) =$$

$$= \int_X \int_I \frac{t}{h \circ T(x)} |F(Tx,t)|^2 \mathrm{d}\mu_{Tx}(t)\mathrm{d}m(x) =$$

$$= \int_H \int_I t|F(x,t)|^2 \mathrm{d}\mu_x(t)\mathrm{d}m(x) \leqslant$$

$$\leqslant \|h\|_\infty \|F\|^2.$$

For $G \in L^2_\nu$, let $G_t(x) = G(x,t)$. Then

$$(WF,G) = \int_X \int_I \frac{t}{h(Tx)} F(Tx,t)\overline{G(x,t)} \mathrm{d}\mu_x(t)\mathrm{d}m(x) =$$

$$= \int_X \int_I F(Tx,t)\overline{G_t(x)} \mathrm{d}\mu_{Tx}(t)\mathrm{d}m(x) =$$

$$= \int_X \int_I F(x,t) h\overline{G_t(x)} \circ T^{-1} \mathrm{d}\mu_x(t)\mathrm{d}m(x) =$$

$$= \int_X \int_I F(x,t)\overline{(C^* G_t)(x)} \mathrm{d}\mu_x(t)\mathrm{d}m(x).$$

Thus $(W^*G)(x,t) = (C^*G_t)(x)$. Now we identify $L^2(X,\Sigma,m)$ with $\{F \varepsilon L^2(\nu) : : F(x,t) \equiv f(x)\}$ and note that for such an F, $\|F\|_\nu = \|f\|_m$. It then follows that W^* is an extension of C^*. A straightforward calculation shows that WW^* is the operator of multiplication by the second independent variable t, while W^*W is multiplication by $t \cdot \chi_H$ and consequently $W^*(WW^*) = (WW^*)W^*$, so that W^* is quasinormal. ■

To prove the necessity of the conditions for subnormality of C^* in Theorem 5, we used the classical relation between moment sequence and complete positivity. It should be noted that a more operator-theoretic proof can be given, using the fact that A is subnormal exactly when $\{A^{*n}A^n\}$ is a moment sequence of operator [4].

COROLLARY 6. *If C^* is subnormal, then the quasinormal extension W^* constructed in Theorem 5, is the minimal quasinormal extension of C^*. Furthermore, W is normal if and only if $X = H$.*

Proof. The domain of the minimal quasinormal extension of C^* by [5] is

$$Y = \{W^k W^{*k} f_k : f_k \in L^2(m)\}.$$

But $(W^k W^{*k} f_k)(x,t) = t^k f_k(x)$ for f_k in $L^2(m)$ so that Y contains all functions of the form $\chi_{A\times J}$ with $m(A) < \infty$, $A \in \Sigma$ and $J \in B$. Therefore $Y = L^2(\nu)$ and W^* is the minimal quasinormal extension of C^*. Furthermore if $X = H$, then W^*W is multiplication by t, as is WW^*. ■

The extension W of C, constructed in Theorem 5, is almost identical with the one given in [13] to obtain a quasinormal extension of C (when C is subnormal). There the measure ν is constructed using the moment sequence $\{h_n(x)\}$ and the extension of C is $(WF)(x,t) = F(Tx,t)$, an unweighted composition operator. In [6] W is shown to be the minimal quasinormal extension of C by an argument similar to that given in Corollary 6.

It would be interesting to know if each subnormal C^* admits a minimal normal extension which is itself the adjoint of a weighted composition operator. We do not know if this is the case. However, we can show that for C^* subnormal, C^* admits a normal extension which is the adjoint of a weighted composition operator. This particular construction does not in general lead to a minimal extension.

Theorem 7. *W^* has a normal extension whose adjoint is a weighted composition operator.*

Proof. Let J be the countable direct sum of copies of $L^2(\nu)$ and let Z be the operator on J defined by the matrix

$$\begin{bmatrix} W & 0 & 0 & 0 & \cdots \\ V & 0 & 0 & 0 & \ddots \\ 0 & V & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

where $Vf = \sqrt{t}\chi_{X\text{-}H}f$. Then direct calculation shows that Z^* is a normal extension of W^*. Now let γ be the counting measure on the nonnegative integers and let $\lambda = \gamma \times \nu$. For $\langle f_i \rangle$ in J, let $u(n, x, t) = f_n(x, t)$. One verifies in a routine manner that $u \in L^2(\lambda)$ and that $\|\langle f_i \rangle\| = \|u\|_{L^2(\lambda)}$. Let U be the isometry from J to $L^2(\lambda)$ so defined. U is in fact surjective. We will show that U induces a unitary equivalence between Z and a weighted composition operator. Define the transformation S on $\mathbb{Z}^+ \times X \times I$ by

$$S(n, x, t) = \begin{cases} (0, Tx, t) & ; \quad n = 0 \\ (n-1, x, t) & ; \quad n \geqslant 1 \end{cases}$$

and let the function r be defined by

$$r(n, x, t) = \begin{cases} \dfrac{t}{h \circ T(x)} & ; \quad n = 0 \\ \sqrt{t} \cdot \chi_{X\text{-}H}(x) & ; \quad n \geqslant 1 \end{cases}.$$

It follows that $UZ = RU$, where R is defined by $Rf = r \cdot f \circ S$. ■

The transformation S may be visualized as follows. Consider a tower of copies $X \times I$. S projects points vertically downward so long as there is a "downward". A point $(0, x, t)$ on the bottom level is sent by S to $(0, Tx, t)$. In some sense this is the reverse of the more standard tower construction.

4. EXAMPLES AND APPLICATIONS

1. From the measure-theoretic point of view, one of the simplest examples of a composition operator induced by a noninvertible transformation is that of the adjoint of a unilateral weighted shift. (See [15] for a thorough treatment of weighted shifts.) C. Berger has presented the following elegant characterization of subnormal weighted shifts.

Proposition (C. Berger; See [15]). *S is subnormal if and only if $\{\beta_n^2\}$ is a moment sequence (where $\{\beta_n\}$ is defined below.)*

This result follows as a special case of Theorem 5:

Let S be the weighted shift on ℓ^2_+ with weight sequence $\{\alpha_1, \alpha_2, \ldots\}$ (all weights positive) and let $\beta_0 = 1$, $\beta_n = \prod_{i=1}^{n} \alpha_i$, $n \geqslant 1$. S is unitarily equivalent to the unweighted unilateral shift on the weighted ℓ^2 space with mass $m(i) = \beta_i^{-2}$ at each nonnegative integer i. Using functional notation rather than the more usual sequence notation, we see that S^* is given by the action $(S^* f)(k) = f(k+1)$, that is S is the adjoint of the composition operator associated with the transformation $T: k \to k+1$ on $\mathbb{Z}^+$. One sees immediately that in this case $T^{-1}\Sigma = \Sigma$, and so S is subnormal if and only if $\{h_n \circ T^n(k)\}_{n=1}^{\infty}$ is a moment sequence for each k in $\mathbb{Z}^+$. But

$$h_n \circ T^n(k) = \frac{m \circ T^{-n}\{n+k\}}{m\{n+k\}} = \frac{m_k}{m_{n+k}} = \frac{\beta_{n+k}^2}{\beta_k^2}.$$

It now readily follows from Theorem 5 that C^* is subnormal if and only if $\{\beta_k^2\}$ is a moment sequence. ■

2. A continuous semigroup of cosubnormal composition operators on $L^2(\mathbb{R}^+, \mathrm{d}x)$:

Let

$$(S_t f)(x) = \begin{cases} \sqrt{\dfrac{\varphi(x)}{\varphi(x-t)}} & ; \quad x > t \\ 0 & ; \quad x \leqslant t \end{cases},$$

where φ is strictly positive and continuous.

It was shown in [7] that this semigroup consists of subnormal operator if and only if φ is the Laplace-Stieltjes transform of a probability measure; i.e. for some probability measure μ on $\mathbb{R}^+$,

$$(*) \qquad \varphi(x) = \int e^{-xt}\mathrm{d}\mu(t).$$

Let v be the absolutely continuous measure: $\mathrm{d}v = \dfrac{1}{\varphi}\mathrm{d}x$. Then the canonical unitary operator U from $L^2(\mathrm{d}x)$ to $L^2(\mathrm{d}v)$ is given by $Uf = \sqrt{\varphi} \cdot f$. Now direct calculation shows that

$$(S_t f)(x) = \sqrt{\frac{\varphi(x+t)}{\varphi(x)}} f(x+t); \qquad f \in L^2(\mathrm{d}x).$$

It then follows that for f in $L^2(\mathrm{d}v)$, $\left(US_t^* U^{-1} f\right)(x) = f(x+t)$. Thus for φ as in $(*)$ and C_t defined by $(C_t f)(x) = f(x+t)$ on $L^2\left(\dfrac{1}{\varphi}\mathrm{d}x\right)$, each C_t^* is subnormal. It may prove informative to see how the characterization of subnormality given for general composition operator adjoints applies in this particular case. We shall examine the case $t=1$, there being no substantive difference for arbitrary t. Here $dm = \frac{1}{\varphi}dx$ and $T(x) = x+1$ on $\mathbb{R}^+$. Thus $T^{-1}\Sigma = \Sigma$ and $H = [1, \infty)$. Now

$$h(x) = \frac{\frac{dm \circ T^1}{dx}}{\frac{dm}{dx}} = \begin{cases} \frac{\varphi(x)}{\varphi(x-1)} & ; \quad x \geqslant 1 \\ 0 & ; \quad 0 \leqslant x < 1 \end{cases}.$$

It follows that $h \circ T^k(x) = \frac{\varphi(x+k)}{\varphi(x+k-1)}$, and consequently

$$h_n \circ T^n(x) = \prod_{k=1}^{n} h \circ T^k(x) = \frac{\varphi(x+n)}{\varphi(x)} = \frac{1}{\varphi(x)}\int_0^\infty e^{(-x+n)t}d\mu(t) =$$

$$= \int_0^\infty e^{-nt}d\mu_x(t) \qquad \left(d\mu_x(t) = \frac{e^{-xt}}{\varphi(x)}d\mu(t)\right).$$

A simple change of variables reduces this to a Hausdorff moment sequence. ■

REMARK. It is clear that when C^* is hyponormal, C^* resembles a weighted shift in at least two ways: It is power hyponormal and it is centered. Since the kernel of a hyponormal operator is a reducing subspace for the operator, and the kernel of C^* is $L^2(\Sigma) \ominus L^2(T^{-1}\Sigma) = [L^2(\Sigma_\infty)]^\perp$, we may study C^* in terms of its restriction to $L^2(\Sigma_\infty)$. When viewed as a composition operator, the underlying measure space for the shift has $\Sigma = \Sigma_\infty$. Thus the resemblence of C^* to a shift is strengthened. If the transformation T is ergodic (that is, the only sets invariant under T^{-1} are $\emptyset$ and X (mod 0)) and $\Sigma = \Sigma_\infty$, there is a sequence of sets $\{K_n\}$ such that

$$X - H = K_0 = T^{-1}K_1 = T^{-2}K_2 = \cdots, \text{ where } K_n \subset H_n, \ n = 1, 2, \ldots .$$

Since $T^{-1}K_0 = \emptyset$, the ergodicity assumption guarantees that $\bigcup_{n=0}^{\infty} K_n = X$. It is easy to verify that the K_n's are mutualy disjoint. It then follows that C^* is unitarily equivalent to an operator-valued weighted shift on $\sum^\oplus L^2(K_n)$.

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