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A SPECTRAL THEORY OF THE KLEIN-GORDON EQUATION INVOLVING A HOMOGENOUS ELECTRIC FIELD

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1. INTRODUCTION

The Klein-Gordon equation describing the motion of a relativistic spinless charged particle in a static electric potential q reads formally

$$\left(\mathrm{i}\bar{h}\frac{\partial}{\partial t} - q\right)^2 \psi = \left(c^2p^2 + m^2c^4\right)\psi. \tag{1.1}$$

Here $q = q(x)$, $\psi = \psi(x,t)$ where t is the time, $x \in \mathbb{R}^n$ the space variable, and $p = \mathrm{i}\bar{h}\nabla$. The constants $\bar{h}, m, c > 0$ denote the Planck constant, the mass of the particle, and light velocity, respectively. The substitution

$$\psi_1 = \psi, \quad \psi_2 = \left(\mathrm{i}\bar{h}\frac{\partial}{\partial t} - q\right)\psi$$

transforms (1.1) to the Hamiltonian form:

$$\mathrm{i}\bar{h}\dot{\Psi} = A_0\Psi, \quad A_0 = \begin{pmatrix} q & 1 \\ \varepsilon(p)^2 & q \end{pmatrix}, \tag{1.2}$$

where

$$\varepsilon(p)^2 = c^2p^2 + m^2c^4, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

The operator A_0 is not symmetric in L_2 (not even for $q \equiv 0$). Consider the scale of Hilbert spaces $\mathcal{H}_\alpha$, defined by the scalar products

$$\langle \Psi, \Phi\rangle_\alpha = \left(\varepsilon^{\frac{1}{2}-\alpha}\psi_1, \varepsilon^{\frac{1}{2}-\alpha}\varphi_1\right) + \left(\varepsilon^{-\frac{1}{2}-\alpha}\psi_2, \varepsilon^{-\frac{1}{2}-\alpha}\varphi_2\right), \tag{1.3}$$

where $\alpha \in \mathbb{R}$ and $(\cdot,\cdot)$ denotes the L_2-scalar product. The most interesting scalar products are given by

$$\begin{cases} \alpha = -\frac{1}{2} & (\text{“energy” norm}) \\ \alpha = \ 0 & (\text{“number” norm}) \\ \alpha = \ \frac{1}{2} & (\text{“negative energy” norm}) \end{cases}$$

The number norm is of particular interest because of its connection with the number operator in the second quantized theory. Najman [5] showed that an appropriate extension of A_0 generates a uniformly bounded group with respect to any of the scalar products in (1.3) if, for example

(1.4) $$\|q\psi\| \leqslant b\,\|\varepsilon\psi\|, \quad 0 < b < 1^{*)}$$

(See [1], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] for earlier results also in the second quantized case.) The condition (1.4) in fact insures the positive definiteness of the energy form $c^2p^2 + m^2c^4 - q^2$, a key technical tool of the theory. Independently of q the operator A_0 is formally symmetric with respect to the indefinite form

$$[\Psi, \Phi] = (\psi_1, \varphi_2) + (\psi_2, \varphi_1).$$

Now, just in the number-norm space $\mathcal{H}_0(\psi, \Psi)$ gives rise to a Krein space structure since

$$[\Psi, \Phi] = \langle J\Psi, \Phi\rangle_0,$$

where J is a selfadjoint operator in H_0 with $J^2 = I$ (see e. g. [11]).

If the condition (1.4) is violated then the evolution group may fail to be uniformly bounded in *all spaces* $\mathcal{H}_\alpha$. Indeed, then a q from the Schwarz space $\mathcal{S}$ of smooth functions can be found such that the operator A_0 has an eigenvector in $\mathcal{S}^2$ with a non-real eigenvalue. A construction of such q follows a well-known calculation in [14], made for the square well potential.

We are interested in the question: what properties of the potential q are responsible for the breaking down of the uniform boundedness in time. Simplest ε- unbounded potentials q are those for which $q(x)$ is unbounded at infinity.

In this note we consider the potential

(1.5) $$q(x) = \eta x_1$$

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*) In fact, the substitution $\psi \to e^{iat/h}\psi$ in (1.1) shows that (1.4) can be replaced by $\|(q-a)\psi\| \leqslant b\,\|\varepsilon\psi\|$ for a real a.

where, for simplicity, $\eta > 0$. This potential cannot be considered as ε- small in any sense.

In Section 2 we present the spectral theory of the Klein-Gordon operator in some detail since it defines the unperturbed dynamics in a Stark effect theory and gives a spectral theoretic interpretation e. g. of the "eigenvalue approach" of the resonance theory as developed by Graffi et.al. [2].

Our main result is that in a homogenous electric field (1.5) the Klein-Gordon operator generates a uniformly bounded group with respect to the *number norm* ($\alpha = 0$) and *none else* from (1.3). Since the energy form $c^2p^2 + m^2p^4 - \eta^2 x_1^2$ is deeply indefinite, we were not able to use common operator theoretical tools, based on any form of selfadjointness. Instead, we apply oscillatory integrals as they are used in the asymptotics of functions of parabolic cylinder ([13]). In fact, our key technical result can be interpreted as a series of uniform asymptotic estimates for such functions. The mentioned Krein-space selfadjointness played only a minor role - it was used to rule out all spaces $\mathcal{H}_\alpha$ for $\alpha \neq 0$. (Lemma 2.3).*)

Our result seems to suggest that the uniform boundedness in time will hold (at least in $\mathcal{H}_0$) if the potential has *no strong local oscillations.* To prove or to disprove this conjecture more powerful spectral theoretical techniques seem to be needed.

If we interpret the norm $\|\Psi\|_0$ as the number of particles in a state Ψ then our result means that the *total number of created pairs is bounded through the whole time history.* In fact, according to the machinery of our proof, the increase of the number norm is caused by the acceleration of the classical relativistic particle (see (2.32) below). The existence of a limiting velocity (velocity of light) now implies that the acceleration goes to zero for $|t| \to 0$. This gives a certain physical plausibility to our result.

On the other hand, in Section 3 we show that the obtained evolution group is not implementable in the second quantization theory, so that it is not clear at present whether or in which form this result will enter a definite rigorous quantum field theory of the homogeneous electric field.

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*) In fact, the operator shows to have a "highly mixed spectrum" i. e. the form $[\cdot,\cdot]$ is indefinite on any spectral subspace. The general spectral theory of such operators in Krein spaces is rather poor and it would be interesting if the Klein-Gordon equation would offer the possibility to define abstract classes of such operators enjoying a stable spectral theory.

2. THE SPECTRAL DECOMPOSITION

Instead of working with one operator A_0 in the family of scalar products (1.3) we can equivalently consider the family of operators

$$B_{0\alpha} = Z_\alpha^{-1} A_0 Z_\alpha, \tag{2.1}$$

$$Z_\alpha = Z_\alpha(p) = \begin{pmatrix} \varepsilon^{-\frac{1}{2}+\alpha} & 0 \\ 0 & \varepsilon^{\frac{1}{2}+\alpha} \end{pmatrix} \tag{2.2}$$

defined on the Schwartz subspace $\mathcal{S}$ of $\mathcal{L}_2$

$$B_{0\alpha} = \begin{pmatrix} \varepsilon^{\frac{1}{2}-\alpha} q \epsilon^{-\frac{1}{2}+\alpha} & \varepsilon \\ \varepsilon & \varepsilon^{-\frac{1}{2}-\alpha} q \epsilon^{\frac{1}{2}+\alpha} \end{pmatrix} \tag{2.3}$$

Using (1.5) we have, at least on $\mathcal{S}$,

$$\varepsilon^{\frac{1}{2}-\alpha} \eta x_1 \varepsilon^{-\frac{1}{2}+\alpha} = \eta x_1 + \mathrm{i}\left(-\tfrac{1}{2}+\alpha\right) \eta \bar{h} \frac{c^2 p_1}{\epsilon^2},$$

$$\varepsilon^{-\frac{1}{2}-\alpha} \eta x_1 \varepsilon^{\frac{1}{2}+\alpha} = \eta x_1 + \mathrm{i}\left(\tfrac{1}{2}+\alpha\right) \eta \bar{h} \frac{c^2 p_1}{\epsilon^2}.$$

Thus,

$$B_{0\alpha} = H_0 + K$$

$$H_0 = \begin{pmatrix} \eta x_1 & \varepsilon \\ \epsilon & \eta x_1 \end{pmatrix}, \quad K = \mathrm{i}\bar{h}\eta \frac{p_1 c^2}{\epsilon^2} \begin{pmatrix} -\frac{1}{2}+\alpha & 0 \\ 0 & \frac{1}{2}+\alpha \end{pmatrix}.$$

Here obviously H_0 is symmetric and K is bounded.

2.1. LEMMA. *The operator H_0 is essentially selfadjoint on C_0^∞ in the momentum space.**) *The operator*

$$B_\alpha = H + K \tag{2.4}$$

generates a strongly continuous group of bounded operators. Here B_α, H are the closures of $B_{0\alpha}$, H_0, respectively.

Proof. Here x_1 is represented by $\mathrm{i}\bar{h}\,\partial/\partial p_1$. Set

$$\xi = \xi(p) = \int_0^{p_1} \varepsilon(\hat{p}_1, p')\,\mathrm{d}\hat{p}_1, \tag{2.5}$$

*) From now on all our considerations are made in the momentum space.

where we use the notation

$$(p_1, p') = p, \quad p' \in \mathbb{R}^{n-1}.$$

Then

$$H_0 = U_0 \eta x_1 U_0^{-1}, \quad U_0(p) = \exp\left(\mathrm{i}\frac{1}{\hbar\eta}\xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \tilde{U}_0(\xi). \tag{2.6}$$

Now x_1 is essentialy selfadjoint on C_0^∞ since this subspace is invariant under the p_1-translation group

$$\mathrm{e}^{\mathrm{i}\eta x_1 t/\hbar} = \mathrm{e}^{-t\eta \frac{\partial}{\partial p_1}}$$

(cf. Reed and Simon [8], Theorem X. 49). Since C_0^∞ is also invariant for U_0 the essential selfadjointness follows. The rest of the proof is trivial. Q.E.D.

The proof of the Lemma 2.1 contains already the idea of the spectral decomposition of the operator B_α. We shall show that

$$B_\alpha = V_\alpha \eta x_1 V_\alpha^{-1} \tag{2.7}$$

where $x_1 = \mathrm{i}\hbar\, \partial/\partial p_1$ and V_α is the multiplication operator defined by a matrix valued function $V_\alpha(p)$. Because of the non-commutativity of the matrix functions entering (2.4), V_α will not be a simple matrix exponential but a solution of an appropriate differential equation. Indeed, let $V_\alpha(p)$ be the solution of the differential equation

$$\mathrm{i}\frac{\partial V_\alpha(p)}{\partial p_1} = \left[-\frac{1}{\eta\hbar}\begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} + \frac{\mathrm{i}}{2}\frac{c^2 p_1}{\varepsilon^2}\begin{pmatrix} 1-2\alpha & 0 \\ 0 & -1-2\alpha \end{pmatrix}\right] V_\alpha(p) \tag{2.8}$$

with the initial condition

$$V_\alpha(0, p') = I. \tag{2.9}$$

Since (2.8) has real analytic coefficients, the solution is real analytic on $\mathbb{R}^n$. By (2.8) we have, at least on C_0^∞

$$V_\alpha^{-1} B_{0\alpha} V_\alpha = V_\alpha(p)^{-1}\left(\mathrm{i}\hbar\eta\,\frac{\partial}{\partial p_1}\right) V_\alpha(p) +$$

$$+ V_\alpha(p)^{-1}\left\{\begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} + \mathrm{i}\hbar\eta\,\frac{p_1 c^2}{2\varepsilon^2}\begin{pmatrix} -1+2\alpha & 0 \\ 0 & 1+2\alpha \end{pmatrix}\right\} V_\alpha(p) =$$

$$= \mathrm{i}\hbar\eta\,\frac{\partial}{\partial p_1}.$$

Thus, with V_α defined by (2.8), (2.9) the relation (2.7) is valid at least on C_0^∞. Clearly, B_α will generate a uniformly bounded group if both $V_\alpha(p)$ and $V_\alpha(p)^{-1}$ are bounded on $\mathbb{R}^n$. More precisely, we have

2.2. LEMMA: *The group* $\exp(-itB_\alpha)$ *is uniformly bounded if and only if the suprema*

$$\sup_{p\in\mathbb{R}^n} \|V_\alpha(p)\|, \qquad \sup_{p\in\mathbb{R}^n} \left\|V_\alpha(p)^{-1}\right\|$$

are finite. Here $\|\ \|$ *denotes the operator norm of a matrix.*

Proof. We prove first the identity

$$\exp(-\mathrm{i}B - \alpha t/\bar{h}) = V_\alpha \exp(-\mathrm{i}x_1 t\eta/\bar{h}) V_\alpha^{-1}. \tag{2.10}$$

Indeed, the right hand side of (2.10) is defined at least on C_0^∞. For $\Psi \in C_0^\infty$ we have immediately

$$\left(V_\alpha \exp(-\mathrm{i}x_1 t\eta/\bar{h}) V_\alpha^{-1} \Psi\right)(p) = V_\alpha(p) V_\alpha(p + e_1\eta t)^{-1} \Psi(p + e_1\eta t), \tag{2.11}$$

with $e_1^T = (1, 0, \cdots, 0)$. Thus, the right hand side of (2.10) maps C_0^∞ into C_0^∞. Differentiating (2.11) with respect to t we obtain

$$\begin{aligned}
&\mathrm{i}\frac{\partial}{\partial t}\left(V_\alpha \exp(-\mathrm{i}x_1 t\eta/\bar{h}) V_\alpha^{-1} \Psi\right)(p) = \\
&= V_\alpha(p)\left(\mathrm{i}\eta\frac{\partial}{\partial p_1}\right) V_\alpha(p)^{-1} V_\alpha(p) V_\alpha(p + e_1\eta t)^{-1} \Psi(p + e_1\eta t) = \\
&= \left(B_\alpha V_\alpha \exp(-\mathrm{i}x_1 t\eta/\bar{h}) V_\alpha^{-1} \Psi\right)(p).
\end{aligned}$$

Since the functions V_α, V_α^{-1}, Ψ are smooth and the support of Ψ is compact, the above differentiation is valid in the Hilbert space L_2. By the uniqueness of the solution of the differential equation

$$\mathrm{i}\dot{\chi} = B_\alpha \chi$$

in L_2 we conclude that (2.10) holds. We have

$$\left\|\exp\left(-\mathrm{i}tB_\alpha/\bar{h}\right)\psi\right\|^2 = \int \left\|V_\alpha\left(p - e_1\eta t\right) V_\alpha(p)^{-1} \Psi(p)\right\|^2 \mathrm{d}p. \tag{2.12}$$

The uniform boundedness of the group $\exp\left(-itB_\alpha/\bar{h}\right)$ means

$$M = \sup_{\substack{p\in\mathbb{R}^n \\ t\in\mathbb{R}}} \left\|V_\alpha\left(p - e_1\eta t\right) V_\alpha(p)^{-1}\right\| < \infty. \tag{2.13}$$

Taking here $p_1 = 0$ we obtain

$$\sup_{p\in\mathbb{R}^n} \|V_\alpha(p)\| \leqslant M \tag{2.14}$$

whereas $\eta t = p_1$ gives

$$\sup_{p\in\mathbb{R}^n} \left\|V_\alpha(p)^{-1}\right\| \leqslant M. \tag{2.15}$$

Conversely, let (2.14) and (2.15) hold for some $M > 0$. Then, by (2.11),

$$\left\|\exp\left(-itB_\alpha/\overline{h}\right)\right\| \leqslant M^2.$$

This proves the lemma.

2.3. LEMMA. *The group* $\exp\left(-\mathrm{i}tB_\alpha/\overline{h}\right)$ *is uniformly bounded for no* $\alpha \neq 0$.

Proof. Note first the following two identities

$$V_\alpha(p) = [\varepsilon(0,p')/\varepsilon(p)]^\alpha V_0(p) \tag{2.16}$$

and

$$V_0(p)^{-1} = JV_0(p)^* J \tag{2.17}$$

with

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.18}$$

The identity (2.16) is verified by a direct substitution into (2.8) whereas the identity (2.17), which is called the J-unitarity of V_0, follows again from (2.8) and the J-hermiticity*) of the coefficient matrix of the equation (2.8)

$$-\frac{1}{\eta\overline{h}}\begin{pmatrix} 0 & \varepsilon \\ \epsilon & 0 \end{pmatrix} + \frac{\mathrm{i}}{2}\frac{c^2 p_1}{\varepsilon^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.19}$$

for the case $\alpha = 0$. From (2.16) and (2.17) it follows immediately

$$V_\alpha(p_1,0)^{-1} = \epsilon(p_1,0)^{2\alpha} JV_\alpha(p_1,0)^* J/\left(mc^2\right)^{2\alpha} \tag{2.20}$$

and

$$\left\|V_\alpha\left(p_1,0\right)^{-1}\right\| = \varepsilon\left(p_1,0\right)^{2\alpha}\|V_\alpha\left(p_1,0\right)\| / \left(mc^2\right)^{2\alpha}$$

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*) A matrix A is J-hermitean if $A^* = JAJ$

for any p_1. The simultaneous boundedness of V_α and V_α^{-1} would imply e. g. for $\alpha > 0$

$$V_\alpha(p_1, 0) \to 0, \quad |p_1| \to \infty$$

and then also by (2.16)

$$V_0(p_1, 0) \to 0 \quad |p_1| \to \infty.$$

This is impossible by the estimate

$$\|V_0(p)\| \geqslant 1$$

due to the J-unitarity of V_0. The reasoning for $\alpha < 0$ is analogous and the lemma is proved.

Thus, the only case which remains to be studied is that of the number norm, characterized by $\alpha = 0$.

We make the following substitution

$$V_0(p) = S(p)W(z) \tag{2.21}$$

where

$$S(p) = \begin{pmatrix} 1 & 0 \\ i\eta\bar{h}\dfrac{p_1c^2}{2\varepsilon(p)^3} & 1 \end{pmatrix}, \tag{2.22}$$

and $z = z(p) = (\xi(p), p')$, and ξ is given by (2.5). Inserting (2.21) into (2.8) we obtain the following differential equation for W (the calculation is a little tedious, but straightforward)

$$i\frac{\partial W}{\partial z_1} + \frac{1}{\eta\bar{h}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W = \eta\bar{h}h(z)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} W, \tag{2.23}$$

$$W(0, z_2, \cdots, z_n) \equiv I \tag{2.24}$$

with

$$h(z) = \frac{c^2}{2\varepsilon^4} - \frac{5p_1^2c^4}{4\varepsilon^6} = \frac{c^2\varepsilon(0,p')^2}{2\varepsilon^6} - \frac{3p_1^2c^4}{4\varepsilon^6}$$

and

$$\int_{-\infty}^{\infty} |h(z)|\, dz_1 \leqslant \frac{2c}{3\varepsilon(0,p')^2} + \frac{c}{\varepsilon(0,p')^2} = \frac{5c}{3\varepsilon(0,p')^2} \leqslant \frac{5}{3m^2c^3}\,. \tag{2.25}$$

We now use the standard conversion of (2.23), (2.24) into an integral equation. We put

$$W = U_0 W_1,$$

where $U_0 = U_0(z)$ is given by (2.6) and obtain

$$W_1(z) = I + \frac{\eta\bar{h}}{i} \int_0^{z_1} h(\xi_1, z') \tilde{U}_0(-\xi_1) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tilde{U}_0 W_1(\xi_1, z') \, d\xi_1 \tag{2.26}$$

The iterates of this integral equation converge uniformly for z_1 from any compact interval and for all $z' \in \mathbb{R}^{n-1}$ and its solution W_1 satisfies the inequality

$$\|W_1(z)\| \leqslant \exp \eta\bar{h}\tilde{h}(z), \quad \tilde{h}(z) = \int_0^{z_1} |h(\xi_1, z')| \, d\xi_1. \tag{2.27}$$

Thus, by (2.25)

$$\begin{aligned} \|W(z) - U(z)\| &\leqslant e^{\eta\bar{h}\tilde{h}} - 1 \leqslant e^{5\eta\bar{h}c/3\varepsilon(0,p')^2} - 1 \leqslant \\ &\leqslant e^{5\eta\bar{h}/3m^2c^3} - 1 \end{aligned} \tag{2.28}$$

and, by (2.21), (2.28)

$$\begin{aligned} \|V_0(p) - U_0(p)\| &\leqslant \left(1 + \frac{\eta\bar{h}p_1c^2}{2\varepsilon^3}\right) e^{\eta\bar{h}\tilde{h}} - 1 \leqslant \\ &\leqslant \left(1 + \frac{\eta\bar{h}p_1c^2}{2\varepsilon(p)^3}\right) e^{5\eta\bar{h}c/3\varepsilon(0,p')^2} - 1 \leqslant \\ &\leqslant e^{5\eta\bar{h}c/3\varepsilon(0,p')^2} - 1 + \frac{\eta\bar{h}p_1c^2}{2\varepsilon(p)^3} e^{5\eta\bar{h}/3m^2c^3} \leqslant \\ &\leqslant \left(\frac{5\eta\bar{h}c}{3\varepsilon(0,p')^2} + \frac{\eta\bar{h}p_1c^2}{2\varepsilon(p)^3}\right) e^{5\eta\bar{h}/3m^2c^3} \leqslant \\ &\leqslant K \frac{\eta\bar{h}c}{\varepsilon(0,p')^2} e^{5\eta\bar{h}/3m^2c^3} \leqslant K \frac{\eta\bar{h}}{m^2c^3} e^{5\eta\bar{h}/3m^2c^3}, \end{aligned} \tag{2.29}$$

with

$$K = \frac{5}{3} + \frac{1}{24\sqrt{3}}. \tag{2.30}$$

We have therefore immediately the

2.4. THEOREM. *The operator B_0 from (2.4) generates a uniformly bounded group. More precisely, a new scalar product, topologically equivalent to the number* *norm can be introduced in which B_0 is selfadjoint with an absolutely continuous spectrum covering the whole real axis. An upper bound for* $\exp itB_0/\bar{h}$ *in the number operator norm is given by*

$$\|\exp itB_0/\bar{h}\| \leqslant \left(1 + K\frac{\eta\bar{h}}{m^2c^3}e^{5\eta\bar{h}/3m^2c^3}\right)^2. \tag{2.31}$$

The key quantity in the estimate above is

$$\alpha = \frac{\eta\bar{h}}{m^2c^4} = \frac{\eta}{m}/\frac{c^2}{\lambda} \tag{2.32}$$

where $\lambda = \frac{\bar{h}}{mc}$ is the Compton wave length. Obviously α is dimensionless and measures the classical acceleration of the field in the natural unit c^2/λ.

So, if the field is weak or the mass is large, the growth of the number norm in time will be small. A more careful inspection of the chain of inequalities (2.29) reveals that the states with a large transversal momentum p' behave as if they had the larger, "transversal mass" m' defined by

$$m' = \left(m^2 + \frac{p'^2}{c^2}\right)^{1/2}. \tag{2.33}$$

For a state ψ having its transversal momentum support outside the ball of the radius $|p'|$ the estimates (2.29) yield

$$\left\|\exp\frac{itB_0}{\bar{h}}\psi\right\| \leqslant K\frac{\eta\bar{h}}{m'^2c^3}e^{5\eta\bar{h}/3m'^2c^3} =$$

$$= K\frac{\eta\bar{h}c}{\varepsilon(0,p')^2}e^{5\eta\bar{h}c/3\varepsilon(0,p')^2}.$$

Thus, for particles with a large transversal momentum the increase of the number norm is small.

REMARK. The result contained in Theorem 2.4 can be also derived by computing the eigenfunctions of the equation (1.1). This leads to the investigation of the asymptotic behaviour of the functions of the parabolic cylinder. However, in order to insure the uniformity of the asymptotic estimates with respect to several parameters involved a transition to an integral equation equivalent to (2.26) is needed there, too.

3. PROPERTIES OF THE EVOLUTION GROUP

Define the orthogonal projections

$$P_{\pm} = \frac{1}{2}(I \pm J) \tag{3.1}$$

where J is given by (2.18). Then

$$P_+ + P_- = I, \quad P_+ - P_- = J.$$

As it is well known (see e. g. [9]) the unitary implementability of the group $\exp\left(-itB_0/\overline{h}\right)$ in the second quantized theory is equivalent to the Hilbert-Schmidt property of both of the operators

$$P_+ \exp\left(-itB_0/\overline{h}\right) P_-, \quad P_- \exp\left(-itB_0/\overline{h}\right) P_+ \tag{3.2}$$

or of

$$\exp\left(-ix_1 t\eta/\overline{h}\right) P_{\pm} \exp\left(-itB_0/\overline{h}\right) P_{\mp}.$$

The latter are multiplication operators by

$$P_{\pm} V_0\left(p - e_1 \eta t\right) V_0(p)^{-1} P_{\mp}$$

and can be obviously Hilbert-Schmidt only if they vanish i.e. if

$$V_0\left(p - e_1 \eta t\right) V_0(p)^{-1}$$

commutes with J for any $p \in \mathbb{R}^n$. Taking $p_1 = 0$ it follows that

$$V_0\left(-\eta t, p'\right) J - J V_0\left(-\eta t, p'\right) = 0 \tag{3.3}$$

for all $p' \in \mathbb{R}^{n-1}$. Now V_0 being a real analytic function of $p \in \mathbb{R}^n$, (3.3) can hold either for a discrete set of t's or for all real t's. The latter posibility contradicts the non-hermiticity of the coefficient matrix (2.19) occuring in (2.8) for $\alpha = 0$. Thus, the operators (3.3) are Hilbert-Schmidt for at *most a discrete set of* t's.

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