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MINIMAL PROJECTIONS IN THE REDUCED GROUP C^*-ALGEBRA OF $\mathbf{Z}_n * \mathbf{Z}_m$

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INTRODUCTION

There are several important unsolved problems concerning equivalence and comparability of projections in simple C^*-algebras. Some of the most important of these can be combined into a single question, which may be called the Fundamental Comparability Question for projections in simple C^*-algebras:

(1) If A is a simple C^*-algebra, p and q are projections in A with

$$\tau(p) < \tau(q)$$

for every nonzero trace τ on A, is p equivalent to a subprojection of q? (Note that $<$ cannot be replaced by $\leqslant$, even in AF algebras [1, 7.6.2]).

If A is a stably finite simple unital C^*-algebra, then $\mathrm{K}_0(A)$ is a simple ordered group with its natural ordering. A tracial state τ on A induces a state (i.e., a normalized order-preserving homomorphism) $\tau_* : \mathrm{K}_0(A) \to \mathbb{R}$. Two natural questions arise in this setting:

(2) If x is in $\mathrm{K}_0(A)$ and $\tau_*(x) > 0$ for every trace τ, is $x > 0$? (Is $\mathrm{K}_0(A)$ weakly unperforated and does every state on $\mathrm{K}_0(A)$ come from a trace on A?)

(3) If x is in $\mathrm{K}_0(A)$ and $0 \leqslant x \leqslant [1_A]$, is there a projection p in A with $x = [p]$?

Question (3) is a weak form of cancellation (if $[p] = [q]$ in $\mathrm{K}_0(A)$ then $p \sim q$). The questions (2) and (3) are consequences of (1). See [1] for a complete discussion of these and related questions, and of ordered K_0-groups.

The present state of knowledge about these questions is rudimentary. It is straightforward to verify (1) for simple AF algebras and factors using known structure results (see [1] for more details). In addition Rieffel [8] has recently shown that (1) holds for simple (and even nonsimple, nonrational) noncommutative tori. Question (1) is also true for the Cuntz algebras [5]; if A is not stably finite then (1) simply says that every nonzero projection in A is infinite. Counterexamples to the analogous questions for non-simple C^*-algebras have led some experts to doubt the validity of (1); but evidence in favor of (1) has been gradually accumulating, to the point of being at least suggestive.

The results of this paper add to the empirical evidence for (1). We study projections in the C^*-algebra $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$, the reduced group C^*-algebra of the free product of two finite cyclic groups. Recall that $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ is a simple unital C^*-algebra with unique trace if $n, m \geqslant 2$ and $n + m \geqslant 5$ [6]. We write u and v for the unitary generators of $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$, so that $u^n = v^m = 1$, and denote the minimal spectral projections of u (resp. v) by $p_1, \ldots, p_n$ (resp. $q_1, \ldots, q_m$).

Cuntz [4] showed that $\mathrm{K}_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))$ is generated by $[p_1], \ldots [p_n]$, $[q_1], \ldots, [q_m]$ subject to the single (obvious) relation

$$[p_1] + \ldots + [p_n] = [q_1] + \ldots + [q_m].$$

Thus

$$\mathrm{K}_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)) \approx \mathbf{Z}^{n+m-1} \approx \mathbf{Z}^{n+m}/\mathbf{Z}(1, \ldots, 1, -1, \ldots, -1)$$

where there are n 1's and m $-$ 1's in the denominator. Several interesting properties of $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ are implicit in this result. For example it follows that the p_i's and q_j's are pairwise inequivalent. Since $\tau(p_i) = 1/n$ and $\tau(q_j) = 1/m$ we also get that the range of τ_* on $\mathrm{K}_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))$ is $(1/\mu)\mathbf{Z}$, where $\mu = \ell\mathrm{cm}(n, m)$ is the least common multiple of n and m. Hence if r is a nonzero projection in $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$, then $\tau(r)$ is a multiple of $1/\mu$, and therefore $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ contains minimal projections.

If $n = m$, then each p_i and q_j is a minimal projection, and Cuntz [4] conjectured that the p_i and q_j are also minimal when $n \neq m$. This conjecture was inconsistent with (1), which says in the case of $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ that a projection is minimal if and only if its trace is $1/\mu$ (and that every element of $\mathrm{K}_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))$ of trace $1/\mu$ is the image of a projection in $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$). While we cannot completely verify (1) (or even (2) or (3)) for these algebras, we do show that many of the elements x in $\mathrm{K}_0(C_r^*(\mathbf{Z}_n * \mathbf{Z}_m))$ with $0 < \tau_*(x) < 1$ are represented by the projections in $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ and are therefore positive. We also verify some of the comparability relations implied by (1). In particular, we show that there always exist projections of trace $1/\mu$ under any p_i and q_j; so these projections are *not* minimal if $n \neq m$. We believe that (1) is true (at least) for these algebras, and a modification of our techniques may be sufficient to prove it. (Note that by the results of [12], [13], and [14], τ_* is the only state on $K_0(C_r^*(\mathbb{Z}_n * \mathbb{Z}_m))$, so that if $x \in K_0(C_r^*(\mathbb{Z}_n * \mathbb{Z}_m))$ with $\tau_*(x) > 0$, then some multiple of x is positive, i.e. represented by a projection [1, 6.8.5].)

Our results are obtained by explicit norm and spectrum calculations for certain operators which we believe may be of independent interest (cf. Section 2). These calculations are inspired by the work of Cartwright and Soardi [2] in which the spectra of related operators in $C_r^*(\mathbb{Z}_n * \mathbb{Z}_m)$ are computed by a different method. Also, although not explicitly stated, our work in Section 2 uses Voiculescu's notion of the reduced free product of C^*-algebras introduced in [10]. Finally we note that related results have been obtained (via different techniques) by Voiculescu in [11].

1. MINIMAL PROJECTIONS IN CHOI'S ALGEBRA

In the case $n = 2, m = 3$ the algebra $C_r^*(\mathbb{Z}_n * \mathbb{Z}_m)$ was studied in detail by Choi [3]. He showed that one may assume the unitary generators have the form

$$u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 & t^* \\ s & ts^* \end{pmatrix}$$

where s and t are isometries satisfying $ss^* + tt^* = 1$ (i.e., s and t generate the Cuntz algebra $\mathcal{O}_2$). We write

$$C = C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3).$$

Note that since lcm(2,3)=6,

$$\tau_*(K_0(C)) = \frac{1}{6}\mathbb{Z}$$

and if e is a nontrivial projection ($e \neq 0,\ e \neq 1$) in C then

$$\tau(e) \in \{1/6, 1/3, 1/2, 2/3, 5/6\}.$$

The spectral projections of u (resp. v) are p_1 and p_2 (resp. q_1, q_2 and q_3) and we have that

$$u = p_1 - p_2, \qquad v = q_1 + \omega q_2 + \omega^2 q_3,$$

where ω is a primitive third root of unity. Thus $\tau(p_i) = 1/2$ and $\tau(q_j) = 1/3$, since $\tau(u) = \tau(v) = 0$.

In the next section we shall analyze (among others) elements of the form $p_i q_j p_i$ in detail. In particular we will compute the spectrum of such elements exactly and show that 0 is always an isolated point. From this it follows easily that C contains projections of trace 1/6. This calculation is somewhat involved, however. Our purpose here is to show that for Choi's algebra one may exploit the fact that 2 and 3 are "small" to get the existence of projections of trace 1/6 quite easily. We begin by presenting two easy (known) results. We use $\tilde{A}$ to denote the C^*-algebra obtained by adjoining a unit to the C^*-algebra A.

PROPOSITION 1. *If e and f are projections in C^*-algebra A with*

$$\|efe - e\| < 1,$$

then e is homotopic to a subprojection d of f. Hence e and d are unitarily equivalent in $\tilde{A}$. Moreover, e is orthogonal to $f - d$.

Proof. For $0 \leqslant t \leqslant 1$ write

$$e_t = te + (1-t)f$$

and note that

$$\|ee_te - e\| \leqslant \|efe - e\| = \alpha < 1.$$

Put

$$a_t = \sqrt{(ee_te)^{-1}}$$

where the inverse is taken in eAe. We have then that $\|a_t\| \leqslant (1-\alpha)^{-(1/2)}$ and therefore the map

$$t \mapsto a_t$$

is continuous in norm. Now set $b_t = \sqrt{e_t}$ and put

$$w_t = b_t a_t.$$

Note that $t \mapsto w_t$ is continuous, $w_t^* w_t = e$ for all t, $w_0 w_0^* = d \leqslant f$ and $w_1 = w_1 w_1^* = e$. Hence $w_t w_t^*$ gives a path of projections from d to e and the first part of the proposition is proved.

Since norm close projections are unitarily equivalent, the second part follows from the first in conjunction with a routine compactness argument. For the final assertion note that

$$ede = ew_0 w_0^* e = ef(efe)^{-1} fe = efe$$

so that $e(f-d)e = 0$. ■

PROPOSITION 2. *Suppose A is a C^*-algebra containing projections p and q such that A is generated (as a C^*-algebra) by p, q and families $\{x_\alpha\}$, $\{y_\beta\}$ of positive elements. If in addition*

i) $\|pq\| = 1$,

ii) $px_\alpha = 0$ *for all* α *and* $qy_\beta = 0$ *for all* β, *then there is a complex homomorphism φ on A with $\varphi(p) = \varphi(q) = 1$.*

Proof. We have that $\|pqp\| = \|pq\|^2 = 1$ and therefore there is a state φ on A with $\varphi(pqp) = 1$. As $pqp \leqslant p \leqslant 1$ we get that $\varphi(p) = 1$. By the Cauchy-Schwartz inequality $\varphi(pxp) = \varphi(x)$ for all x in A. In particular $\varphi(q) = 1$ and $\varphi(qyq) = \varphi(y)$ for all y in A. Hence $\varphi(x_\alpha) = \varphi(px_\alpha) = 0$ for all α, $\varphi(y_\beta) = 0$ for all β and φ is multiplicative on A. ∎

THEOREM 3. *Suppose A is a simple, unital C^*-algebra containing the nontrivial projections p and q. If A is generated by p,q and some other positive elements each of which is orthogonal to either p or q then q is homotopic (hence equivalent) to a subprojection of $1-p$.*

Proof. By Proposition 1 it suffices to show that $\|q(1-p)q - q\| < 1$. Suppose this is not the case so that

$$\|pq\|^2 = \|qpq\| = \|q(1-p)q - q\| = 1.$$

Then by Proposition 2 there is a complex homomorphism on A. As A is simple we must have that $A = \mathbb{C}$, the complex numbers. Since p and q are nontrivial projections this is impossible. ∎

COROLLARY 4. *C contains projections of trace 1/6.*

Proof. Since C is simple and generated by $\{p_1, p_2, q_1, q_2, q_3\}$ the hypotheses of Theorem 3 are satisfied with $p = p_2$ and $q = q_1$. Hence there is a subprojection s of $p_1 = 1 - p_2$ that is homotopic (hence equivalent) to q_1. Therefore $\tau(s) = \tau(q_1) = 1/3$. So if we put $r = p_1 - s$, then $\tau(r) = 1/2 - 1/3 = 1/6$. ∎

REMARKS. 1) Clearly, the argument in Corollary 4 works for any pair (p_i, q_j). It follows that the elements in $\mathrm{K}_0(C)$ corresponding to

$$(1,0,-1,0,0),\ (1,0,0,-1,0),\ (1,0,0,0,-1)$$

and

$$(0,1,-1,0,0),\ (0,1,0,-1,0),\ (0,1,0,0,-1)$$

are all represented by projections in C.

2) Suppose r, s, p_1 and q_1 are as in the proof of Corollary 4 so that $r + s = p_1$, s is equivalent to q_1 and (by Proposition 1) r is orthogonal to q_1. We have that

$$\|rq_3r\| = \|rp_1q_3p_1r\| \leqslant \|p_1q_3p_1\| < 1.$$

As r is orthogonal to q_1 we get that r is equivalent to a subprojection of q_2 in the C^*-algebra generated by r, q_2 and q_3 (with unit q_2+q_3). Thus q_2 majorizes a projection r_2 with $r_2 \sim r$. Hence in $K_0(C)$ we have

$$[r_2] = [r] = [p_1] - [q_1].$$

Note that $\tau(q_2 - r_2) = 1/3 - 1/6 = 1/6$ so that $q_2 - r_2$ is also minimal in $C^*_r(\mathbf{Z}_n * \mathbf{Z}_m)$. Moreover,

$$[q_2 - r_2] = [q_2] + [q_1] - [p_1] = [p_2] - [q_3].$$

Arguing in like manner with q_1 and q_3 interchanged we get that q_2 also majorizes a minimal projection s_2 such that

$$[s_2] = [p_1] - [q_3] \quad \text{and} \quad [q_2 - s_2] = [p_2] - [q_1].$$

3) By applying the argument of 2) to the remaining q_i's one obtains 2 orthogonal decompositions of each q_i into minimal projections $r_i,\ q_i - r_i$ and $s_i,\ q_i - s_i$ such that for each i these 4 projections determine distinct elements of $K_0(C)$. Thus the identity in C decomposes as the sum of 6 orthogonal minimal projections in several different ways. Note that

$$[r_1 + r_2 + r_3] = [p_1] - [q_3] + [p_1] - [q_1] + [p_1] - [q_2] = (3, 0, -1, -1, -1)$$

and so $(2, -1, 0, 0, 0) = (3, 0, -1, -1, -1)$ is positive; in fact it is represented by a projection in C.

4) Write $e_1,\ e_2$ and $f_1,\ f_2$ for the projections giving the two orthogonal decompositions of q_1 in 3) above. We claim that each compression of the form $e_i f_j e_i$ or $f_j e_i f_j$ has full spectrum in $C^*(e_1, f_1, 1)$. Indeed, we must have that $\|e_1 f_1 e_1\| = 1$. For otherwise e_1 would be equivalent to a subprojection of f_2 and since f_2 is minimal e_1 and f_2 would be equivalent. This is impossible since $[e_1] \neq [f_2]$. Similarly $\|e_1(q_1 - f_1)e_1\| = \|e_1 f_2 e_1\| = 1$ and it follows that 0 and 1 are in the spectrum of $e_1 f_1 e_1$. If this spectrum were disconnected then e_1 would not be minimal. Hence $e_1 f_1 e_1$ has full spectrum. Similarly all compressions have full spectrum. As we shall see in the next section it follows that

$$C^*(e_1, f_1, q_1) \sim C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2).$$

Arguing in like manner for q_2 and q_3 we get

$$C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2) \oplus C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2) \oplus C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$$

embeds unitally into C.

5) It is possible to get an upper bound for the norm of $p_1q_1p_1$ as follows. We have

$$p_1 = \frac{1+u}{2} \quad \text{and} \quad \frac{1+v+v^2}{3}$$

or in terms of Choi's matrix representations for u and v

$$p_1 = (1/2)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad q_1 = (1/3)\begin{pmatrix} 1 & s^*+t^* \\ s+t & 1+st^*+ts^* \end{pmatrix}.$$

If we conjugate p_1 and q_1 by the unitary

$$x = (1/\sqrt{2})\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

we get

$$xp_1x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and that the (1,1) entry of $xp_1q_1p_1x$ is

$$\frac{2+s+t^*+s^*+t+ts^*+st^*}{6} =$$

$$= \frac{1+2\mathrm{Re}(s+t)+(1+st^*+ts^*)}{6} =$$

$$= \frac{1+2\sqrt{2}\mathrm{Re}(w)+2ww^*}{6}$$

where $\sqrt{2}w = s+t$ and w is an isometry. Hence

$$||p_1q_1p_1|| \leqslant \frac{3+2\sqrt{2}}{6} \approx .97 < 1.$$

We shall see in the next section (Remark 15) that the spectrum of $p_1q_1p_1$ is

$$\{0\} \cup \{t : (3-2\sqrt{2})/6 \leqslant t \leqslant (3+2\sqrt{2})/6\}.$$

so that in fact this estimate is sharp.

2. THE GENERAL CASE

Throughout this section n and m will denote fixed integers such that $n \geqslant 2$, $m \geqslant$ $\geqslant 2$ and $n+m \geqslant 5$. We denote the unitary generators of $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ by u and v so that

$$u^n = v^m = 1.$$

We shall view $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ as acting on $\ell^2(\mathbf{Z}_n * \mathbf{Z}_m)$ via the left regular representation. If x denotes a linear combination of words in u and v then x simultaneously determines an element of $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ and $\ell^2(\mathbf{Z}_n * \mathbf{Z}_m)$. As usual we use the same symbol for both. We write $\|x\|_2$ for the norm in $\ell^2(\mathbf{Z}_n * \mathbf{Z}_m)$ and $\langle x, y\rangle$ for the inner product. Recall that $\langle x, y\rangle = \tau(y^* x)$.

Throughout this section p and q will denote fixed spectral projections of u and v, respectively. Set

$$\tau(p) = \alpha \quad \text{and} \quad \tau(q) = \beta.$$

We are interested in studying $C^*(p, q, 1)$, the C^*-subalgebra of $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ generated by p, q and 1. Replacing p by $1 - p$ and q by $1 - q$ if necessary we may assume that $\alpha \leqslant 1/2$ and $\beta \leqslant 1/2$. Also, we may assume $\beta \leqslant \alpha$. So from now on we assume

$$0 < \beta \leqslant \alpha \leqslant 1/2.$$

We now construct an orthonormal subset of $\ell^2(\mathbf{Z}_n * \mathbf{Z}_m)$ that will be useful in the sequel. Put

$$a = \frac{1}{\sqrt{\alpha(1-\alpha)}}(p - \alpha 1) \quad \text{and} \quad b = \frac{1}{\sqrt{\beta(1-\beta)}}(q - \beta 1).$$

Note that since p is a spectral projection of u and $\tau(p) = \alpha$ we have

$$p = \alpha 1 + \sum_{i=1}^{n-1} \lambda_i u^i.$$

Similarly,

$$q = \beta 1 + \sum_{j=1}^{m-1} \mu_j v^j.$$

Thus a (resp. b) is a linear combination of nonzero powers of u (resp. v). Let $\mathbf{B}$ denote all words of the form 1, $(ab)^k$, $(ab)^{k-1}a$, $(ba)^k$ or $(ba)^{k-1}b$ for $k = 1, 2, \ldots$. Note that if $w \neq 1$ is a word in $\mathbf{B}$ then w is a linear combination of products of nonzero powers of u and v and therefore $\tau(w) = 0$. It is easy to check that

$$a^2 = \gamma a + 1, \quad \text{where } \gamma = (1 - 2\alpha)/(\alpha(1-\alpha))^{1/2}. \tag{1}$$

and

$$b^2 = \delta b + 1, \quad \text{where } \delta = (1 - 2\beta)/(\beta(1-\beta))^{1/2}. \tag{2}$$

If $w \in \mathbf{B}$ then we write for the "adjoint" word in $\mathbf{B}$. Thus

$$((ab)^k)^* = (ba)^k, \quad ((ab)^{k-1}a)^* = (ab)^{k-1}a, \quad \text{etc.}$$

LEMMA 5. *If $k \geqslant 1$ then*

$$(ab)^k(ba)^k = \delta\left(\sum_{i=1}^{k}(ab)^{2i-1}\right)a + \gamma\left(\sum_{j=1}^{k}(ab)^{2(j-1)}\right)a + 1 \tag{3}$$

$$(ab)^k a(ab)^k a = \gamma(ab)^{2k}a + (ab)^k(ba)^k. \tag{4}$$

Analogous formulas hold for $(ba)^k(ab)^k$ and $(ba)^k b(ba)^k b$.

Proof. We proceed by induction. Using (1) and (2) above we have

$$(ab)(ba) = ab^2a = \delta aba + a^2 = \delta(ab)a + \gamma a + 1$$

and so the first equality is true for $k = 1$. Suppose that it is true for $k \geqslant 1$. For $k+1$ we have

$$(ab)^{k+1}(ba)^{k+1} = (ab)^k(ab^2a)(ba)^k =$$
$$= \delta(ab)^k(aba)(ba)^k + \gamma(ab)^k a(ba)^k + (ab)^k(ba)^k =$$
$$= \delta(ab)^{2k+1}a + \gamma(ab)^{2k}a + (ab)^k(ba)^k$$

and the first formula follows from the induction hypothesis. For the second we have

$$(ab)^k a(ab)^k a = (ab)^k a^2(ba)^k = \gamma(ab)^{2k}a + (ab)^k(ba)^k.$$

The analogous formulas follow by symmetry. ■

LEMMA 6. *The set $\mathbf{B}$ is orthonormal in $\ell^2(\mathbf{Z}_n * \mathbf{Z}_m)$.*

Proof. Fix w in $\mathbf{B}$. If $w = 1$, then $\|w\|_2^2 = \tau(w^*w) = \tau(1) = 1$. If $w \neq 1$, then as observed above $\tau(w) = 0$. This together with (3) and (4) (and their analogues) imply that $\|w\|_2^2 = 1$. To see that $\mathbf{B}$ is orthogonal it is necessary to examine various cases. Since the proofs are similar we shall only treat the case $w = (ba)^k$ in detail. Fix x in $\mathbf{B}$ with $x \neq w$. We must show

$$\langle w, x\rangle = \tau(x^*w) = 0.$$

If x begins with a then x^* ends with a and x^*w is in $\mathbf{B}$. Hence we are reduced to the case where x begins with b; i.e.,

$$x = (ba)^j \quad\text{or}\quad x = (ba)^j b.$$

Suppose $x = (ba)^j$ and $j > k$. In this case we have

$$x^*w = (ab)^j(ba)^k = (ab)^{j-k}(ab)^k(ba)^k = (ab)^{j-k}(a[*]+1) \qquad \text{by (3)}$$

and therefore $\tau(x^*w) = 0$. The proof is similar when $x = (ba)^j$, $j < k$ or $x = (ba)^j b$, $j \geqslant k$. We are left with the case $x = (ba)^j b$, $j < k$. With this choice we get

$$x^*w = (ba)^j b(ba)^k = b(ab)^j (ba)^j (ba)^{k-j} = b(a[*]a + \gamma a + 1)(ba)^{k-j}$$

and so

$$\tau(x^*w) = \tau(b(ba)^{k-j}) = \tau(b^2(ab)^{k-j-1}a) = \delta\tau((ba)^{k-j}) + \tau(((ab)^{k-j-1})a) = 0.$$

■

Write $\mathcal{H}$ for closed linear subspace of $\ell^2(\mathbf{Z}_n * \mathbf{Z}_m)$ spanned by $\mathbf{B}$ and note that (by (1) and (2)) $\mathcal{H}$ is left invariant by multiplication by a or b. Thus restriction to $\mathcal{H}$ gives a representation of $C^*(p,q,1)$. We next show that the matrix for $ab|_{\mathcal{H}}$ has surprisingly simple form. Define a partition of $\mathbf{B}$ as follows. Set

$$\mathbf{B}_1 = \{1\} \cup \{(ab)^k : k = 1,2,\ldots\}, \quad \mathbf{B}_2 = \{(ab)^{k-1}a : k = 1,2,\ldots\},$$

$$\mathbf{B}_3 = \{(ba)^k : k = 1,2,\ldots\} \quad \text{and} \quad \mathbf{B}_4 = \{(ba)^{k-1}b : k = 1,2,\ldots\}$$

and write $\mathcal{H}_i$ for the subspace of $\mathcal{H}$ spanned by $\mathbf{B}_i$; $i = 1,2,3,4$.

LEMMA 7. *The matrix for $ab|_{\mathcal{H}}$ in the descomposition defined above has the form*

$$ab|_{\mathcal{H}} = \begin{pmatrix} S & 0 & * & * \\ 0 & S & * & * \\ 0 & 0 & S^* & 0 \\ 0 & 0 & 0 & S^* \end{pmatrix},$$

where S denotes the unilateral shift and the $*$'s stand for unspecified entries.

Proof. It is clear that ab acts as the shift on the elements of $\mathbf{B}_1$ and $\mathbf{B}_2$. Hence the first 2 columns are as asserted. Similarly, $(ab)^* = ba$ acts as the shift on $\mathbf{B}_3$ and $\mathbf{B}_4$; hence the bottom 2 rows are asserted. ■

REMARK. In fact a more careful analysis shows that

$$ab|_{\mathcal{H}} = \begin{pmatrix} S & 0 & \delta S + \gamma P & 1-P \\ 0 & S & 1-P & \delta S + \gamma 1 \\ 0 & 0 & S^* & 0 \\ 0 & 0 & 0 & S^* \end{pmatrix},$$

where $P = SS^*$. Thus the restriction of ab to $\mathcal{H}$ is equivalent to an element of $\mathbf{M}_4(C^*(S))$.

THEOREM 8. *The spectral radius of ab in $C^*_r(\mathbf{Z}_n * \mathbf{Z}_m)$ is 1. Moreover, the spectrum of ab contains the unit circle.*

Proof. As noted above, restriction to $\mathcal{H}$ gives a representation of $C^*(p,q,1)$. Since the trace τ is faithful and $\mathbf{B}$ contains 1, this representation is faithful on $C^*(p,q,1)$; so we may calculate the spectrum of ab in $\mathcal{H}$. It follows from Lemma 7 that, on $\mathcal{H}$, ab has the form

$$ab|_{\mathcal{H}} = \begin{pmatrix} T & X \\ 0 & T^* \end{pmatrix},$$

where T is an isometry. An easy induction shows

$$\|(ab|_{\mathcal{H}})^k\| = \left\| \begin{pmatrix} T & X \\ 0 & T^* \end{pmatrix}^k \right\| \leqslant 1 + k\|x\|.$$

Hence, the spectral radius of ab is at most 1. To complete the proof it suffices to show that the spectrum of ab contains the unit circle. By Lemma 7 we have that the restriction of ab to $\mathcal{H}_1$ is the shift S. Recall that the unit circle is in the approximate point spectrum of S. That is, if λ is a complex number of modulus 1, then there is a sequence of unit vectors $\{\eta_n\}$ such that

$$\|(S - \lambda 1)\eta_n\| \to 0$$

as $n \to \infty$. Hence ab also contains the unit circle in its approximate point spectrum. ∎

As this point it is convenient to record some (well known) facts about $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$. It is an easy exercise to construct an invariant mean on $\ell^2(\mathbf{Z}_2 * \mathbf{Z}_2)$ using a Banach limit. Hence $\mathbf{Z}_2 * \mathbf{Z}_2$ is amenable and $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ is isomorphic to the full group C^*-algebra of $\mathbf{Z}_2 * \mathbf{Z}_2$ [7, 7.3.9]. It follows that $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ is the universal C^*-algebra generated by 2 projections and the identity in the sense that if $A = C^*(e, f, 1)$ where e and f are projections, then A is a quotient of $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$. Let us denote the canonical projections in $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ by P and Q so that

$$C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2) = C^*(P, Q, 1).$$

We shall be interested in the irreducible representations of $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$. It is clear that $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ has 4 one-dimensional representations; we denote them by

$$\pi_{0,0}, \pi_{0,1}, \pi_{1,0}, \quad \text{and} \quad \pi_{1,1}.$$

Here the subscripts indicate the effect of the map on P and Q. So for example

$$\pi_{1,0}(P) = 1 \quad \text{and} \quad \pi_{1,0}(Q) = 0.$$

For $0 < t < 1$ we define the two dimensional representation π_t by

$$\pi_t(P) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \pi_t(Q) = \begin{pmatrix} 1-t & \sqrt{t-t^2} \\ \sqrt{t-t^2} & t \end{pmatrix}.$$

It is a straightforward exercise (using [9, p. 306-8] for example) to check that

$$\{\pi_{0,0}, \pi_{0,1}, \pi_{1,0}, \pi_{1,1}\} \cup \{\pi_t : 0 < t < 1\}$$

forms a complete set of representatives from the equivalence classes of irreducible representations of $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$.

If e and f are projections then as noted above $C^*(e, f, 1)$ is a quotient of $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$. Hence the irreducible representations of $C^*(e, f, 1)$ determine a subset of those of $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$; this subset, in turn, determines the structure of $C^*(e, f, 1)$.

DEFINITION 9. If π denotes one of the irreducible representations of $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ defined above we say that π is in the *support* of $C^*(e, f, 1)$ if there is a representation ρ of $C^*(e, f, 1)$ with

$$\rho(e) = \pi(P) \quad \text{and} \quad \rho(f) = \pi(Q).$$

Also we write

$$\sigma(e, f) = \{t : \pi_t \text{ is in the support of } C^*(e, f, 1),\ 0 < t < 1\}.$$

Note that $\sigma(e, f)$ is relatively closed in (0,1); also if 0 (resp. 1) is in the closure of $\sigma(e, f)$ then $\pi_{0,0}$ and $\pi_{1,1}$ (resp. $\pi_{0,1}$ and $\pi_{1,0}$) are in the support of $C^*(e, f, 1)$. Since the reduced atomic representation is faithful [7, 4.3.15], it follows that $C^*(e, f, 1)$ is isomorphic to $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ if and only if $\sigma(e, f) = (0, 1)$. In particular $C^*_r(\mathbf{Z}_2 * * \mathbf{Z}_2)$ is isomorphic to the subalgebra of $M_2(C([0, 1]))$ consisting of the matrices that are diagonal at 0 and 1. Also $C^*(e, f, 1) \approx C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ if and only if each of the compressions efe, $e(1-f)e$, fef, $f(1-e)f$ have full spectrum.

PROPOSITION 10. *If $\beta = \tau(q) < 1/2$ then the representation $\pi_{0,0}$ lies in the support of $C^*(p, q, 1)$. If $\beta < \alpha$ then $\pi_{1,0}$ also lies in the support of $C^*(p, q, 1)$.*

Proof. For the first assertion it is sufficient to show that $\|(1-p)(1-q)(1-p)\| = 1$ when $\beta < 1/2$. For then, as in the proof of Proposition 2, it follows that the required complex homomorphism exists. If $\|(1-p)(1-q)(1-p)\| = \|(1-p)q(1-p)-(1-p)\| < 1$, then by Proposition 1 $1-p$ would be equivalent to a subprojection of q and we would have

$$1/2 \leqslant \tau(1-p) \leqslant \tau(q) < 1/2$$

which is impossible. Hence if $\beta < 1/2$ then $\pi_{0,0}$ is in the support of $C^*(p, q, 1)$. The proof of the second assertion is easier, since $\pi_{1,0}$ is the only irreducible representation π of $C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$ for which $\mathrm{Tr}(\pi(Q)) < \mathrm{Tr}(\pi(P))$. ■

REMARK. We shall see later that if $\alpha = \beta = 1/2$ then $C^*(p, q, 1) \approx C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$.

Our main goal in this section is to show that

$$\sigma(p,q) = [t_1, t_2] \cap (0,1),$$

where $0 \leqslant t_1 < t_2 \leqslant 1$. (See below for a precise definition of the t_i's.)

We shall now analyze the spectrum of $(p-\alpha 1)(q-\beta 1)$ in some detail. Write

$$c(\alpha,\beta) = \sqrt{\alpha\beta(1-\alpha)(1-\beta)}$$

and for $0 < t < 1$ let $\lambda_-(t)$ and $\lambda_+(t)$ denote the eigenvalues of

$$\pi_t((P-\alpha 1)(Q-\beta 1))$$

with the convention that $|\lambda_-(t)| \leqslant |\lambda_+(t)|$. Define real numbers t_1 and t_2 as follows

$$t_1 = (\sqrt{(1-\alpha)(1-\beta)} - \sqrt{\alpha\beta})^2 = \alpha\beta + (1-\alpha)(1-\beta) - 2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}$$

and

$$t_2 = (\sqrt{(1-\alpha)(1-\beta)} + \sqrt{\alpha\beta})^2 = \alpha\beta + (1-\alpha)(1-\beta) + 2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}.$$

Note that $0 \leqslant t_1 \leqslant t_2 \leqslant 1$. In fact if we select θ and φ so that $\alpha = \sin^2\theta$ and $\beta = \sin^2\varphi$ then we have

$$t_1 = \cos^2(\theta+\varphi) \quad \text{and} \quad t_2 = \cos^2(\theta-\varphi).$$

Also $t_1 = 0$ if and only if $\alpha = \beta = 1/2$ and $t_2 = 1$ if and only if $\alpha = \beta$.

LEMMA 11. *Fix $0 < t < 1$. If $t < t_1$ or $t > t_2$ then*

$$|\lambda_-(t)| < c(\alpha,\beta) < |\lambda_+(t)|.$$

If $t_1 \leqslant t \leqslant t_2$ then

$$\lambda_\pm(t) = c(\alpha,\beta)e^{\pm i\theta},$$

where $0 \leqslant \theta \leqslant \pi$ and θ is related to t by the formula

$$(*) \qquad t = \alpha\beta + (1-\alpha)(1-\beta) - 2\cos\theta c(\alpha,\beta).$$

Thus as θ varies from 0 to π, t increases from t_1 to t_2.

Proof. We have

$$\pi_t((P-\alpha 1)(Q-\beta 1)) = \begin{pmatrix} 1-\alpha & 0 \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} 1-\beta-t & (t-t^2)^{1/2} \\ (t-t^2)^{1/2} & t-\beta \end{pmatrix}.$$

Note that since the entries are real the eigenvalues must be real or else complex conjugates. A calculation shows that the determinant of this product is $\alpha\beta(1-\alpha)(1-\beta)$ for all t. Hence

$$(**) \qquad \lambda_-(t)\lambda_+(t) = \alpha\beta(1-\alpha)(1-\beta) = c(\alpha,\beta)^2$$

for all t and therefore

$$|\lambda_1(t)| = |\lambda_2(t)| = c(\alpha,\beta)$$

if and only if the eigenvalues are equal or complex. Now $\lambda_\pm(t)$ satisfy

$$(***) \qquad \lambda^2 - (\alpha\beta + (1-\alpha)(1-\beta) - t)\lambda + \alpha\beta(1-\alpha)(1-\beta) = 0.$$

Thus the eigenvalues are equal or complex if and only if

$$4\alpha\beta(1-\alpha)(1-\beta) \geqslant (\alpha\beta + (1-\alpha)(1-\beta) - t)^2.$$

Another calculation shows that this inequality is satisfied if and only if $t_1 \leqslant t \leqslant t_2$. Hence if $t < t_1$ or $t > t_2$ then the eigenvalues are real and distinct and our first assertion follows from $(**)$. If t and θ are related as in $(*)$ then from $(***)$ we get that $\lambda_\pm(t)$ satisfy

$$\lambda^2 - 2\cos\theta c(\alpha,\beta)\lambda + c(\alpha,\beta)^2 = 0.$$

Hence,

$$\lambda_\pm(t) = c(\alpha,\beta)(\cos\theta \pm \mathrm{i}\sin\theta).$$

■

THEOREM 12. a) *If $\beta < \alpha$ then $0 < t_1 < t_2 < 1$ and*

$$\sigma(p,q) = [t_1, t_2].$$

Moreover, $\pi_{0,1}$ and $\pi_{1,1}$ are not in the support of $C^*(p,q,1)$.

b) *If $\alpha = \beta < 1/2$ then $0 < t_1 = (1-2\alpha)^2$, $t_2 = 1$ and*

$$\sigma(p,q) = [t_1, 1).$$

Moreover, $\pi_{1,1}$ is not in the support of $C^*(p,q,1)$.

c) *If $\alpha = \beta = 1/2$, then $t_1 = 0$, $t_2 = 1$ and*

$$\sigma(p,q) = (0,1).$$

Hence $C^*(p,q,1) \approx C^*_r(\mathbf{Z}_2 * \mathbf{Z}_2)$.

Proof. First suppose $0 < \beta \leqslant \alpha \leqslant 1/2$ and fix t in $\sigma(p,q)$. If ρ is a representation of $C^*(p,q,1)$ with $\rho((p-\alpha 1)(q-\beta 1)) = \pi_t((P-\alpha 1)(Q-\beta 1))$ then the spectral radius of $\rho((p-\alpha 1)(q-\beta 1))$ is $\leqslant c(\alpha,\beta)$ by Theorem 8. (Note that $(p-\alpha 1)(q-\beta 1) = c(\alpha,\beta)ab$.) Hence $t \in [t_1,t_2]$ by Lemma 11 and therefore $\sigma(p,q) \subset [t_1,t_2] \cap (0,1)$ for all cases. For the reverse inclusions it is enough to show $(t_1,t_2) \subset \sigma(p,q)$ because $\sigma(e,f)$ is relatively closed in (0,1). So fix t in (t_1,t_2) and select θ so that $0 < \theta < \pi$ and

$$t = \alpha\beta + (1-\alpha)(1-\beta) - 2\cos\theta c(\alpha,\beta).$$

In this case $c(\alpha,\beta)e^{i\theta}$ is complex and by the Theorem 8 it is in the spectrum of $(p-\alpha 1)(q-\beta 1)$; hence there is an irreducible representation ρ of $C^*(p,q,1)$ such that $c(\alpha,\beta)e^{i\theta}$ is a eigenvalue for $\rho((p-\alpha 1)(q-\beta 1))$. If ρ were one-dimensional then we would have that $\rho((p-\alpha 1)(q-\beta 1)) = c(\alpha,\beta)e^{i\theta}$ is real; so ρ must have dimension 2. Hence ρ is equivalent to π_s for some $0 < s < 1$. By Lemma 11, $s = t$ and so $t \in \sigma(p,q)$.

Now suppose $\beta < \alpha$. If $\pi_{0,1}$ were in the support of $C^*(p,q,1)$ then $-\alpha(1-\beta)$ would be in the spectrum of $(p-\alpha 1)(q-\beta 1)$ and by Theorem 8 we would get

$$\alpha(1-\beta) \leqslant c(\alpha,\beta)$$

so that

$$\alpha(1-\beta) \leqslant (1-\alpha)\beta$$

and this would imply that $\alpha \leqslant \beta$ wich is not the case. Similarly, $\pi_{1,1}$ is not in the support of $C^*(p,q,1)$ if $\beta < 1/2$. ■

REMARK. Note that our calculations show that the spectrum of $(p-\alpha 1)(q-\beta 1)$ is

$$\{\alpha\beta, -\beta(1-\alpha)\} \cup \{c(\alpha,\beta)e^{i\theta} : 0 \leqslant \theta < 2\pi\}$$

when $0 < \beta < \alpha \leqslant 1/2$. In the case where $0 < \beta = \alpha < 1/2$ the spectrum is

$$\{\alpha^2\} \cup \{\alpha(1-\alpha)e^{i\theta} : 0 \leqslant \theta < 2\pi\}.$$

THEOREM 13. a) *If $\beta < \alpha$ then*

$$C^*(p,q,1) \approx \mathbf{C} \oplus \mathbf{M}_2(C([t_1,t_2])) \oplus \mathbf{C},$$

where

$$p \approx 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 1 \quad \textit{and} \quad q \approx 0 \oplus \begin{pmatrix} 1-t & (t-t^2)^{1/2} \\ (t-t^2)^{1/2} & t \end{pmatrix} \oplus 0.$$

(Here t denotes the identity map in $C([t_1, t_2])$). Moreover,

$$p^\perp \wedge q = p \wedge q = 0,$$

$$p \wedge q^\perp,\ p^\perp \wedge q^\perp \in C^*(p, q, 1)$$

and

$$\tau(p \wedge q^\perp) = \alpha - \beta, \quad \tau(p^\perp \wedge q^\perp) = 1 - \alpha - \beta.$$

b) *If $\beta = \alpha < 1/2$ then $t_1 = (1-2\alpha)^2, t_2 = 1$ and*

$$C^*(p, q, 1) \approx \mathbb{C} \oplus A,$$

where A denotes the subalgebra of $\mathbf{M}_2(C[t_1, 1])$ of matrices that are diagonal at 1. We have

$$p \approx 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q \approx 0 \oplus \begin{pmatrix} 1-t & (t-t^2)^{1/2} \\ (t-t^2)^{1/2} & t \end{pmatrix}.$$

Moreover,

$$p^\perp \wedge q = p \wedge q^\perp = p \wedge q = 0,$$

$$p^\perp \wedge q^\perp \in C^*(p, q, 1) \quad \text{and} \quad \tau(p^\perp \wedge q^\perp) = 1 - 2\alpha.$$

c) *If $\alpha = \beta = 1/2$, then*

$$C^*(p, q, 1) \approx C_r^*(\mathbf{Z}_2 * \mathbf{Z}_2).$$

Proof. Due to Proposition 10 and Theorem 12 everything is clear except the assertions concerning the trace. In the case $\beta < \alpha$, let $\lambda_1, \lambda_2, \lambda_3$ be the traces of the three central projections $r_1 = p^\perp \wedge q^\perp$, $r_2 = 1 - p \wedge q^\perp - p^\perp \wedge q^\perp$, and $r_3 = p \wedge q^\perp$ from the isomorphism in a). Then $\lambda_2/2 = \beta$ and $\lambda_2/2 + \lambda_3 = \alpha$, so $\lambda_3 = \alpha - \beta$; $1 = \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + \beta + \alpha$, so $\lambda_1 = 1 - \alpha - \beta$. The proof in part b) is similar.

COROLLARY 14. *The spectrum of $pqp|_{p\mathcal{H}} = \{0\} \cup [1 - t_2, 1 - t_1]$ (so that 0 is an isolated point when $\beta < \alpha$) and $\|pqp\| = 1 - t_1$.*

Proof. This is immediate from the preceding results. ■

REMARK 15. Note that if p and q are in Choi's algebra C and $\alpha = 1/2$, $\beta = 1/3$, then

$$\|pqp\| = 1 - t_1 = 1 - (\sqrt{1/3} - \sqrt{1/6})^2 = \frac{3 + 2\sqrt{2}}{6}$$

and therefore the estimate in Remark 5) of Section 1 is sharp.

We are now in a position to verify that (1) of the introduction holds for spectral projections of u and v. If e and f are projections in a C^*-algebra and e is equivalent to a subprojection of f, we write $e \prec f$.

COROLLARY 16. a) *If $\beta < \alpha$ so that q has smallest trace among p, q, $1-p$, $1-q$ then within $C^*(p,q,1)$*

$$q \prec p, \quad q \prec 1-p, \quad p \prec 1-q \quad \text{and} \quad 1-p \prec 1-q;$$

hence, $q \prec 1-q$.

b) *If $\alpha = \beta < 1/2$ then within $C^*(p,q,1)$*

$$p \prec 1-q \quad \text{and} \quad q \prec 1-p.$$

Proof. This follows immediately from Theorem 13. ■

COROLLARY 17. *If e and f are spectral projections of u and v respectively and $\tau(e) < \tau(f)$ then $e \prec f$. If $\tau(f) < \tau(e)$ then $f \prec e$.*

Proof. This follows by examining all possible cases and using Corollary 16. ■

REMARK 18. (a) If $\beta \leqslant \alpha$ then p is not equivalent to a subprojection of $1-p$ within $C^*(p,q,1)$. If $\alpha = \beta$ then it is also the case that q is not equivalent to a subprojection of $1-q$ within $C^*(p,q,1)$. However, if $\alpha < 1/2$, then $p \prec 1-p$ within $C^*_{\mathrm{r}}(\mathbf{Z}_n * \mathbf{Z}_m)$. For if we let $w = v^*uv$, then u and w generate a subgroup of $\mathbf{Z}_n * \mathbf{Z}_m$ isomorphic to $\mathbf{Z}_n * \mathbf{Z}_n$. If $r = v^*pv$, then r is a spectral projection of w, and $p \prec 1-r$ within $C^*(p,r,1)$ by Corollary 16 (b), and hence $p \prec 1-p = v(1-r)v^*$. Similarly, if $\alpha = \beta < 1/2$ then $q \prec 1-q$ within $C^*_{\mathrm{r}}(\mathbf{Z}_n * \mathbf{Z}_m)$. The same argument shows more generally that if e and f are spectral projections of u (or of v) with $\tau(e) < \tau(f)$, then $e \prec f$ within $C^*_{\mathrm{r}}(\mathbf{Z}_n * \mathbf{Z}_m)$.

(b) Note that if e and f are spectral projections of u and v respectively such that

$$\tau(e) + \tau(f) < 1,$$

then $\|ef\| < 1$. Indeed, if $\tau(f) < \tau(e)$ then $\tau(f) < 1/2$ and $\tau(f) < \tau(1-e)$ so that we may argue as in the proof of part a) of Corollary 16 with $q = f$ and $p = e$ or $1-e$. If $\tau(e) < \tau(f)$ then the same reasoning applies after interchanging n and m (and e and f). If $\tau(e) = \tau(f) = \beta$, then $\beta < 1/2$ and we may use the argument in the proof of part b) of Corollary 16.

LEMMA 19. *If n and m are positive integers with least common multiple μ and ν is a integer such that $1 < \nu < \mu$, then there are integers i and j such that*

$$0 \leqslant i < n, \quad 0 \leqslant j < m$$

and either

$$\frac{i}{n} - \frac{j}{m} = \frac{\nu}{\mu} \quad \text{or} \quad \frac{i}{n} - \frac{j}{m} = 1 - \frac{\nu}{\mu}.$$

Proof. We have

$$\mu = kn - lm$$

for some positive integers k and l. Since μ is the least common multiple of n and m, k and l are relatively prime. Hence there are integers i and j such that

$$ik - jl = \nu.$$

Write

$$i = i_1 + i_2 n \quad \text{and} \quad j = j_1 + j_2 m,$$

where

$$0 \leqslant i_1 < n \quad \text{and} \quad 0 \leqslant j_1 < m.$$

We have now that

$$i_1 k - j_1 l = \nu + (i_2 - j_2)\mu.$$

By our assumptions the left hand side is the difference of 2 non-negative integers that are strictly less than μ. Hence $i_2 - j_2 = 0$ or -1. Replacing i by i_1 and j by j_1 we get

$$\frac{i}{n} - \frac{j}{m} = \frac{ik - jl}{\mu} \in \left\{\frac{\nu}{\mu}, \frac{\nu}{\mu} - 1\right\}.$$

To complete the proof note that if

$$\frac{i}{n} - \frac{j}{m} = \frac{\nu}{\mu} - 1$$

then we may replace i by $n - i$ and j by $m - j$ to get the desired equality. ■

THEOREM 20. *If μ denotes the least common multiple of n and m and ν is an integer with $0 < \nu < \mu$ then there is a projection r in $C^*_r(\mathbf{Z}_n * \mathbf{Z}_m)$ with $\tau(r) = \nu/\mu$.*

Proof. By Lemma 15 there are integers i and j with $1 \leqslant i < n$ and $0 \leqslant j < m$ such that either

$$\frac{i}{n} - \frac{j}{m} = \frac{\nu}{\mu} \quad \text{or} \quad \frac{i}{n} - \frac{j}{m} = 1 - \frac{\nu}{\mu}.$$

Select spectral projections e and f of u and v respectively such that $\tau(e) = i/n$ and $\tau(f) = j/m$. By Corollary 18 there is a projection s in $C^*(p, q, 1)$ such that $s \leqslant e$ and s is equivalent to f. So if we write $r = e - s$,

$$\tau(r) = \frac{i}{n} - \frac{j}{m}.$$

If $i/n - j/m = \nu/\mu$, we are done. If $i/n - j/m = 1 - \nu/\mu$, the proof is completed by using $1 - r$ in place of r. ■

Note that if question (1) in the introduction is true for $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ then every minimal projection has trace $1/\mu$. By Theorem 20 we know that $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ contains projections of trace $1/\mu$. In our final result we show that every spectral projection of u and v majorizes such a projection. We need the following simple lemma.

LEMMA 21. *If μ denotes the least common multiple of n and m then there are integers i and j such that*

$$0 < i < n, \ \ 0 \leqslant j < m \ \text{ and } \ \frac{i}{n} - \frac{j}{m} = \frac{1}{\mu}.$$

Moreover we have that either $j/m \geqslant 1/2$ or else $i/n \leqslant 1/2$.

Proof. If $\mu = n$ the result is trivial. If $\mu > n$, Lemma 19 yields i and j with $i/n - j/m = 1/\mu$ or $1 - 1/\mu$ and $0 \leqslant i < n,\ 0 \leqslant j \leqslant m$. But $i/n - j/m = 1 - 1/\mu$ is impossible since $i/n \leqslant (n-1)/n < 1 - 1/\mu$, so $i/n - j/m = 1/\mu$ and $i > 0$.

For the final assertion note that since $\mu = kn$ and $\mu \neq n,\ k \geqslant 2$. On the other hand if we had

$$(*) \qquad \frac{i}{n} > \frac{1}{2} > \frac{j}{m},$$

then

$$\frac{1}{kn} = \frac{i}{n} - \frac{j}{m} > \frac{i}{n} - \frac{1}{2} > 0$$

and

$$0 < k(2i - n) < 2.$$

But in this case $k = 1$, which is impossible. Hence $(*)$ cannot occur. ■

THEOREM 22. *Each spectral projection of u and v majorizes a (minimal) projection of trace $1/\mu$ in $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$.*

Proof. It suffices to show the result for a minimal spectral projections. Fix a minimal projection q_0 of v. We first show that there is a projection r in $C_r^*(\mathbf{Z}_n * \mathbf{Z}_m)$ with $r \leqslant q_0$ and $\tau(r) = 1/\mu$. Note that if $\mu = m$, then q_0 is itself minimal in $C_r^*(\mathbf{Z}_n * {*\mathbf{Z}_m})$. Hence we may assume $\mu \neq m$. Also, if $\mu = n$ (so that $n > m$) then applying Corollary 16 with $\tau(q_0) = 1/m = \alpha$ and $\tau(p) = 1/\mu = \beta$ (and n and m interchanged) we have that p is equivalent to a subprojection of q_0. So we may assume that $\mu \neq n, m$. By Lemma 21 there are $0 < i < n$ and $0 < j < m$ such that

$$\frac{i}{n} - \frac{j}{m} = \frac{1}{\mu}$$

and $i/n \leqslant 1/2$ or $j/m \geqslant 1/2$.

Select a spectral projection f of v with $\tau(f) = j/m$ and such that $fq_0 = 0$. We are assured that f exists because $j/m < i/n < 1$. Also pick a spectral projection e of u with $\tau(e) = i/n$. Since $j/m \leqslant 1/2$ or $i/n \geqslant 1/2$, either j/m or $1 - i/n$ is the smallest of the four numbers $\{i/n, 1-i/n, j/m, 1-j/m\}$, and so by Theorem 13 $\|f(1-e)\|^2 = \|f-fef\| < 1$. Thus by Proposition 1, f is equivalent to a subprojection s' of e with $s = e - s'$ orthogonal to f. Write $f' = 1 - f - q_0$ and note that

$$\tau(e) + \tau(f') = 1 + 1/\mu - 1/m < 1$$

because $\mu > m$. Hence by Remark 17 $\|ef'\| < 1$ and since $s \leqslant e$, $\|sf'\| < 1$. Thus

$$\|sq_0s - s\| = \|s(1 - f - q_0)s\| = \|f's\|^2 < 1$$

because $s \leqslant 1 - f$. By Proposition 1, s is equivalent to a subprojection of q_0. We have shown that every spectral projections of v majorizes a projection of trace $1/\mu$. That the same is true for spectral projections of u follows upon noting that n and m may be interchanged in our argument. ■

3. OPEN QUESTIONS

Questions 1-5 below concern Choi's algebra C. Analogous questions hold for $C_r^*(\mathbb{Z}_n * \mathbb{Z}_m)$.

1) If $(i, j, k, l, m) \in \mathbb{Z}^5$ and

$$0 < \frac{i+j}{2} + \frac{k+l+m}{3}$$

then is there a projection p in $\mathbf{M}_n(C)$ such that $[p] \sim (i, j, k, l, m)$?

Equivalently: if $x \in \mathrm{K}_0(C)_+$ with $\tau_*(x) > 0$, is x represented by a projection in a matrix algebra over C? (By [12], [13], [14], some multiple of x is represented by a projection in a matrix algebra.)

2) If $x \in \mathrm{K}_0(C)$ and $0 < \tau_*(x) < 1$, is there a projection p in C with $[p] = x$?

3) Is there a projection p in C with $[p] \sim (2, -1, -1, 0, 0)$?

4) Does C contain an infinite number of inequivalent projections?

5) What is the maximum number of orthogonal *equivalent* projections in C? (By K-theory there are at most 5.)

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