J. OPERATOR THEORY
25(1991), 43–51

HYPER-REFLEXIVITY OF ISOMETRIES AND WEAK CONTRACTIONS

MICHAL ZAJAC

1. INTRODUCTION

Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. In [14] a characterization of isometries $V \in \mathcal{B}(\mathcal{H})$ whose lattice of hyperinvariant subspaces is generated by the kernels and the closures of the ranges of the operators in double commutant of V was given. In this paper it is shown that this is also a characterization of isometries that are hyper-reflexive. The hyper-reflexivity of weak contractions is also investigated. We use the terminology of the monograph [11] concerning contractions.

Recall that the set $\operatorname{Lat}\mathcal{H}$ of all closed linear subspaces of $\mathcal{H}$ is a complete lattice. The lattice operations are the intersection $\bigcap$ and the closed linear span $\vee$. For $T \in \mathcal{B}(\mathcal{H})$ the commutant of T is $\{T\}' = \{X \in \mathcal{B}(\mathcal{H}) : TX = XT\}$, the double commutant of T is $\{T\}'' = \bigcap\{\{X\}' : X \in \{T\}'\}$. $\operatorname{Lat}T = \{L \in \operatorname{Lat}\mathcal{H} : TL \subset \subset L\}$ and $\operatorname{Hyplat}T = \bigcap\{\operatorname{Lat}X : X \in \{T\}'\}$ are the lattices of all invariant and all hyperinvariant subspaces of T, respectively. More generally, for $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, $\operatorname{Lat}\mathcal{A} = = \bigcap\{\operatorname{Lat}X : X \in \mathcal{A}\}$. $\operatorname{Alg}\mathcal{A}$ means the smallest weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing $\mathcal{A}$ and the identify I. For $\mathcal{L} \subset \operatorname{Lat}\mathcal{H}$ we denote $\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(\mathcal{H}) : \mathcal{L} \subset \subset \operatorname{Lat}T\}$.

DEFINITION. $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is called *reflexive* if $\operatorname{Alg}\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$.

The investigation of reflexivity was started by D. Sarason [10], who proved that any set of mutually commuting normal operators is reflexive. He proved also that the unilateral shift S of multiplicity one is reflexive. This shows also that $\{S\}'$ is reflexive because $\{S\}' = \operatorname{Alg}S$. Every isometry $V \in \mathcal{B}(\mathcal{H})$ is reflexive [4], [12]. In

[8] it was proved that any subnormal operator is reflexive using the technique of dual algebras introduced by S. Brown in [2]. Many other reflexivity results were obtained by investigation of the structure of the dual algebras (See [3] and references cited there).

An operator $T \in \mathcal{B}(\mathcal{H})$ is called *hyper-reflexive* if its commutant $\{T\}'$ is a reflexive algebra. Hyper-reflexivity was much less investigated. In [1] a characterization of hyper-reflexive contractions of class C_0 in terms of their Jordan models was given. L. Kérchy [7] has shown that every contraction of class C_{11} is hyper-reflexive. It is very easy to show that every normal operator is hyper-reflexive:

PROPOSITION 1.1. *Let $N \in \mathcal{B}(\mathcal{H})$ be normal. Then $\{N\}'$ is reflexive,* i.e. *N is hyper-reflexive.*

Proof. According to well-known Fuglede's theorem $\{N\}'$ is a von Neumann algebra and so it is reflexive [9, Theorem 9.17].

2. UNILATERAL SHIFTS

An isometry $V \in \mathcal{B}(\mathcal{H})$ is called a *unilateral shift* if it is completly non-unitary or equivalently if there exists $L \in \operatorname{Lat}\mathcal{H}$ such that $V^n L$ is orthogonal to $V^m L$ for all pairs of non-negative integers $n \neq m$ and $\mathcal{H} = \bigoplus_{n=0}^{\infty} V^n L$ ($\oplus$ means the orthogonal sum). The dimension of L is the multiplicity of the shift V. We shall use the following theorem due to V. Müller (private communication):

THEOREM 2.1. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unilateral shift. Let $A \in \{V\}'$ and $T \in \operatorname{AlgLat} A$. Then $T \in \{V\}'$.*

Proof. For the unilateral shift of multiplicity 1 this was proved by D. Sarason [10, p. 514]. A simple modification of his idea gives the required result: Let $A \in \{V\}'$ and $T \in \operatorname{Alg}\operatorname{Lat} A$. Then $A^* \in \{V^*\}'$ and $T^* \in \operatorname{Alg}\operatorname{Lat} A^*$. Any complex $\lambda : |\lambda| < 1$ is an eigenvalue of V^*. Let us denote the corresponding eigenspace by $\mathcal{H}(\lambda)$. $\mathcal{H}(\lambda) \in \operatorname{Hyplat} V^* \subset \operatorname{Lat} A^* \subset \operatorname{Lat} T^*$. It follows for every $x \in \mathcal{H}(\lambda)$ $T^*V^*x = \lambda . T^*x = V^*T^*x$. Since $\bigvee_{|\lambda|<1} \mathcal{H}(\lambda) = \mathcal{H}$, this means $T^* \in \{V^*\}'$ and equivalently $T \in \{V\}'$.

In fact, we have proved the following more general theorem:

THEOREM 2.2. *Any unilateral shift is hyper-reflexive.*

3. $U \oplus S$ IS NOT HYPER-REFLEXIVE

Let $V \in \mathcal{B}(\mathcal{H})$ be an arbitrary isometry. By the Wold decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ and $\mathcal{H}_1$ reduce V and $V_0 = V \mid \mathcal{H}_0$ is unitary, $V_1 = V \mid \mathcal{H}_1$ is a unilateral shift. We have shown that both $\{V_0\}'$ and $\{V_1\}'$ are reflexive. Now we shall show that $\{V\}'$ need not to be reflexive. More precisely, we shall show that the operator $V = U \oplus S$, where U is the bilateral shift and S is the unilateral shift, both of multiplicity one, has not a reflexive commutant. As in [14] we shall use the functional model of V on the space $\mathcal{H} = L^2 \oplus H^2$, where L^2 is the space of all square-integrable functions on the unit circle and Hardy space H^2 is the subspace of L^2 consisting of those $f \in L^2$ which have the Fourier coefficients with negative indices zero. The measure considered is the normalized Lebesgue measure m on the unit circle C. The operator V is then:

$$V(f \oplus \varphi)(z) = zf(z) \oplus z\varphi(z) \quad (z \in C).$$

As shown in [5] $L \in \operatorname{Hyplat} V$ if and only if either $L = L^2 \oplus L_1$ or $L = L_0 \oplus (0)$, where $L_0 \in \operatorname{Hyplat} U$ and $L_1 \in \operatorname{Hyplat} S$. Let $T \in \mathcal{B}(\mathcal{H})$ be defined by

$$T(f \oplus \varphi) = \varphi(0) \oplus 0,$$

where $\varphi(0)$ means the function identically equal to $\varphi(0)$.

It is easy to see that $T \in \operatorname{Alg}\operatorname{Hyplat} T$, but $T \notin \{V\}'$ and so the commutant of V is not reflexive.

4. GENERAL ISOMETRY

Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. As in [4], [5] and [14] we consider the unique decomposition $\mathcal{H} = \mathcal{H}_{0s} \oplus \mathcal{H}_{0a} \oplus \mathcal{H}_1$, where $\mathcal{H}_{0s}$, $\mathcal{H}_{0a}$, $\mathcal{H}_1$ belong to $\operatorname{Lat}(V)$, $V_{0s} = V \mid \mathcal{H}_{0s}$ is a singular unitary operator, $V_{0a} = V \mid \mathcal{H}_{0a}$ is an absolutely continuous unitary operator and $V_1 = V \mid \mathcal{H}_1$ is a unilateral shift. In [16] it was proved that $\{V\}'$ is reflexive if and only if both $\{V_{0s}\}'$ and $\{V_{0a} \oplus V_1\}'$ are reflexive. Combining this result with Proposition 1.1, $\{V\}'$ is reflexive if and only if $\{V_{0a} \oplus V_1\}'$ is reflexive. Again as in [14] we consider the functional model of $V_{0a} \oplus V_1$, i.e. the operator of multiplication by the independent variable z on the space

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \quad \text{where } \mathcal{H}_0 = \bigoplus_{i=0}^{\infty} L^2(\omega_i),\ \mathcal{H}_1 = \bigoplus_{j=0}^{n-1} \mathcal{H}_j,$$

and $\omega_0 \supset \omega_1 \supset \dots$ are measurable subsets of the unit circle $L^2(\omega_i) = \{f \in L^2 : f(z) = 0 \text{ for almost all } z \notin \omega\}$, $\mathcal{H}_j = H^2$ for all $j : 0 \leqslant j < n$, n is the multiplicity of V_1.

Similarly as for the special case $U \oplus S$ treated in the preceding paragraph it can be shown that if ω_0 has positive Lebesgue measure, $\chi(\omega_0)$ is the characteristic function of ω_0 and $n \geqslant 1$, then the operator

$$T\left[\left(\bigoplus_{i=0}^{\infty} f_i\right) \oplus \left(\bigoplus_{j=0}^{n-1} \varphi_j\right)\right] = [\ldots \oplus 0 \oplus \chi(\omega_0)\varphi(0)] \oplus (0)$$

belongs to $\operatorname{Alg}\operatorname{Hyplat}(V_{0a} \oplus V_1)$ but it does not commute with $V_{0a} \oplus V_1$. So we have proved:

THEOREM 4.1. *An arbitrary isometry $V \in \mathcal{B}(\mathcal{H})$ is hyper-reflexive if and only if either V is unitary, or the absolutely continuous unitary part of V is zero.*

Let us recall the following definition (See [13], [14], [15]):

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the property (L) if $\operatorname{Hyplat} T$ is the smallest complete lattice containing all subspaces of the forms $\operatorname{Ker} S$ and $\overline{S\mathcal{H}}$ for $S \in \{T\}''$. Combining the Theorem 4.1 with the results of [14] we obtain:

COROLLARY 4.2. *An isometry is hyper-reflexive if and only if it has the property* (L).

5. WEAK CONTRACTIONS

Let $T \in \mathcal{B}(\mathcal{H})$ be a weak contraction. (For the definition and basic properties of weak contractions we refer to [11, Chapter VIII]). It is easy to show [16] that T is hyper-reflexive if and only if so is its absolutely continuous part T_{ac}. According to [15, Lemma 3] T_{ac} is similar to a completely non-unitary (c.n.u.) weak contraction T'. Moreover, the C_0 part of T and the C_0 part of T' coincide. Since similarity (even quasi-similarity [1, Proposition 4.1]) preserves hyper-reflexivity, it does not restrict generality if we suppose that T is c.n.u.

THEOREM 5.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a* c.n.u. *weak contraction and let $T_0 \in \mathcal{B}(\mathcal{H}_0)$ and $T_1 \in \mathcal{B}(\mathcal{H}_1)$ be its C_0 part and C_{11} part, respectively. Then*

T is hyper-reflexiv if and only if

T_0 is hyper-reflexive.

Proof. Recall that [13, Lemma 2] there exists operators R, S from $\{T\}''$ such that

$$\mathcal{H}_0 = \operatorname{Ker} R = \overline{S\mathcal{H}} \quad \mathcal{H}_1 = \operatorname{Ker} S = \overline{R\mathcal{H}}.$$

Let T be hyper-reflexive and let $A_0 \in \operatorname{Alg}\operatorname{Hyplat} T_0$. If $\mathcal{L} \in \operatorname{Hyplat} T$ then [13, Lemma 2] $\mathcal{L} = (\mathcal{L} \cap \mathcal{H}_0) \vee (\mathcal{L} \cap \mathcal{H}_1)$ and $\mathcal{L} \cap \mathcal{H}_0 \in \operatorname{Hyplat} T_0$. Therefore

$$A_0 S(\mathcal{L} \cap \mathcal{H}_0) \subset A_0(\mathcal{L} \cap \mathcal{H}_0) \subset \mathcal{L} \cap \mathcal{H}_0 \quad \text{and} \quad A_0 S(\mathcal{L} \cap \mathcal{H}_1) = \{0\}.$$

This means $A_0 S\mathcal{L} \subset \mathcal{L}$ and so $A_0 S \in \{T\}'$. It follows that

$$T_0 A_0 S = T A_0 S = A_0 S T = A_0 T S = A_0 T_0 S.$$

Since $\mathcal{H}_0 = \overline{S\mathcal{H}}$ this means $A_0 \in \{T_0\}'$ and T_0 is hyper-reflexive.

To prove the other implication let us suppose that T_0 is hyper-reflexive. T_1 is a C_{11} contraction, therefore it is hyper-reflexive, too. If $A \in \operatorname{Alg}\operatorname{Hyplat} T$, then $A_0 = A \mid \mathcal{H}_0$ belongs to $\operatorname{Alg}\operatorname{Hyplat} T_0$ and $A_1 = A \mid \mathcal{H}_1 \in \operatorname{Alg}\operatorname{Hyplat} T_1$: To prove this it sufficies to observe that $\operatorname{Hyplat} T_0 \subset \operatorname{Hyplat} T$ and $\operatorname{Hyplat} T_1 \subset \operatorname{Hyplat} T$. Therefore

$$A_0 T_0 = T_0 A_0, \quad A_1 T_1 = T_1 A_1.$$

Since $\mathcal{H} = \mathcal{H}_0 \vee \mathcal{H}_1$ this means $AT = TA$ and T is hyper-reflexive.

COROLLARY 5.2. *Let T be a weak contraction and let m be the minimal function of T_0 (the C_0 part of T). Then T is hyper-reflexive if and only if $S(m)$,* i.e. *the first operator in the Jordan model of T_0 is hyper-reflexive.*

Proof. The corollary is a direct consequence of [1, Theorem B] and Theorem 5.1.

6. COMPARISON OF HYPER-REFLEXIVITY AND THE PROPERTY (L)

In Corollary 4.2. we have shown that an isometry has the property (L) if and only if it is hyper-reflexive. It is a natural question whether the property (L) and hyper-reflexivity are equivalent for other classes of operators. In general the answer to both implications in question are negative.

(L) does not imply hyper-reflexivity because every operator on a finite-dimensional space has the property (L) [6], but the operator $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is not hyper-reflexive. It can be seen by easy computation that $B \in \{A\}'$ if and only if $B = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ for some complex numbers a, b. It follows that the only non-trivial hyper-invariant subspace for A is the second coordinate space: $\left\{ \begin{pmatrix} 0 \\ c \end{pmatrix} : c \text{ — complex number} \right\}$. Therefore the operator $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ belongs to $\operatorname{Alg}\operatorname{Hyplat} T$ without being in $\{T\}'$.

H^∞ means the algebra of all bounded analytic functions in the unit disk. We shall consider H^∞ as a subalgebra of the algebra L^∞ of all essentially bounded functions on the unit circle.

In the following example we shall construct a contraction T of class C_{11} and therefore hyper-reflexive with the bicommutant property:

$$(*) \qquad \{T\}'' = \{\varphi(T) : \varphi \in H^\infty\}.$$

The operator T does not have the property (L) because it has many non-trivial hyperinvariant subspaces but according to [11, Proposition III.4.1] $\operatorname{Ker}\varphi(T) = \{0\}$ and $\overline{\varphi(T)\mathcal{H}} = \mathcal{H}$ for all non-zero $\varphi \in H^\infty$.

EXAMPLE 6.1. (See [11, Chapter VI.4.2]). *There exists a hyper-reflexive operator not having the property* (L).

Proof. For $n = 1, 2, \ldots$ let θ_n be the (constant) outer function

$$\theta_n(e^{it}) = \sqrt{1/(n+1)}.$$

Let us consider the functional model for θ_n given by [11, Theorem VI.3.1]:

$$\Delta_n(e^{it}) = \sqrt{n/(n+1)}, \quad K_n = H^2 \oplus \sqrt{n/(n+1)}L^2 \tag{1}$$

$$G_n = \left\{ w\sqrt{1/(n+1)} \oplus w\sqrt{n/(n+1)} : w \in H^2 \right\} = \{ w \oplus w\sqrt{n} : w \in H^2 \}. \tag{2}$$

$$H_n = G_n^\perp = K_n \ominus G_n \tag{3}$$

$$U_n \in \mathcal{B}(K_n) \quad U_n(u \oplus v) = e^{it}u \oplus e^{it}v \quad (u \oplus v \in K_n)$$

$T_n = P_n U_n \mid H_n$, where P_n denotes the orthogonal projection from K_n onto H_n.

Since every T_n is a C_{11} contraction so is the operator

$$T = \bigoplus_{n=1}^{\infty} T_n \quad \text{on the space } H = \bigoplus_{n=1}^{\infty} H_n. \tag{4}$$

Let D, D' be operators on $\mathcal{H}$, $\mathcal{H}'$, respectively. $I(D, D')$ denotes the set of intertwining operators:

$$I(D, D') = \{ X \in \mathcal{B}(\mathcal{H}, \mathcal{H}') : XD = D'X \}.$$

($\mathcal{B}(\mathcal{H}, \mathcal{H}')$ denots the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}'$.)

Let $A \in \{T\}'$. We consider the matrix representation of A corresponding to the decomposition (4):

$$A = (A_{rs}), \quad \text{where } A_{rs} \in I(T_s, T_r).$$

Note that

$$L^2 = \bigcap_{k=0}^{\infty} U_n^k K_n, \quad n = 1, 2, \ldots \tag{5}$$

According to the lifting theorem [11, Theorem II.2.3] there exists

$$(6) \qquad B_{rs} \in I(U_s, U_r)$$

satisfying

$$(7) \qquad B_{rs}G_s \subset G_r, \quad \text{or equivalently} \quad P_r B_{rs} P_s = P_r B_{rs}$$

$$A_{rs} = P_r B_{rs} \mid H_s \text{ and } \|B_{rs}\| = \|A_{rs}\| \leqslant \|A\|.$$

Using the matrix representation of B_{rs} corresponding to the decomposition (1) we obtain from (5), (6)

$$B_{rs} = \begin{pmatrix} \varphi_{rs} & 0 \\ g_{rs} & f_{rs} \end{pmatrix} \quad \text{for some} \quad \varphi_{rs} \in H^\infty \text{ and } f_{rs},\ g_{rs} \in L^\infty$$

satisfying

$$(8) \qquad \varphi_{rs} = \sqrt{1/r}\, g_{rs} + \sqrt{s/r}\, f_{rs}$$

(8) is the necessary and sufficient condition for B_{rs} to satisfy (7). We denote the operator of multiplication by a function by the same letter as the function itself.

Suppose that $A \in \{T\}''$. For any pair (m,n) of natural numbers let $X \in \{T\}'$ be given by a matrix with the only non-zero entry $X_{mn} = P_m Y_{mn} \mid H_n$, where

$$(9) \qquad Y_{mn} = \begin{pmatrix} \varphi & 0 \\ g & f \end{pmatrix}, \quad \varphi \in H^\infty,\ f,\ g \in L^\infty, \varphi = \sqrt{1/m}\, g + \sqrt{n/m}\, f$$

$AX = XA$ implies:

$$(10) \qquad A_{jm} X_{mn} = 0 \quad \text{for all } j : j \neq m$$

$$(11) \qquad A_{mm} X_{mn} = X_{mn} A_{nn}.$$

Note that for $u \in H^2$, $v \in L^2$, $u \oplus v \in H_n$ means for all $w \in H^2$, $u \oplus v \perp w \oplus \sqrt{n}\, w$, i.e.

$$(12) \qquad u \oplus v \in H_n \iff u + \sqrt{n}\, v \perp H^2.$$

In particular, if in (12) $u = 0$, $v = \mathrm{e}^{-it}$ and in (9) $g = 0$, $f = 1$, $\varphi = \sqrt{n/m}$, then by (7), (10)

$$P_j B_{jm} P_m Y_{mn} (0 \oplus \mathrm{e}^{-it}) = P_j B_{jm} Y_{mn} (0 \oplus \mathrm{e}^{-it}) = 0,$$

by (3) this means $B_{jm}Y_{mn}(0\oplus e^{-it}) = 0\oplus f_{jm}e^{-it} \in G_j$. By (2) it follows $f_{jm} = 0$ and it is easy to see that this means $A_{jm} = 0$. By (11) $(B_{mm}Y_{mn} - Y_{mn}B_{nn})(0\oplus e^{it}) \in G_m$ and this means

$$f_{mm} = f_{nn} = f_0. \tag{13}$$

Observe that the operator $A_{nn} = P_n B_{nn} \mid H_n = 0$ if $f_{nn} = 0$. Therefore for a given fixed $f_{nn} = f_0 \in L^\infty$ the operator A_{nn} does not depend on the choice of $\varphi \in H^\infty, g \in L^\infty$ satisfying (9): $g = \sqrt{n}(\varphi - f_0)$. Let us denote by $\|u\|$ the L_2 norm and by $\|u\|_\infty$ the L^∞ norm of a function $u \in L^\infty$. If P_- denotes the orthogonal projection from L^2 onto $(H^2)^\perp$, then

$$P_-(\varphi - f_0) = -P_- f_0 \text{ and } \|P_- f_0\| < \|\varphi - f_0\| \leqslant \|\varphi - f_0\|_\infty.$$

Since $\|g\|_\infty \leqslant \|B_{nn}\| \leqslant \|A\|$ this is possible only if $P_- f_0 = 0$, i.e. $f_0 \in H^\infty$ and $A = f_0(T)$. This means that T has the required bicommutant property $(*)$.

Acknowledgements. The author wishes to thank the referee for his help to avoid an error in the proof of Example 6.1.

REMARK. After submitting the paper the author obtained an unpublished note of Katsutoshi Takahashi: Bicommutants of some C_{11}-contractions, in which a more general example of a C_{11}-contraction satisfying $(*)$ can be found.

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MICHAL ZAJAC
Department of Mathematics,
Faculty of Electrical Engineering,
Slovak Technical University,
81219 Bratislava,
Czechoslovakia.

Received June 6, 1989; revised August 9, 1991.