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ON THE INDEX OF THE INFINITESIMAL GENERATOR OF A FLOW

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Dedicated to my teacher Manfred Breuer in honour of his sixtieth birthday

1. INTRODUCTION

Toeplitz C^*-algebras occur in various contexts. Let us recall first the construction of the classical Toeplitz C^*-algebra on the unit disc. Denote by P the orthogonal Szegö projection onto the Hardy space $H^2(\mathbb{S}^1)$. Then for $f \in C(\mathbb{S}^1)$, the Toeplitz operator associated to f is defined by $T_f := PM_fP$, where M_f is the operator of multiplication by f. The Toeplitz C^*-algebra $\mathcal{T}(\mathbb{S}^1)$, generated by all Toeplitz operators T_f, fits into an exact sequence

$$0 \to \mathcal{KL}(H^2(\mathbb{S}^1)) \to \mathcal{T}(\mathbb{S}^1) \xrightarrow{\sigma} C(\mathbb{S}^1) \to 0$$

and the classical Gohberg-Krein index theorem states that T_f is Fredholm if and only if f is invertible and $\mathrm{ind}(T_f)$ is the negative of the winding number of f around zero.

We can look at this theorem from different points of view. First, the Hardy space $H^2(\mathbb{S}^1)$ comes out of function theory. The Toeplitz construction is possible on every strongly pseudoconvex or bounded symmetric domain and there are several papers on this ([2], [11]).

On the other hand, P is the spectral projection onto the positive spectral subspace of the elliptic differential operator $\frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}$ (φ the angular function on $\mathbb{S}^1$). Therefore the Gohberg-Krein index theorem is a special case of the so called odd index theorem [1].

But $\frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}$ is also a vector field and generates a flow on $\mathbb{S}^1$. This is the point of view of this note. To a flow α on a C^*-algebra $\mathcal{A}$ we construct a Toeplitz extension

$$(*) \qquad 0 \to \mathcal{C} \to \mathcal{T} \to \mathcal{A} \to 0.$$

If there is an α-invariant trace on $\mathcal{A}$, above this would be the surface measure on $\mathbf{S}^1$, we can prove an index theorem analogous to the Gohberg-Krein index theorem. In general, our index is real-valued and so we make use of the Fredholm theory of Breuer [3].

The sequence (*) contains as special cases the classical Toeplitz extension, the almost periodic Toeplitz extensions [12], especially Toeplitz extensions on the n--torus [8], the Toeplitz extensions for flows on compact spaces [4], but also "non--commutative" analogues, for example flows on non-commutative tori.

Even in the commutative case we are slightly more general than [4], because we do not need any minimality assumptions.

In [5] a pseudo-differential calculus for flows is developed and an index theorem is stated. Our index theorem can be viewed as the odd analogue to this. As a corollary, we get another proof of a theorem of Connes ([6], Theorem III.3) on the behaviour of dual traces under the Thom isomorphism.

2. TOEPLITZ C^*-ALGEBRAS

We consider a C^*-dynamical system $(\mathcal{A}, \mathbf{R}, \alpha)$ and a covariant representation (π, λ), i.e. π is a representation on the Hilbert space $\mathcal{H}$ and λ is a one parameter group of unitaries on $\mathcal{H}$ such that for $t \in \mathbf{R}$ and $a \in \mathcal{A}$

$$\lambda_t \pi(a) \lambda_{-t} = \pi(\alpha_t(a)). \tag{2.1}$$

For $f, g \in L^1(\mathbf{R}, \mathcal{A})$ one defines

$$\begin{aligned} f * g(s) &:= \int_{\mathbf{R}} f(t)\alpha_t(g(s-t))\mathrm{d}t \\ f^*(s) &:= \alpha_s(f(-s)^*). \end{aligned} \tag{2.2}$$

With this product and involution $L^1(\mathbf{R}, \mathcal{A})$ is a $*$-algebra, the twisted convolution algebra of the C^*-dynamical system $(\mathcal{A}, \mathbf{R}, \alpha)$. The universal enveloping C^*-algebra of $L^1(\mathbf{R}, \mathcal{A})$ is the covariance algebra or crossed product $\mathcal{A} \rtimes_\alpha \mathbf{R}$.

(π, λ) gives rise to a representation $\pi \times \lambda$ of $L^1(\mathbf{R}, \mathcal{A})$ and hence of $\mathcal{A} \rtimes_\alpha \mathbf{R}$ on $\mathcal{H}$. For $f \in L^1(\mathbf{R}, \mathcal{A})$ one has

$$(\pi \times \lambda)(f) := \int_{\mathbf{R}} \pi(f(t))\lambda_t \mathrm{d}t. \tag{2.3}$$

An important special covariant representation is the following one. Let π be a representation of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$. For $\xi \in L^2(\mathbf{R}, \mathcal{H})$ one defines

$$\begin{aligned} \tilde{\pi} : \mathcal{A} &\longrightarrow \mathcal{L}(L^2(\mathbf{R},\mathcal{H})) \\ (\tilde{\pi}(a)\xi)(t) &:= \pi(\alpha_{-t}(a))(\xi(t)) \\ \mu_t\xi &:= \xi(\cdot - t). \end{aligned} \tag{2.4}$$

Since $\mathbf{R}$ is abelian, thus amenable, the corresponding representation $\tilde{\pi} \times \mu$ of $\mathcal{A} \rtimes_\alpha \mathbf{R}$ is known to be faithful if π is faithful.

We turn back to an arbitrary covariant representantion (π, λ) on $\mathcal{H}$. By Stone's theorem, there is a unique (unbounded) self-adjoint operator D on $\mathcal{H}$, such that

$$\lambda_t = \mathrm{e}^{2\pi \mathrm{i} t D}. \tag{2.5}$$

D can be defined by

$$Dh = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \lambda_t h \bigg|_{t=0}$$

for h in a suitable dense subspace of $\mathcal{H}$.

D is related to the infinitesimal generator of the flow α. Let $\mathcal{A}^\infty$ be the subalgebra consisting of those elements $a \in \mathcal{A}$, for which $t \mapsto \alpha_t(a)$ is C^∞. $\mathcal{A}^\infty$ is a dense local C^*-algebra. Define

$$\begin{aligned} \delta : \mathcal{A}^\infty &\longrightarrow \mathcal{A}^\infty \\ a &\mapsto \lim_{t\to 0} \frac{1}{t}(\alpha_t(a) - a). \end{aligned} \tag{2.6}$$

δ is a $*$-derivation of $\mathcal{A}^\infty$. Now, for $a \in \mathcal{A}^\infty$ and h in the domain of D we have

$$\begin{aligned} \pi(\delta(a))h &= \frac{\mathrm{d}}{\mathrm{d}t} \pi(\alpha_t(a))h \bigg|_{t=0} = \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{2\pi \mathrm{i} t D} \pi(a) \mathrm{e}^{-2\pi \mathrm{i} t D} h \bigg|_{t=0} = \\ &= 2\pi \mathrm{i}[D, \pi(a)]h. \end{aligned}$$

D plays the role of the elliptic differential operator $\frac{1}{\mathrm{i}} \frac{\partial}{\partial \varphi}$ in the introduction. As a self-adjoint operator, D admits a Borel functional calculus. Let

$$P := 1_{[0,\infty)}(D) \tag{2.7}$$

be the spectral projection onto the positive spectral subspace of D. For $a \in \mathcal{A}$ we define the Toeplitz operator T_a by

$$T_a := P\pi(a)P. \tag{2.8}$$

DEFINITION 2.1. The Toeplitz C^*-algebra $\mathcal{T}(\pi,\lambda)$ is the C^*-algebra generated by the operators T_a, $a \in \mathcal{A}$. By $\mathcal{C}(\pi,\lambda)$ we denote the closed $*$-ideal generated by $T_aT_b - T_{ab}$, $a,b \in \mathcal{A}$.

As in [8], it is convenient to consider a sort of "smoothed" version of $\mathcal{T}(\pi,\lambda)$. One reason is the following: If D has discrete spectrum $\left(\text{e.g. } D = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}\right)$, $1_{[0,\infty)}(D)$ can be realized by a smooth function. In general, this is not possible and we get another Toeplitz algebra by considering a smooth function

$$h : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \mapsto \begin{cases} 0, & t \leqslant t_0 \\ 1, & t \geqslant t_1 \\ \text{increasing}, & t_0 \leqslant t \leqslant t_1 \end{cases}, \quad t_0 < t_1. \tag{2.9}$$

h' is C^∞ with compact support.

DEFINITION 2.2. $\mathcal{T}_h(\pi,\lambda)$ is the C^*-algebra generated by $(\pi\times\lambda)(\mathcal{A}\rtimes_\alpha \mathbb{R})$ and the operators $h(D)\pi(a)h(D)$, $a \in \mathcal{A}$.

Let $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, $f \mapsto \int_{\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}x\cdot} f(x)\mathrm{d}x$ be the Fourier transform, where the integral converges in the L^2-sense. By $\mathcal{F}$ we also denote the Fourier transform on $L^1(\mathbb{R})$ or, more generally, on tempered distributions.

LEMMA 2.3. 1. *For $g \in L^1(\mathbb{R})$ we have*

$$(\overline{\mathcal{F}}g)(D) = \int_{\mathbb{R}} g(t)\lambda_t \mathrm{d}t.$$

2. *For h as in (2.9) and $a \in \mathcal{A}^\infty$ we have*

$$[h(D),\pi(a)] = \frac{1}{2\pi i}\int_{\mathbb{R}} (\mathcal{F}(h'))(t)\pi\left(\frac{\alpha_t(a)-a}{t}\right)\lambda_t \mathrm{d}t,$$

in particular, $[h(D),\pi(a)] \in (\pi\times\lambda)(\mathcal{A}\rtimes_\alpha\mathbb{R})$.

3. *For $T \in (\pi\times\lambda)(\mathcal{A}\rtimes_\alpha \mathbb{R})$ or $T \in \mathcal{C}(\pi,\lambda)$ we have*

$$\lim_{N\to\infty} T1_{[N,\infty)}(D) = 0.$$

Proof. 1. Let $(e_s)_{s\in\mathbb{R}}$ be the spectral measure of D. Then

$$(\overline{\mathcal{F}}g)(D) = \int_{\mathbb{R}} (\overline{\mathcal{F}}g)(s)\mathrm{d}e_s =$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{2\pi\mathrm{i}st} g(t)\mathrm{d}t\mathrm{d}e_s =$$

$$= \int_{\mathbb{R}} g(t) \int_{\mathbb{R}} e^{2\pi i s t} de_s dt =$$

$$= \int_{\mathbb{R}} g(t) e^{2\pi i t D} dt =$$

$$= \int_{\mathbb{R}} g(t) \lambda_t dt.$$

2. Suppose first that h lies in the Schwartz space $\mathcal{S}(\mathbf{R})$. With the identity

$$2\pi i t(\mathcal{F}h)(t) = (\mathcal{F}(h'))(t)$$

we get

$$[h(D), \pi(a)] = \int_{\mathbb{R}} (\mathcal{F}h)(t)[\lambda_t, \pi(a)] dt =$$

$$= \int_{\mathbb{R}} (\mathcal{F}h)(t)\pi(\alpha_t(a) - a)\lambda_t dt =$$

$$= \frac{1}{2\pi i} \int_{\mathbb{R}} (\mathcal{F}(h'))(t)\pi\left(\frac{\alpha_t(a) - a}{t}\right) \lambda_t dt.$$

Now, let h be as in (2.9) and h_N be a C^∞-function with compact support and

$$h_N(x) = \begin{cases} h(x), & x \leqslant N \\ 0, & x \geqslant N + \dfrac{1}{N} \end{cases}, \quad |h_N(x)| \leqslant 1.$$

$h_N(D)$ converges to $h(D)$ strongly, hence

$$\begin{aligned} [h(D), \pi(a)] &= \lim_{N\to\infty} [h_N(D), \pi(a)], \text{ strongly} \\ &= \lim_{N\to\infty} \frac{1}{2\pi i} \int_{\mathbb{R}} (\mathcal{F}(h'_N))(t)\pi\left(\frac{\alpha_t(a) - a}{t}\right) \lambda_t dt, \text{ strongly} \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} (\mathcal{F}(h'))(t)\pi\left(\frac{\alpha_t(a) - a}{t}\right) \lambda_t dt. \end{aligned}$$

3. Let $T \in (\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$. It suffices to check the assertion for T in a dense subspace of $(\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$, i.e. for

$$T = \int_{\mathbb{R}} \pi(\varphi(t))\lambda_t dt$$

with $\varphi \in C_c(\mathbb{R},\mathcal{A}) = \{f : \mathbb{R} \to \mathcal{A} | f \text{ continuous with compact support}\}$. Since $C_c(\mathbb{R}) \otimes \mathcal{A}$ is dense in $C_c(\mathbb{R},\mathcal{A})$ (in the L^1-norm), it suffices to consider $\varphi = \psi \otimes a$, $\psi \in C_c(\mathbb{R})$, $a \in \mathcal{A}$. Then

$$\int_{\mathbb{R}} \pi((\psi \otimes a)(t))\lambda_t dt 1_{[N,\infty)}(D) = \pi(a)(\overline{\mathcal{F}}\psi)(D)1_{[N,\infty)}(D) \tag{2.10}$$

and as N tends to infinity, $(\overline{\mathcal{F}}\psi)1_{[N,\infty)}$ tends to zero uniformly, hence

$$\lim_{N\to\infty} (\overline{\mathcal{F}}\psi)(D)1_{[N,\infty)}(D) = 0.$$

Though one should expect, that the proof for $T \in \mathcal{C}(\pi,\lambda)$ is as straightforward as for $T \in (\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbb{R})$, we only could find a proof which is not difficult but lengthy. Therefore, we postpone it to the appendix. ∎

3. EXACT SEQUENCES

PROPOSITION 3.1. *Let (π,λ) be a covariant representation of $(\mathcal{A},\mathbb{R},\alpha)$ such that π is faithful for $\mathcal{A}$ and $\pi \times \lambda$ is faithful for $\mathcal{A} \rtimes_\alpha \mathbb{R}$.*

1. *$\mathcal{T}_h(\pi,\lambda)$ is independent of h and (π,λ). Therefore, we refer to it as $\mathcal{T}_{\text{sm}}$ (sm for smooth). There is an exact sequence*

$$0 \to \mathcal{A} \rtimes_\alpha \mathbb{R} \to \mathcal{T}_{\text{sm}} \xrightarrow{\sigma_{\text{sm}}} \mathcal{A} \to 0. \tag{3.1}$$

For any h as in (2.9), $s : a \mapsto h(D)\pi(a)h(D)$ is a completely positive cross-section of σ_{sm}.

2. *There is an exact sequence*

$$0 \to \mathcal{C}(\pi,\lambda) \to \mathcal{T}(\pi,\lambda) \xrightarrow{\sigma} \mathcal{A} \to 0 \tag{3.2}$$

with complętly positive cross-section $a \mapsto T_a$.

REMARK 3.2. 1. In [4], sequence (3.2) is derived for $\mathcal{A} = C(X)$, X compact, and α minimal. We do not need the assumption that α is minimal. But the price we pay is that the proof is rather long.

2. One should expect that $\mathcal{T}(\pi,\lambda)$ is independent of (π,λ), too. But the author could not find a proof in this generality. For minimal, uniquely ergodic flows on commutative C^*-algebras the independence of (π,λ) is a consequence of the much stronger result [9], Theorem 1.

Proof. 1. $(\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$ is a $*$-ideal in $\mathcal{T}_h(\pi,\lambda)$. To show this, let $\varphi = \psi \otimes a$, $\psi \in \mathcal{S}(\mathbf{R})$, $a \in \mathcal{A}$ and $b \in \mathcal{A}$. Then

$$(\pi \times \lambda)(\varphi)h(D)\pi(b)h(D) = \pi(a)(\overline{\mathcal{F}}\psi)(D)h(D)\pi(b)h(D) = \quad cf.(2.10)$$
$$= \pi(a)((\overline{\mathcal{F}}\psi)h)(D)[\pi(b),h(D)] + \pi(a)((\overline{\mathcal{F}}\psi)h^2)(D)\pi(b).$$

$(\overline{\mathcal{F}}\psi)h \in \mathcal{S}(\mathbf{R})$, so $\pi(a)((\overline{\mathcal{F}}\psi)h)(D) \in (\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$ and by Lemma 2.3 the first summand lies in $(\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$. Since the elements of A are multipliers of $A \rtimes_\alpha \mathbf{R}$, the second summand lies in $(\pi \times \lambda)(A \rtimes_\alpha \mathbf{R})$, too. Since finite sums of elements of the form $\psi \otimes a$ are dense in $(\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$, it is a right ideal. Since it is a C^*-algebra, it is a $*$-ideal.

For h_1, h_2 as in (2.9), $h_1 - h_2$ is C^∞ with compact support and hence we have $h_1(D) - h_2(D) \in (\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$, which implies $\mathcal{T}_{h_1}(\pi,\lambda) = \mathcal{T}_{h_2}(\pi,\lambda)$.

By Lemma 2.3, s induces a $*$-homomorphism

$$s' : \mathcal{A} \to \mathcal{T}_h(\pi,\lambda)/(\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R}),$$

which is obviously surjective.

Now consider $a \in \operatorname{Ker}(s')$. This is equivalent to $h(D)\pi(a)h(D) \in (\pi \times \lambda)\cdot(\mathcal{A} \rtimes_\alpha \mathbf{R})$.

$$h(D)\pi(a)h(D) = [h(D),\pi(a)]h(D) + \pi(a)(h(D)^2 - h(D)) + \pi(a)h(D),$$

thus by Lemma 2.3 we have $\pi(a)h(D) \in (\pi \times \lambda)(\mathcal{A} \rtimes_\alpha \mathbf{R})$ and

$$\lim_{N\to\infty} \pi(a)1_{[N,\infty)}(D) = 0. \tag{3.3}$$

Now we consider smooth functions

$$f_{N,k} : \mathbf{R} \to \mathbf{R},\ t \mapsto \begin{cases} 1, & N \leqslant t \leqslant N+2k \\ 0, & t \leqslant N - \frac{1}{N},\ t \geqslant N+2k+\frac{1}{N} \\ \text{increasing}, & N - \frac{1}{N} \leqslant t \leqslant N \\ \text{decreasing}, & N+2k \leqslant t \leqslant N+2k+\frac{1}{N}. \end{cases}$$

The norm of $\pi(a)f_{N,k}(D)$ can be estimated by considering the representation $\tilde{\pi} \times \mu$ (cf. (2.4)) of $\mathcal{A} \rtimes_\alpha \mathbf{R}$ on $L^2(\mathbf{R},\mathcal{H})$. For $h \in \mathcal{H}$ we get

$$((\tilde{\pi} \times \mu)(\pi \times \lambda)^{-1}(\pi(a)f_{N,k}(D)))((\mathcal{F}1_{[N,N+2k]})h)(s) =$$
$$= \left(\tilde{\pi}(a)\int_{\mathbf{R}}(\mathcal{F}f_{N,k}(t)\mu_t \mathrm{d}t)((\mathcal{F}1_{[N,N+2k]})h)\right)(s) =$$
$$= \int_{\mathbf{R}}(\mathcal{F}f_{N,k})(t)(\mathcal{F}1_{[N,N+2k]})(s-t)\mathrm{d}t\pi(\alpha_{-s}(a))h =$$
$$= ((\mathcal{F}f_{N,k}) * (\mathcal{F}1_{[N,N+2k]}))(s)\pi(\alpha_{-s}(a))h =$$
$$= (\mathcal{F}1_{[N,N+2k]})(s)\pi(\alpha_{-s}(a))h,$$

since $f_{N,k}1_{[N,N+2k]} = 1_{[N,N+2k]}$.

Choose $h \in \mathcal{H}$, $\|h\| = 1$, such that $\|\pi(a)h\| \geqslant (1-\epsilon)\|a\|$ and choose k, such that

$$\|\pi(\alpha_{-t}(a))h - \pi(a)h\| \leqslant \epsilon\|a\| \quad \text{for} \quad |t| \leqslant \frac{1}{8k}.$$

An easy calculation shows that

$$(\mathcal{F}1_{[N,N+2k]})(s) = e^{-2\pi i s(N+k)}\frac{\sin(2\pi sk)}{\pi s}$$

and for $|s| \leqslant \frac{1}{8k}$ we have

$$|(\mathcal{F}1_{[N,N+2k]})(s)| \geqslant 8k\frac{\sin(\pi/4)}{\pi} = \frac{8k}{\sqrt{2}\pi}.$$

Now, $\|\mathcal{F}1_{[N,N+2k]}\|_2^2 = \|1_{[N,N+2k]}\|_2^2 = 2k$ and hence

$$\begin{aligned}
\|\pi(a)f_{N,k}(D)\|^2 &\geqslant \left\|(\tilde{\pi}\times\mu)(\pi\times\lambda)^{-1}(\pi(a)f_{N,k}(D))\left(\frac{1}{\sqrt{2k}}(\mathcal{F}1_{[N,N+2k]})h\right)\right\|_2^2 = \\
&= \frac{1}{2k}\int_{\mathbb{R}} \|\pi(\alpha_{-s}(a))h\|_{\mathcal{H}}^2 \, |\mathcal{F}1_{[N,N+2k]})(s)|^2 ds \geqslant \\
&\geqslant \frac{1}{2k}\int_{-1/8k}^{1/8k} \|\pi(\alpha_{-s}(a))h\|_{\mathcal{H}}^2 ds \frac{32k^2}{\pi^2} \geqslant \\
&\geqslant \frac{16k}{\pi^2}\int_{-1/8k}^{1/8k} \left|\, \|\pi(\alpha_{-s}(a))h - \pi(a)h\|_{\mathcal{H}} - \|\pi(a)h\|_{\mathcal{H}}\, \right|^2 ds \geqslant \\
&\geqslant \frac{16k}{\pi^2}(1-2\epsilon)^2\|a\|^2\frac{1}{4k} = \\
&= \frac{4}{\pi^2}(1-2\epsilon)^2\|a\|^2,
\end{aligned}$$

thus (3.3) can be fulfilled only if $a = 0$.

The independence of $\mathcal{T}_{\mathrm{sm}}(\pi,\lambda)$ from (π,λ) is clear: let (π_u,λ_u) be the universal representation of $\mathcal{A} \rtimes_\alpha \mathbb{R}$. Then, since (π,λ) is contained in (π_u,λ_u), we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{A}\rtimes_\alpha\mathbb{R} & \to & \mathcal{T}_{\mathrm{sm}}(\pi_u,\lambda_u) & \to & \mathcal{A} & \to & 0 \\
 & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \mathrm{id} & & \\
0 & \to & \mathcal{A}\rtimes_\alpha\mathbb{R} & \to & \mathcal{T}_{\mathrm{sm}}(\pi,\lambda) & \to & \mathcal{A} & \to & 0
\end{array}$$

and the assertion follows from the five-lemma.

2. As in 1. we have a surjective $*$-homomorphism

$$s' : \mathcal{A} \to \mathcal{T}(\pi, \lambda)/\mathcal{C}(\pi, \lambda).$$

Consider $a \in \mathrm{Ker}(s')$, i.e. $1_{[0,\infty)}(D)\pi(a)1_{[0,\infty)}(D) \in \mathcal{C}(\pi, \lambda)$ and consider h as in (2.9) with $h(t) = 0$ for $t \leqslant 0$. Then by Lemma 2.3 we have

$$\lim_{N\to\infty} 1_{[0,\infty)}(D)\pi(a)1_{[N,\infty)}(D) = 0$$

and thus

$$\lim_{N\to 0} h(D)\pi(a)1_{[N,\infty)}(D) = 0.$$

Now, the same computation as in 1. shows that (3.3) is satisfied for a, which is only possible for $a = 0$. ∎

4. A REAL VALUED INDEX THEOREM

From now on, we assume that $\mathcal{A}$ is unital and admits a faithful finite α-invariant trace τ, which we normalize by $\tau(1_A) = 1$. $\mathcal{A}$ admits a Hilbert algebra structure with scalar product

$$(a \mid b) := \tau(a^*b). \tag{4.1}$$

We denote the Hilbert space completion of $\mathcal{A}$ with respect to this scalar product by $L^2(\mathcal{A}, \tau)$. The left regular representation of $\mathcal{A}$ on $L^2(\mathcal{A}, \tau)$ is faithful since τ is faithful.

According to (2.4) this representation induces a covariant representation (π, λ) on the Hilbert space $L^2(\mathbf{R}, L^2(\mathcal{A}, \tau)) \cong L^2(\mathbf{R}) \otimes L^2(\mathcal{A}, \tau)$ by

$$\begin{aligned} (\pi(a)\xi)(t) &:= \alpha_{-t}(a)\xi(t) \\ \lambda_s\xi &:= \xi(\cdot - s), \quad \xi \in L^2(\mathbf{R}, \mathcal{A}). \end{aligned} \tag{4.2}$$

The corresponding representation $\pi \times \lambda$ of the twisted convolution algebra $L^1(\mathbf{R}, \mathcal{A})$ can also be interpreted as a left regular representation. First, $L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$ is a Hilbert algebra with scalar product

$$(f \mid g) := \tau((f^* * g)(0)). \tag{4.3}$$

The completion is, of course, $L^2(\mathbf{R}) \otimes L^2(\mathcal{A}, \tau)$. Now we define for $f \in L^2(\mathbf{R}, \mathcal{A})$

$$(Vf)(t) := \alpha_t(f(t)). \tag{4.4}$$

Since τ is α-invariant, V extends to a unitary operator on $L^2(\mathbf{R}) \otimes L^2(\mathcal{A}, \tau)$ and for $\varphi \in L^1(\mathbf{R}, \mathcal{A})$, $f \in L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$ and $a \in \mathcal{A}$ we have

$$\begin{aligned}
(V(\pi \times \lambda)(\varphi)V^*)(f) &= (V(\pi \times \lambda)(\varphi))(\alpha_{-\cdot}(f)) = \\
&= V \int_{\mathbf{R}} \alpha_{-\cdot}(\varphi(t))\alpha_{-\cdot+t}(f(\cdot - t))\mathrm{d}t = \\
&= \int_{\mathbf{R}} \varphi(t)\alpha_t(f(\cdot - t))\mathrm{d}t = \varphi * f, \\
(V\pi(a)V^*)(f) &= af =: M_a f, \\
\mu_t(f) := (V\lambda_t V^*)(f) &= V\lambda_t\alpha_{-\cdot}(f) = V\alpha_{-\cdot+t}(f(\cdot - t)) = \alpha_t(f(\cdot - t)).
\end{aligned}$$

V intertwines $\pi \times \lambda$ and the left regular representation of $L^1(\mathbf{R}, \mathcal{A})$ on the Hilbert algebra $L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$. Hence, the left regular representation extends to a faithful representation of the covariance algebra $\mathcal{A} \rtimes_\alpha \mathbf{R}$. From now on, we identify $\mathcal{A} \rtimes_\alpha \mathbf{R}$ with the image under this representation.

To a Hilbert algebra corresponds in a natural way a von Neumann algebra, the so called left von Neumann algebra. The left von Neumann algebra of $\mathcal{A}$ we denote by $L^\infty(\mathcal{A}, \tau)$. Of course, the left von Neumann algebra of $L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$ is the W^*-crossed product $L^\infty(\mathcal{A}, \tau) \rtimes_\alpha \mathbf{R}$.

By the general theory of Hilbert algebras, these von Neumann algebras admit natural traces. The trace on $L^\infty(\mathcal{A}, \tau)$ extends τ and the trace on $L^\infty(\mathcal{A}, \tau) \rtimes_\alpha \mathbf{R}$ extends the dual trace $\hat{\tau}$ on $\mathcal{A} \rtimes_\alpha \mathbf{R}$. So we denote these traces by τ and $\hat{\tau}$, too. Instead of $L^\infty(\mathcal{A}, \tau) \rtimes_\alpha \mathbf{R}$ now we can also write $L^\infty(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau})$.

For $1 \leqslant p < \infty$ we denote by

$$k_p := k_p(L^\infty(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau})) := L^\infty(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau}) \cap L^p(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau})$$

the space of p-summable operators in $L^\infty(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau})$. k_1 is the space of trace class operators. The norm closure

$$k_\infty := \overline{k}_1 = \overline{k}_p$$

is the Breuer ideal of (relatively) compact operators.

With Breuer [3] we call an operator $T \in L^\infty(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau})$ Fredholm if and only if it is invertible modulo the ideal k_∞. If T is Fredholm, the null projections $N(T)$ and $N(T^*)$ are trace class and we define the real-valued index of T

$$\operatorname{ind}_{\hat{\tau}}(T) := \hat{\tau}(N(T)) - \hat{\tau}(N(T^*)). \tag{4.5}$$

This real-valued index has the same properties which one knows from ordinary Fredholm theory. Especially, the index is locally constant on the space of Fredholm operators.

Note that there is a one-one correspondence between the elements of k_2 and the bounded elements in the completion $L^2(\mathbf{R}) \otimes L^2(\mathcal{A}, \tau)$ of the Hilbert algebra $L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$. We recall that an $f \in L^2(\mathbf{R}) \otimes L^2(\mathcal{A}, \tau)$ is called bounded, if and only if there is an operator $K_f \in L^\infty(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau})$ such that for $\xi \in L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$

$$K_f \xi = f * \xi,$$

where $\cdot * \xi$ is the operator of right multiplication by ξ. $f \mapsto K_f$ gives the above correspondence ([7], Theorem I.6.1). Thus we have

LEMMA 4.1. *An operator $K \in L^\infty(\mathcal{A} \rtimes_\alpha \mathbf{R}, \hat{\tau})$ lies in k_2 if and only if there is an $f \in L^2(\mathbf{R}, L^2(\mathcal{A}, \tau))$ such that for $\xi \in L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$*

$$(K\xi)(t) = \int_{\mathbf{R}} f(x) \alpha_x(\xi(t-x)) \mathrm{d}x.$$

For $K_f, K_g \in k_2$ we have

$$\hat{\tau}(K_f^* K_g) = (f \mid g)_{L^2(\mathbf{R}, L^2(\mathcal{A}, \tau))} = \int_{\mathbf{R}} \tau(f(x)^* g(x)) \mathrm{d}x.$$

Now let D be the self-adjoint operator corresponding to μ_t, i.e. $\mu_t = \mathrm{e}^{2\pi i t D}$ (cf. 2.5).

LEMMA 4.2. *For $1 \leqslant p < \infty$ and $f \in L^\infty(\mathbf{R}) \cap L^p(\mathbf{R})$ we have $f(D) \in k_p$. In the case $p = 1$ we have*

$$\hat{\tau}(f(D)) = \int_{\mathbf{R}} f.$$

Proof. Let $p = 2$: since μ_t is diagonalized by V and the Fourier transform, it is easy to see that for $\xi \in L^1(\mathbf{R}, \mathcal{A}) \cap L^2(\mathbf{R}, \mathcal{A})$

$$(f(D)\xi)(t) = \int_{\mathbf{R}} (\mathcal{F} f)(x) \alpha_x(\xi(t-x)) \mathrm{d}x,$$

hence $f(D) \in k_2$ by the preceding lemma.

For $f \in L^\infty(\mathbf{R}) \cap L^p(\mathbf{R})$ we have

$$|f(D)|^{p/2} = |f|^{p/2}(D) \in k_2,$$

thus $f(D) \in k_p$.

If $p = 1$ we write $f = \overline{g}h$ with $g, h \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, then

$$\begin{aligned}\hat{\tau}(f(D)) &= \hat{\tau}(K^*_{\mathcal{F}g} K_{\mathcal{F}h}) = \\ &= \int_{\mathbb{R}} \tau(\overline{(\mathcal{F}g)(x)}(\mathcal{F}h)(x))\mathrm{d}x = \\ &= (\mathcal{F}g \mid \mathcal{F}h)_{L^2(\mathbb{R})} = \\ &= (g \mid h)_{L^2(\mathbb{R})} = \\ &= \int_{\mathbb{R}} f(x)\mathrm{d}x.\end{aligned}$$

■

LEMMA 4.3. *Let $a \in \mathcal{A}^\infty$, $H := 2P - I$, $P := 1_{[0,\infty)}(D)$, $H_f := 2f(D) - I$, f a function as in (2.9). Then we have*

$$[H, M_a],\ [H_f, M_a] \in k_2.$$

Proof. By Lemma 2.3 we have

$$\begin{aligned}[H_f, M_a] &= 2[f(D), M_a] = \\ &= \frac{1}{\pi i}\int_{\mathbb{R}} (\mathcal{F}(f'))(t) M_{\frac{1}{t}(\alpha_t(a)-a)}\mu_t \mathrm{d}t.\end{aligned}$$

Since $a \in \mathcal{A}^\infty$, the function $t \mapsto \mathcal{F}(f')\frac{1}{t}(\alpha_t(a) - a)$ lies in $L^2(\mathbb{R}, \mathcal{A})$ and the assertion for H_f follows from Lemma 4.1. Since

$$H_f - H = 2(f - 1_{[0,\infty)})(D)$$

and $f - 1_{[0,\infty)} \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, we are done by the preceding lemma. ■

For $a, b \in \mathcal{A}^\infty$ we define the function

$$\varphi_{a,b} : \mathbb{R} \to \mathbb{C}, \quad t \mapsto \tau\left(\frac{a(\alpha_t(b) - b)}{t}\right).$$

$\varphi_{a,b}$ lies in $C_0(\mathbb{R})$.

PROPOSITION 4.4. *Let $a, b \in \mathcal{A}^\infty$. Then*

$$T_a T_b - T_{ab} \in k_1(L^\infty(\mathcal{A} \rtimes_\alpha \mathbb{R}, \hat{\tau})),$$
$$\mathcal{F}(\varphi_{a,b}) \in L^1(\mathbb{R})$$

and

$$\hat{\tau}(T_aT_b - T_{ab}) = \frac{1}{2\pi i}\int_0^\infty \mathcal{F}(\varphi_{a,b})(t)\mathrm{d}t.$$

REMARK. This proposition is a generalization of [4], Theorem 22.2 to the non--commutative case. It is the key to the proof of the index theorem.

Proof. $H_a := PM_a(I-P) = P[P, M_a] \in k_2$ by Lemma 4.3, thus

$$T_aT_b - T_{ab} = -PM_a(I-P)M_bP = \\ = -H_aH_{b^*}^* \in k_1.$$

$\varphi_{a,b}$ is a function of the type $\frac{f}{\mathrm{id}}$ with $f \in C^\infty(\mathbf{R})$, $f(0) = 0$, f, f' bounded. Moreover,

$$\varphi'_{a,b} = \frac{f' - \frac{1}{\mathrm{id}}f}{\mathrm{id}} =: \frac{g}{\mathrm{id}}$$

with $g \in C^\infty(\mathbf{R})$, $g(0) = 0$, g bounded. Thus we have $\varphi_{a,b}, \varphi'_{a,b} \in L^2(\mathbf{R})$ and hence $\mathcal{F}(\varphi_{a,b})$, $\mathrm{id}\,\mathcal{F}(\varphi_{a,b}) \in L^2(\mathbf{R})$. Now for any function $\psi \in L^2(\mathbf{R})$ with $\mathrm{id}\,\psi \in L^2(\mathbf{R})$ we obtain by the Cauchy-Schwarz inequality

$$\int_{\mathbf{R}} |\psi(x)|\mathrm{d}x = \int_{|x|\leqslant\frac{1}{2}} |\psi(x)|\mathrm{d}x + \int_{|x|>\frac{1}{2}} |\psi(x)|\mathrm{d}x \leqslant$$

$$\leqslant \left(\int_{|x|\leqslant\frac{1}{2}} |\psi(x)|^2\mathrm{d}x\right)^{1/2} + \left(\int_{|x|>\frac{1}{2}} \frac{\mathrm{d}x}{x^2}\right)^{1/2}\left(\int_{|x|>\frac{1}{2}} x^2|\psi(x)|^2\mathrm{d}x\right)^{1/2} < \infty,$$

and hence $\mathcal{F}(\varphi_{a,b}) \in L^1(\mathbf{R})$.

Choose $f_\epsilon, g_{N,\epsilon} \in C^\infty(\mathbf{R})$ with

$$f_\epsilon(t) = \begin{cases} 0, & t \leqslant 0 \\ \text{increasing}, & 0 \leqslant t \leqslant \epsilon \\ 1, & t \geqslant \epsilon \end{cases}$$

and

$$g_{N,\epsilon} = \begin{cases} 1, & |\,t\,| \leqslant N \\ \text{increasing}, & -N-\epsilon \leqslant t \leqslant -N \\ \text{decreasing}, & N \leqslant t \leqslant N+\epsilon \\ 0, & |\,t\,| \geqslant N+\epsilon. \end{cases}$$

As $\epsilon \to 0$ $f_\epsilon(D)$ converges to P in the trace norm. Since $\mathcal{F}(g_{N,\epsilon})$ is an approximate unit in $L^1(\mathbf{R})$, we have for $T \in k_1$

$$\lim_{N\to\infty} g_{N,\epsilon}(D)T = T$$

in the trace norm.

Let $h_{N,\epsilon} := g_{N,\epsilon} f_\epsilon^2$. Then we have

$$\begin{aligned}\hat{\tau}(T_a T_b - T_{ab}) &= \lim_{\epsilon\to 0} \hat{\tau}(f_\epsilon(D) M_a [f_\epsilon(D), M_b] f_\epsilon(D)) = \\ &= \lim_{\epsilon\to 0} \lim_{N\to\infty} \hat{\tau}(g_{N,\epsilon}(D) f_\epsilon(D)^2 M_a [f_\epsilon(D), M_b]) = \\ &= \lim_{\epsilon\to 0} \lim_{N\to\infty} \hat{\tau}(h_{N,\epsilon}(D) M_a [f_\epsilon(D), M_b]) = \\ &= \lim_{\epsilon\to 0} \lim_{N\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} (\mathcal{F} h_{N,\epsilon})(t)(\mathcal{F} f_\epsilon')(t) \varphi_{a,b}(t) \mathrm{d}t = \\ &= \lim_{\epsilon\to 0} \lim_{N\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\mathbf{R}} (h_{N,\epsilon} * f_\epsilon')(t)(\mathcal{F}\varphi_{a,b})(t) \mathrm{d}t,\end{aligned}$$

by Lemma 2.3 and Lemma 4.1. Moreover,

$$h_{N,\epsilon} * f_\epsilon'(t) = \int_0^\epsilon h_{N,\epsilon}(t-x) f_\epsilon'(x) \mathrm{d}x = \begin{cases} 0, & t \leqslant 0 \\ 1, & 2\epsilon \leqslant t \leqslant N \\ 0, & t \geqslant N + 2\epsilon. \end{cases}$$

So, applying Lebesgue's theorem two times, we obtain the formula. ■

COROLLARY 4.5. *Let $a, b \in \mathcal{A}^\infty$ be commuting elements. Then*

$$[T_a, T_b] \in k_1$$

and

$$\hat{\tau}([T_a, T_b]) = \frac{1}{2\pi i} \tau(a\delta(b)).$$

Proof. Denote by id the identify function on $\mathbf{R}$. τ is α-invariant, hence we have

$$\begin{aligned}\int_0^\infty \mathcal{F}\left(\tau\left(\frac{b(\alpha_{\mathrm{id}}(a) - a)}{\mathrm{id}}\right)\right)(t)\mathrm{d}t &= \int_{-\infty}^0 \mathcal{F}\left(\tau\left(\frac{b(\alpha_{-\mathrm{id}}(a) - a)}{-\mathrm{id}}\right)\right)(t)\mathrm{d}t = \\ &= -\int_{-\infty}^0 \mathcal{F}\left(\tau\left(\frac{a(\alpha_{\mathrm{id}}(b) - b)}{\mathrm{id}}\right)\right)(t)\mathrm{d}t.\end{aligned}$$

Since a and b commute, by the preceding proposition we get

$$\begin{aligned}\hat{\tau}([T_a, T_b]) &= \frac{1}{2\pi \mathrm{i}} \int_0^\infty (\mathcal{F}\varphi_{a,b})(t)\mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \int_0^\infty (\mathcal{F}\varphi_{b,a})(t)\mathrm{d}t = \\ &= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^\infty (\mathcal{F}\varphi_{a,b})(t)\mathrm{d}t = \\ &= \frac{1}{2\pi \mathrm{i}} \varphi_{a,b}(0) = \\ &= \frac{1}{2\pi \mathrm{i}} \tau(a\delta(b)).\end{aligned}$$

■

LEMMA 4.6. *For* $T \in \mathcal{A} \rtimes_\alpha \mathbf{R}$ *we have* $\lim_{n\to\infty} \| T1_{[n,n+1]}(D) \|_{2,\hat{\tau}} = 0$.

Proof. It suffices to check this for T in a dense subspace of $\mathcal{A} \rtimes_\alpha \mathbf{R}$ and hence it suffices to consider $T = K_f$ with $f(t) = a\varphi(t)$, $a \in \mathcal{A}$, $\varphi \in L^1(\mathbf{R}) \cap L^2(\mathbf{R})$. Then $T = M_a\varphi(D)$ and

$$\begin{aligned}\| T1_{[n,n+1]}(D) \|_{2,\hat{\tau}}^2 &= \hat{\tau}(1_{[n,n+1]}(D)\overline{\varphi}(D)M_{a^*a}\varphi(D)) = \\ &= \hat{\tau}(1_{[n,n+1]}(D)|\varphi|^2(D)M_{a^*a}) = \\ &= \int_n^{n+1} |\varphi|^2(t)\mathrm{d}t\tau(a^*a).\end{aligned}$$

Since $\varphi \in L^2(\mathbf{R})$, as n tends to infinity this expression tends to zero. ∎

THEOREM 4.7. *$T \in \mathcal{T}$ (resp. $\mathcal{T}_{\rm sm}$) is Breuer-Fredholm if and only if $\sigma(T)$ (resp. $\sigma_{\rm sm}(T)$) is invertible in $\mathcal{A}$.*

Proof. We have to prove that $\mathcal{T} \cap k_\infty = \mathcal{C}$ and $\mathcal{T}_{\rm sm} \cap k_\infty = \mathcal{A} \rtimes_\alpha \mathbf{R}$. $\mathcal{C} \subset k_\infty$ and $\mathcal{A} \rtimes_\alpha \mathbf{R} \subset k_\infty$ follows from Lemma 4.3 and Proposition 4.4. Now we let

$$T = f_\epsilon(D)M_a f_\epsilon(D) + C \in \mathcal{T}_{\rm sm} \cap k_\infty, \quad C \in \mathcal{A} \rtimes_\alpha \mathbf{R}.$$

Then we have

$$f_\epsilon(D)M_a f_\epsilon(D) \in k_\infty.$$

As in the proof of Proposition 3.1 this implies $M_a f_\epsilon(D) \in k_\infty$. As in the proof of Lemma 4.6 we get

$$\|M_a f_\epsilon(D)1_{[n,n+1]}(D)\|_{2,\hat{\tau}}^2 = \int_n^{n+1} |f_\epsilon|^2(t)\mathrm{d}t\tau(a^*a) = \tau(a^*a), \quad n \geqslant \epsilon.$$

Since τ is faithful, Lemma 4.6 implies $a = 0$. $1_{[0,\infty)}(D) - f_\epsilon(D) \in k_\infty$, thus $T_a \in k_\infty$ implies $f_\epsilon(D)M_a f_\epsilon(D) \in k_\infty$ and the theorem is proved. ∎

To compute the Breuer index we introduce the following Lemma, which has been used by several authors for the computation of Fredholm indices (cf. [9], Theorem 25.2 and [10], p.126).

LEMMA 4.8. *Let $\mathcal{M}$ be a von Neumann algebra with semifinite normal trace φ. For a (relative) Fredholm operator $T \in \mathcal{M}$ with $T^*T - I \in k_1(\mathcal{M}, \varphi)$, we have $[T, T^*] \in k_1(\mathcal{M}, \varphi)$ and the Breuer index is given by*

$$\mathrm{ind}_\varphi(T) = \varphi([T, T^*]).$$

Proof. Let $T = U|T|$ be the polar decomposition of T. Then the null projections of T and T^* are

$$N(T) = I - U^*U, \quad N(T^*) = I - UU^*.$$

Hence we have $[U, U^*] \in k_1(\mathcal{M}, \varphi)$ and

$$\mathrm{ind}_\varphi(T) = \varphi(N(T)) - \varphi(N(T^*)) = \varphi([U, U^*]).$$

Furthermore we get

$$\begin{aligned} T - U &= U(|T| - I) = U(|T| + I)^{-1}(|T|^2 - I) = \\ &= U(|T| + I)^{-1}(T^*T - I) \in k_1(\mathcal{M}, \varphi), \end{aligned}$$

thus $[T - U, T^* + U^*] \in k_1(\mathcal{M}, \varphi)$ and

$$\varphi([T - U, T^* + U^*]) = 0.$$

An easy calculation yieds

$$[T - U, T^* + U^*] = [T, T^*] - [U, U^*]$$

and the assertion follows. ■

THEOREM 4.9. *Let $T \in \mathcal{T}$ (resp. $\mathcal{T}_{\mathrm{sm}}$) with $a := \sigma(T)$ (resp. $\sigma_{\mathrm{sm}}(T)) \in \mathcal{A}^\infty$ invertible. Then the Breuer index of T is given by*

$$\mathrm{ind}_{\hat{\tau}}(T) = -\frac{1}{2\pi i}\tau(a^{-1}\delta(a)). \tag{4.6}$$

Proof. The index is a homotopy invariant and the right hand side is a homotopy invariant, too (cf. [5], p. 601). $T - T_a \in \mathrm{Ker}(\sigma) \subset k_\infty$, hence

$$T_t := T_{a(a^*a)^{-t/2}} + (1 - t)(T - T_a), \quad 0 \leqslant t \leqslant 1$$

is a family of Fredholm operators by Theorem 4.7. We have $T_0 = T$ and $b := \sigma(T_1) = = a(a^*a)^{-1/2}$ is unitary. Since $\mathcal{A}^\infty$ is a local C^*-algebra, b lies in $\mathcal{A}^\infty$, too. We have by Proposition 4.4

$$T_1^*T_1 - I = T_{b^*}T_b - T_{b^*b} \in k_1.$$

Hence Corollary 4.5, Lemma 4.8 and the homotopy invariance of both sides of (4.6) yield

$$\begin{aligned} \mathrm{ind}_{\hat{\tau}}(T) &= \mathrm{ind}_{\hat{\tau}}(T_1) = \hat{\tau}([T_b, T_{b^*}]) = \\ &= \frac{1}{2\pi i}\tau(b\delta(b^*)) = \\ &= -\frac{1}{2\pi i}\tau(b^{-1}\delta(b)) = \end{aligned}$$

$$= -\frac{1}{2\pi i}\tau(a^{-1}\delta(a)).$$

■

REMARK 4.10. As in [8], §2 one checks that the extension (3.1) defines a Thom class in $KK^1(\mathcal{A}, \mathcal{A} \rtimes_\alpha \mathbf{R})$. Therefore the boundary maps in the six term K-theory exact sequence of (3.2)

$$\partial_i : K_i(\mathcal{A}) \to K_{i+1}(\mathcal{A} \rtimes_\alpha \mathbf{R})$$

are the Thom isomorphisms of Connes [6]. This implies

$$K_*(\mathcal{T}_{sm}) = 0.$$

Let $\hat{\tau}_* : K_0(\mathcal{A} \rtimes_\alpha \mathbf{R}) \to \mathbf{R}$ be the homomorphism induced by the trace $\hat{\tau}$ (cf. [6], Appendix 4). Then for a (relative) Fredholm operator $T \in \mathcal{T}_{sm}$ one has by naturality of boundary maps

$$\mathrm{ind}_{\hat{\tau}}(T) = \hat{\tau}_*(\partial_1(\sigma_{sm}(T))),$$

hence we have for an invertible $a \in \mathcal{A}^\infty$ (cf. Proposition 3.1)

$$-\frac{1}{2\pi i}\tau(\delta(a)a^{-1}) = \mathrm{ind}_{\hat{\tau}}(s(a)) = \hat{\tau}_*(\partial_1(a)).$$

Since ∂_1 is the Thom isomorphism, Theorem 4.9 implies [6], Theorem III.3.

APPENDIX

In this appendix we prove the second part of Lemma 2.3.3 in several steps. For a Borel subset $I \subset \mathbf{R}$ let

$$E_I := 1_I(D) \quad \text{and} \quad P_t := E_{[t,\infty)}.$$

If it is convenient, we identify E_I with the subspace $E_I(\mathcal{H})$. We define

$$\begin{aligned}\mathcal{D} :=&\{T \in \mathcal{L}(\mathcal{H}) \mid \text{there is a constant } k \in \mathbf{R}, \text{ such that for} \\ &\text{any interval } I \subset \mathbf{R}, \text{ there is an interval } J \supset I \text{ with} \\ &\mathrm{Vol}(J \setminus I) \leqslant k,\ T(E_I) \subset E_J \text{ and } T^*(E_I) \subset E_J\} \\ \mathcal{E} :=&\{T \in \mathcal{D} \mid \exists\, t \in \mathbf{R} : P_tT = TP_t = 0\} \\ \mathcal{K} :=&\overline{\mathcal{E}} = \text{norm closure of } \mathcal{E} \text{ in } \mathcal{L}(\mathcal{H}).\end{aligned}$$

LEMMA 1. 1. *$\mathcal{D}$ is a $*$-algebra.*

2. *$\mathcal{E}$ is a $*$-ideal in $\mathcal{D}$.*

3. *Let $T \in \mathcal{L}(\mathcal{H})$. T lies in $\mathcal{K}$ if and only if $T \in \overline{\mathcal{D}}$ and*

$$\lim_{t\to\infty} P_t T = \lim_{t\to\infty} TP_t = 0.$$

Proof. 1. is obvious.

2. It is clear, that $\mathcal{E}$ is a $*$-invariant subspace of $\mathcal{D}$. Now let $S \in \mathcal{E}$ and $T \in \mathcal{D}$. There is a $t \in \mathbf{R}$, such that $P_t S = P_t S^* = 0$. This implies $P_t ST = 0$ and $\mathrm{Im}(S^*) \subset \subset E_{(-\infty,t)}$. Since $T \in \mathcal{D}$ there is a $r > t$, such that $T^*(E_{(-\infty,t)}) \subset E_{(-\infty,r)}$, hence $E_{[r,\infty)}T^*S^* = 0$ and thus $STE_{[r,\infty)} = 0$.

3. Consider $T \in \overline{\mathcal{D}}$ with $\lim_{t\to\infty} P_t T = \lim_{t\to\infty} TP_t = 0$. For $\epsilon > 0$ there is an $S \in \mathcal{D}$ such that

$$\|S - T\| < \epsilon$$

and there is a $t_0 \in \mathbf{R}$ such that for $t \geqslant t_0$

$$\|TP_t\|,\ \|P_t T\| < \epsilon.$$

Now for $t \geqslant t_0$ we have $(I - P_t)S(I - P_t) \in \mathcal{E}$ and

$$\begin{aligned}\|T - (I - P_t)S(I - P_t)\| &\leqslant \\ \leqslant \|T - S\| + \|P_t S\| + 2\|SP_t\| &\leqslant \\ \leqslant \epsilon + \|P_t(S - T)\| + \|P_t T\| + 2\|(S - T)P_t\| + 2\|TP_t\| &< 7\epsilon.\end{aligned}$$

The converse is trivial. ∎

LEMMA 2. *For $S, T \in \mathcal{D}$ we have*

$$P_0 S P_0 T P_0 - P_0 S T P_0 \in \mathcal{E}.$$

Proof. There is a $t > 0$, such that $T(P_t) \subset P_0$, hence $P_0 S P_0 T P_t = P_0 S T P_t$. Replacing S by T^* and T by S^* the assertion follows. ∎

DEFINITION 3. $\mathcal{D}(\mathcal{A}) := \{a \in \mathcal{A} \mid \pi(a) \in \mathcal{D}\}$.

PROPOSITION 4. *$\mathcal{D}(\mathcal{A})$ is a dense $*$-subalgebra of $\mathcal{A}$.*

From this proposition we can conclude the lacking part of Lemma 2.3.3.

COROLARY 5. *For $T \in \mathcal{C}(\pi, \lambda)$ we have $\lim_{t\to\infty} TP_t = 0$.*

Proof. Since $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{A}$, $\mathcal{C}(\pi, \lambda)$ is the closed $*$-ideal in $\mathcal{T}(\pi, \lambda)$, generated by

$$P_0\pi(a)P_0\pi(b)P_0 - P_0\pi(ab)P_0, \quad a, b \in \mathcal{D}(\mathcal{A}).$$

Thus $\mathcal{C}(\pi, \lambda) \subset \mathcal{K}$ by Lemma 2 and the assertion follows from Lemma 1.3. ∎

Proof of Proposition 4. We only have to show that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{A}$. For a function φ in the Schwartz space $\mathcal{S}(\mathbf{R})$ with $\varphi \geqslant 0$, $\int_{\mathbf{R}} \varphi = 1$, $\mathcal{F}\varphi \in \mathcal{D}(\mathbf{R})$,

$$\varphi_t := \frac{1}{t}\varphi\left(\frac{\cdot}{t}\right), \quad t > 0$$

is an approximate unit for $L^1(\mathbf{R})$. Thus the elements of the form

$$\alpha(\varphi)(a) := \int_{\mathbf{R}} \varphi(s)\alpha_s(a)\mathrm{d}s, \quad a \in \mathcal{A},\ \varphi \in \mathcal{S}(\mathbf{R}),\ \mathcal{F}\varphi \in \mathcal{D}(\mathbf{R})$$

are dense in $\mathcal{A}$. Hence it suffices to show that all the $\alpha(\varphi)(a)$ lie in $\mathcal{D}(\mathcal{A})$.

Consider $I = [c, d]$, $c \in \mathbf{R} \cup \{-\infty\}$, $d \in \mathbf{R} \cup \{\infty\}$ and let $(e_s)_{s \in \mathbf{R}}$ be the spectral measure of D. We have

$$e_s = 1_{(-\infty, s]}(D) = E_{(-\infty, s]}.$$

Now

$$\begin{aligned}
\pi(\alpha(\varphi)(a))E_I &= \int_{\mathbf{R}} \varphi(s)\pi(\alpha_s(a))\mathrm{d}sE_I = \\
&= \int_{\mathbf{R}} \varphi(s)\mathrm{e}^{2\pi \mathrm{i} sD}\pi(a)\mathrm{e}^{-2\pi \mathrm{i} sD}\mathrm{d}sE_I = \\
&= \int_{\mathbf{R}} \varphi(s)\int_{\mathbf{R}} \mathrm{e}^{2\pi \mathrm{i} s\mu}\mathrm{d}e_\mu \pi(a)\int_{\mathbf{R}} \mathrm{e}^{-2\pi \mathrm{i} s\lambda}\mathrm{d}e_\lambda \mathrm{d}sE_I = \\
&= \int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} \varphi(s)\mathrm{e}^{-2\pi \mathrm{i} s(\lambda-\mu)}\mathrm{d}s\mathrm{d}e_\mu \pi(a)\mathrm{d}e_\lambda E_I = \\
&= \int_c^d \int_{\mathbf{R}} (\mathcal{F}\varphi)(\lambda - \mu)\mathrm{d}e_\mu \pi(a)\mathrm{d}e_\lambda.
\end{aligned}$$

Hence, if $\mathrm{supp}(\mathcal{F}\varphi) \subset [c_1, d_1]$, we have

$$\pi(\alpha(\varphi)(a))(E_I) \subset E_{[c-d_1, d-c_1]}.$$

∎

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Added in proofs: After this paper had been revised, the paper [13] of Ji appeared, in which he proves Theorem 3.1 for the special representation (2.4). We emphasize that our definition of P via Stone's Theorem and the Borel functional calculus of D works for arbitrary covariant representations and seems to be more natural and perspicuous. The use of the Fourier transform in [13] is just a concrete way of realizing D in that special case.