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BOUNDED MODULE MAPS AND PURE COMPLETELY POSITIVE MAPS

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0. INTRODUCTION

Let A and B be two C^*-algebras, φ a completely positive map from A to B. The map φ gives a Hilbert B-module H_φ and a $*$-homomorphism π_φ from A into the C^*--algebra of all bounded B-module maps with adjoints on H_φ. In the case that $B = \mathbf{C}$, it is well-known that φ is pure if and only if π_φ is irreducible. It is then natural to ask whether it is also true for general C^*-algebras B. In this note we give a negative answer to the problem in general. We also show that for many C^*-algebras B, the maps φ are never pure and π_φ are never irreducible. However, for some C^*-algebras B, the purity of φ does imply the irreducibility of π_φ and for some C^*-algebras B, the irreducibility of π_φ implies φ is pure.

In [5], Kasparov showed that if H is a Hilbert module over C^*-algebra B, then $L(H)$, the C^*-algebra of all bounded module maps with adjoints on H, is isomorphic to $M(K(H)$, the multiplier algebra of $K(H)$), where $K(H)$ is the C^*-algebra of "compact" module maps. In this note we show that $B(H)$, the Banach algebra of all bounded module maps on H, is isometric isomorphic to $LM(K(H))$, the Banach algebra of left multipliers of $K(H)$, and $B(H, H^\#)$, the set of bounded module maps from H to $H^\#$, is isometric isomorphic to $QM(K(H))$, the quasi-multipliers of $K(H)$.

$B(H) \cong LM(K(H))$ was established in [10] in the case that H is countably generated. It plays an important role in studying the self-duality of countably generated Hilbert modules. It turns out that the isomorphism $B(H, H^\#) \cong QM(K(H))$ has its role in studying completely positive maps.

Suppose that A is a C^*-algebra, $M(A)$, $LM(A)$, $RM(A)$, $QM(A)$ will denote the multiplier algebra, left multiplier algebra, right multiplier algebra and the set of quasi-multipliers of A respectively.

1. BOUNDED MODULE MAPS

Recall the definition of Hilbert modules over a C^*-algebra B ([5]).

DEFINITION 1.1 Let H be a linear space over the complex field $\mathbb{C}$ with the structure of a right B-module. We suppose that $\lambda(xb) = (\lambda x)b = x(\lambda b)$, for all $\lambda \in \mathbb{C}$, $x \in H$, $b \in B$. The space H is called a *pre-Hilbert B-module* if there exists an inner product $H \times H \to A$ satisfying for every $x, y, z \in H$, $b \in B$, $\lambda \in \mathbb{C}$ the following conditions:

(1) $$\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle;\ \langle x, \lambda y\rangle = \langle x, y\rangle\lambda$$

(2) $$\langle x, yb\rangle = \langle x, y\rangle b$$

(3) $$\langle y, x\rangle = \langle x, y\rangle^*$$

(4) $$\langle x, x\rangle \geqslant 0;\ \text{if } \langle x, x\rangle = 0,\ \text{then } x = 0.$$

For $x \in H$, put $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. This is a norm on H. If H is complete, H is called a *Hilbert module over B*. H is called *full*, if the closed ideal generated by $\{\langle x, y\rangle : x, y \in H\}$ is B.

We will use the notation H_B for the Hilbert module of those sequences $\{b_n\}$ such that $b_n \in B$ and $\sum_{n=1}^{N} b_n^* b_n$ norm converge as $N \to \infty$. For a Hilbert B-module H, we let $H^\#$ denote the set of bounded B-module maps of H into B. Each $h \in H$ gives rise to a map $h^\wedge \in H^\#$ defined by $h^\wedge(x) = \langle h, x\rangle$ for $x \in H$, we call H self-dual if $H = H^\#$, i.e. if every map in $H^\#$ arises by taking B-valued inner products with some fixed $h \in H$. If we define scalar multiplication on $H^\#$ by $(\lambda x)(h) = \lambda^* x(h)$ for $\lambda \in \mathbb{C}$, $x \in H^\#$, $h \in H$ and add maps in $H^\#$ pointwise, then $H^\#$ becomes a linear space. $H^\#$ becomes a right B-module if we set $(xb)(h) = b^* x(h)$ for $x \in H^\#$, $b \in B$, $h \in H$. The map $h \to h^\wedge$ is then an one-to-one module map from H into $H^\#$. We regard H as a submodule of $H^\#$ by identifying H with $H^\wedge$. Instead of using notation $h^\wedge$ we will write h everywhere in this note.

If B is a W^*-algebra, then by [12, 4.3], the inner product $\langle\cdot,\cdot\rangle$ extends to $H^\# \times H^\#$ in such a way as to make $H^\#$ into a self-dual Hilbert B-module. Moreover, $x(h) = \langle x, h\rangle$ for x in H, h in H. For the details of $H^\#$ readers are reffered to [12].

DEFINITION 1.2. ([12, 3]) For Hilbert B-module H we denote by $L(H)$ the set of such bounded module maps $T : H \to H$ that there exists $T^* : H \to H$ satisfying the condition:

$$\langle T(x), y\rangle = \langle x, T^* y\rangle \quad \text{for all } x, y \in H.$$

With the operator norm, $L(H)$ forms a C^*-algebra.

For $x, y \in H$, put $\theta_{x,y}(h) = x\langle y, h\rangle$ for $h \in H$. Then $\theta_{x,y}$ is in $L(H)$. The closure of the linear span of $\{\theta_{x,y} : x, y \in H\}$ in $L(H)$ will be denoted by $K(H)$. $K(H)$ is an ideal of the C^*-algebra of $L(H)$. In [5], Kasparov showed that $L(H) \cong M(K(H))$. Let $B(H)$ denote the set of all bounded maps from H into H. With the operator norm, $B(H)$ forms a Banach algebra. We denote by $B(H, H^\#)$ the set of bounded module maps from H into $H^\#$. With the operator norm, $B(H, H^\#)$ forms a Banach space.

DEFINITION 1.3. Let H be a Hilbert module over a C^*-algebra A. The algebraic tensor product $H \otimes A^{**}$ becomes a right A^{**}-module if we set $(h \otimes a) \cdot a_1 = h \otimes aa_1$ for $h \in H$, $a, a_1 \in A^{**}$. Define $[\cdot,\cdot] : H \otimes A^{**} \times H \otimes A^{**}$ by

$$\left[\sum_i h_i \otimes a_i,\ \sum_j x_j \otimes \alpha_j\right] = \sum_{ij} a_i^* \langle h_i, x_j\rangle \alpha_j .$$

Set $N = \{z \in H \otimes A^{**} : [z, z] = 0\}$. As in [12,4], $H \otimes A^{**}/N$ becomes a pre-Hilbert A^{**}-module containing H as an A-submodule. We will denote by $H^\sim$ the self-dual Hilbert A^{**}-module $[(H \otimes A^{**}/N)^-]^\#$. For more details about $H^\sim$, please see [12, 4].

If $T \in B(H)$, a staightforward computation shows that T extends uniquely to a module map $\tilde{T}$ on $H \otimes A^{**}/N$ (with $\|\tilde{T}\| = \|T\|$). Therefore, by [12, 3.6], $\tilde{T}$ extends uniquely to a module map in $B(H^\sim)$. If $T \in B(H, H^\#)$, for any $h \in H$, $Th \in H^\#$. However, it is clear that $H^\#$ is an A-submodule of $H^\sim$. We define

$$\left[T\left(\sum_i h_i \otimes a_i\right),\ \sum_j x_j \otimes \alpha_j\right] =$$

$$= \sum_{ij} a_i^* [T(h_i)(x_j)] \alpha_j .$$

Then T becomes an element in $B((H \otimes A^{**}/N)^-, H^\sim)$. So, by [12, 3.6], T extends a map $\tilde{T}$ in $B(H^\sim)$. The extension from T to $\tilde{T}$ is unique. Indeed, if $\tilde{T}|_H = 0$, take $\tilde{T}^*$, the adjoint of $\tilde{T}$ in $B(H^\sim)$ ($B(H^\sim) = L(H^\sim)$ by [12, 3.4]). Then

$$\left[\tilde{T}^*\left(\sum_i h_i \otimes a_i)\right), \sum_j x_j \otimes \alpha_j\right] =$$

$$= \sum_{ij} a_i^* \langle h_i, Tx_j\rangle \alpha_j = 0$$

Therefore $\tilde{T}^* = 0$. So $\tilde{T} = 0$.

Since $H^{\sim}$ is self-dual and $B(H^{\sim}) = L(H^{\sim})$ is a W^*-algebra, it is sometime pleasent to work in $H^{\sim}$ and $B(H^{\sim})$. In what follows, we will work in $H^{\sim}$ whenever we want to and identify T with $\tilde{T}$ without warning (with some exceptions, of course).

LEMMA 1.4. *Let H be a Hilbert module over a C^*-algebra A, $x \in H$ and $\varphi \in H^{\#}$. Then there are $x_k \in (xA)^-$ such that $\|x_k\| \leqslant 1$ and*

$$\langle x_k, x_k\rangle \varphi(x) \to \varphi(x) \text{ in norm.}$$

Proof. For each $a \in A$, we define $U(x \cdot a) = \langle x, x\rangle^{\frac{1}{2}} a$. Clearly, U extends a module map from $(xA)^-$ onto R, where $R = [\langle x, x\rangle^{\frac{1}{2}} A]^-$ such that

$$[U(y)]^* U(z) = \langle y, z\rangle \quad \text{for all } \ y, z \in (xA)^-.$$

Let $x = u\langle x, x\rangle^{\frac{1}{2}}$ be the polar decomposition in $H^{\sim}$ (see [12, 3.11]) and $x_n = u\langle x, x\rangle^{\frac{1}{n}}$, $n = 1, 2 \ldots$. Since

$$\varphi(u\langle x, x\rangle^{\frac{1}{2}+\frac{1}{n}}) \to \varphi(x) \text{ in norm,}$$

we have

$$\varphi(x_n)\langle x, x\rangle^{\frac{1}{2}} \to \varphi(x) \text{ in norm.}$$

Clearly $\varphi(x_n) \in R^*$. Let $v_n = [\varphi(x_n)]^* \in R$, $y_n = U^{-1}(v_n)$, $n = 1, 2 \ldots$. Then

$$\langle y_n, x\rangle = v_n^* \langle x, x\rangle^{\frac{1}{2}} \to \varphi(x) \text{ in norm.}$$

Let $y_n = v_n\langle y_n, y_n\rangle^{\frac{1}{2}}$ be the polar decomposition for y_n in $H^{\sim}$. Set $\omega_{n,m} = v_n\langle y_n, y_n\rangle^{\frac{1}{m}}$, $n, m = 1, 2, \ldots$. Then, for fixed n,

$$\langle \omega_{n,m}, \omega_{n,m}\rangle\langle y_n, x\rangle =$$

$$= \langle y_n, y_n\rangle^{\frac{1}{m}} \langle y_n, x\rangle \to \langle y_n, x\rangle \text{ in norm.}$$

Therefore there is a sequence $\{x_k\} \subset (xA)^{-1}$ such that $\|x_k\| \leqslant 1$ and

$$\langle x_k, x_k\rangle > \varphi(x) \to \varphi(x) \text{ in norm.}$$

THEOREM 1.5. *Let A be a C^*-algebra and H a Hilbert A-module. Then there is an isometric isomorphism Φ from the Banach algebra $B(H)$ onto the Banach algebra $LM(K(H))$. Moreover, the restriction of Φ on $L(H)$ gives an isomorphism from the C^*-algebra $L(H)$ onto the C^*-algebra $M(K(H))$.*

Proof. H can be viewed as an $\tilde{A}$-module, if we define $x \cdot 1 = x$ for all $x \in H$. Thus we may assume that A is unital.

For every $T \in B(H)$, define $\Phi(T)$ by

$$\Phi(T)(k) = T \cdot k \quad \text{for } k \in K(H).$$

It is easy to see that Φ is a linear map from $B(H)$ into $LM(K(H))$ and $\|\Phi\| \leqslant 1$. If $T, S \in B(H)$,

$$[\Phi(T) \cdot \Phi(S)](k) = \Phi(T)(Sk) = T(Sk) = \Phi(TS)(k).$$

So Φ is a homomorphism. Since $\Phi(T)(\theta_{x,y}) = \theta_{Tx,y}$ for all $x, y \in H$, if $x \in H$

$$\|\Phi(T)(\theta_{x,Tx})\| = \|\theta_{Tx,Tx}\| = \|Tx\|^2.$$

Since $\|\theta_{x,Tx}\| = \|\langle x,x\rangle^{\frac{1}{2}}\langle Tx,Tx\rangle^{\frac{1}{2}}\|$, we conclude that $\|\Phi(T)\| = \|T\|$. To show that Φ is surjective, we put for $T_1 \in LM(K(H))$, $x \in H$

$$\psi(T_1)(x) = \lim_{n\to\infty}(T_1\theta_{x,x})(x)\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}.$$

As in [7, Theorem 1], it is easy to see that the limit exists. Moreover, one sees that $\psi(T_1)$ is a linear map on H. Now we assume that $x \in H$, $\|x\| \leqslant 1$ and $x = u\langle x,x\rangle^{\frac{1}{2}}$, the polar decomposition for x in $H^{\sim}$ (see [12, 3.11]). Set $y = u\langle x,x\rangle^{3\alpha-\frac{1}{2}}$, $z = u\langle x,x\rangle^{\frac{1}{2}-\alpha}$, where $\frac{1}{4} < \alpha < \frac{1}{2}$. Then $y, z \in H$ and $\theta_{x,x} = \theta_{z,z}\cdot\theta_{zy}$. Put $\psi_n(T_1)(x) = (T_1\theta_{xx})(x)\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}$, $n = 1, 2, \ldots$. For any n

$$\begin{aligned}
&\langle\psi_n(T_1)(x), \psi_n(T_1)(x)\rangle = \\
&= \left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}\langle(T_1\theta_{z,z})(\theta_{z,y})(x), (T_1\theta_{z,z})(\theta_{z,y})(x)\rangle\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1} = \\
&= \left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}\langle y,x\rangle^*\langle(T_1\theta_{z,z})(z), (T_1\theta_{z,z})(z)\rangle\langle y,x\rangle\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1} \leqslant \\
&\leqslant \left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}\langle x,x\rangle^{3\alpha}\|(T_1\theta_{z,z})\|^2\langle z,z\rangle\langle x,x\rangle^{3\alpha}\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}.
\end{aligned}$$

(See 2.8(ii) and 2.9 in [12] for the last inequality.) Let $n \to \infty$, we have

$$\langle\psi(T_1)(x), \psi(T_1)(x)\rangle \leqslant \|T_1\|^2\langle x,x\rangle^{4\alpha-1}$$

for $\frac{1}{2} < \alpha < \frac{1}{2}$. Let $\alpha \to \frac{1}{2}$, we have

$$\langle\psi(T_1)(x), \psi(T_1)(x)\rangle \leqslant \|T_1\|^2\langle x,x\rangle$$

for all $x \in H$ with $\|x\| \leqslant 1$. It follows from [12, 2.8 (ii)] that $\psi(T_1)$ is a (bounded) module map.

It remains to show that for any $x, y \in H$, $T_1\theta_{x,y} = \theta_{\psi(T_1)x,y}$. Suppose that $x = u\langle x, x\rangle^{\frac{1}{2}}$ is the polar decomposition for x in $H^{\sim}$. Let $w = u\langle x, x\rangle^{\frac{1}{6}}$. Then

$$T_1\theta_{\omega,\omega}(\omega)\langle x, x\rangle \left[\langle x, x\rangle + \frac{1}{n}\right]^{-1}$$

converges to $T_1\theta_{\omega,\omega}(\omega)$ in norm. Therefore

$$\begin{aligned}\theta_{\psi(T_1)(x),y} &= \lim_{n\to\infty} \theta_{(T_1\theta_{x,x})(x)[\langle x,x\rangle+\frac{1}{n}]^{-1},y} = \\ &= \lim_{n\to\infty} \theta_{(T_1\theta_{\omega,\omega})(\theta_{\omega,x})(x)[\langle x,x\rangle+\frac{1}{n}]^{-1},y} = \\ &= \lim_{n\to\infty} \theta_{(T_1\theta_{\omega,\omega})(\omega)\langle x,x\rangle[\langle x,x\rangle+\frac{1}{n}]^{-1},y} = \\ &= \theta_{(T_1\theta_{\omega,\omega})(\omega),y}.\end{aligned}$$

On the other hand,

$$T_1\theta_{x,y} = (T_1\theta_{\omega,\omega})\theta_{\omega,y} = \theta_{(T_1\theta_{\omega,\omega})(\omega),y}.$$

So $T_1\theta_{x,y} = \theta_{\psi(T_1)(x),y}$.

Therefore Φ is an isometric isomorphism from the Banach algebra $B(H)$ onto the Banach algebra $LM(K(H))$. It is clear that the restriction of Φ on $L(H)$ is exactly the same map defined in [7, Theorem 1]. Therefore the restriction of Φ on $L(H)$ gives an isomorphism from the C^*-algebra $L(H)$ onto the C^*-algebra $M(K(H))$.

THEOREM 1.6. *Let A be a C^*-algebra and H be a Hilbert A-module. Then there is an isometric linear map Φ_1 from $B(H, H^{\#})$ onto $QM(K(H))$. Moreover the restriction of Φ_1 on $B(H)$ is the map described in Theorem 1.5.*

Proof. Define a map Φ_1 from $B(H, H^{\#})$ into $QM(K(H))$ as follows:

$$\theta_{x',y'}\Phi_1(T)\theta_{x,y} = \theta_{x',y(T(x)(y'))} \quad \text{for } T \in B(H, H^{\#})$$

and $x, y, x', y' \in H$. Suppose that $x = u\langle x, x\rangle^{\frac{1}{2}}$ be the polar decomposition of x in $H^{\sim}$. Set $\omega = u\langle x, x\rangle^{\varepsilon}$, where $\frac{1}{2} > \varepsilon > 0$. For $z \in H$, (notice that $\varphi\cdot a(y) = a^*\varphi(y)$ for all $\varphi \in H^{\#}$, $a \in A$ and $y \in H$. We also use the inequality $[T(\omega)(y')]^*[T(\omega)(y')] \leqslant \|T(\omega)\|^2\langle y', y'\rangle$ (see [12, 2.8 (ii) and 2.9])),

$$\begin{aligned}&\|\theta_{x',y(T(x)(y'))}(z)\|^2 = \\ &= \|\langle z, y\rangle(T(x)(y'))\langle x', x'\rangle(T(x)(y'))^*\langle y, z\rangle\| = \\ &= \|\langle z, y\rangle\langle x, x\rangle^{\frac{1}{2}-\varepsilon}[T(\omega)(y')]\langle x', x'\rangle[T(\omega)(y')]^*\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\| \leqslant \\ &\leqslant \|\langle x', x'\rangle^{\frac{1}{2}}[T(\omega)(y')]^*\|^2\|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\|^2 = \\ &= \|\langle x', x'\rangle^{\frac{1}{2}}[T(\omega)(y')]^*[T(\omega)(y')]\langle x', x'\rangle^{\frac{1}{2}}\|\,\|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\|^2 \leqslant \\ &\leqslant \|T(\omega)\|^2\|\langle x', x'\rangle^{\frac{1}{2}}\langle y', y'\rangle\langle x', x'\rangle^{\frac{1}{2}}\|\,\|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\langle z, y\rangle\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\| \leqslant \\ &\leqslant \|T(\omega)\|^2\|\langle x', x'\rangle^{\frac{1}{2}}\langle y', y'\rangle^{\frac{1}{2}}\|^2\|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, y\rangle^{\frac{1}{2}}\|^2\|z\|^2.\end{aligned}$$

Let $\varepsilon \to 0$, we have

$$\|\theta_{x'y(T(x)(y'))}(z)\| \leqslant \|T\| \, \|\langle x', x'\rangle^{\frac{1}{2}} \langle y', y'\rangle^{\frac{1}{2}}\| \, \|\langle x, x\rangle^{\frac{1}{2}} \langle y, y\rangle^{\frac{1}{2}}\| \, \|z\|.$$

Therefore

$$\|\theta_{x'y'} \Phi_1(T)\theta_{x,y}\| \leqslant \|T\| \, \|\theta_{x',y'}\| \, \|\theta_{x,y}\|$$

for all $x', y', x, y \in H$.

So $\Phi_1(T)$ defines a quasi-multiplier of $K(H)$ and

$$\|\Phi_1(T)\| \leqslant \|T\| \quad \text{for all } T \in B(H, H^{\#}).$$

To show that $\|\Phi_1(T)\| = \|T\|$, we assume that $\|x\| \leqslant 1$, $\|y\| \leqslant 1$ and $\|y'\| \leqslant 1$. Let $\xi = y(T(x)(y'))$ and $\xi = \nu\langle\xi, \xi\rangle^{\frac{1}{2}}$ be the polar decomposition of ξ in $H^{\sim}$. Set $x' = \nu\langle\xi, \xi\rangle^{\alpha}/\|\langle\xi, \xi\rangle^{\alpha}\|$ for $\alpha > 0$. So $\|x'\| \leqslant 1$.

$$\|\theta_{x',y'} \Phi_1(T)\theta_{x,y}\| =$$

$$= \|\theta_{x',y'(T(x)(y'))}\| =$$

$$= \|\langle x', x'\rangle^{\frac{1}{2}} [T(x)(y'))^{*} \langle y, y\rangle (T(x)(y'))]^{\frac{1}{2}}\|.$$

For any $\varepsilon > 0$, there are $x, y' \in H$ with $\|x\| \leqslant 1$ and $\|y'\| \leqslant 1$ such that

$$\|T(x)(y')\| > \|T\| - \varepsilon.$$

By Lemma 1.3, if we chose y in the unit ball of H properly, when α is small enough, we have

$$\|\theta_{x',y'} \Phi_1(T)\theta_{x,y}\| > \|T\| - \varepsilon.$$

This implies that $\|\Phi_1(T)\| = \|T\|$. So Φ_1 is an isometry from $B(H, H^{\#})$ into $QM(K(H))$.

We now show that Φ_1 is surjective. Suppose that $T_1 \in QM(K(H))$. For any $k \in K(H)$, $k \cdot T_1 \in LM(K(H))$. Let ψ be the same notation used in Theorem 1.2. For each $x, y \in H$, we define

$$(\psi_1(T_1))(x)(y) =$$

$$= \lim_{n\to\infty} \langle \psi(\theta_{y,y}T_1)(x), y\rangle \left[\langle y, y\rangle + \frac{1}{n}\right]^{-1}.$$

It is now routine to show that the limit exists and for each $x \in H$, $\psi_1(T_1)(x)$ is a linear map from H into A. Let $y = u\langle y, y\rangle^{\frac{1}{2}}$ be the polar decomposition of y in $H^{\sim}$.

Set $z_1 = u\langle y,y\rangle^{3\alpha-\frac{1}{2}}$ and $z_2 = u\langle y,y\rangle^{\frac{1}{2}-\alpha}$, $\frac{1}{4} < \alpha < \frac{1}{2}$. For $n = 1, 2, \dots$,

$$
\begin{aligned}
&\langle \psi(\theta_{y,y}T_1)(x), y\rangle \left[\langle y,y\rangle + \frac{1}{n}\right]^{-1} = \\
&= \langle \psi(\theta_{z_1,z_2}\theta_{z_2,z_2}T_1)(x), y\rangle \left[\langle y,y\rangle + \frac{1}{n}\right]^{-1} = \\
&= \langle \psi(\theta_{z_1,z_2})\psi(\theta_{z_2,z_2}T_1)(x), y\rangle \left[\langle y,y\rangle + \frac{1}{n}\right]^{-1} = \\
&= \langle \theta_{z_1,z_2} \cdot \psi(\theta_{z_2,z_2}T_1)(x), y\rangle \left[\langle y,y\rangle + \frac{1}{n}\right]^{-1} = \\
&= \langle \psi(\theta_{z_2,z_2}T_1)(x), z_2\rangle\langle z_1, y\rangle \left[\langle y,y\rangle + \frac{1}{n}\right]^{-1} = \\
&= \langle \psi(\theta_{z_2,z_2}T_1)(x), z_2\rangle\langle y,y\rangle^{3\alpha} \left[\langle y,y\rangle + \frac{1}{n}\right]^{-1}.
\end{aligned}
$$

By the arguments used in the proof of Theorem 1.5, we can show that $\psi_1(T_1)(x)$ is a bounded module map. Since ψ is an isometry from $LM(K(H))$ onto $B(H)$, we see that $\psi_1(T_1)$ is a bounded module map in $B(H, H^{\#})$.

To show that Φ_1 is surjective, it is suffices to show that $\theta_{x',y'}T_1\theta_{x,y} = \theta_{x',y(\psi_1(T_1)(x)(y'))}$ for $T_1 \in QM(K(H))$ and $x, y, x', y' \in H$. Let $x = u\langle x,x\rangle^{\frac{1}{2}}$, $y' = \nu\langle y',y'\rangle^{\frac{1}{2}}$ be the polar decomposition of x and y' in $H^{\sim}$, respectively. Set $\omega_1 = u\langle x,x\rangle^{\frac{1}{6}}, \omega_2 = u\langle y',y'\rangle^{\frac{1}{6}}$. From the proof of Theorem 1.5, we know that for $S \in LM(K(H))$, $\psi(S)(x) = (S\theta_{\omega_1,\omega_1})(\omega_1)$. As in the proof of Theorem 1.2, we have

$$
\begin{aligned}
&\theta_{x',y(T_1)(x)(y'))} = \\
&= \lim_{n\to\infty} \theta_{x',y\langle\psi(\theta_{y',y'}T_1)(x),y'\rangle[\langle y',y'\rangle+\frac{1}{n}]^{-1}} = \\
&= \lim_{n\to\infty} \theta_{x',y\langle\psi(\theta_{\omega_2,\omega_2}T_1)(x),\omega_2\rangle\langle y',y'\rangle[\langle y',y'\rangle+\frac{1}{n}]^{-1}} = \\
&= \theta_{x',y\langle\psi(\theta_{\omega_2,\omega_2}T_1)(x),\omega_2\rangle} = \\
&= \theta_{x',y\langle\theta_{\omega_2,\omega_2}T_1\theta_{\omega_1,\omega_1}(\omega_1),\omega_2\rangle}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\theta_{x',y'}T_1\theta_{x,y} &= \theta_{x',\omega_2}\theta_{\omega_2,\omega_2}T_1\theta_{\omega_1,\omega_1}\theta_{\omega_1,y} = \\
&= \theta_{x',y\langle\theta_{\omega_2,\omega_2}T_1\theta_{\omega_1,\omega_1}(\omega_1),\omega_2\rangle}.
\end{aligned}
$$

So $\theta_{x',y'}T_1\theta_{x,y} = \theta_{x',y(\psi_1(T_1)(x)(y'))}$. Hence Φ_1 is surjective. It is easy to see that the restriction of Φ_1 on $L(H)$ is Φ defined in 1.5.

REMARK 1.7. The ideal of 1.5 and 1.6 is surely taken from [7, Theorem 1]. The proofs of 1.5 and 1.6 are more complicated than we first thought. In [7, Theorem 1], when $T \in M(K(H))$, the equality $\langle Tx, y\rangle = \langle x, T^*y\rangle$ guarantees that T is a module map. In both 1.5 and 1.6, some additional efforts have to be made to show that the map $\psi(T_1)$ and $\psi_1(T_1)$ are module maps. One may also notice that in general $\|\theta_{x,y}\| = \|\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}\| \neq \|x\|\,\|y\|$. So one has to take more care about the norm of $\psi_1(T_1)$. One may not need the polar decomposition in $H^\sim$, if one shows a version of [13, 1.4.5]. On the other hand, if one works in $H^\sim$ and $B(H^\sim)$ freely, some routine computation can be avoided. However, the present proofs are more direct and elementary. Further applications of 1.5 and 1.6 may be found in the author's successive paper.

2. COMPLETELY POSITIVE MAPS.

Let A and B be two C^*-algebras and H a Hilbert B-module. Given a $*$-homomorphism $\pi : A \to L(H)$ (henceforth called a representation of A on H) and an element $e \in H^\#$, we may define a linear map $\varphi : A \to B$ by $\varphi(a) = \langle e, \pi(a)e\rangle$ provided $\langle e, \pi(a)e\rangle \in B$ for every $a \in A$, where the inner product is the one on $H^\sim$. An easy computation (as in [12, 5.1]) shows that φ is completely positive. In the case that both A and B are unital, Paschke showed (he actually showed more than this) in [12, 5.2] that every completely positive maps of A into B arises in this way.

THEOREM 2.1 (Cf. [12, 5.2]). *Let A and B be two C^*-algebras and $\varphi : A \to B$ a completely positive map. There is a Hilbert B-module H_φ, a representation π_φ of A on H_φ and an element $e \in H_\varphi^\#$ such that $\varphi(a) = \langle e, \pi_\varphi(a)e\rangle$ for all $a \in A$ and the set $\{\pi_\varphi(a)(e\cdot b) : a \in A,\ b \in B\}$ spans a dense subspace H_0 of H_φ. Furthermore, if A is unital, we may take $e \in H_\varphi$.*

Proof. It follows from [12, 5.2] that we need only to show the existence of the element $e \in H_\varphi^\#$. We will follow notations in the proof of [12, 5.2] with the following exceptions: we will use H_φ instead of X, H_0 instead of X_0 and N_φ instead of N. One may also notice our definition of inner product on Hilbert module is the conjugate of the one defined in [12, 2.1].

As before, we may assume that B is unital. Suppose that $\{u_\lambda\}$ is an approximate identity for A. Set $e_\lambda = u_\lambda \otimes 1 + N_\varphi$. For any

$$x = \sum_{i=1}^{n} a_i \otimes b_i + N_\varphi$$

where $a_i \in A,\ b_i \in B$,

$$\langle e_\lambda, x\rangle = \sum_{i=1}^{n} \varphi(u_\lambda a_i)b_i \to \sum_{i=1}^{n} \varphi(a_i)b_i$$

in norm. Therefore there is $e \in H^\#$ such that

$$\langle e, x\rangle = \lim\langle e_\lambda, x\rangle \quad \text{for all } x \in H.$$

Since $\pi_\varphi(a)e_\lambda \cdot b = au_\lambda \otimes b + N$, we have

$$\pi_\varphi(a)e \cdot b = a \otimes b + N_\varphi.$$

Hence the linear span of the set

$$\{\pi_\varphi(a)(e \cdot b) : a \in A,\ b \in B\}$$

is dense in H and

$$\langle e, \pi_\varphi(a)e\rangle = \lim\langle u_\lambda \otimes 1,\ a \otimes 1\rangle = \lim \varphi(u_\lambda a) = \varphi(a) \quad \text{for } a \in A.$$

Let $\varphi : A \to B$ be a completely positive map. If H, π_φ and e are as in Theorem 1.7 and $T \in B(H, H^\#)$, define φ_T by $\varphi_T(a) = \langle e, T\pi_\varphi(a)e\rangle$ (notice that $\pi_\varphi(a)e \in H$) for $a \in A$. If T commutes with $\pi_\varphi(a)$ (in $B(H^\sim)$) for every $a \in A$ and $1 \geqslant T \geqslant 0$ (we say $T \geqslant 0$ if $\langle x, Tx\rangle \geqslant 0$ for every $x \in H$) φ_T is a completely positive map from A to B and $0 \leqslant \varphi_T \leqslant \varphi$. The next theorem can be thought of as a Radon-Nikodym theorem. The proof of the following theorem is similar to that of [12, 5.4]. The key difference is that the Hilbert B-module may not be self-dual. Thus the set $B(H, H^\#)$ has to be considered. This is one of the reason why $B(H, H^\#)$ catchs our attention.

THEOREM 2.2 (Cf. [12, 5.4]). *The map $T \to \varphi_T$ is an affine order isomorphism of*

$$\{T \in B(H, H^\#) : 0 \leqslant T \leqslant I_H,\ T \text{commutes with every element in } \pi_\varphi(A)\}$$

onto $[0, \varphi]$.

Proof. First we show that $T \to \varphi_T$ is one-to-one. Indeed, if $T \in B(H, H^\#_\varphi)$, T commutes with each $\pi(a)$ for $a \in A$ and $\varphi_T = 0$, we have, (by working in $H^\sim$ if necessary) $\langle \pi(a_1)e \cdot b_1,\ T\pi(a_2)e \cdot b_2\rangle = 0$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. So $T(x)(y) = 0$ for all $x, y \in H_0$, whence for all $x, y \in H_\varphi$. Thus $T = 0$. That φ_T is completely positive follows exactly as the argument used in the proof of [12, 5.3]. As in [12, 5.3], we see that $T \to \varphi_T$ is an affine order isomorphism of $\{T \in B(H, H^{\#}) : : 0 \leqslant T \leqslant I,\ T$ commutes with every element in $\pi_\varphi(A)\}$ into $[0, \varphi]$.

To show that this isomorphism is surjective, take $\psi \in [0, \varphi]$. From 2.1, we get a $*$-representation ρ of A on a Hilbert B-module E and an element $d \in E^{\#}$ such that $\psi(a) = \langle d, \rho(a)d\rangle$ for all $a \in A$ and the set $\{\rho(a)(d \cdot b) : a \in A,\ b \in B\}$ spans a dense subspace E_0 of E.

Fix $a_0 \in A$ and $b_0 \in B$, define

$$\psi_{a_0,b_0}(\pi(a)e \cdot b) = \langle \rho(a_0)d \cdot b_0, \rho(a)d \cdot b\rangle$$

for $a \in A,\ b \in B$. Clearly ψ_{a_0,b_0} is a module map. Moreover, since $\psi \leqslant \varphi$,

$$\begin{aligned}\|\psi_{a_0,b_0}(\pi(a)e \cdot b)\|^2 &\leqslant \|\langle \rho(a_0) \cdot b_0, \rho(a_0)d \cdot b_0\rangle\| \, \|\langle \rho(a)d \cdot b, \rho(a)d \cdot b\rangle\| = \\ &= \|b_0^* \psi(a_0^* a_0) b_0\| \, \|b^* \psi(a^* a) b\| \leqslant \\ &\leqslant \|b_0^* \varphi(a_0^* a_0) b\| \, \|b^* \varphi(a^* a) b\| = \\ &= \|\pi(a_0) e b_0\|^2 \|\pi(a)e \cdot b\|^2.\end{aligned}$$

So ψ_{a_0,b_0} is a bounded module map and

$$\|\psi_{a_0,b_0}\| \leqslant \|\pi(a_0) e b_0\|.$$

For $a \in A,\ b \in B$, define

$$S(\pi(a)e \cdot b) = \psi_{a,b}.$$

Then S is a module map and $\|S\| \leqslant 1$. Thus $S \in B(H_\varphi, H_\varphi^{\#})$. Let $T = S^*$. We have

$$\langle \pi(a)e \cdot b, T\pi(a)e \cdot b\rangle =$$

$$= \langle \rho(a)d \cdot b, \rho(a)e \cdot b\rangle = b^* \psi(a^* a) b \geqslant 0.$$

So $0 \leqslant T \leqslant 1$.For any $a_1, a_2, a_3 \in A$ and $b_1, b_2 \in B$,

$$\begin{aligned}&\langle \pi(a_1)e \cdot b_1, T\pi(a_2)\pi(a_3)e \cdot b_2\rangle = \\ &= \langle \rho(a_1)d \cdot b_1, \rho(a_2)\rho(a_3)d \cdot b_2\rangle = \\ &= b_1^* \psi(a_1^* a_2 a_3) b_2 = b_1^* \psi(a_1^* (a_2^*)^* a_3) b_2 = \\ &= \langle \rho(a_2^*)\rho(a_1)d \cdot b_1, \rho(a_3)d \cdot b_2\rangle = \\ &= \langle \pi(a_2)^* \pi(a_1)e \cdot b_1, T\pi(a_3)e \cdot b_2\rangle.\end{aligned}$$

Therefore T commutes with $\pi(a)$ for each $a \in A$.

Finally, we have

$$\begin{aligned} \varphi_T(a) &= \langle e, T\pi(a)e\rangle = \\ &= \lim\langle e, T\pi(u_\lambda)\pi(a)e\rangle = \\ &= \lim\langle \pi(u_\lambda)e, T\pi(a)e\rangle = \\ &= \lim \psi(u_\lambda a) = \psi(a) \end{aligned}$$

where $\{u_\lambda\}$ is an approximate identity for A. This completes the proof.

DEFINITION 2.3(Cf. [1, 1.4]). Let A and B be two C^*-algebras. We denote by $CP(A,B)$ the set of completely positive maps from A to B. A completely positive map $\varphi \in CP(A,B)$ is called *pure* if, for every $\psi \in CP(A,B)$, $\psi \leqslant \varphi$ implies that ψ is a scalar multiple of φ. A completely positive map φ in $CP(A,B)$ is called *full* if the closed ideal generated by $\varphi(A)$ is B.

DEFINITION 2.4. Let A and B be two C^*-algebras, H a Hilbert B-module. A representation π of A on H is said to be *irreducible* if for every $\pi(A)$-invariant submodule H_0 of H, either $H_0 = \{0\}$ or $H_0^\perp = \{0\}$.

DEFINITION 2.5. Let A be a C^*-algebra. If the zero ideal of A is a prime ideal, i.e. if I_1 and I_2 be two ideals of A, $I_1 I_2 = \{0\}$ implies $I_1 = \{0\}$ or $I_2 = \{0\}$, then we say that A is a prime C^*-algebra. A C^*-algebra A is said to be *weakly primitive* if $C^b(\hat{A}) \cong \mathbb{C}$, where $C^b(\hat{A})$ is the set of bounded continuous functions on the spectrum $\hat{A}$ of A.

PROPOSITION 2.6. *Let A and B be two C^*-algebras, H a Hilbert B-module and $\pi : A \to L(H)$ is a representation of A on H. If π is irreducible, then $\pi(A)$ is a prime C^*-algebra.*

Proof. If π is irreducible and I_1 and I_2 are two ideals of $\pi(A)$ such that $I_1 \cdot I_2 = \{0\}$. Clearly $I_1 H$ is an invariant submodule of $\pi(A)$. Since $I_2 H \subset (I_1 H)^\perp$, we conclude that either $I_2 = \{0\}$ or $I_2 = \{0\}$. So $\pi(A)$ is a prime C^*-algebra.

Let A be a C^*-algebra. Recall that an open projection p in A^{**} is called *dense* if for any $a \in A$, $ap = 0$ implies $a = 0$.

PROPOSITION 2.7. *Let $\varphi \in CP(A,B)$. Then*

(1) φ is pure if and only if

$$\{T \in B(H_\varphi, H_\varphi^\#) : 0 \leqslant T \leqslant 1,\ T \text{commutes with every element in } \pi_\varphi(A)\}$$

is scalar multiples of the identity;

(2) *π_φ is irreducible if and only if the only possible open projections of $K(H_\varphi)$ commuting with $\pi(a)$ for every $a \in A$ are dense open projections or zero.*

Proof. (1) is a immediate consequence of Theorem 2.5. Suppose that p is a non-zero open projection of $K(H_\varphi)$ commuting with $\pi_\varphi(a)$ for every $a \in A$. Let K_1 be the hereditary C^*-subalgebra of $K(H_\varphi)$ corresponding to p. Since $\pi_\varphi(A)K_1 \subset K_1$, $K_1 H_\varphi$ is an invariant submodule for $\pi_\varphi(A)$. If p is not dense, then there is a non-zero open projection q of $K(H_\varphi)$ such that $q \leqslant 1-p$. Let K_2 be the hereditary C^*-subalgebra of $K(H_\varphi)$ corresponding to q. Then $K_2 H_\varphi \neq \{0\}$ and $K_2 H_\varphi \subset (K_1 H_\varphi)^\perp$. Hence π_φ is not irreducible.

For the converse, suppose that H_0 is a non-zero (closed) submodule of H, $H_0^\perp \neq$ $\neq \{0\}$, and is invariant under $\pi_\varphi(A)$. Clearly, $H_0^\perp$ is also invariant under $\pi_\varphi(A)$. Set

$$K_0 = \{k \in K(H_\varphi) : kH_0^\perp = 0 \text{ and } k^* H_0^\perp = 0\}.$$

Clearly $K_0 K(H_\varphi) K_0 \subset K_0$. Therefore $(K_0 K(H_\varphi) K_0)^- = K_0$. This implies that K_0 is a hereditary C^*-subalgebra of $K(H_\varphi)$. For every $a \in A$, $\pi_\varphi(a)k \in K_0$ and $k\pi_\varphi(a) \in K_0$ for all k in K_0. Let p be the open projection of $K(H_\varphi)$ corresponding to K_0 and $\{k_\lambda\}$ an approximate identity for K_0. Then $\pi_\varphi(a)k_\lambda = p\pi_\varphi(a)k_\lambda$ for all a in A and λ. Then $\pi_\varphi(a)p = p\pi_\varphi(a)p$ for all a in A. Therefore $\pi_\varphi(a)$ commutes with p. Suppose that $x \in H_0^\perp$, then $p\theta_{x,x} = \theta_{x,x}p = 0$. This implies that p is not dense.

PROPOSITION 2.8. *Let H be a Hilbert module over a C^*-algebra A. Then the identity representation of $K(H)$ is irreducible if and only if $K(H)$ is a prime C^*-algebra.*

Proof. Suppose that $K(H)$ is a prime C^*-algebra and H_0 is a non-trivial invariant submodule for $K(H)$. If $H_1 = (H_0)^\perp \neq \{0\}$, then H_1 is also a non-trivial invariant submodule for $K(H)$. It follows from the proof of Proposition 2.7 that there are two orthogonal central open projections of $K(H)$. Therefore $K(H)$ has two orthogonal non-zero ideals. This is impossible, since $K(H)$ is prime.

The converse immediately follows from Proposition 2.6.

COROLLARY 2.9. *A C^*-algebra A is a prime if and only if there is a faithful irreducible representation $\pi : A \to L(H)$, for some Hilbert module H.*

Proof. The "if" part follows from 2.6. Suppose that A is prime. Let $H = A$, then H is a Hilbert A-module. Define $\pi(a) \cdot b = ab$ for $a, b \in A$. It easy to check that $K(H) = A$ and the representation π is irreducible by 2.8.

REMARK 2.10. In spite of the obvious resemblance between primitivity and primeness staded in 2.9, the author is not sure if 2.9 is at all relevant to the problem whether every prime C^*-algebra is primitive.

THEOREM 2.11. *Let A be a C^*-algebra. Then the identity map $i : A \to A$ is pure (in $CP(A, A)$) if and only if A is weakly primitive.*

Proof. If $i : A \to A$ is pure, by Proposition 2.7,

$$\{T \in B(H_i, H_i^{\#}) : 0 \leqslant T \leqslant 1,\ T \text{ commutes with } \pi_i(a) \text{ for every } a \in A\}$$

is scalar multiples of identity. In particular, $\{T \in L(H_i) : 0 \leqslant T \leqslant 1,\ T$ commutes with $\pi_i(a)$ for every $a \in A\}$ is isomorphic to $\mathbb{C}$. Since $\pi_i(A) = K(H_i)$, it follows from [7, Theorem 2] that the center Z of $M(K(H_i)) = M(A) \cong \mathbb{C}$. By the Dauns-Hofmann Theorem (see [13, 4.4.8]), A is weakly primitive. Conversely, if A is weakly primitive, then by Dauns-Hofmann, the center Z of $M(A)$ $(= M(K(H_i)))$ is isomorphic to $\mathbb{C}$. Suppose that T is in $QM(K(H_i))$ and commutes with every $a \in K(H_i) = \pi_i(A)$. Let $a = u|a|^{\alpha}$, where $u \in K(H_i)$, $0 < \alpha < 1$ (see[15, 1.4.5]). Then $Ta = Tu|a|^{\alpha} = uTa^{\alpha} \in K(H_i)$. Hence $T \in LM(K(H_i))$. Similarly $T \in RM(K(H_i))$. Therefore $T \in M(K(H_i))$, where $T = \lambda I$ for some $\lambda \in \mathbb{C}$. It follows from Theorem 1.5 and 1.6 that

$$\{T \in B(H_i, H_i^{\#}) : 0 \leqslant T \leqslant 1,\ T \text{ commutes with } \pi_i(a) \text{ for every } a \in A\}$$

is isomorphic to $\mathbb{C}$. By Proposition 2.6, i is pure.

PROPOSITION 2.12. *Every prime C^*-algebra is weakly primitive. There are weakly primitive C^*-algebras that are not prime.*

Proof. Suppose that A is a prime C^*-algebra. If A is not weakly primitive, then $C^b(A)$ is not isomorphic to the complex field $\mathbb{C}$. By Dauns-Hoffmann's theorem, the center Z of $M(A)$ is not the complex field. Therefore there are $x, y \in Z_+$, $x \neq 0$, $y \neq 0$, $xy = yx = 0$. Let I_1 be the closure of xA, I_2 the closure of yA. Then I_1 and I_2 are two ideals and $I_1 \cdot I_2 = \{0\}$. This contradicts the fact that A is prime.

Let B_1 be a non-elementary simple C^*-algebra. We may assume that $B_1 \subset B(\ell^2)$. Then $B_1 \cap K = \{0\}$, where K is the C^*-algebra of compact operators on ℓ^2. Suppose that Φ is the canonical homomorphism from $B(\ell^2)$ onto $B(\ell^2)/K$. Then $\Phi|_{B_1}$ is faithful. Let B_2 be the C^*-algebra of $\Phi^{-1}[\Phi(B_1)]$, the preimage of $\Phi(B_1)$, $B_3 = B_2 \oplus B_2$ and Φ_1 the canonical homomorphism from $B_2 \oplus B_2$ onto $B_2 \oplus B_2 / K \oplus K$. Then $\Phi_1(B_3) \cong B_2 \oplus B_1$. Set

$$B_4 = \{z \in B_1 \oplus B_1 : z = x \oplus x,\ x \in B_1\}.$$

Then $B_4 \cong B_1$. Now put $A = \Phi_1^{-1}(B_4)$. A is an extension of the following form

$$0 \to K + K \to A \to B_1 \to 0.$$

By the construction above, the only primitive ideals of A are $K \oplus 0$, $0 \oplus K$ and $K \oplus K$. Let $t_1 = K \oplus 0$, $t_2 = 0 \oplus K$, $t_3 = K \oplus K$ and $f \in C^b(\hat{A})$. If $f(t_1) = \lambda$, since $t_3 \in \bar{t_1}$, $f(t_3) = \lambda$. Since $t_3 \in \bar{t_2}$, $f(t_2) = \lambda$. Therefore f is a constant function. This implies that $C^b(\hat{A}) \cong \mathbb{C}$. Hence A is weakly primitive. However A is obviously not a prime C^*-algebra.

REMARK 2.13. Let $i : A \to A$ be the identity map. Then $\pi_i(A) = K(H_i)$. If A is prime then π_i is irreducible and i is pure (in $CP(A, A)$). However if A is weakly primitive but not prime then i is pure but π_i is not irreducible. Therefore we have the following conclusion:

A pure completely positive map φ may not give an irreducible representation.

THEOREM 2.14. *Let A be a separable C^*-algebra, B a C^*-algebra such that every ideal of B is σ-unital, and $\varphi : A \to B$ a completely positive.*

(1) *If φ is pure, then the ideal generated by $\varphi(A)$ must be weakly primitive.*

(2) *If π_φ is irreducible, then the ideal generated by $\varphi(A)$ must be prime.*

(3) *If B has no ideal which is a weakly primitive C^*-algebra, then $CP(A, B)$ has no pure elements.*

(4) *If B has no ideal which is a prime C^*-algebra, then every representation π of A on a Hilbert B-module is reducible.*

Proof. (1) We may assume that φ is full. If φ is pure, then the center of $L(H_\varphi)$ must be isomorphic to $\mathbb{C}$. Therefore $K(H_\varphi)$ must be a weakly primitive C^*-algebra. Since A is separable, H_φ is countably generated. Since φ is full, by the construction in [12, 5.2], H_φ is full. By [8, 1.5], $K(H_\varphi)$ has a strictly positive element. It follows from [6] that $K(H_\varphi)$ is stably isomorphic to B. By [14, Theorem 1.6], for any C^*-algebra C, $\hat{C}$ is homeomorphic to $(C \otimes K)^\wedge$, where K is the C^*-algebra of compact operators on an infinite dimensional separable Hilbert space. Therefore B is also weakly primitive.

(2) We may again assume that φ is full. If π_φ is irreducible, by 2.8, $K(H_\varphi)$ is a prime C^*-algebra. As in (1), $K(H_\varphi) \otimes K$ is isomorphic to $B \otimes K$. Since the only ideals of $B \otimes K$ are those $I \otimes K$, where the I's are ideals of B, we see clearly that B is also a prime C^*-algebra.

The conclusions of (3) and (4) are immediate consequences of (1) and (2), respectively.

REMARK 2.15. If, in both cases (1) and (2), we assume that φ is full, then we only need to assume that B is σ-unital.

EXAMPLE. The C^*-algebra $A = C[0, 1]$ is an example that all ideals of A are neither weakly primitive nor prime.

THEOREM 2.16. *Let A be a separable C^*-algebra, B a separable type I C^*-algebra and $\varphi : A \to B$ a completely positive map. If π_φ is irreducible, then φ is pure.*

Proof. Let I be the ideal of B generated by $\varphi(A)$. Then I is also a separable and type I C^*-algebra. Thus we may assume that φ is full. If there are two non-zero ideals I_1 and I_2 of $K(H_\varphi)$ such that $I_1 \cdot I_2 = \{0\}$, then $I_1 H$ is a $\pi_\varphi(A)$-invariant submodule such that $I_2 H_\varphi \subset (I_1 H)^\perp$. Since π_φ is irreducible, this is impossible. Therefore $K(H_\varphi)$ must be a prime C^*-algebra. By [3, 1.2], $K(H_\varphi)$ is stably isomorphic to B. Thus $K(H_\varphi)$ is a separable, primitive and type I C^*-algebra, since every separable prime C^*-algebra is primitive (see [13, 4.3.6], for example).

Let ρ be a faithful irreducible representation of $K(H_\varphi)$ on a Hilbert space H_ρ. It follows from [15,] that the extension of ρ is faithful on $M(K(H_\rho))$, whence on $\pi_\varphi(A)$. Let K be the C^*-algebra of all compact operators on H_ρ. Then $K \subset \subset \rho(K(H_\varphi))$. If $\rho[\pi_\varphi(A)]$ has a non-trivial invariant subspace of H_ρ, there is a non-trivial open projection p of K such that p commutes with $\rho(\pi_\varphi(A))$ for every $a \in A$. Notice that ρ is faithful. There is an open projection $\bar{p}$ of $\rho^{-1}(K)$ ($\cong K$) such that $\rho(\bar{p}) = p$, (we use the notation ρ for its normal extension of $\rho^{-1}(K)^{**}$). Then $\bar{p}$ commutes with $\pi_\varphi(a)$ for all a in A. Since $\rho^{-1}(K) \subset K(H_\varphi)$, $\bar{p}$ is an open projection of $K(H_\varphi)$. Since p is non-trivial, so is $\bar{p}$. By Proposition 2.7, this is impossible. Therefore $\rho[\pi_\varphi(A)]$ is irreducible. Since ρ is faithful on $K(H_\varphi)$, by [13, 3.12.5], ρ is faithful on $QM(K(H_\varphi))$, whence $\rho|_{\pi_\varphi(A)}$ is also faithful. Moreover $\rho[\pi_\varphi(A)]$ is weakly dense in $B(H_\rho)$. If $T \in QM(K(H_\varphi))$, $1 \geqslant T \geqslant 0$ such that T commutes with every element in $\pi_\varphi(A)$, then $\rho(T)$ commutes with every element in $\rho[\pi_\varphi(A)]$. Since $\rho[\pi_\varphi(A)]$ is weakly dense, $\rho(T)$ commutes with every element in $\rho(K(H_\varphi))$. Therefore T commutes with $K(H_\varphi)$. By the argument used in Theorem 2.9, $T \in M(K(H_\varphi)) = L(H_\varphi)$. Since $K(H_\varphi)$ is primitive, $T = \lambda I$ for some $\lambda \in \mathbb{C}$. It follows from Theorem 1.5 and Proposition 2.7 that φ is pure.

Let $\varphi : A \to B$, be a completely positive map. If B is a W^*-algebra, π_φ may be extended to a representation of A on $H^\#$. We denote this extension by $\pi_\varphi^\#$.

The following lemma follows easily from [12, 3.4] or from [2, 1.7 and 1.8 (ii)] ($H_0^\#$ is the strong closure of H_0).

LEMMA 2.17. *Let A be a W^*-algebra, H a self-dual Hilbert A-module and $H_0 \subset \subset H$. Then $H = H_0^\# \oplus (H_0^\#)^\perp$.*

THEOREM 2.18. *Let A be a C^*-algebra, B a W^*-algebra and $\varphi : A \to B$ a completely positive map. Then φ is pure if and only if $\pi_\varphi^\#$ is irreducible.*

Proof. Suppose φ is pure. If $T \in B(H_\varphi, H_\varphi^\#)$ and T commutes with $\pi_\varphi(a)$ for every $a \in A$, then $T = \lambda I$ for some $\lambda \in \mathbb{C}$. Therefore the set

$$\{T \in B(H_\rho, H_\rho^\#) : T \text{ commutes with } \pi_\varphi(a) \text{ for every } a \in A\}$$

is isomorphic to $\mathbb{C}$. If $H_0 \subset H_\varphi^\#$ is a non-zero invariant (under $\pi_\varphi^\#(A)$) closed submodule of $H_\varphi^\#$, then $H_0^\# \subset H_\varphi^\#$ (by Lemma 2.14) is invariant under $\pi_\varphi^\#(A)$. Let p be the projection from $H_\varphi^\#$ onto $H_0^\#$. Then p commutes with $\pi_\varphi^\#(a)$ for every $a \in A$. Since $p \in B(H_\varphi^\#, H_\varphi^\#)$, $p = 1$. Thus $H_0^\# = H_\varphi^\#$. So $H_0^\perp = \{0\}$. Consequently $\pi_\varphi^\#$ is irreducible.

For the converse, suppose that φ is not pure, by Proposition 2.7, there is $T \in$ $\in B(H, H^\#)$, $1 \geqslant T \geqslant 0$, $T \neq \lambda$ for any $\lambda \in \mathbb{C}$ such that T commutes with $\pi_\varphi(a)$ for any $a \in A$. It follows from [12, 3.6] that T can be uniquely extended to a map T in $B(H_\varphi^\#, H_\varphi^\#)$. T commutes with $\pi_\varphi^\#(a)$ for every $a \in A$. Therefore there are $T_1, T_2 \in B(H_\varphi^\#, H_\varphi^\#)$ such that $T_1 \neq 0$, $T_2 \neq 0$, $T_1T_2 = T_2T_1 = 0$ and T_1, T_2 commutes with $\pi_\varphi^\#(a)$ for every $a \in A$. This implies that $\pi_\varphi^\#(A)$ is not irreducible. This completes the proof.

COROLLARY 2.19. *Let A be a C^*-algebra B a W^*-algebra and $\varphi : A \to B$ a pure completely positive map. Then π_φ is irreducible.*

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