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THE BAND METHOD FOR SEVERAL POSITIVE EXTENSION PROBLEMS OF NON-BAND TYPE

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0. INTRODUCTION

There exists a large variety of positive extension problems for matrices, matrix functions, integral operators, etc. These problems are mostly of band type. In the matrix case the latter means that one has given a band matrix A and the problem is to find positive definite matrices B such that B coincides with A on the given band. In the papers [3,4; 6,7,8] a general scheme, called the *band method*, has been developed which allows one to solve these band extension problems from one point of view. For finite matrices, as has been shown by R. Grone, C. R. Johnson, E. M. de Sá and H. Wolkowitz [5], positive extension problems have similar results for more general patterns than a band. The aim of the present paper is to develop further the band method in order to cover examples of positive extension problems of non-band type in different concrete algebras.

Recall that the band method concerns the following general structure. Let $\mathcal{M}$ be an algebra with involution * and unit e which admits a direct sum decomposition

$$\mathcal{M} = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0 \dot{+} \mathcal{M}_4 \tag{0.1}$$

satisfying

(i) $\mathcal{M}_4 = \mathcal{M}_1^*$, $\mathcal{M}_3^0 = (\mathcal{M}_2^0)^*$, $\mathcal{M}_d^* = \mathcal{M}_d$,

(ii) $e \in \mathcal{M}_d$.

The algebra $\mathcal{M}$ is called an *algebra with band structure* if, in addition, the following multiplication table is fulfilled:

(0.2)

	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_-^0$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}_c$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$

Here $\mathcal{M}_+^0 = \mathcal{M}_1\dot{+}\mathcal{M}_2^0$, $\mathcal{M}_-^0 = \mathcal{M}_3^0\dot{+}\mathcal{M}_4$ and $\mathcal{M}_c = \mathcal{M}_2^0\dot{+}\mathcal{M}_d\dot{+}\mathcal{M}_3^0$. The projections of $\mathcal{M}$ onto $\mathcal{M}_c$ along the natural complementary subspace in (0.1) is denoted by P_c. Similarly, P_2, P_3 and P_d denote the natural projections of $\mathcal{M}$ onto $\mathcal{M}_2(= \mathcal{M}_2^0\dot{+}\mathcal{M}_d)$, $\mathcal{M}_3(= \mathcal{M}_3^0\dot{+}\mathcal{M}_d)$ and $\mathcal{M}_d$, respectively. An element $a \in \mathcal{M}$ is called *positive* if $a = c^*c$ with c invertible in $\mathcal{M}$.

The main results of the band method consist of three theorems. The first reads as follows.

THEOREM 0.1. *Let $\mathcal{M}$ be an algebra with band structure, and let $k = k^* \in \mathcal{M}_c$. Then there exists a positive element b in $\mathcal{M}$ such that*

$$P_c(b) = k, \quad b^{-1} \in \mathcal{M}_c \tag{0.3}$$

if and only if the equation

$$P_2(kx) = e \tag{0.4}$$

has a solution x with the following properties:

(i) $x \in \mathcal{M}_2$,

(ii) x *is invertible and* $x^{-1} \in \mathcal{M}_+ = \mathcal{M}_+^0\dot{+}\mathcal{M}_d$,

(iii) $P_d x = d^*d$ *for some invertible element* d *in* $\mathcal{M}_d$.

In that case, b is given by a right spectral factorization, namely $b = (u^*)^{-1}u^{-1}$ with $u = xd^{-1}$.

We call a positive element $b \in \mathcal{M}$ with the properties (0.3) a *(positive) band extension* of k.

Theorem 0.1 has a second version in which equation (0.3) is replaced by the equation

$$P_3(ky) = e \tag{0.5}$$

and the solution y is required to have the following properties:

(i)' $y \in \mathcal{M}_3$,

(ii)' y is invertible and $y^{-1} \in \mathcal{M}_- = \mathcal{M}_-^0 \dot{+} \mathcal{M}_d$,

(iii)' $P_d y = g^* g$ for some invertible g in $\mathcal{M}_d$.

In this case, the positive extension is given by a left spectral factorization, namely $b = (v^*)^{-1} v^{-1}$, where $v = yg^{-1}$.

The second main theorem, which assumes that the equations (0.4) and (0.5) both have a solution with the properties mentioned above, describes all positive elements a in $\mathcal{M}$ with the property that $P_c(a) = k$ by a linear fractional map. This second theorem holds provided the algebra $\mathcal{M}$ satisfies some additional axioms (see [6,7] for details).

The third main theorem, which also requires some additional properties for $\mathcal{M}$, identifies the band extension among all positive extensions by an extremal property of so-called multiplicative diagonals (see [8] for details).

An analysis of the proof of Theorem 0.1 shows that the requirements on the multiplication table (0.2) can be relaxed considerably. In fact, in this paper we show that Theorem 0.1 also holds if the table (0.2) is replaced by the following new one.

(0.6)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_c \dot{+} \mathcal{M}_1$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_+^0$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_c \dot{+} \mathcal{M}_4$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_-^0$	$\mathcal{M}_4$

This table is not symmetric anymore, and for the second version of Theorem 0.1, which involves equation (0.5), one needs the reflected version of the table (0.6). With the new table and the usual additional axioms on $\mathcal{M}$ the third main theorem also holds. These results appear in Chapter I. By introducing additional axioms on $\mathcal{M}$ the requirements on the multiplication table can even be weakened further to the following table (0.7).

(0.7)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_+^0$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0 \dot{+} \mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_1 \dot{+} \mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_-^0$	$\mathcal{M}_-^0$

This version also appears in Chapter I. In both these more general settings we cannot give a description of all positive extensions.

The new versions of Theorem 0.1, based on the tables (0.6) and (0.7), allows one to solve new positive extension problems in which the given data do not have a band structure. Infinite dimensional examples of such problems, both for the continuous and discrete case, are treated in Chapter II. Included are extension problems for semi-infinite operator matrices and for integral operators.

Chapter III concerns the case of finite operator matrices. It contains the applications of the general theory for this case and a further analysis of the multiplication table in terms of the pattern underlying the set of given data and the associated graph. To be more specific, let $\mathcal{M}$ be the algebra of all $n \times n$ matrices $[A_{ij}]_{i,j=1}^{n}$ whose entries are operators acting on a Hilbert space $\mathcal{H}$. The involution is the usual matrix adjoint and the unit in $\mathcal{M}$ is the $n \times n$ identity operator matrix. In this case the space $\mathcal{M}_c$ is determined by an index set S of pairs (i, j). The set S is assumed to contain all pairs (i, i), $i = 1, \ldots, n$, and is symmetric with respect to the main diagonal. By definition the space $\mathcal{M}_c$ consists of all matrices $[A_{ij}]_{i,j=1}^{n}$ such that $A_{ij} = 0$ whenever $(i, j) \notin S$. The spaces $\mathcal{M}_+$ and $\mathcal{M}_-$ are usual spaces consisting of the upper and lower triangular operator matrices in $\mathcal{M}$, respectively. For the induced decomposition of $\mathcal{M}$ the multiplication table (0.6) holds if and only if

$$(i, j) \in S,\ i \leqslant k \leqslant j\ \Rightarrow\ (i, k) \in S.$$

In other words, the graph associated with index set S is a so-called interval graph. In this way we show that Theorem 0.1 yields the construction of the band extension for patterns induced by interval graphs. For the induced decomposition of $\mathcal{M}$ multiplication table (0.7) holds if and only if

$$(i, k) \in S,\ (j, k) \in S,\ i, j \leqslant k\ \Rightarrow\ (i, j) \in S.$$

This corresponds precisely to the case when the associated graph is chordal. An adjusted version of Theorem 0.1 yields now the construction of the band extension for patterns induced by chordal graphs.

We like to mention that the recent paper of M. Bakonyi [1] triggered the research which led to the present paper. In conclusion we wish to thank C. R. Johnson and M. E. Lundquist for sharing their expertise on chordal graphs with us.

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CHAPTER I: THE GENERAL SCHEME

I.1. THE BAND EXTENSION

Let $\mathcal{M}$ be an algebra with a unit e and an involution *. We suppose that $\mathcal{M}$ admits a direct sum decomposition of the form

$$\mathcal{M} = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0 \dot{+} \mathcal{M}_{\rm d} \dot{+} \mathcal{M}_3^0 \dot{+} \mathcal{M}_4, \tag{1.1}$$

where $\mathcal{M}_1, \mathcal{M}_2^0, \mathcal{M}_{\rm d}, \mathcal{M}_3^0$ and $\mathcal{M}_4$ are linear subspaces of $\mathcal{M}$ and the following conditions are satisfied:

(i) $e \in \mathcal{M}_{\rm d}$, $\mathcal{M}_1 = \mathcal{M}_4^*$, $\mathcal{M}_2^0 = (\mathcal{M}_3^0)^*$, $\mathcal{M}_{\rm d} = \mathcal{M}_{\rm d}^*$

(ii) the following partial multiplication table describes some additional rules on the noncommutative multiplication in $\mathcal{M}$:

(0.7)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_{\rm d}$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_c \dot{+} \mathcal{M}_1$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_+^0$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_c \dot{+} \mathcal{M}_4$
$\mathcal{M}_{\rm d}$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_{\rm d}$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_-^0$	$\mathcal{M}_4$

Here and in the sequel we use the notation

$$\begin{gathered} \mathcal{M}_+^0 := \mathcal{M}_1 \dot{+} \mathcal{M}_2^0, \quad \mathcal{M}_-^0 := \mathcal{M}_3^0 \dot{+} \mathcal{M}_4, \\ \mathcal{M}_c := \mathcal{M}_2^0 \dot{+} \mathcal{M}_{\rm d} \dot{+} \mathcal{M}_3^0, \\ \mathcal{M}_+ := \mathcal{M}_+^0 \dot{+} \mathcal{M}_{\rm d}, \quad \mathcal{M}_- := \mathcal{M}_-^0 \dot{+} \mathcal{M}_{\rm d} \\ \mathcal{M}_2 := \mathcal{M}_2^0 \dot{+} \mathcal{M}_{\rm d}, \quad \mathcal{M}_3 := \mathcal{M}_3^0 \dot{+} \mathcal{M}_{\rm d}. \end{gathered} \tag{1.3}$$

So, for instance, we require

$$\mathcal{M}_1 \mathcal{M}_2^0 \subset \mathcal{M}_1, \quad \mathcal{M}_2^0 \mathcal{M}_1 \subset \mathcal{M}_+^0.$$

Recall that in earlier papers [6–8] the multiplication table was commutative and more restrictive. If $\mathcal{M}$ has a decomposition (1.1) satisfying (i) and the multiplication rules in (1.2) we say that $\mathcal{M}$ is an *algebra structured by the multiplication table* (1.2).

If $\mathcal{A}$ is an algebra with a unit and an involution *, we say that an element $a \in \mathcal{A}$ is *nonnegative definite* in $\mathcal{A}$ (notation: $a \geqslant_{\mathcal{A}} 0$) if there exists an element $c \in \mathcal{A}$ such that $a = c^*c$. The element $a \in \mathcal{A}$ is called *positive definite* in $\mathcal{A}$ (notation: $a >_{\mathcal{A}} 0$) if there exists an invertible element $c \in \mathcal{A}$ such that $a = c^*c$. We shall write $b \geqslant_{\mathcal{A}} a$ instead of $b - a \geqslant_{\mathcal{A}} 0$, and $b >_{\mathcal{A}} a$ instead of $b - a >_{\mathcal{A}} 0$. When $\mathcal{A} = \mathcal{M}$ we shall omit the subscript $\mathcal{M}$.

If $b \in \mathcal{M}$ is positive definite, then b is said to admit a *right spectral factorization* (relative to the decomposition (1.1)) if $b = b_+^* b_+$ for some $b_+ \in \mathcal{M}_+$ with $b_+^{-1} \in \mathcal{M}_+$.

We shall use the symbols P_i ($i = 1,2,3,4$), P_i^0 ($i = 2,3$), $P_\pm$, $P_\pm^0$, P_c and P_d to denote the natural projections of $\mathcal{M}$ onto the subspaces of the same index along the natural complement in $\mathcal{M}$. For instance,

$$P_- = P_-^0 + P_d, \quad P_c = P_2^0 + P_d + P_3^0.$$

Let $k = k^* \in \mathcal{M}_c$. An element $b \in \mathcal{M}$ is called *positive extension* of k if $P_c b = k$ and b is positive definite in $\mathcal{M}$. A positive extension b of k is called a *(positive) band extension* of k if in addition $b^{-1} \in \mathcal{M}_c$. In what follows we shall just speak about a band extension and omit the adjective positive.

THEOREM I.1.1. *Let $\mathcal{M}$ be an algebra structured by the multiplication table (1.2), and let $k = k^* \in \mathcal{M}_c$. The element k has a band extension which admits a right spectral factorization if and only if the equation*

$$P_2(kx) = e \tag{1.4}$$

has a solutions x with the properties

(i) $x \in \mathcal{M}_2$,

(ii) *x is invertible, $x^{-1} \in \mathcal{M}_+$,*

(iii) *$P_d x$ is positive definite in $\mathcal{M}_d$.*

In that case such a band extension b is given by

$$b = x^{*-1}(P_d)x^{-1} \tag{1.5}$$

Moreover, all band extensions which admit a right spectral factorization are obtained in this way.

LEMMA I.1.2. *If $b_+ \in \mathcal{M}_+$ is invertible with inverse in $\mathcal{M}_+$, and $b = b_+ b_+^*$ belongs to $\mathcal{M}_c$, then $b_+ \in \mathcal{M}_2$.*

Proof. Since $b_+ = bb_+^{*-1}$, $b \in \mathcal{M}_c$ and $b_+^{*-1} \in \mathcal{M}_-$, we get by the multiplication table (1.2) that $b_+ \in \mathcal{M}_c + \mathcal{M}_-$. Consequently, $b_+ \in \mathcal{M}_+ \cap (\mathcal{M}_c + \mathcal{M}_-) = \mathcal{M}_2$. ■

LEMMA I.1.3. *Let $x_+ \in \mathcal{M}_+$ with $x_+^{-1} \in \mathcal{M}_+$. Then $P_d x_+$ is invertible and $(P_d x_+)^{-1} = P_d(x_+^{-1})$.*

Proof. Write $x_+ = P_d x_+ + P_+^0 x_+$ and $x_+^{-1} = P_d(x_+^{-1}) + P_+^0(x_+^{-1})$. Since

$$e = x_+ x_+^{-1} = (P_d x_+)(P_d(x_+^{-1})) + m_+^0 \tag{1.6}$$

with $m_+^0 = (P_d x_+)(P_+^0(x_+^{-1})) + (P_+^0 x_+)(P_d(x_+^{-1})) + (P_+^0 x_+)(P_+^0(x_+^{-1})) \in \mathcal{M}_+^0$, we get that $m_+^0 = 0$ and $e = (P_d x_+)(P_d(x_+^{-1}))$. ■

LEMMA I.1.4. *If $d \in \mathcal{M}_d$ is invertible, then $d^{-1} \in \mathcal{M}_d$.*

Proof. Write $d^{-1} = P_1(d^{-1}) + P_2^0(d^{-1}) + P_d(d^{-1}) + P_3^0(d^{-1}) + P_4(d^{-1})$. Writing out $e = dd^{-1} = d^{-1}d$ and projecting both side on $\mathcal{M}_d$ we obtain that $e = dP_d(d^{-1}) = P_d(d^{-1})d$, and thus $P_d(d^{-1}) = d^{-1}$. ■

Proof of Theorem I.1.1. Let b be a band extension, and let $b = b_+^* b_+ (b_+^{\pm} \in \mathcal{M}_+)$ be a right spectral factorization for b. Since $b^{-1} = b_+^{-1} b_+^{*-1} \in \mathcal{M}_c$ we get by Lemma I.1.2 that $b_+^{-1} \in \mathcal{M}_2$. Put $x := b_+^{-1}(P_d b_+)^{*-1}$. Then $x \in \mathcal{M}_2$, and $x^{-1} \in \mathcal{M}_+$. Further, since $P_c b = k$, we have that $b = P_1 b + k + P_4 b$. So using multiplication table (1.2)

$$\begin{aligned} P_2(kx) &= P_2(bx - (P_1 b)x - (P_4 b)x) = \\ &= P_2(bx) = P_2\left(b_+^*(P_d b_+)^{*-1}\right) = e. \end{aligned}$$

Thus every band extension b of k admitting a right spectral factorization appears as in (1.5) where x satisfies (1.4), (i), (ii) and (iii).

Conversely, suppose that a solution x to (1.4) with properties (i), (ii) and (iii) exist. Let $\hat{b}$ be defined by $\hat{b} = b_1 + k + b_1^*$, where $b_1 = -P_1(kx)x^{-1} \in \mathcal{M}_1$. Then $\hat{b}x = -P_1(kx) + kx + b_1^* x$. Since $b_1^* x \in \mathcal{M}_-^0$ we get that $P_1(\hat{b}x) = -P_1(kx) + P_1(kx) = 0$ and $P_2(\hat{b}x) = P_2(kx) = e$. So $\hat{b}x \in e + \mathcal{M}_-^0$. But then $x^*\hat{b}x \in P_d x^* + \mathcal{M}_-^0 = P_d x + \mathcal{M}_-^0$. Also $x^*\hat{b}x$ is symmetric, which yields that $x^*\hat{b}x = P_d x$. But then

$$\hat{b} = x^{*-1}(P_d x)x^{-1}$$

is a band extension of k, and since $P_d x >_{\mathcal{M}_d} 0$ the element $\hat{b}$ admits a right spectral factorization. ■

I.2 A MAXIMUM ENTROPY PRINCIPLE

Let $\mathcal{M}$ be an algebra structured by the multiplication table (1.2). We introduce the following notion. Let b be a positive definite element of $\mathcal{M}$ which admits a right spectral factorization $b = b_+^* b_+$, $b_+^{\pm 1} \in \mathcal{M}_+$. We define the *right multiplicative diagonal* $\Delta_r(b)$ of b to be the element

$$\Delta_r(b) := b_{+d}^* b_{+d},$$

where $b_+ = b_{+d} + b_{+0}$ with $b_{+d} \in \mathcal{M}_d$ and $b_{+0} \in \mathcal{M}_+^0$. It is straightforward to check that $\Delta_r(b)$ is independent of the choice of the spectral factorization (see also [8]). Note that we can write

$$b = (e + b_+^0)^{*-1} \Delta_r(b)(e + b_+^0)^{-1}, \tag{2.1}$$

where $b_+^0 = b_{+d}^{-1} b_{+0} \in \mathcal{M}_+^0$. It is straightforward to check that the factorization (2.1) is unique (see also [8]).

Recall that an element $a \in \mathcal{M}$ is called nonnegative definite in $\mathcal{M}$ if there is an element $c \in \mathcal{M}$ such that $a = c^* c$. In order to derive a 'maximum entropy principle' we require that $\mathcal{M}$ satisfies the following two axioms.

AXIOM 1. The element $P_d(c^* c)$ is nonnegative definite for all $c \in \mathcal{M}$.

AXIOM 2. If $P_d(c^* c) = 0$ then $c = 0$.

THEOREM I.2.1. *Let $\mathcal{M}$ be an algebra structured by the multiplication table (1.2), and assume that Axioms 1 and 2 hold true. Let $k = k^* \in \mathcal{M}_c$ have a band extension b which admits a right spectral factorization. Then for any positive extension a of k which admits a right spectral factorization*

$$\Delta_r(b) \geqslant \Delta_r(a). \tag{2.2}$$

Furthermore, equality holds in (2.2) if and only if $a = b$.

Proof. Let k have a band extension b, let a be a positive extension of k, and suppose that both admit a right spectral factorization

$$a = (e + a_+)^* \Delta_r(a)(e + a_+), \quad b = (e + b_+)^* \Delta_r(b)(e + b_+),$$

with $a_+, b_+ \in \mathcal{M}_+^0$, and $(e + a_+)^{-1}, (e + b_+)^{-1} \in \mathcal{M}_+$. Since $b^{-1} \in \mathcal{M}_c$, Lemma I.1.2 implies that $(e + b_+)^{-1} \in \mathcal{M}_2$. Write $a = b + a - b$, and observe that

$$\begin{aligned}(e + b_+)^{*-1}(e + a_+)^* \Delta_r(a)(e + a_+)(e + b_+)^{-1} = \\ = \Delta_r(b) + (e + b_+)^{*-1}(a - b)(e + b_+)^{-1}.\end{aligned} \tag{2.3}$$

Since a and b are both positive extension of k,

$$a - b = m_1 + m_1^*,$$

for some $m_1 \in \mathcal{M}_1$. Then $(e + b_+)^{*-1} m_1 \in \mathcal{M}_+^0$. From this we obtain that

$$P_d\left((e + b_+)^{*-1}(a - b)(e + b_+)^{-1}\right) = 0.$$

Write $(e + a_+)(e + b_+)^{-1} = e + w$, with $w \in \mathcal{M}_-^0$. Applying P_d on equation (2.3) gives

$$\Delta_r(b) = P_d((e + w)^* \Delta_r(a)(e + w)) = \Delta_r(a) + P_d(w \Delta_r(a) w^*) \geqslant \Delta_r(a).$$

Here we use that $P_d(w^* \Delta_r(a)) = 0$, and Axiom 1. Furthermore, if $\Delta_r(a) = \Delta_r(b)$ then $P_d(w \Delta_r(a) w^*) = 0$. Since $\Delta_r(a) > 0$, we obtain from Axiom 2 that this can only happen when $w = 0$. But then $a = b$. ■

It is clear from Theorem I.2.1 that uniqueness of the band extension with a right spectral factorization follows. Indeed, if b_1 and b_2 are band extensions of $k \in M_c$ which allow a right spectral factorization Theorem I.2.1 yields that.

$$\Delta_r(b_1) \geqslant \Delta_r(b_2) \quad \text{and} \quad \Delta_r(b_2) \geqslant \Delta_r(b_1)$$

and consequently $\Delta_r(b_1) = \Delta_r(b_2)$. But then $b_1 = b_2$. Thus we proved the following result.

COROLLARY I.2.2. *Let $\mathcal{M}$ be an algebra structured by the multiplication table* (1.2), *and assume that Axioms* 1 *and* 2 *hold. Let $k = k^* \in \mathcal{M}_c$. Then k has at most one band extension which admits a right spectral factorization.*

The complete section allows a left analogue. In fact, they are the same results only now one introduces the multiplication $\times_\ell$, defined by

$$a \times_\ell b = ba,$$

and make the assumptions (i) and (ii) on the *-algebra $(\mathcal{M}_+, \times_\ell, ^*)$. The details are left to the reader.

I.3 AN ALTERNATIVE GENERAL SCHEME

In this section we present a version of the general scheme in which the requirements on the multiplication table are weakened. But, in order to obtain similar results as in the previous sections, we need in this case to add additional axioms on $\mathcal{M}$ which are automatically satisfied if the stronger table (1.2) holds.

Let $\mathcal{M}$ be an algebra with a unit e and an involution *. We suppose that $\mathcal{M}$ admits a direct sum decomposition of the form

$$\mathcal{M} = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0 \dot{+} \mathcal{M}_4,$$

where $\mathcal{M}_1, \mathcal{M}_2^0, \mathcal{M}_d, \mathcal{M}_3^0$ and $\mathcal{M}_4$ are linear subspaces of $\mathcal{M}$ and the following conditions are satisfied:

(i) $e \in \mathcal{M}_d, \mathcal{M}_1 = \mathcal{M}_4^*, \mathcal{M}_2^0 = (\mathcal{M}_3^0)^*, \mathcal{M}_d = \mathcal{M}_d^*$

(ii) the following partial multiplication table describes some additional rules on the noncommutative multiplication in $\mathcal{M}$:

(0.7)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_+^0$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0 \dot{+} \mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_1 \dot{+} \mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_-^0$	$\mathcal{M}_-^0$

We use the same notations as in Section I.1. Also the definitions of nonnegative/positive definite elements, spectral factorization, extensions, multiplicative diagonals remain the same as before. In addition to (i) and (ii) we require also $\mathcal{M}$ to satisfy

(iii) if $a \in \mathcal{M}_c$ is a positive element in $\mathcal{M}$ that admits a left spectral factorization $a = a_-^* a_-$ then one may choose $a_- \in \mathcal{M}_3$.

(iv) if $k = k^* \in \mathcal{M}_c$ and $x \in \mathcal{M}_2$ with $x^{-1} \in \mathcal{M}_+$ and $P_d x >_{\mathcal{M}_d} 0$ then $P_2(kx) = 0$ implies $k = 0$.

We now have the following results.

THEOREM I.3.1. *Let $\mathcal{M}$ be an algebra structured by the multiplication table* (3.1) *and assume in addition that* (iii) *and* (iv) *hold. Let $k = k^* \in \mathcal{M}_c$. The element k has a band extension which admits a right spectral factorization if and only if the equation*

$$P_2(kx) = e \tag{3.2}$$

has a solutions x with the properties

(i) $x \in \mathcal{M}_2$,

(ii) *x is invertible, $x^{-1} \in \mathcal{M}_+$,*

(iii) *$P_d x$ is positive definite in $\mathcal{M}_d$.*

In that case such a band extension b is given by

$$b = x^{*-1}(P_d x)x^{-1} \tag{3.3}$$

Moreover, all band extensions which admit a right spectral factorization are obtained in this way.

Proof of Theorem I.3.1. Let b be a band extension admitting a right spectral factorization. Since (iii) holds $b^{-1} \in \mathcal{M}_c$ allows a left spectral factorization $b^{-1} = a_-^* a_-$ with $a_- \in \mathcal{M}_3$. Put $x := a_-^*(P_d a_-)$. Then $x \in \mathcal{M}_2, x^{-1} \in \mathcal{M}_+$ and $P_d x >_{\mathcal{M}_d} 0$. Now one can proceed as in the proof of Theorem I.1.1 and obtain that

$$P_2(kx) = P_2\left(a_-^{-1}(P_d a_-)^{-1}\right) = e.$$

This concludes the proof of the necessity.

For the converse, let $x \in \mathcal{M}_2$ be a solution to $P_2(kx) = e$ with $x^{-1} \in \mathcal{M}_+$ and $P_d x >_{\mathcal{M}_d} 0$. Let $b := x^{*-1}(P_d x)x^{-1}$. Then $P_2(bx) = e$. Moreover, writing $b = P_1 b + P_c b + P_4 b$ and using the multiplication table (3.1) one obtains that

$$P_2(bx) = P_2((P_1 b)x + (P_c b)x + (P_4 b)x) = P_2((P_c b)x).$$

Consequently

$$P_2((k - (P_c b)x) = e - e = 0.$$

Using (iv) we obtain that $k = P_c b$. Since clearly b is positive definite we get that b is a positive extension of k. Further, $b^{-1} = x(P_d x)x^* \in \mathcal{M}_c$ by the multiplication table (1.3). ■

THEOREM I.3.2. *Let $\mathcal{M}$ be an algebra structured by the multiplication table* (3.1), *satisfying* (iii) *and* (iv), *and assume that Axioms* 1 *and* 2 *hold true. Let $k = k^* \in \mathcal{M}_c$ have a band extension b which admits a right spectral factorization. Then for any positive extension a of k which admits a right spectral factorization*

$$\Delta_r(b) \geqslant \Delta_r(a). \tag{3.4}$$

Furthermore, equality holds in (3.4) *if and only if $a = b$.*

Proof. The proof is completely analogous to the proof of Theorem I.2.1 only instead of using Lemma I.2.1 one needs to use (iii). ■

We can draw the same conclusion as before.

COROLLARY I.3.3. *Let $\mathcal{M}$ be an algebra structured by the multiplication table* (3.1), *satisfying* (iii) *and* (iv), *and assume that Axioms* 1 *and* 2 *hold. Let $k = k^* \in \mathcal{M}_c$. Then k has at most one band extension which admits a right spectral factorization.*

CHAPTER II: TWO EXAMPLES

II.1. SEMI-INFINITE OPERATOR MATRICES

Let $\mathcal{L}$ denote the linear space of all semi-infinite operator matrices $V = (V_{jk})_{j,k=1}^{\infty}$ such that

$$\sum_{v=-\infty}^{\infty} \sup_{k-j=v} \|V_{jk}\| < \infty.$$

The entry V_{jk} of V is assumed to be an operator from the Hilbert $\mathcal{H}_k$ into the Hilbert space $\mathcal{H}_j$. The space $\mathcal{L}$ is an algebra under the usual operations of addition and multiplication for infinite matrices. For $V = (V_{jk})_{j,k=1}^{\infty}$ we define

$$V^* = (V_{k,j}^*)_{j,k=1}^{\infty},$$

and this operation * is an involution on $\mathcal{L}$. The element $E = \operatorname{diag}(I_{\mathcal{H}_j})_{j=1}^{\infty}$ is the unit in $\mathcal{L}$.

We write Z for the Hilbert space $\bigoplus_{j=1}^{\infty} \mathcal{H}_j$. Thus Z consists of all square summable sequences $(\eta_j)_{j=1}^{\infty}$ with $\eta_j \in \mathcal{H}_j$. Note that each element $V \in \mathcal{L}$ induces a bounded linear operator on Z.

We are interested in the following extension problem. Given are operators $A_{ij} = A_{ij}^*$ for (i, j) in the symmetric set of indices

$$S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i = 1 \text{ or } j = 1 \text{ or } |j - i| \leqslant m\}. \tag{1.1}$$

Thus the given data is centered around the main diagonal in a band of width m and is located in the first row and column. We are looking for $V = (V_{ij})_{i,j=1}^{\infty} \in \mathcal{L}$ with the properties

(a) V induces a positive definite operator on Z;

(b) $V_{ij} = A_{ij}$, $(i, j) \in S$;

(c) $(V^{-1})_{ij} = 0$, $(i, j) \notin S$.

Such an infinite matrix $V \in \mathcal{L}$ will be called a *(positive) band extension* of $\{A_{ij} \mid (i, j) \in S\}$. If V only satisfies (a) and (b) we call V a *positive extension* of $\{A_{ij} \mid (i, j) \in S\}$.

THEOREM II.1.1. *Let $S \subset \mathbb{N} \times \mathbb{N}$ be the index set (1.1) and let $A_{ij} = A_{ij}^*$, $(i, j) \in S$, be an operator from the Hilbert space $\mathcal{H}_j$ into the Hilbert space $\mathcal{H}_i$. The given data $\{A_{ij} \mid (i, j) \in S\}$ has a positive extension if and only if*

$$\sum_{v=1}^{\infty} \|A_{1v}\| < \infty, \tag{1.2}$$

the operator matrices

$$H_j := \begin{pmatrix} A_{11} & \dots & A_{1j} \\ \vdots & & \vdots \\ A_{j1} & \dots & A_{jj} \end{pmatrix}, \quad j = 1, \dots, m+1,$$

(1.3)

$$H_j := \begin{pmatrix} A_{11} & A_{1,j-m} & \dots & A_{1,j} \\ A_{j-m,1} & A_{j-m,j-m} & \dots & A_{j-m,j} \\ \vdots & \vdots & & \vdots \\ A_{j1} & A_{j,j-m} & \dots & A_{jj} \end{pmatrix}, \quad j = m+2, \dots$$

are positive definite, and

(1.4) $$\|H_j\|, \|H_j^{-1}\| \leqslant M, \quad j = 1, 2, \dots,$$

for a fixed M. Let $\hat{X} = (\hat{X}_{ij})_{i,j=1}^{\infty}$ be given via

$$H_j \begin{bmatrix} \hat{X}_{1j} \\ \vdots \\ \hat{X}_{jj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \quad j = 1, 2, \dots, m+1,$$

(1.5)

$$H_j \begin{bmatrix} \hat{X}_{1j} \\ \hat{X}_{j-m,j} \\ \vdots \\ \hat{X}_j \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \quad j = m+2, \dots,$$

and $\hat{X}_{ij} = 0$ otherwise. Then

$$B := X^{*-1} X^{-1},$$

with

$$X = (X_{ij})_{i,j=1}^{\infty}, \quad X_{ij} = \hat{X}_{ij} \hat{X}_{jj}^{-1/2},$$

is the unique positive band extension of $\{A_{ij} \mid (i,j) \in S\}$.

Proof. We will obtain this theorem as a special case of Theorem I.1.1. Let $\mathcal{M}$ be the algebra $\mathcal{L}$, and put

$$\mathcal{M}_1 := \mathcal{M}_4^* := \{ (V_{ij})_{i,j=1}^{\infty} \in \mathcal{L} \mid V_{ij} = 0, (i,j) \in S \text{ and } i \geqslant j \}$$
$$\mathcal{M}_2^0 := (\mathcal{M}_3^0)^* := \{ (V_{ij})_{i,j=1}^{\infty} \in \mathcal{L} \mid V_{ij} = 0, (i,j) \in S \text{ and } i \geqslant j \}$$
$$\mathcal{M}_{\mathrm{d}} := \{ (V_{ij})_{i,j=1}^{\infty} \in \mathcal{L} \mid V_{ij} = 0, i \neq j \}.$$

It is easy to see that $\mathcal{M}$ is an algebra structured by the multiplication table (1.2).

Assume that (1.2) and (1.4) hold true, and that $H_j > 0$ for $j \in \mathbb{N}$. Put $A = (A_{ij})_{i,j=1}^{\infty}$, where A_{ij} for $(i,j) \in S$ are given matrices and $A_{ij} = 0$ for $(i,j) \notin S$.

Since (1.2) and (1.4) hold true $A \in \mathcal{M}$, and thus $A \in \mathcal{M}_c$. Equation (I.1.4) with $k = A$ has the unique solution $x = \hat{X}$, where $\hat{X}$ is as in the theorem. It follows from the uniformly boundedness of $H_k^{-1} > 0$ that

$$P_d x = \operatorname{diag}\left((A_{kk} - A_{k1}A_{11}^{-1}A_{1k})^{-1}\right)_{k=1}^{\infty}$$

is positive definite.

It remains to show that $\hat{X}^{-1} \in \mathcal{M}_+$. When this is proven Theorem II.1.1 follows directly from Theorem I.1.1.

Denote

$$H_j = \begin{bmatrix} A_{11} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{bmatrix}, \quad X_1^{(j)} = \hat{X}_{1j}$$

$$X_2^{(j)} = \begin{bmatrix} \hat{X}_{2j} \\ \vdots \\ \hat{X}_{jj} \end{bmatrix}, \quad j = 1, \ldots, m+1; \quad X_2^{(j)} = \begin{bmatrix} \hat{X}_{j-m,j} \\ \vdots \\ \hat{X}_{jj} \end{bmatrix}, \quad j \geqslant m+2$$

and

$$P = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}.$$

Then (1.5) becomes

$$\begin{bmatrix} A_{11} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix} \begin{bmatrix} X_1^{(k)} \\ X_2^{(k)} \end{bmatrix} = \begin{bmatrix} 0 \\ P \end{bmatrix}, \quad k = 1, 2, \ldots$$

Using the invertibility of $A_{22}^{(k)}$ we may rewrite this as

$$\text{(1.6)} \qquad \begin{aligned} &\begin{pmatrix} I & A_{12}^{(k)}A_{22}^{(k)-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} - A_{12}^{(k)}A_{22}^{(k)-1}A_{21}^{(k)} & 0 \\ 0 & A_{22}^{(k)} \end{pmatrix} \cdot \\ &\begin{pmatrix} I & 0 \\ A_{22}^{(k)-1}A_{21} & I \end{pmatrix} \begin{pmatrix} X_1^{(k)} \\ X_2^{(k)} \end{pmatrix} = \begin{pmatrix} 0 \\ P \end{pmatrix}. \end{aligned}$$

Put

$$\begin{pmatrix} Y_1^{(k)} \\ Y_2^{(k)} \end{pmatrix} = \begin{pmatrix} X_1^{(k)} \\ A_{22}^{(k)-1}A_{21}^{(k)}X_1^{(k)} + X_2^{(k)} \end{pmatrix}.$$

With this notation (1.6) becomes

$$\begin{pmatrix} A_{11} - A_{12}^{(k)}A_{22}^{(k)-1}A_{21}^{(k)} & 0 \\ 0 & A_{22}^{(k)} \end{pmatrix} \begin{pmatrix} Y_1^{(k)} \\ Y_2^{(k)} \end{pmatrix} = \begin{pmatrix} -A_{12}^{(k)}A_{22}^{(k)-1}P \\ P \end{pmatrix}.$$

Thus

$$Y_1^{(k)} = -\left(A_{11} - A_{12}^{(k)} A_{22}^{(k)-1} A_{21}^{(k)}\right)^{-1} A_{12}^{(k)} A_{22}^{(k)-1} P,$$
$$Y_2^{(k)} = A_{22}^{(k)-1} P.$$

Let $Y = (Y_{ij})_{i,j=1}^{\infty}$ be given via

$$\begin{bmatrix} Y_{11} \\ \vdots \\ Y_{jj} \end{bmatrix} = Y_2^{(j)}, \quad j = 1, \ldots, m+1,$$
$$\begin{bmatrix} Y_{j-m,j} \\ \vdots \\ Y_{jj} \end{bmatrix} = Y_2^{(j)}, \quad j = m+2, \ldots$$

and $Y_{ij} = 0$ otherwise. We know from [7, §II.3] that Y is invertible in $\mathcal{M}$ and that $Y^{-1} \in \mathcal{M}_+$. Write

$$\hat{X} = \begin{pmatrix} \hat{X}_{11}(p) & \hat{X}_{12}(p) \\ 0 & \hat{X}_{22}(p) \end{pmatrix}, \quad Y = \begin{pmatrix} \hat{Y}_{11}(p) & \hat{Y}_{12}(p) \\ 0 & \hat{Y}_{22}(p) \end{pmatrix}$$

where the decomposition is described by requiring to have in the upper left corner of $\hat{X}_{22}(p)(\hat{Y}_{22}(p))$ the entry $\hat{X}_{pp}(Y_{pp})$. Note now that

$$\|\hat{X}_{22}(p) - \hat{Y}_{22}(p)\| \leqslant \sum_{k=p}^{\infty} \|A_{22}^{(k)-1} A_{21}^{(k)} X_1^{(k)}\| =$$
$$= \sum_{k=p}^{\infty} \left\| A_{22}^{(k)-1} A_{21}^{(k)} \left(A_{11} - A_{12}^{(k)} A_{22}^{(k)-1} A_{21}^{(k)}\right)^{-1} A_{12}^{(k)} A_{22}^{(k)-1} P \right\| \leqslant$$
$$\leqslant \sum_{k=p}^{\infty} \|A_{22}^{(k)-1}\| \|A_{21}^{(k)}\| \|H_k^{-1}\| \|P\| \leqslant M^2 \sum_{k=p}^{\infty} \|A_{21}^{(k)}\|,$$

where M is such that $\|H_k^{-1}\| \leqslant M$ and $\|A_{22}^{(k)-1}\| \leqslant M$ for all k. Since

$$\sum_{k=p}^{\infty} \|A_{21}^{(k)}\| \leqslant m \sum_{k=p-m}^{\infty} \|A_{1,k}\|$$

and $\sum_{k=1}^{\infty} \|A_{1,k}\| < \infty$ we get that for an arbitrary $\varepsilon > 0$

$$\|\hat{X}_{22}(p) - \hat{Y}_{22}(p)\| < \varepsilon$$

when p is large enough. But since $\hat{Y}_{22}(p)$ is invertible and its inverse is upper triangular, the same holds for $\hat{X}_{22}(p)$ for p large enough. Clearly $\hat{X}_{11}(p)$ is invertible (its diagonal elements are) and $\hat{X}_{11}(p)^{-1}$ is upper triangular. Buth then $\hat{X}$ is invertible and

$$\hat{X}^{-1} = \begin{pmatrix} X_{11}(p)^{-1} & -\hat{X}_{11}(p)^{-1}\hat{X}_{12}(p)\hat{X}_{22}(p)^{-1} \\ 0 & \hat{X}_{22}(p)^{-1} \end{pmatrix}$$

is upper triangular. Further, it is easy to see that $\hat{X}$ has diagonals which are summable in norm. But then the same holds for $\hat{X}^{-1}$.

This concludes the proof of Theorem II.1.1. ■

The maximum entropy principle for this case read as follows.

THEOREM II.1.3. *Let $S \subset \mathbb{N}\times\mathbb{N}$ be the index set* (1.1) *and let $A_{ij} = A_{ji}^*$, $(i,j) \in S$, be an operator from the Hilbert space $\mathcal{H}_j$ into the Hilbert space $\mathcal{H}_i$. Suppose that the operator matrices H_j, $j = 1, 2, \ldots$, defined in* (1.3), *are positive definite, and that* (1.2) *and* (1.4) *hold true. Put*

$$M_i = [0 \ldots 0 I] H_i^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \quad i = 1, 2, \ldots .$$

If $C = (C_{ij})_{i,j=1}^{\infty}$ is a positive extension of $\{A_{ij} \mid (i,j) \in S\}$, then

$$\text{(1.7)} \qquad C_{11} \leqslant M_1^{-1}, \quad \Delta_k^r(C) \leqslant M_k^{-1}, \quad k = 2, 3, \ldots,$$

where

$$\Delta_k^r(C) = C_{kk} - [C_{k1} \ldots C_{k,k-1}] \begin{bmatrix} C_{11} & \ldots & C_{1k-1} \\ \vdots & & \vdots \\ C_{k-1,1} & \ldots & C_{k-1,k-1} \end{bmatrix}^{-1} \cdot \begin{bmatrix} C_{1k} \\ \vdots \\ C_{k-1,k} \end{bmatrix}, \quad k = 2, 3, \ldots$$

Equality holds in (1.7) *if and only if C is the unique positive band extension of the given data.*

Proof. Let $\mathcal{M}$ be the same algebra structured by the multiplication table (1.2) as in the proof of Theorem II.1.1. It is easy to check that $\mathcal{M}$ satisfies the Axioms 1 and 2 in Section I.2.

Let B be the band extension of the given band. From the factorization $B = X^{*-1}X^{-1}$ in Theorem II.1.1 it follows that diag(M_i^{-1}) is precisely the right multiplicative diagonal of B. The right multiplicative diagonal of C is given by diag $(\Delta_i^r(C))$ (with $\Delta_1^r(C) = C_{11}$). But then the theorem follows directly from Theorem I.2.1. ■

II.2. FREDHOLM INTEGRAL OPERATORS

In this section we apply the abstract results of Section I.1 to functions f which may be viewed as kernels of integral operators.

Let $D \subset [0,1] \times [0,1]$ be the domain

$$D = \{(t,s) \in [0,1]^2 \mid t - \tau < s < t + \tau \text{ or } s < \alpha \text{ or } t < \alpha\}.$$

Here α and τ are fixed numbers between 0 and 1.

Introduce $\mathcal{F} = \mathcal{F}_D$ to be the class of $n \times n$ matrix valued functions $f(t,s)$ which are defined on the square $[0,1] \times [0,1]$, continuous on each of the open regions

$$\Delta_1 = \{(t,s) \in [0,1]^2 \mid t < s \text{ and } (t,s) \notin D\},$$
$$\Delta_2 = \{(t,s) \in [0,1]^2 \mid t < s \text{ and } (t,s) \in D\},$$
$$\Delta_3 = \{(t,s) \in [0,1]^2 \mid t < s \text{ and } (t,s) \in D\},$$
$$\Delta_4 = \{(t,s) \in [0,1]^2 \mid t < s \text{ and } (t,s) \notin D\},$$

and the restriction f_i of f to Δ_i extends continuously to the closure $\overline{\Delta}_i$. The set $\mathcal{F}$ is an algebra with the usual addition of functions, and the multiplication defined by

$$(f * g)(t,s) = \int_0^1 f(t,u)g(u,s)\mathrm{d}s.$$

Also, $\mathcal{F}$ has a natural involution *, namely

$$f^*(t,s) := f(s,t)^*. \tag{2.1}$$

Then * in the right hand side of (2.1) is the usual adjoint of a matrix. We shall say that $f \in \mathcal{F}$ is *regular* in $\mathcal{F}$ if there exists a $g \in \mathcal{F}$ such that

$$f + g + f * g = 0, \quad g + f + g * f = 0.$$

In that case g is uniquely determined by f and denoted by $f^\dagger$.

Given $f \in \mathcal{F}$ we shall write F for the integral operator on $L_n^2[0,1]$ with kernel f. Thus

$$(F\varphi)(t) = \int_0^1 f(t,s)\varphi(s)\mathrm{d}s, \quad 0 < t < 1.$$

Similarly, G stands for the integral operator with kernel g. If f is regular in $\mathcal{F}$, then $f^\dagger$ is precisely the kernel of the integral operator $(I - F)^{-1} - I$; in other words, $f^\dagger$ is the resolvent kernel. Furthermore, F^* is the integral operator with kernel f^*.

We shall deal with the following extension problem. Let

$$k \in \mathcal{F}_c := \{f \in \mathcal{F} \mid f(t,s) = 0,\ (t,s) \in \Delta_1 \cup \Delta_4\}.$$

A matrix valued function $g \in \mathcal{F}$ is called a *positive extension* of k if $k(t,s) = g(t,s)$, $(t,s) \in \Delta_2 \cup \Delta_3$ and $I - G$ is a positive operator on $L_n^2[0,1]$. To find such a g we need some additional notation. For $\xi \in [0,1]$ let J_ξ denote the open set in $[0,1]$ given by

$$J_\xi = \{t : t < \xi,\ (t,\xi) \in D\}.$$

For $k \in \mathcal{F}_c$ and $\xi \in [0,1]$ let $A_{k,\xi}$ denote the integral operator on $L_n^2(\bar{J}_\xi)$ which is defined by

$$(A_{k,\xi}\varphi)(t) = \varphi(t) - \int_{J_\xi} k(t,s)\varphi(s)\mathrm{d}s, \quad t \in \bar{J}_\xi. \tag{2.2}$$

THEOREM II.2.1. *Let $k \in \mathcal{F}_c$ be given, and suppose that for every ξ in the interval $0 < \xi < 1$ the operator $A_{k,\xi}$ in (2.2) is positive definite. Let x be given by*

$$x(t,s) - \int_{J_s} k(t,u)x(u,s)\mathrm{d}u = k(t,s), \tag{2.3}$$

for $t \in J_s$, $0 < s < 1$, and $x(t,s) = 0$ elsewhere. Then x is regular in $\mathcal{F}$ and the function $f \in \mathcal{F}$ given by

$$-f = x^\dagger + (x^\dagger)^* + (x^\dagger)^* * x^\dagger$$

is the unique positive extension $f \in \mathcal{F}$ of k with $f^\dagger \in \mathcal{F}_c$.

Proof. We will obtain this theorem as a special case of Theorem I.1.1. Let $\mathcal{M}$ be the direct linear span of $\{\mathcal{F}, I_n\}$ where I_n denotes the $n \times n$ identity matrix. The multiplication on $\mathcal{M}$ is defined by

$$(\lambda I_n + f)(\mu I_n + g) := \lambda\mu I_n + \lambda g + \mu f + f * g.$$

The unit in $\mathcal{M}$ is I_n and the involution * is defined by $(\lambda I_n + f)^* := \bar{\lambda} I_n + f^*$. Let

$$\mathcal{M}_1 := \{f \in \mathcal{M};\ f|\Delta_j = 0,\ j = 2,3,4\}$$
$$\mathcal{M}_2^0 := \{f \in \mathcal{M};\ f|\Delta_j = 0,\ j = 1,3,4\}$$
$$\mathcal{M}_d := \{\lambda I_n;\ \lambda \in \mathbb{C}\}$$
$$\mathcal{M}_3^0 := (\mathcal{M}_2^0)^*, \quad \mathcal{M}_4 := \mathcal{M}_1^*.$$

Then clearly (I.1.1) holds and the decomposition satisfies (i) and (ii). In order to see that indeed the multiplication table (I.1.2) holds true one needs to make several straightforward calculations. Let us write out one of them, and prove that $\mathcal{M}_2^0\mathcal{M}_3^0 \subset \subset \mathcal{M}_c$. So let $f \in \mathcal{M}_2^0$ and $g \in \mathcal{M}_3^0$ and take $(t,s) \notin D$. We have to show that $(f*g)(t,s) = 0$. Indeed, since $f \in \mathcal{M}_2^0$ we have that $\operatorname{supp} f \subset \Delta_2 \subset D$ and likewise $\operatorname{supp} g \subset \Delta_3 \subset D$. But then

$$(f*g)(t,s) = \int_0^1 f(t,u)g(u,s)\mathrm{d}u = \int_{\max\{t,s\}}^1 f(t,u)g(u,s)\mathrm{d}u = 0$$

since $(t,s) \notin D$, and $u \geqslant s$ and $u \geqslant t$ imply that either $(t,u) \notin D$ or $(u,s) \notin D$ (depending on whether $s \geqslant t$ or $t \geqslant s$).

Now we can apply Theorem I.1.1 on the element $I_n - k \in \mathcal{M}_c$. This means that we have to solve the equation

$$P_2((I_n - k)(I_n + x)) = I_n,$$

which precisely comes down to equation (2.3). The main difficulty is to show that x belongs to the algebra $\mathcal{M}$. As soon as this is done, clearly $I_n + x \in \mathcal{M}_2$, $P_d(I_n + x) = = I_n$ is positive definite and x is regular with $x^\dagger \in \mathcal{M}_+$. The latter holds since x is the kernel belonging to a Volterra operator. Thus it remains to show that $x \in \mathcal{M}$. This follows from the next proposition.

PROPOSITION II.2.2. *Let*

$$J_\xi = (\max\{0, \xi - \tau\}, \min\{1, \xi\}) \cup (\alpha, 1),$$

and $k \in \mathcal{F}_c$ and suppose that for $\xi \in [0,1]$ the operator A_ξ on $L_2^n(\bar{J}_\xi)$ defined by

$$(A_\xi \varepsilon)(t) := \varphi(t) - \int_{J_\xi} k(t,s)\varphi(s)\mathrm{d}s, \quad t \in J_\xi, \tag{2.4}$$

is invertible. Then, for each $s \in [0,1]$, the equation

$$x_s(t) - \int_{J_s} k(t,u)x_s(u)\mathrm{d}u = k(t,s), \quad t \in J_s, \tag{2.5}$$

has a unique solution x_s, and

$$x(t,s) := x_s(t)$$

belongs to $\mathcal{M}$ (or, in fact, to $\mathcal{M}_2^0$).

Proof. The fact that (2.4) has a unique solution follows directly from the invertibility of A_ξ. We have to prove $x \in \mathcal{M}$, i.e., $x|\Delta_i$ is a continuous and allows a continuous extension to $\bar{\Delta}_i$, $i = 1, 2, 3, 4$. First note that

$$x(t,s) = \begin{cases} \gamma(t,s), & t \in J_s, \\ 0, & t \notin J_s, \end{cases}$$

where γ_ξ denotes the kernel of the integral operator $A_\xi^{-1} - I$. Thus it suffices to prove that $\gamma_\xi(t,s)$ is jointly continuous in the variables (ξ, t, s) on $[0,1] \times \bar{\Delta}_i$ providing that $(t,s) \in [0,1]^2 \cap (\bar{J}_\xi \times \bar{J}_\xi)$. The latter follows from the following adjustment of Theorem 3.3 in [2].

PROPOSITION II.2.3. *Let $f \in \mathcal{M}$ and introduce for $0 \leqslant a < b \leqslant c \leqslant d \leqslant 1$ the operator Λ_{abcd} on $L_n^2([a,b] \cup [c,d])$ by*

$$(\Lambda_{abcd}\varphi)(t) = \varphi(t) - \int_a^b f(t,s)\varphi(s)\mathrm{d}s - \int_c^d f(t,s)\varphi(s)\mathrm{d}s.$$

Suppose that Λ_{abcd} is invertible on $L_n^2([a,b] \cup [c,d])$. Then there exists a $\delta > 0$ such that $\Lambda_{a'b'c'd'}$ is invertible on $L_n^2([a',b'] \cup [c',d'])$ for every $0 \leqslant a' < b' \leqslant c' < d' \leqslant 1$ with $|a - a'| < \delta$, $|b - b'| < \delta$, $|c - c'| < \delta$ and $|d - d'| < \delta$, and the kernels $\gamma_{a'b'c'd'}$ of $\Lambda_{a'b'c'd'} - I$ are uniformly bounded:

$$|\gamma_{a'b'c'd'}(t,s)| \leqslant M < \infty$$

for $t, s \in [a',b'] \cup [c',d']$, $|a-a'| < \delta$, $|b-b'| < \delta$, $|c-c'| < \delta$ and $|d-d'| < \delta$. Moreover, $\gamma_{a'b'c'd'}(t,s)$ jointly continuous in all six variables in the appropriate domains: given any $\varepsilon > 0$ there exists a $\delta' \leqslant \delta$ such that

$$|\gamma_{abcd}(t,s) - \gamma_{a'b'c'd'}(t',s')| < \varepsilon,$$

whenever $|a-a'| < \delta'$, $|b-b'| < \delta'$, $|c-c'| < \delta'$, $|d-d'| < \delta'$, $(t,s) \in \Delta_i \cap ([a,b] \cup [c,d])^2$, and $(t',s') \in \bar{\Delta}_i \cap ([a',b'] \cup [c',d'])^2$, for $i = 1, 2, 3, 4$.

Proof. The proof of Theorem 3.3 in [2] generalizes straightforwardly to this case. ∎

The proof of Theorem II.2.1 is now complete. ∎

CHAPTER III. MATRICES AND GRAPHS

III.1 PATTERNS AND TABLES

Let $\mathcal{M}$ be the algebra of $n \times n$ operator matrices $(A_{ij})_{i,j=1}^n$, where A_{ij} is an operator from the (nontrivial) Hilbert space $\mathcal{H}_j$ into the (nontrivial) Hilbert space $\mathcal{H}_i$. Given a *pattern* $S \subset \underline{n} \times \underline{n}$, i.e., S is a symmetric index set containing the diagonal ($(i,i) \in S$, and $(i,j) \in S$ implies $(j,i) \in S$), we can make an additive decomposition of $\mathcal{M}$ as follows:

$$\mathcal{M} = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0 \dot{+} \mathcal{M}_{\mathrm{d}} \dot{+} \mathcal{M}_3^0 \dot{+} \mathcal{M}_4, \tag{1.1}$$

where

$$\begin{aligned}
\mathcal{M}_1 &= \left\{(A_{ij})_{i,j=1}^n \mid A_{ij} = 0 \text{ for } i \geqslant j \text{ and } (i,j) \in S\right\} \\
\mathcal{M}_2^0 &= \left\{(A_{ij})_{i,j=1}^n \mid A_{ij} = 0 \text{ for } i \geqslant j \text{ and } (i,j) \notin S\right\} \\
\mathcal{M}_{\mathrm{d}} &= \left\{(A_{ij})_{i,j=1}^n \mid A_{ij} = 0 \text{ for } i \neq j\right\} \\
\mathcal{M}_3^0 &= \left\{(A_{ij})_{i,j=1}^n \mid A_{ij} = 0 \text{ for } i \leqslant j \text{ and } (i,j) \notin S\right\} \\
\mathcal{M}_4 &= \left\{(A_{ij})_{i,j=1}^n \mid A_{ij} = 0 \text{ for } i \leqslant j \text{ and } (i,j) \in S\right\}
\end{aligned}$$

We shall refer to this decomposition as the decomposition of $\mathcal{M}$ *induced by the pattern* S.

It is obvious that different types of patterns lead to different types of multiplication rules between the subspaces in the induced decomposition. The following proposition makes the connection between certain types of patterns and multiplication tables.

We call a pattern S *row-diagonally connected* if $(i,j) \in S$ and $i \leqslant k \leqslant j$ imply $(i,k) \in S$. We call S *column-diagonally connected* if $(i,j) \in S$ and $i \leqslant k \leqslant j$ imply $(k,j) \in S$. Pattern which are both row- and column-diagonally connected are called *generalized block banded*. Such a pattern is characterized by the condition that $(i,j) \in S$ and $i \leqslant k, l \leqslant j$ imply $(k,l) \in S$. A pattern S is called *perfect* if $(i,k) \in$ $\in S, (j,k) \in S$ and $i,j \leqslant k$ imply $(i,j) \in S$.

We introduce the following multiplication tables:

(MR)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_c \dot{+} \mathcal{M}_1$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_+^0$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_c \dot{+} \mathcal{M}_4$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_-^0$	$\mathcal{M}_4$

(MC)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}$	$\mathcal{M}_-^0$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_c \dot{+} \mathcal{M}_1$	$\mathcal{M}_c$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_-^0$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_c \dot{+} \mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$

(MB)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_-^0$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}_c$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$

(MP)

left\right	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_+^0$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0 \dot{+} \mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_1 \dot{+} \mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_-^0$	$\mathcal{M}_-^0$

PROPOSITION III.1.1. *Let S be a $\underline{n} \times \underline{n}$ be a pattern. Then*

(i) *S is row-diagonally connected if and only if the induced decomposition of $\mathcal{M}$ satisfies the multiplication table* (MR).

(ii) *S is column-diagonally connected if and only if the induced decomposition of $\mathcal{M}$ satisfies the multiplication table* (MC).

(iii) *S is generalized block banded if and only if the induced decomposition of $\mathcal{M}$ satisfies the multiplication table* (MB).

(iv) *S is perfect if and only if the induced decomposition of $\mathcal{M}$ satisfies the table* (MP).

Proof. To prove (i) let S be row-diagonally connected. It is straightforward to check that the decomposition of $\mathcal{M}$ induced by S satisfied (MR). As an illustration we prove that $\mathcal{M}_3^0\mathcal{M}_1 \subset \mathcal{M}_+^0$. Let $F = (F_{ij})_{i,j=1}^n \in \mathcal{M}_3^0$ and $G = (G_{ij})_{i,j=1}^n \in \mathcal{M}_1$. Then $F_{ij} = 0$ for $i \leqslant j$ and $(i,j) \notin S$ and $G_{ij} = 0$ for $i \geqslant j$ and $(i,j) \in S$. Let now $i \geqslant j$. Then

$$(FG)_{ij} = \sum_{q=1}^{n} F_{iq}G_{qj}.$$

Suppose $F_{iq} \neq 0$. Then $i > q$ and $(i,q) \in S$. If $j \leqslant q$ then $G_{qj} = 0$. If $j > q$, then, since S is row-diagonally connected, we have $(q,j) \in S$. But then also in this case $G_{qj} = 0$. Consequently, $(FG)_{ij} = 0$. This proves $\mathcal{M}_3^0\mathcal{M}_1 \subset \mathcal{M}_+^0$.

For the converse, suppose that the decomposition (1.1) satisfies (MR). Then, in particular, $\mathcal{M}_1\mathcal{M}_+^0 \subset \mathcal{M}_1$. Suppose now that $(i,k) \notin S$, and let $j > k$. Introduce $A = (A_{pq})_{p,q=1}^n, B = (B_{pq})_{p,q=1}^n$ with $A_{pq} = 0$ for $(p,q) \neq (i,k), B_{pq} = 0$ for $(p,q) \neq$ $\neq (k,j)$ and $A_{ik}B_{kj} \neq 0$. Then $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_+^0$. Thus $AB \in \mathcal{M}_1$. Since $(AB)_{pq} = 0$ for $(p,q) \in S$ we must have that $(i,j) \notin S$. But now we may conclude that S is row-diagonally connected.

The proof of (ii) is similar to that of (i).

For (iii) note that if S is a generalized block banded the pattern S is both row- and column-diagonally connected. Thus the induced decomposition satisfies both (MR) and (MC), and consequently (MB). Conversely, if a decomposition (1.1) satisfied (MB), then it satisfies in particular both (MC) and (MR). But then by (i) and (ii) S must be both row- and column-diagonally connected, implying that S is generalized block banded.

It is easy to prove that if S is perfect the induced decomposition satisfies (MP). Let us prove the converse. If (MP) holds then $\mathcal{M}_2\mathcal{M}_3 \subset \mathcal{M}_c$. Let $(i,k) \in S, (k,j) \in$ $\in S$ and $i,j < k$. Put $A = (A_{pq})_{p,q=1}^n, B = (B_{pq})_{p,q=1}^n$ with $A_{pq} = 0$ for $(p,q) \neq$ $\neq (i,k), B_{pq} = 0$ for $(p,q) \neq (k,j)$ and $A_{ik}B_{kj} \neq 0$. Then $A \in \mathcal{M}_2^0$, $B \in M_3^0$ so AB must be in $\mathcal{M}_c$. Since $(AB)_{ij} = A_{ik}B_{kj} \neq 0$ clearly we have $(i,j) \in S$. ∎

III.2. THE EXTENSION PROBLEM

The following theorem generalizes the result of [2] for band patterns to row-diagonally patterns.

THEOREM III.2.1. *Let $S \subset \underline{n} \times \underline{n}$ be a row-diagonally connected pattern, and let $A_{ij} = A_{ji}^*, (i,j) \in S$, be an operator from the Hilbert space $\mathcal{H}_j$ into the Hilbert space $\mathcal{H}_i$. Then there exists a positive definite block matrix $B = (B_{ij})_{i,j=1}^n$ with $B_{ij} = A_{ij}, (i,j) \in S$, if and only if the operator matrices*

$$H_k := (A_{ij})_{i,j \in S_k}, \quad k = 1, \dots, n, \tag{2.1}$$

are positive definite. Here

$$S_k = \{p \in \underline{n} \mid (p,k) \in S \text{ and } p \leqslant k\}.$$

In that case, if $S_k = \{p_1, \dots, p_{s_k}\}$ for $k = 1, \dots, n$ with $p_1 \leqslant \dots \leqslant p_{s_k} = k$, put

$$\begin{bmatrix} \hat{X}_{p_1,k} \\ \vdots \\ \hat{X}_{p_{s_k},k} \end{bmatrix} := H_k^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \quad k = 1, \dots, n, \tag{2.2}$$

and let $X = (X_{ik})_{i,k=1}^n$ be given by

$$X_{ik} = \begin{cases} 0, & i \notin S_k; \\ \hat{X}_{ik}\hat{X}_{kk}^{-1/2}, & i \in S_k. \end{cases} \tag{2.3}$$

Then

$$B := X^{*-1}X^{-1}$$

is the unique positive definite operator matrix $B = (B_{ij})_{i,j=1}^n$ with $B_{ij} = A_{ij}, (i,j) \in$ $\in S$ and $(B^{-1})_{ij} = 0, (i,j) \notin S$.

Proof. We will obtain this theorem as a special case of Theorem I.1.1. Let $\mathcal{M}$ be the algebra of operator matrices $F = (F_{ij})_{i,j=1}^n$ with $A_{ij}: \mathcal{H}_j \to \mathcal{H}_i$. The unit in $\mathcal{M}$ is the identity block matrix $\operatorname{diag}(I_{\mathcal{H}_i})_{i=1}^n$ and the involution * on $\mathcal{M}$ is the usual adjoint of operator matrices. Make the decomposition (1.1) of $\mathcal{M}$ induced by S. Since S is row-diagonally connected the subspaces satisfy the multiplication (I.1.2) (Proposition III.1.1).

Put $A = (A_{ij})_{i,j=1}^n$, where A_{ij} for $(i,j) \in S$ are given operators and $A_{ij} = 0$ for $(i,j) \notin S$. Then $A = A^* \in \mathcal{M}_c$. The equation (I.1.5) with $k = A$ has the unique solution $x = \hat{X}$, where $\hat{X} = (\hat{X}_{ij})_{i,j=1}^n$, with $\hat{X}_{ij}$ for $(i,j) \in S$ and $i \leqslant j$ given by (2.1) and $\hat{X}_{ij} = 0$ otherwise. Since the matrices $H_k, k = 1, \dots, n$, are positive definite and $\hat{X}_{kk}$ is the right lower entry of H_k^{-1}, we have that

$$P_d x = \operatorname{diag}(\hat{X}_{kk})_{k=1}^n$$

is positive definite. Further, since $\hat{X}$ is upper triangular with an invertible diagonal, $x^{-1} = \hat{X}^{-1} \in \mathcal{M}_+$. But then theorem follows immediately from Theorem I.1.1, where the uniqueness of the extension B follows form the uniqueness of the solution x to the equation $P_2(kx) = e$. ■

We refer to the operator matrix B in Theorem III.2.1 as the *band extension* of the given data $\{A_{ij} \mid (i,j) \in S\}$.

THEOREM III.2.2. *Let $S \subset \underline{n} \times \underline{n}$ be a row-diagonally connected pattern, and let $A_{ij} = A_{ji}^*$ be an operator from the Hilbert space $\mathcal{H}_j$ into the Hilbert space $\mathcal{H}_i$. Suppose that the operator matrices H_j, defined in (2.1), are positive definite ($j = 1, \ldots, n$). Put*

$$M_i := [0 \cdots 0 I] H_i^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \quad i = 1, \ldots, n.$$

Then, if $C = (C_{ij})_{i,j=1}^n$ is a positive definite operator matrix with $C_{ij} = A_{ij}, (i,j) \in S$, then

$$\Delta_k^r(C) \leqslant M_i^{-1}, \quad k = 1, \ldots, n, \tag{2.4}$$

where

$$\Delta_k^r(C) = C_{kk} - [C_{k1} \cdots C_{k,k-1}] \begin{bmatrix} C_{11} & \ldots & C_{1,k-1} \\ \vdots & & \vdots \\ C_{k-1,1} & \ldots & C_{k-1,k-1} \end{bmatrix}^{-1} \cdot \begin{bmatrix} C_{1k} \\ \vdots \\ C_{k-1,k} \end{bmatrix}, \quad k = 1, \ldots, n.$$

Moreover, equality holds for $k = 1, \ldots, n$ in (2.4) if and only if C is the unique band extension of the given data.

Proof. The proof is similar to the proof of Theorem II.1.3. ■

Theorem III.2.2 has the following corollary.

COROLLARY III.2.3. *Let $S \subset \underline{n} \times \underline{n}$ be row-diagonally connected and let $A_{ij} = A_{ji}^*$ be matrices of size $v_i \times v_j$ for $(i,j) \in S$. Suppose that the matrices $H_j, j = 1, \ldots, n$, defined in (2.1), are positive definite. Then the unique positive definite block matrix $B = (B_{ij})_{i,j=1}^n$ with $B_{ij} = A_{ij}, (i,j) \in S$, and $(B^{-1})_{ij} = 0, (i,j) \notin S$, has the property that*

$$\det B \geqslant \det C, \tag{2.5}$$

where $C = (C_{ij})_{i,j=1}^{n}$ is any positive definite block matrix with $C_{ij} = A_{ij}$ for $(i,j) \in S$. Moreover, equality holds in (2.5) if and only if $B = C$.

Proof. Use that the determinant of a positive definite block matrix equals the determinant of its right multiplicative diagonal, and the fact that $F \geqslant G$ and $F \neq G$ implies that $\det F > \det G$. ■

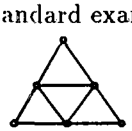
Using the results in Section I.3 one can prove similar results as Theorem III.2.1 and III.2.2 for the case of perfect patterns as well. This requires among others to check the requirements (iii) and (iv) in Section I.3 on the algebra $\mathcal{M}$ of finite operator matrices. In the paper[1] it is shown that for the case of perfect patterns these requirements are fulfilled. The analogs of Theorems III.2.1 and III.2.2 for this case also appear in [1].

III.3. PATTERNS AND GRAPHS

Let $S \subset \underline{n} \times \underline{n}$ be a pattern. We associate with S an undirect graph G without loops or multiple edges as follows. Let the set of vertices V be $\underline{n}$ and there is an edge between vertex i and vertex j if $i \neq j$ and $(i,j) \in S$. Conversely, if $G = (V, E)$ is an undirect graph without loops or multiple edges then after choosing a numbering of the nodes we can associate a pattern as follows. If the vertices are numbered from 1 to n, then the associated pattern $S \subset \underline{n} \times \underline{n}$ is given by

$$S = \{(i,i) \mid i \in \underline{n}\} \cup \{(i,j) \mid \text{there is an edge between } i \text{ and } j \ \}.$$

A graph is called *chordal* if every cycle of lenght strictly greater than 3 (i.e., a sequence of pairwise distinct vertices $v_1, \ldots, v_s$ with $s > 3$ with the property that there is an edge between v_i and $v_{i+1}, i = 1, \ldots, s$ (here $v_{s+1} = v_1$)) possesses a chord, that is an edge joining two nonconsecutive vertices of the cycle. An undirect graph is called an *interval graph* if its vertices can be put into one-to-one correspondence with a set of closed finite intervals $\mathcal{F}$ of the real line $\mathbb{R}$ such that two vertices are connected by an edge of G if and only if their corresponding intervals have a nonempty intersection. It is known that an interval graph is chordal (see [9], and also the appendix in [10]). The converse is not true, and the standard example is the graph



An interval graph is called a *proper interval graph* if each interval may be taken to have unit lenght.

The following proposition translates the properties of patterns, introduced in Section III.1., into properties of the associated graph and vice versa.

PROPOSITION III.3.1. (i) *If a pattern $S \subset \underline{n} \times \underline{n}$ is perfect then its associated graph is a chordal graph. Conversely, if G is a chordal graph then there exists a numbering of the vertices such that the associated pattern is perfect.*

(ii) *If a pattern $S \subset \underline{n} \times \underline{n}$ is row-(column-)diagonally connected, then its associated graph is an interval graph. Conversely, if G is an interval graph then there is a numbering of the vertices such that the associated pattern is row-(column-)diagonally connected.*

(iii) *If a pattern $S \subset \underline{n} \times \underline{n}$ is generalized block banded then its associated graph is a proper interval graph. Conversely, if G is a proper interval graph then there exists a numbering of the vertices such that the associated pattern is generalized block banded.*

Proof. (i) This is a classical result (see [9]). The appropriate ordering is usually referred to as a perfect elimination scheme.

(ii) Let (V, E) be an interval graph. Number the vertices in such a way so that the intervals $I_j \subset \mathbf{R}$ corresponding to vertex j have the property that

$$\min I_j \leqslant \min I_{j+1}, \quad j = 1, \ldots, |V| - 1.$$

If there is an edge between vertex i and j and $i \leqslant j$, then $\min I_j \in I_i$. But then for $i \leqslant k \leqslant j$

$$\min I_k \in [\min I_i, \min I_j] \subset I_i$$

so that $I_k \cap I_i \neq \emptyset$. Thus there is an edge between i and k. But then it follows that the corresponding pattern is row-diagonally connected. In order to make the pattern column-diagonally connected one should require that

$$\max I_i \leqslant \max I_{i+1}, \quad i = 1, \ldots, |V| - 1.$$

For the converse, let S be row-diagonally connected. Put

$$I_i := [\beta(i), i], \quad i = 1, \ldots, n,$$

where $\beta(i) = \min\{j \mid (i,j) \in S\}$. It is easy to see that $(i,j) \in S$ if and only if $I_i \cap I_j \neq \emptyset$. But then the graph associated with S is an interval graph.

For (iii) note that if the intervals have unit lenght, then

$$\min I_j \leqslant \min I_{j+1}, \quad j = 1, \ldots, |V| - 1, \tag{3.1}$$

implies

$$\max I_j \leqslant \max I_{j+1}, \quad j = 1, \ldots, |V| - 1. \tag{3.2}$$

This implies that for a suitable ordering the pattern associated with a proper interval graph is generalized block banded.

For the converse, remark that if a set of finite intervals $I_1, \ldots, I_n$ have the properties (3.1) and (3.2) then one can make the intervals of unit lenght without changing the existence of an overlap between pairs of intervals. ∎

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