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ON THE THEORY OF INDEX FOR TYPE III FACTORS

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0. INTRODUCTION

In [8], V. Jones developed the theory of index for type II_1 factors and since then index theory has led to many other important discoveries of properties of type II_1 subfactors (cf. [15]), and has also found applications in the study of knots and mathematical physics [9].

In [10], H. Kosaki extended the notion of an index to any (normal faithful) expectation from a factor onto a subfactor. While Jones' definition of the index is based on the coupling constant introduced by Murray and von Neumann, Kosaki's definition of the index of an expectation relies on the notion of spatial derivatives due to A. Connes as well as the theory of operator-valued weights due to U. Haagerup.

In [10], it was shown that many fundamental properties of the Jones index in the type II_1 case such as the basic construction, the range of the value of the index, the local index formula, etc., can be extended to the general setting.

For convenience, we shall say that N is a subfactor of finite index of M if there is a normal faithful conditional expectation of M onto N that has finite index. Given a subfactor N of finite index in M, how closely related are N in M? It follows from the definition N is of finite index in M if and only if M' is so with respect to N'. Thus a result of J. Tomiyama (cf. [18]) asserts that in this case, M and N must have the same algebraic type. By the work of A. Connes, type III factors are further classified into type III_λ factors for $0 \leqslant \lambda \leqslant 1$, a natural question is then: given a III_λ factor M, what subfactors of finite index can M contain? In the II_1 case, if the Jones index of N is finite with respect to M, then M and N share many common properties such as hyperfiniteness, property Γ, property T, as shown in [15] and [16]. It is thus reasonnable to expect that there ought to be a close relationship between the types of a given type III factor and its subfactors of finite index.

Another question that arises naturally is that, since a type III_λ factor, $0 < \lambda \leqslant 1$, is the crossed product of a type II_∞ factor by an action of $\mathbb{Z}$ or $\mathbb{R}$, one may wonder if the theory of index for type III factors is related to that for type II_1 factors.

This paper grew out of an attempt to study the theory of index along these lines. Parts of it have been announced in [12]. The organization of the paper is as follows:

In section 1, after recalling the basic facts and properties about index theory from [10], we present some elementary examples such as inclusions coming from crossed products and fixed point algebras, inclusions of type I factors, etc.

In section 2, we will prove the main results of the paper. First we establish the stability result concerning the type of a subfactor N of finite index of a type III_λ factor M, $0 \leqslant \lambda \leqslant 1$. The idea behind the proof is to compare the modular theory of N to that of M, via the conditional expectation E. It turns out that in the case $0 < \lambda < 1$, a certain discrepancy between the T-sets of M and N gives rise to an obstruction for N to have finite index in M. Then by using the discrete and continuous decompositions for type III_λ factors, $\lambda \neq 0$, we show how the theory of index for type III_λ factors, $\lambda \neq 0$, can be reduced essentially to the theory of Jones index for type II_1 factors. Finally we show that a subfactor of type III_{λ^m} or $\mathrm{III}_{\lambda^{1/m}}$ of index m contained in the hyperfinite type III_λ factor $\mathcal{R}_\lambda$, for $\lambda \neq 0, 1$, is unique up to conjugacy.

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1. PRELIMINARIES

In this section, we recall some basic facts from the theory of index as developed in [8] and [10]. Other useful references are [5] and [15].

All von Neumann algebras in this paper are always assumed to be separable. Let $N \subset M$ be a pair of factors, let E be a normal faithful conditional expectation from M onto N, (the set of such expectations will be denoted by $\mathcal{E}(M, N)$). Then there is a uniquely defined operator-valued weight E^{-1} from N' into M' that satisfies the following equations of spatial derivatives:

$$\frac{\mathrm{d}\varphi \circ E}{\mathrm{d}\psi'} = \frac{\mathrm{d}\varphi}{\mathrm{d}\psi' \circ E^{-1}}$$

for all normal faithful semi-finite weights φ on N and ψ' on M'.

The index of E is defined to be (see [10]): $\mathrm{Ind}E = E^{-1}(1)$. Since N is a factor, $\mathrm{Ind}E$ is a scalar (possibly ∞). E is said to have finite index if $\mathrm{Ind}E < \infty$.

If E has finite index, then $cE^{-1} \in \mathcal{E}(N', M')$, where $c = (\mathrm{Ind}E)^{-1}$, and $\mathrm{Ind}(cE^{-1}) = \mathrm{Ind}E$.

We recall some of the main properties of $\mathrm{Ind}E$. (See [8] and [10] for proofs.)

(1) $\mathrm{Ind}E$ does not depend on the Hilbert space on which M and N act.

(2) $\mathrm{Ind}E \in \{4\cos^2\left(\frac{\pi}{n}\right); n \geqslant 3\} \cup [4, \infty]$.

(3) $\mathrm{Ind}(E \circ F) = \mathrm{Ind}E \cdot \mathrm{Ind}F$, for $E \in \mathcal{E}(N, P)$ and $F \in \mathcal{E}(M, N)$.

(4) The following local index theorem holds.

Let p be a non-zero projection of $N' \cap M$. If $E_p(x) = E^{-1}(p)E(x)p$, for $x \in pMp$, then $E_p \in \mathcal{E}(pMp, pN)$ and:

(a) If $\mathrm{Ind}E < \infty$, then $\mathrm{Ind}E_p \leqslant E(p)E^{-1}(p)$.

(b) If $p \in (N' \cap M)_E$, then $\mathrm{Ind}E_p = E(p)E^{-1}(p)$.

(c) $\{p_i\}$ is a partition of the unit in $(N' \cap M)_E$, then

$$\mathrm{Ind}E = \sum E(p_i)^{-1}\mathrm{Ind}E_{p_i}.$$

Thus $\mathrm{Ind}E < \infty \Rightarrow \dim(N' \cap M)_E < \infty$ and $\mathrm{Ind}E < 4 \Rightarrow N' \cap M = \mathbb{C}$.

Let us now consider some elementary examples where the index can be computed. These examples are probably well-known, they are included here for they will used later.

EXAMPLE 1.1. *Let G be a discrete group of outer automorphisms of the von Neumann algebra A and H a subgroup of G. Set $M = A \rtimes G$ and $N = A \rtimes H$. Suppose that $E \in \mathcal{E}(M, N)$ is the expectation for which $E(u_g) = 0$, for any $g \in G \backslash H$, then $\mathrm{Ind}E = [G\colon H]$.*

Proof. We may assume that M is acting standardly with a cyclic and separating vector ξ such that $\varphi \circ E = \omega_\xi$, where φ is a fixed normal faithful state of N. Let e_N be the projection defined by: $e_N(x\xi) = E(x)\xi$, for $x \in M$.

Write $G = \bigcup_{i \in I} g_i H$ as a disjoint union of H-cosets. Then it is easy to see that $\{u_{g_i} e_N u_{g_i}^*\}_{i \in I}$ is a partition of the unit of $\{M, e_n\}''$.

Define $F = JE^{-1}(J \cdot J)J$. Then F is an operator-valued weight $\{M, e_N\}''$ into M. We have:

$$E^{-1}(1) = F(1) = F\left(\sum_{i \in I} u_{g_i} e_N u_{g_i}^*\right) = \sum_{i \in I} u_{g_i} F(e_N) u_{g_i}^* = \sum_{i \in I} u_{g_i} u_{g_i}^* = [G\colon H],$$

where we have used the facts $Je_N J = e_N$ and $E^{-1}(e_N) = 1$ (see [10]). ∎

As an application of the calculations above, using the discrete decomposition for type III_λ factors, $\lambda \neq 1$, we can construct, for any positive integers m and n, an example of a type $\mathrm{III}_{\lambda^{m/n}}$ subfactor of a type III_λ factor, $\lambda \neq 1$, which has index mn.

EXAMPLE 1.2. *Let M be a factor and G a discrete group of outer automorphisms of M, and M^G the fixed point algebra of G. Then for any $E \in \mathcal{E}(M, M^G)$,* $\mathrm{Ind}E = |G|$.

Proof. We may assume that M is acting standardly. Let $\{u_g, g \in G\}$ be the canonical group of unitaries which implement α. Since $u_g M' u_g^* = M'$, G also acts on M'. Denote this action by α'. As α is outer on M, so is α' on M'. We shall need the following result of Aubert [1, II.4].

With M, α, G as above, if there exists an expectation $T \in \mathcal{E}((M^\alpha)', M')$, then there is an isomorphism Φ from $(M^\alpha)'$ onto $M' \rtimes_{\alpha'} G$, which is the identity on M' and such that $\Phi \circ T \circ \Phi^{-1} = F$, where F is the canonical conditional expectation from $M' \rtimes_{\alpha'} G$ onto M'.

There are two cases:

1) If $|G| < \infty$, then as $M \rtimes_\alpha G$ is the basic construction for $M^\alpha \subset M$, $M \subset \subset M \rtimes_{\alpha'} G$ is anti-isomorphic to $M' \subset (M^\alpha)'$. It follows from Aubert's result that there is an expectation $F \in \mathcal{E}((M^\alpha)', M')$ such that $\mathrm{Ind}F = |G|$. Thus $(|G|)^{-1}F^{-1}$ must agree with E as the relative commutant of M^α in M is trivial. So $\mathrm{Ind}E = = \mathrm{Ind}F = |G|$.

2) If $|G| = \infty$ and suppose that $\mathrm{Ind}E < \infty$, then $T = (\mathrm{Ind}E)^{-1}E^{-1} \in \mathcal{E}((M^\alpha)', M')$ and $\mathrm{Ind}E = \mathrm{Ind}T$. On the other hand, by Aubert's result, $\mathrm{Ind}T = \mathrm{Ind}F = \infty$, where F is canonical expectation from $M' \rtimes_{\alpha'} G$ onto M'. This contradicts our assumption that $\mathrm{Ind}E < \infty$. Hence $\mathrm{Ind}E = \infty$. ■

The following result of [2] will be useful for the next two examples.

LEMMA 1.3 [2, Proposition 2.3]. *Let $N \subset M$ be von Neumann algebras, and P a factor. For each $E \in \mathcal{E}(M \otimes P, N \otimes P)$, there is a unique $F \in \mathcal{E}(M, N)$ such that* $E = F \otimes \mathrm{Id}_P$.

Note that if M, N are factors, then $\mathrm{Ind}E = \mathrm{Ind}F$.

Now suppose that $N \subset M$ are type I factors, and $E \in \mathcal{E}(M, N)$ has finite index. Then there exist Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ such that $N \subset M$ is isomorphic to $B(\mathcal{H}) \otimes \mathbb{C} \subset \subset B(\mathcal{H}) \otimes B(\mathcal{K})$. Thus by 2.3, there is an $F \in \mathcal{E}(B(\mathcal{K}), \mathbb{C})$ such that $E = \mathrm{Id}_{B(\mathcal{H})} \otimes F$. Hence F can be viewed as a normal faithful state on $B(\mathcal{K})$.

As $\mathrm{Ind}F = \mathrm{Ind}E < \infty, F^{-1}$ is a normal faithful functional on $B(\mathcal{K})$. Thus there are positive, non-singular, trace-class operators A and B on $\mathcal{K}$ such that $F = = \mathrm{Tr}_A$, $\mathrm{Tr}(A) = 1$, and $F^{-1} = \mathrm{Tr}_B$. Using 7.3 in [18], we infer that $A = B^{-1}$ and so both B and B^{-1} are of trace-class. This holds if and only if $\mathcal{K}$ is finite dimensional and in which case, $\mathrm{Ind}E = \mathrm{Ind}F = F^{-1}(1) = \mathrm{Tr}(B) = \mathrm{Tr}(A^{-1})$.

We have thus shown the following:

EXAMPLE 1.4. *The inclusion $N \subset M$ of type I factors has finite index if and only if there exist Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ such that $N \subset M$ is isomorphic to $B(\mathcal{H}) \otimes \mathbf{C} \subset \subset B(\mathcal{H}) \otimes B(\mathcal{K})$.*

In particular, we see that in this case, the numbers $4\cos^2\left(\frac{\pi}{n}\right)$ are never taken on by the index.

Next we consider a pair $N \subset M$ of type II_∞ factors, together with an expectation $E \in \mathcal{E}(M, N)$ such that $\mathrm{Tr}_M = \mathrm{Tr}_N \circ E$, where Tr_M and Tr_N are some normal faithful semi-finite traces on M and N. Suppose that $\mathrm{Ind}E < \infty$, then there exist type II_1 factors $B \subset A$ such that $N \subset M$ is isomorphic to $A \otimes B(\mathcal{H}) \subset B \otimes B(\mathcal{H})$. And so by 2.3, E can be identified with $F \otimes \mathrm{Id}_{B(\mathcal{H})}$, where F is the trace conditioning expectation on $B \subset A$. It follows that $\mathrm{Ind}E = \mathrm{Ind}F$, and thus in this context, the type II_∞ index is reduced to just the type II_1 index.

2. INCLUSIONS OF TYPE III FACTORS OF FINITE INDEX

In this section we will study the index theory for an inclusion of type III factors. To this end we will need to recall some facts concerning the notion of an orthonormal basis and some analytical formula for the index due to [15]. We state these results in the setting of inclusions of properly infinite factors, their proofs are an easy adaptation of the original arguments in [15] and can also be found in [12].

PROPOSITION 2.1. *Let M and N be properly infinite factors on the Hilbert space $\mathcal{H}$, and $E \in \mathcal{E}(M, N)$. Let φ be a normal faithful state on N with $\varphi \circ E = \omega_\xi$, where ξ is a cyclic and separating vector for M. Let e_N be the projection defined by: $e_N(x\zeta) = E(x)\xi$, for $x \in M$.*
The following are equivalent:

(1) $\mathrm{Ind}E < \infty$.

(2) *for each $x \in \{M, e_n\}''$, there is a unique $y \in N$ such that $xe_N = ye_N$;*

(3) *there is a $u \in M$ such that $E(u^*u) = 1$, $ue_Nu^* = 1$, and $x = uE(u^*x)$ $\forall x \in \in M$;*

(4) *there is a $u \in M$ such that $ue_Nu^* = 1$.*

PROPOSITION 2.2. *(Pimsner-Popa estimates) Let M and N be properly infinite factors and $E \in \mathcal{E}(M, N)$ such that $\mathrm{Ind}E < \infty$, then the following hold:*

$$\begin{aligned}(\mathrm{Ind}E)^{-1} &= \sup\{c \geqslant 0; E(x) \geqslant cx, \forall x \in M_+\}\\ &= \inf\{\|E(x)\|; \forall x \in M^+ \textit{ with } \|x\| = 1\}\\ &= \inf\{\|E(f)\|; \forall \textit{ nonzero projection } f \in M\}\end{aligned}$$

The converse of Proposition 2.1 was shown to hold in some special cases in [13], and a simple proof in the general case has been obtained in [11].

The Pimsner-Popa estimates are remarkably useful alternatives to the original definition of the index. Indeed, they have been proposed in [15] as an alternate definition for the index. For instance, using these estimates, one can show (cf. [12]) that the relative commutant is always finite dimensional for any inclusion of factors with finite index.

Let $N \subset M$ be an inclusion of type III factors of finite index, we are going to show the restriction on the type of the subfactor N. We consider the III_0 case first.

PROPOSITION 2.3. *Let $N \subset M$ be factors such that M is of type* III_0. *If there exists an expectation $E \in \mathcal{E}(M,N)$ having finite index, then N is of type* III_0.

Proof. The fact that N is of type III is a consequence of Tomiyama's theorem. If N is of type III_λ with $0 < \lambda \leqslant 1$, then there exists a normal faithful semi-finite weight φ on N such that the centralizer N^φ is a factor (cf. 21.6, 23.9, 29.12 in [18]). If we set $\psi = \varphi \circ E$, then since $\sigma_t^\psi \circ E = E \circ \sigma_t^\psi$ for all t, $E\colon \mathcal{Z}(M^\psi) \to \mathcal{Z}(N^\psi) = \mathbb{C}$.

By the Pimsner-Popa estimates in Proposition 2.2, we have:

$$0 < (\mathrm{Ind}E)^{-1} \leqslant \inf\{\|E(f)\|; 0 \neq f \in \mathcal{Z}(M^\psi)\}.$$

But since $\mathcal{Z}(M^\psi)$ is diffuse (cf. [18]) and $\|E(f)\| = E(f)\ \forall f \in \mathcal{Z}(M^\psi)$, the above inequality is violated. Thus N must be of type III_0. ∎

To consider the remaining cases where $0 < \lambda \leqslant 1$, we need the following result which shows that, for an inclusion $N \subset M$ of type III factors, a certain discrepancy between the T-sets of M and N will imply that the index is infinite.

PROPOSITION 2.4. *Let $N \subset M$ be type* III *factors and $E \in \mathcal{E}(M,N)$. If there exists a number $t \in T(N)$ such that $nt \notin T(M)$ for all nonzero integers n, then* $\mathrm{Ind}E = \infty$.

Proof. Suppose on the contrary that $\mathrm{Ind}E < \infty$. Then $N' \cap M$ has finite dimension. Let p be a minimal projection in $N' \cap M$ and consider $E_p \in \mathcal{E}(pMp, Np)$ defined by:

$$E_p(pxp) = (E(p))^{-1}E(pxp)p,\ x \in M.$$

It follows from the local index formula that $\mathrm{Ind}E_p \leqslant \mathrm{Ind}E < \infty$ and $(Np)' \cap pMp = \mathbb{C}$. Moreover, the T-sets of $Np \subset pMp$ satisfy the same hypothesis as in the proposition. Thus without loss of generality, we may assume that $N' \cap M = \mathbb{C}$.

Let $t \in T(N)$ be as stated. There is then a normal faithful state φ on N such that $\sigma_t^\varphi = \mathrm{Id}_N$. Set $\psi = \varphi \circ E$, then $\alpha = \sigma_t^\psi$ is an aperiodic automorphism of M

such that $N \subset M^\alpha$. Moreover, M^α is a factor (as $N' \cap M = \mathbb{C}$) which is globally invariant under the modular group σ^ψ, hence by Takesaki's theorem (cf. [18]), there exists a conditional expectation $F \in \mathcal{E}(M, M^\alpha)$ such that $\psi = \psi \circ F$ or equivalently, $\varphi \circ E = \varphi \circ E_1 \circ F$, where E_1 denotes the restriction of E on M^α. Hence $E = E_1 \circ F$ by 11.3 of [18] and it follows that $\mathrm{Ind}E = (\mathrm{Ind}E_1)(\mathrm{ind}F) = \infty$ because $\mathrm{Ind}F = \infty$ by Example 1.2. We have reached a contradiction. Thus $\mathrm{Ind}E = \infty$ as was to be proved. ∎

We record in the following a couple easy corollaries of Proposition 2.4.

COROLLARY 2.5. *Let $N \subset M$ be type* III *factors and E a conditional expectation in $\mathcal{E}(M, N)$. If* $\mathrm{Ind}E < \infty$, *then there exist positive integers m and n such that $mT(N) \subset T(M)$ and $nT(M) \subset T(N)$.*

Proof. It suffices to prove the first inclusion. The second one follows from the same arguments applied to the inclusion of commutants.

As $\mathrm{Ind}E < \infty$, we may assume, as in the proof of 2.4, that $N' \cap M = \mathbb{C}$. Also from the proof of 2.4, we see that for each nonzero $t \in T(N)$, there is an integer k with $0 \leqslant k \leqslant \mathrm{Ind}E$ and $kt \in T(M)$. Taking m to be the l.c.m of these integers, we get $mT(N) \subset T(M)$. ∎

COROLLARY 2.6. *Let $N \subset M$ be type* III *factors and E a conditional expectation in $\mathcal{E}(M, N)$. If* $\mathrm{Ind}E < \infty$, *then*

(1) *$T(M)$ is countable if and only if $T(N)$ is;*

(2) *$T(M)$ is dense if and only if $T(N)$ is.*

Since factors of type III_λ, $0 < \lambda \leqslant 1$, are characterized by their T-invariants, by combining Propositions 2.3 and 2.4, we obtain the restriction of the type of subfactors of finite index of a type III factor.

THEOREM 2.7. *Let M be a factor of type* III_λ, $0 \leqslant \lambda \leqslant 1$, *and N a subfactor of M. If there exists a conditional expectation E in $\mathcal{E}(M, N)$ of finite index, then N is of the type* $\mathrm{III}_{\lambda^{m/n}}$, *for some positive integers m and n.*

We should mention that without the extra condition of finite index, Theorem 2.7 doesn't hold. In fact, it is possible to construct (as was done in [13]), for any $0 \leqslant \lambda$, $\mu \leqslant 1$, an inclusion $N \subset M$ such that M is of type III_λ and N is of type III_μ, which is the range of a conditional expectation from M.

We shall see that in the case $0 < \lambda < 1$, an upper bound for the integers m, n can be obtained.

Having now obtained the restriction of the type of subfactors of finite index of any given type III factors, we look into the special case of $0 < \lambda < 1$. Since every type III_λ factor, for $\lambda \neq 0,1$, can be decomposed as the crossed product of a type II_∞ factor by an action of $\mathbb{Z}$, it is then natural to try to relate the index of a given type III_λ inclusion to the type II_1 index.

The next result shows that, under some mild conditions, a type III_λ inclusion can always be decomposed into three inclusions: the top and bottom ones are, respectively, a crossed product and a fixed point algebra by an outer action of a finite group, and the middle one is determined by some inclusion of type II_1 factors.

THEOREM 2.8. *Let M be a factor of type III_λ, $0 < \lambda < 1$, and N a subfactor of M of type $\mathrm{III}_{\lambda^{m/n}}$, where m, n are (coprime) positive integers. Suppose that there is an expectation $E \in \mathcal{E}(M,N)$ with finite index and such that $(N' \cap M)_E$ is a factor. Then there exist:*

(1) *factors of type III_{λ^m} P and Q such that $N \subset Q \subset P \subset M$;*

(2) *normal faithful conditional expectations $F \in \mathcal{E}(M,P)$, $G \in \mathcal{E}(P,Q)$, and $H \in \in \mathcal{E}(Q,N)$ which satisfy:*

(a) *$E = H \circ G \circ F$, $\mathrm{Ind}F = m$, $\mathrm{Ind}H = n$, so that $\mathrm{Ind}E = mn(\mathrm{Ind}G)$.*

(b) *There exists a common discrete decomposition for the pair of III_{λ^m} factors $Q \subset P$, and $\mathrm{Ind}G$ is equal to the Jones index for the pair of type II_1 factors, which appear as the finite tensor components of the pair of type II_∞ factors that give rise to the common discrete decomposition of $Q \subset P$.*

Recall that $(N' \cap M)_E = \{x \in N' \cap M; \sigma_t^E(x) = x \ \forall t \in \mathbb{R}\}$. We will need the following well-known fact. Let G be a finite group of outer automorphisms of the factor M, then M^G is isomorphic to $M \rtimes G$.

Proof. Let φ be a normal faithful state on N such that $\sigma_S^\varphi = \mathrm{Id}_N$, where $S = -\dfrac{n}{m}\dfrac{2\pi}{\log\lambda}$ is a generator for $T(N)$. Set $\psi = \varphi \circ E$. As $mS \in T(M)$, $\sigma_{mS}^\psi = \mathrm{Ad}u$ for some unitary $u \in \mathcal{Z}(M^\psi)$. Since $\sigma_{mS}^\psi | N = \sigma_{mS}^\varphi = \mathrm{Id}_N$, $u \in \mathcal{Z}(M^\psi) \cap N' \subset \subset \mathcal{Z}((N' \cap M)_E) = \mathbb{C}$ by hypothesis. Thus u is a scalar and $\sigma_{mS}^\psi = \mathrm{Id}_M$. This means that if we set $\alpha = \sigma_S^\psi$, then α is an outer automorphism of M with period $= m$. Thus M^α is a subfactor of M of index m and containing N.

We claim that M^α is of type III_{λ^m}. According to Theorem 2.4 and the remark preceding the proof, it suffices to calculate the T-set of $M \rtimes_\alpha \mathbb{Z}_m$. To this end, applying Proposition 2.9 in [17], we see that $t \in T(M \rtimes_\alpha \mathbb{Z}_m)$ if and only if there exists some integer r with $0 \leqslant r < m$ such that $\sigma_t^\psi \circ \alpha^r$ is inner, i.e., σ_{t+rS}^ψ is inner. Note that the second condition in Sauvageot's result is automatically satisfied due to the invariance of ψ with respect to α. As $mS \in T(M)$, the preceding condition is equivalent to the fact that there is an integer r with $0 \leqslant r < m$ such that $\sigma_{t+rS+qmS}^\psi$ is inner for some integer q, or $\sigma_{t+(r+qm)S}^\psi$ is inner for some integers q, r with $0 \leqslant r < m$.

Therefore $t \in T(M \rtimes_\alpha \mathbb{Z}_m)$ if and only if $t + kS \in T(M)$ for some $k \in \mathbb{Z}$;

if and only if $t + kS = l\dfrac{2\pi}{\log\lambda}$ for some $k, l \in \mathbb{Z}$;

if and only if $t = (ml - nk)\dfrac{2\pi}{m\log\lambda}$ for some $k, l \in \mathbb{Z}$.

As m and n are assumed to be coprime, the latter condition occurs if and only if t is a multiple of $\frac{2\pi}{m\log\lambda}$. Hence we obtain $T(M \rtimes_\alpha \mathbb{Z}_m) = \dfrac{2\pi}{m\log\lambda}\mathbb{Z}$ and so M^α is of type III_{λ^m}.

Set $P = M^\alpha$ and let F be the canonical expectation of M onto P, then $\mathrm{Ind}F = m$, $\psi \circ F = \psi$ so that $\varphi \circ \tilde{E} \circ F = \varphi \circ E$, where $\tilde{E}$ is the restriction of E on P. Thus $E = \tilde{E} \circ F$.

We observe that $(N' \cap P)_{\tilde{E}}$ is a factor for $(N' \cap P)_{\tilde{E}} = N' \cap M^\alpha \cap M^\psi = N' \cap M^\psi = (N' \cap M)_E$. We also note that $\mathrm{Ind}\tilde{E} < \infty$.

Passing to the commutants, we have:

$c\tilde{E}^{-1} \in \mathcal{E}(N', P')$, with $c = (\mathrm{Ind}\tilde{E}^{-1})$, N' is of type $\mathrm{III}_\mu (\mu = \lambda^{m/n})$ and P' is of type III_{μ^n}. Moreover, $(N' \cap P)_{c\tilde{E}^{-1}} = (N' \cap P)_{\tilde{E}}$ because $\sigma_t^{c\tilde{E}^{-1}} = \sigma_{-t}^{\tilde{E}}$ for all $t \in \mathbb{R}$.

Applying the same argument as in the first part of the proof to the pair $P' \subset N'$ and the expectation $c\tilde{E}^{-1}$, we infer that there exist:

1) a subfactor Q of P which is of type III_{μ^n}, (i.e., III_{λ^m}) such that $P' \subset Q' \subset N'$;

2) conditional expectations $F_1 \in \mathcal{E}(N', Q')$, $F_2 \in \mathcal{E}(Q', P')$ such that $c\tilde{E}^{-1} = F_1 \circ F_2$, $\mathrm{Ind}F_2 = n$ and $(Q' \cap P)_{F_2}$ is a factor.

Set $G = (\mathrm{Ind}F_2)^{-1}F_2^{-1}$ and $H = (\mathrm{Ind}F_1)^{-1}F_1^{-1}$, we have $N \subset Q \subset P \subset M$, $E = H \circ G \circ F$, $\mathrm{Ind}H = n$ and $(Q' \cap P)_G$ is a factor. Now let us concentrate on the inclusion $Q \subset P$.

Let ω be a generalized trace on Q, i.e., $\omega(1) = \infty$ and $\sigma_T^\omega = \mathrm{Id}_Q$, where $T = \dfrac{-2\pi}{m\log\lambda}$. If $\rho = \omega \circ G$, then $\sigma_T^\rho = \mathrm{Ad}u$, with $u \in Q' \cap \mathcal{Z}(P^\rho) \subset \mathcal{Z}((Q' \cap P)_G) = \mathbb{C}$. So u is a scalar and $\sigma_T^\rho = \mathrm{Id}_P$ and hence ρ is a generalized trace on P. If v denotes the unitary in Q which satisfies $\lambda^m \omega = \omega \circ \mathrm{Ad}v$ as in 30.1 of [18], then $Q = Q^\omega \rtimes_\theta \mathbb{Z}$, where θ is the automorphism defined by $\mathrm{Ad}v$. It is then obvious that $\lambda^m \rho = \rho \circ \mathrm{Ad}v$, and as in the proof of 30.1 of [18], this ensures that $P = P^\rho \rtimes_\theta \mathbb{Z}$ as well. In other words, there is a common discrete decomposition for $Q \subset P$. Note also that $Q^\omega \subset P^\rho$ are type II_∞ factors.

The restriction of G to P^ρ then defines a conditional expectation onto Q^ω which is denoted by $\tilde{G}$. Furthermore, $\tau_1 = \tau_2 \circ \tilde{G}$, where τ_1 and τ_2 are the normal faithful semifinite traces on P^ρ and Q^ω obtained by restricting ρ and ω.

If e is a finite non-zero projection in Q^ω, then e is also finite in P^ρ and thus the pair $Q^\omega \subset P^\rho$ is spatially isomorphic to $(Q^\omega)_e \otimes B(\mathcal{H}) \subset (P^\rho)_e \otimes B(\mathcal{H})$. By Lemma 2.3, $\tilde{G}$ can be identified with an expectation of the form $K \otimes \mathrm{Id}_{B(\mathcal{H})}$, where K is the trace preserving conditional expectation of the pair of type II_1 factors $(Q^\omega)_e \subset (P^\rho)_e$. We have $\mathrm{Ind}\tilde{G} = \mathrm{Ind}K = [(Q^\omega)_e : (P^\rho)_e]$.

As $\mathrm{Ind}G < \infty$, G satisfies the Pimsner-Popa estimates, hence so does the restriction $\tilde{G}$ and thus K. By Theorem 2.2 of [15], this implies that $\mathrm{Ind}K$ and hence $\mathrm{Ind}\tilde{G}$ are finite.

By Proposition 2.1, there exists a $w \in P^\rho$ such that for all $x \in P^\rho$, we have $x = w\tilde{G}(w^*x)$ and $ww^* = \mathrm{Ind}\tilde{G}$. Since $G(v^k) = v^k$ for all integers k, we see that the identity $x = wG(w^*x)$ holds for all finite sums $x = \sum x_k v^k$, where the x_k's are in P^ρ. By the σ-weakly continuity of G , $x = wG(w^*x)$ for all $x \in P$. By Proposition 2.1, we conclude that $\mathrm{Ind}G = ww^* = \mathrm{Ind}\tilde{G} = [(Q^\omega)_e : (P^\rho)_e]$. ■

REMARKS 2.9.

1) The assumption that $(N' \cap M)_E$ is a factor is equivalent to the fact that $N' \cap M$ is a factor and σ^E is trivial. Indeed, suppose that $(N' \cap M)_E$ is a factor. As $\mathrm{Ind}E < \infty$, $\dim(N' \cap M) < \infty$. Since σ^E is the modular group for the normal faithful state $E|N' \cap M$, $\sigma_t^E = \mathrm{Ad}(h^{it})$, $t \in \mathbb{R}$, for some positive invertible element h in $N' \cap M$. But since h is actually in the center of $N' \cap M$, which is trivial, $\sigma_t^E = \mathrm{Id}$ for all $t \in \mathbb{R}$ and so $N' \cap M$ is factor. The converse is obvious.

2) The condition that $(N' \cap M)_E$ is a factor is only needed to show that for the middle inclusion $Q \subset P$, a common discrete decomposition exists, since with a little more effort, we can show that the top inclusion, and hence the bottom one, always exists. For let φ and $\psi = \varphi \circ E$ be as in the Theorem, then $\alpha = \sigma_S^\psi$ is an outer automorphism of M and $\alpha^m = \mathrm{Ad}u$ for some unitary u in $N' \cap \mathcal{Z}(M^\psi)$. Let w be a unitary in $N' \cap \mathcal{Z}(M^\psi)$ such that $w^m = u^*$ and set $\beta = \mathrm{Ad}w \circ \alpha$, then β defines an outer action of $\mathbb{Z}_m$ on M such that $N \subset M^\beta$. Similar computations as in Thorem 2.8 show that M^β is a type III_{λ^m} factor. Set $P = M^\beta$ and let $F \in \mathcal{E}(M, P)$ be defined as before, we have that $E = \tilde{E} \circ F$, where $\tilde{E}$ is the restriction of E on P. Now the preceding argument can be repeated for the pair $P' \subset N'$ and the expectation $c\tilde{E}^{-1}$, $c = (\mathrm{Ind}\tilde{E})^{-1}$.

3) For an inclusion $N \subset M$ of type III_λ factors, $0 < \lambda < 1$, such that there is a common discrete decomposition as in the middle inclusion in Theorem 2.8, it is routine to check that the common discrete decomposition is essentially unique as in the single factor case. And thus one can carry out a classification program of this type of inclusion in the hyperfinite case by classifying the corresponding trace scaling automorphism on a pair of type II_∞ factors. These ideas are taken up further in [14] to classify certain subfactors (up to conjugacy) of the Powers factor with index less that 4.

As a consequence of Theorem 2.8, we obtain an upper bound for the integers m

and n in the following.

COROLLARY 2.10. *Let M be a III_λ factor, and N a subfactor of type $\mathrm{III}_{\lambda^{m/n}}$, where $\lambda \neq 0, 1$, and m and n are coprime. Then for any $E \in \mathcal{E}(M, N)$, one has $mn \leqslant \mathrm{Ind}E$ with equality only if $N' \cap M = \mathbb{C}$.*

Proof. If $\mathrm{Ind}E = \infty$, then there is nothing to be proved. Suppose that $\mathrm{Ind}E$ is finite, then $N' \cap M$ has finite dimension. Let p be a minimal projection in $N' \cap M$ so that $(Np)' \cap pMp = \mathbb{C}$ and let $E_p \in \mathcal{E}(pMp, Np)$ be as in Proposition 2.4. By Theorem 2.8, $mn \leqslant \mathrm{Ind}E_p \leqslant \mathrm{Ind}E$.

If $\mathrm{Ind}E = mn$, then for any nonzero projection f in $N' \cap M$ we have:

$$mn \leqslant \mathrm{Ind}E_f \leqslant E(f)E^{-1}(f) \leqslant mn.$$

This forces $f = 1$, i.e., $N' \cap M = \mathbb{C}$. ■

The next corollary shows that, for $\lambda \neq 0, 1$, the index theory for a pair of type III_λ factors of index < 4 is mostly a type II_1 phenomenon.

COROLLARY 2.11. *Let $N \subset M$ be factors. If M is of type III_λ, $\lambda \neq 0, 1$ and $E \in \mathcal{E}(M, N)$ has index < 4, then one of the following holds:*

1) N is of type III_{λ^2} or $\mathrm{III}_{\lambda^{1/2}}$ and $\mathrm{Ind}E = 2$.

2) N is of type III_{λ^3} or $\mathrm{III}_{\lambda^{1/3}}$ and $\mathrm{Ind}E = 3$.

3) N is of type III_λ and $\mathrm{Ind}E$ is equal to the Jones index for a certain pair of type II_1 factors coming from a common discrete decomposition of $N \subset M$.

Proof. As $N' \cap M = \mathbb{C}$ in this case, Theorem 2.8 and Corollary 2.9 imply that N is of type $\mathrm{III}_{\lambda^{m/n}}$ with $mn < 4$. The rest of the corollary follows easily from 2.8.■

The idea of relating the index of a type III_λ inclusion using crossed product to the index of some type II_1 pair also works for type III_1 inclusions.

Let $N \subset M$ be type III_1 factors and $E \in \mathcal{E}(M, N)$ a conditional expectation. Let φ be a normal, faithful and semi-finite dominant weight on N. Then (cf. [18]) there exists a strongly continuous unitary group $\{u_s\}_{s \in \mathbb{R}}$ in N such that $\sigma_t^\varphi(u_s) = e^{its}u_s$, for all t, $s \in \mathbb{R}$. If $\psi = \varphi \circ E$, then $\sigma_t^\varphi(u_s) = e^{its}u_s$, for all t, $s \in \mathbb{R}$ also, and hence ψ is a dominant weight on M. In other words, the pair $N \subset M$ admits a common discrete decomposition, i.e., $N \subset M$ is isomorphic to the inclusion of continuous products $N^\varphi \rtimes_\theta \mathbb{R} \subset M^\psi \rtimes_\theta \mathbb{R}$, where $\theta_t = \mathrm{Ad}u_t$, for $t \in \mathbb{R}$.

As M and N are type III_1, M^ψ and $N^\varphi$are type II_∞ factors. Furthermore, there exist normal, faithful and semi-finite traces τ_ψ and τ_φ on M^ψ and N^φ, respectively, such that $\psi = \tau_\psi \circ E_\varphi$, and $\varphi = \tau_\varphi \circ E_\varphi$, where E_ψ and E_φ are the operator-valued weights from M and N into M^{ψ} and N^{φ}, respectively defined by:

$$E_{\psi}(x) = \int_{-\infty}^{\infty} \sigma_t^{\psi}(x)\mathrm{d}t, x \in M^{+};$$

$$E_{\varphi}(x) = \int_{-\infty}^{\infty} \sigma_t^{\varphi}(x)\mathrm{d}t, x \in N^{+}.$$

It is straightforward to check that $E \circ E_{\psi} = E_{\varphi} \circ E$. Thus the restriction of E to M^{ψ} is the trace preserving expectation. Using the Pimsner-Popa basis and estimates, an argument similar to that of Theorem 2.8 then shows that Ind$E$ (finite or infinite) is the Jones index for the type II_1 tensor components of $N^{\varphi} \subset M^{\psi}$. We summarize the preceding discussion in the following.

PROPOSITION 2.12. *Let $N \subset M$ be type* III_1 *factors and $E \in \mathcal{E}(M,N)$. Then there always exists a common continuous decomposition for $N \subset M$ such that the index of E is given by the Jones index type* II_1 *tensor components of the pair of type* II_{∞} *factors that give rise to the common continuous decomposition.*

We remark that the common continuous decomposition in 2.12 is also essentially unique as in the single factor case.

The rest of the paper is devoted to prove the uniqueness (up to conjugacy) of subfactors of type III_{λ^m} or $\mathrm{III}_{\lambda^{1/m}}$ of index m of the hyperfinite III_{λ} factor $\mathcal{R}_{\lambda}$.

First we show that an inclusion satisfying the conditions above can be easily described by means of crossed product or fixed point algebra of finite discrete group of outer automorphisms.

PROPOSITION 2.13. *Let M be a factor of type* III_{λ}, $0 < \lambda < 1$, *and N a factor of M such that there is an expectation $E \in \mathcal{E}(M,N)$ with* Ind$E = m$.

1) *If N is of type* III_{λ^m}, *then N is the fixed point algebra of an outer action of $\mathbb{Z}_m$ on M.*

2) *If N is of type* $\mathrm{III}_{\lambda^{1/m}}$, *then M is the crossed product of N by an outer action of $\mathbb{Z}_m$.*

Proof. We note that in either case, $N' \cap M = \mathbb{C}$, by Corollary 2.10.

1) Suppose that N is of type III_{λ^m}. Let φ be a normal faithful state on N with $\sigma_S^{\varphi} = \mathrm{Id}_N$, where $S = \dfrac{-2\pi}{m\log\lambda}$. If $\psi = \varphi \circ E$, then $\alpha = \sigma_S^{\varphi}$ is an outer automorphism of M with period m. As $N \subset M^{\alpha} \subset M$, there is an expectation $F \in \mathcal{E}(M, M^{\alpha})$ with Ind$F = m$, hence we have $M^{\alpha} = N$.

2) Suppose that N is of type $\mathrm{III}_{\lambda^{1/m}}$. Let φ be a normal faithful state on N such that $\sigma_T^{\varphi} = \mathrm{Id}_N$, where $T = -m\dfrac{2\pi}{\log\lambda}$. If $\psi = \varphi \circ E$, then $\sigma_{T/m}^{\varphi} = \mathrm{Ad}u$, for some

unitary u in M^{ψ}. As $\mathrm{Ad}u^m = \sigma_T^{\varphi}$, which is identity on N, u^m is a scalar, which we may assume to be 1. Set $\alpha = \sigma_{T/m}^{\varphi}$, then α determines an outer action of $\mathbb{Z}_m$ on N.

For any $x \in N$, we have $\alpha(x)u = ux$ and so, $\alpha(x)E(u) = E(u)x$. From the outerness of α, $E(u) = 0$. Similarly, $E(u^k) = 0$, for $1 \leqslant k \leqslant m-1$. Hence the subfactor N_1 generated by N and u is isomorphic to the crossed product $N \rtimes_{\alpha} \mathbb{Z}_m$. Thus there is an expectation $F \in \mathcal{E}(N_1, N)$ with $\mathrm{Ind}F = m$. On the other hand, by Takesaki's theorem, there is an expectation $G \in \mathcal{E}(M, N_1)$. As $N' \cap M = \mathbb{C}$, $F \circ G = E$. It follows that $\mathrm{Ind}G = 1$ and therefore $M = N_1$. ∎

REMARK 2.14. In case (1) of Theorem 2.13, there is an alternate description of the structure of $N \subset M$ as follows: let φ be a generalized trace on N, put $\psi = \varphi \circ E$ and $S = -\dfrac{2\pi}{m\log\lambda}$. As $N' \cap M = \mathbb{C}$, $\sigma_{mS}^{\psi} = \mathrm{Id}_M$, hence ψ is a generalized trace on M. By the discrete decomposition theorem of Connes, there is a unitary u in M, such that $\sigma_t^{\psi}(u) = \lambda^{it}u$ for all $t \in \mathbb{R}$ and M is isomorphic to $M^{\psi} \rtimes_{\mathrm{Ad}u} \mathbb{Z}$. Note that $N \subset M^{\alpha}$, where $\alpha = \sigma_S^{\psi}$. Since $\alpha^m = \mathrm{Id}_M$, there is an expectation $F \in \mathcal{E}(M, M^{\alpha})$ with $\mathrm{Ind}F = m$. Thus $N = M^{\alpha}$ and so $M^{\psi} = N^{\varphi}$.

Thus for $x \in M$, $x \in N$ if and only if $\alpha(x) = x$. Writing $x = \sum_{-\infty}^{\infty} x_n u^n$, we have: $x = \alpha(x)$ if and only if $\sum_{-\infty}^{\infty} x_n u^n = \sum_{-\infty}^{\infty} \lambda^{inS} x_n u^n$, because $\alpha(u) = \lambda^{iS}u$. Thus $x = \alpha(x)$ if and only if $x_n = 0$ for all $n \notin m\mathbb{Z}$. Hence $N = \{N^{\varphi}, u^m\}''$, and $N \subset M$ is isomorphic to $N^{\varphi} \rtimes_{\mathrm{Ad}u^m} \mathbb{Z} \subset M^{\psi} \rtimes_{\mathrm{Ad}u} \mathbb{Z}$.

It turns out that with some extra assumption, any pair of factors $N \subset M$ satisfying the conditions stated in Theorem 2.13 is unique up to conjugacy in the following sense: for $i = 1, 2$, let $N_i \subset M_i$ be factors and $E_i \in \mathcal{E}(M_i, N_i)$ be expectations. The pairs $N_i \subset M_i$, $i = 1, 2$, are said to be *conjugate* if there exists an isomorphism $\Phi: M_1 \to M_2$ such that $\Phi \circ E_1 \circ \Phi^{-1} = E_2$. Note that this implies that $\Phi(N_1) = N_2$.

THEOREM 2.15. *For each $i = 1, 2$, let $N_i \subset M_i$ be factors. Suppose that M_i is of type III_{λ}, $\lambda \neq 0, 1$, and there exist expectations $E_i \in \mathcal{E}(M_i, N_i)$ such that $\mathrm{Ind}E_i = m$ for some positive integer m.*

If N_1 and N_2 are of type III_{λ^m} (or $\mathrm{III}_{\lambda^{1/m}}$) and if $M_1 \cong M_2$ (resp. if $N_1 \cong N_2$), then the pairs $N_i \subset M_i$ $i = 1, 2$ are conjugate.

Proof. Note that in both cases, the relative commutant is trivial by Corollary 2.10 and so, it suffices to show that there is an isomorphism from $N_1 \subset M_1$ onto $N_2 \subset M_2$.

1) Suppose that N_1, N_2 are of type III_{λ^m} and $M_1 \cong M_2$. We may then assume $M_1 = M_2 = M$ and will show that there is an automorphism of M mapping N_1 onto N_2. Let $E_i \in \mathcal{E}(M, N_i)$ be the expectations with $\mathrm{Ind}E_i = m$. As shown in 2.13, if φ_i is a generalized trace on N_i, then $N_i = M^{\alpha i}$, where $\alpha_i = \sigma_S^{\psi_i}$, $\psi_i = \varphi_i \circ E_i$, and

$S = -\frac{2\pi}{m\log\lambda}$. Since the relative commutants are trivial, each ψ_i is also a generalized trace on M. Thus there exist some constant $c > 0$ and a unitary u in M such that $\psi_1 = c\psi_2 \circ \mathrm{Ad}u$ by 19.5 of [18]. Hence $\sigma_t^{\psi_1} = \mathrm{Ad}u \circ \sigma_t^{\psi_2} \circ \mathrm{Ad}u^*$ for all $t \in \mathbb{R}$.

In particular, $\mathrm{Ad}u \circ \alpha_2 \circ \mathrm{Ad}u^* = \alpha_1$. Thus $\mathrm{Ad}u$ is the automorphism of M that does the job.

2) Suppose that N_1, N_2 are of type $\mathrm{III}_{\lambda^{1/m}}$ and $\Phi\colon N_1 \to N_2$ is an isomorphism. Let φ_1 be a normal faithful periodic state on N_1 of period $S = -m\frac{2\pi}{\log\lambda}$. Set $\varphi_2 = \varphi_1 \circ \Phi^{-1}$. It follows that φ_2 is a normal faithful periodic state of period S on N_2 and we have $\sigma_t^{\varphi_2} = \Phi \circ \sigma_t^{\varphi_1} \circ \Phi^{-1}$ for all $t \in \mathbb{R}$.

From Theorem 2.13, if $\alpha_i = \sigma_{S/M}^{\varphi_i}$ for each $i = 1, 2$, then M_i is naturally isomorphic to the crossed products $N_i \rtimes_{\alpha_i} \mathbb{Z}_m$. Since α_1 and α_2 are conjugate by Φ, it follows that there is an isomorphism Ψ from $N_1 \rtimes_{\alpha_1} \mathbb{Z}_m$ onto $N_2 \rtimes_{\alpha_2} \mathbb{Z}_m$ such that $\Psi(\pi_{\alpha_1}(x)) = \pi_{\alpha_2}(\Psi(x))$, $x \in N_1$. Hence the pairs $N_1 \subset M_1$ and $N_2 \subset M_2$ are conjugate. ∎

Restricting to the hyperfinite case, we have the following classification result.

COROLLARY 2.16. *For each $\lambda \neq 0, 1$, and for each positive integer m, the hyperfinite type III_λ factor contains a unique subfactor of type III_{λ^m} or $\mathrm{III}_{\lambda^{1/m}}$ having index m.*

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