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CUBIC ALGEBRAS

CARLA FARSI and NEIL WATLING

1. INTRODUCTION

The classification in [8] reduces the study of the fixed point subalgebras of the automorphisms of the rotation algebra $\mathcal{A}_\theta$ induced by $SL(2,\mathbb{Z})$ to the computation of the ones associated to the four automorphisms: σ,τ,ζ,η. When θ is rational, the fixed point subalgebra of $\mathcal{A}_\theta$, under the action of the automorphism $\sigma: u\to u^{-1},\ v\to v^{-1}$, referred to as the "flip", and of the automorphism $\tau: u\to v^{-1},\ v\to u$ referred to as the "square root of the flip" were characterized as trivial C^*-bundles over the 2-sphere in [4] and [6]. We now consider the fixed point subalgebra of the rotation algebra defined using the automorphism $\zeta: u\to e^{-\pi i\theta}u^{-1}v,\ v\to u^{-1}$. The only other relevant example is $\mathcal{A}_\theta^\eta$ which is studied in [7].

Our main result is a characterization of $\mathcal{A}_\theta^\zeta, \theta=p/q,\ p,q$ coprime integers, as a trivial C^*-bundle with generic fibre M_q over the 2-sphere with three "singular points" $\Omega_0,\Omega_1,\Omega_2$. We will not give the precise result here (for this see Theorem 3.4.1), but illustrate it for small q. For example, when $q=1$, $\mathcal{A}_\theta^\zeta$ is the algebra of continuous functions on $\mathbb{S}^2$ and when $q=2$, $\mathcal{A}_\theta^\zeta$ is the algebra of continuous functions from $\mathbb{S}^2$ to M^2 such that the functions take values in the subalgebras $M_1\oplus M_1$ at the three points Ω_i. Interestingly we had to make extensive use of results in analytic number theory and compute explicity several examples of generalized Gaussian sums. Also, as with $\mathcal{A}_\theta^\sigma$ and $\mathcal{A}_\theta^\zeta$, note that $\mathcal{A}_\theta^\zeta$ up to isomorphism, is independent of p. A simple corollary of the main result is the calculation of the K-theory of $\mathcal{A}_\theta^\zeta$.

COROLLARY. *Let $\theta=p/q$, where p,q are comprime positive integers. Then the K-theory of $\mathcal{A}_\theta^\zeta$ is given by,*

$*$	$q=1$	$q=2$	$q=3$	$q>3$
$K_0(A_\theta^\zeta)$	$\mathbb{Z}^2$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^8$
$K_1(A_\theta^\zeta)$	0	0	0	0

We can also consider the related algebra $\mathcal{A}_\theta \rtimes_\zeta \mathbf{Z}_3$, and with computations analogous to those in [6] we have:

THEOREM. *Let $\theta = p/q$, where p, q are coprime positive integers and let Ω_i, $i = 0, 1, 2$ be any three distinct points of the 2-sphere $\mathbb{S}^2$. Then the crossed product algebra $\mathcal{A}_\theta \rtimes_\zeta \mathbf{Z}_3$ is isomorphic to the following subalgebra of the C^*-algebra $C(\mathbb{S}^2, M_{3q})$:*

$$\mathcal{A}_\theta \rtimes_\zeta \mathbf{Z}_3 = \{f \in C(\mathbb{S}^2, M_{3q}) | f(\Omega_i) \text{ commutes with } P_i^j\},$$

where P_i^j, $i = 0, 1, 2$, $j = 0, 1$, are self-adjoint orthogonal progections in M_{3q}. The dimension of P_i^j is q for all i, j.

The format of the paper is as follows. In Section 2 we give the basic notation and definitions we will use. In Section 3 we introduce the automorphism ζ and state the main result on the description of $\mathcal{A}_\theta^\zeta$. In Section 4 we give the scheme of the proof with Section 5 detailing the results on Gaussian sums we need. In Section 6 we compute the dimensions of the projections associated to the points $\Omega_0, \Omega_1, \Omega_2$ while in Section 7 we end the proof of the main theorem.

We would like to take this opportunity to thank Professor George Elliott for helpful discussions and also the referees for useful comments on earlier versions of this paper.

2. THE ROTATION ALGEBRA

2.1. Introduction

This section will give a characterization of the rotation algebra $\mathcal{A}_\theta$, θ rational (see e.g. [4], [10] or [6]) which we will use, together with some additional notation.

2.2. Notation

Assume that $\theta = p/q$, where p, q are comprime positive integers with $1 \leqslant p \leqslant q - 1$. Let $\rho = e^{2\pi i\theta}$, $\omega = e^{\frac{2\pi i}{q}}$ and define the following $q \times q$ matrices:

$$U_0 = (\delta_i^j \rho^j)_{i,j=0,\cdots,q-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \rho & 0 & \cdots & 0 \\ 0 & 0 & \rho^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \rho^{q-1} \end{bmatrix},$$

$$V_0 = (\delta_i^{j-1(\mathrm{mod}q)})_{i,j=0,\cdots,q-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\Gamma_0 = (\delta^j_{q-i(\mathrm{mod}q)})_{i,j=0,\cdots,q-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

$$Z_0 = \frac{1}{\sqrt{q}}(\rho^{-|j-(1+(-1)^{q-1})/2|j/2-ij})_{i,j=0,\cdots,q-1}.$$

Now U_0, V_0, Z_0 are unitary and Γ_0 is a self-adjoint unitary. Also U_0 and V_0 generate M_q, the algebra of $q \times q$ matrices. Hence we can define four automorphisms of M_q, ζ_0, γ_0 and α_i, $i = 1, 2$, by:

$$\begin{aligned} &\zeta_0(U_0) = e^{\pi i[-1+(-1)^{q-1}]p/(2q)} U_0^{-1} V_0, && \zeta_0(V_0) = U_0^{-1}, \\ &\gamma_0(U_0) = U_0^{-1}, && \gamma_0(V_0) = V_0^{-1}, \\ &\alpha_1(U_0) = U_0, && \alpha_1(V_0) = \omega V_0, \\ &\alpha_2(U_0) = \omega U_0, && \alpha_2(V_0) = V_0. \end{aligned}$$

Notice that $\alpha_1\alpha_2 = \alpha_2\alpha_1$, $\zeta_0^3 = \iota$, where ι is the identity, $\zeta_0\alpha_1^{-1} = \alpha_1\alpha_2\zeta_0$, $\zeta_0\alpha_2^{-1} = \alpha_1^{-1}\zeta_0$ and $\alpha_2\zeta_0 = \zeta_0\alpha_1^{-1}\alpha_2^{-1}$.

Then, if we use the convention that, for a unitary matrix U, $\mathrm{Ad}\, U$ denotes the automorphism of M_q given by $(\mathrm{Ad}\, U)(A) = U^*AU$, $A \in M_q$, we have,

$$\zeta_0 = \mathrm{Ad}\, Z_0, \quad \gamma_0 = \mathrm{Ad}\, \Gamma_0,$$

and,

$$\alpha_i = \mathrm{Ad}\, W_i, \quad i = 1, 2,$$

where,

$$W_1 = U_0^{-p'} = (\delta_i^j \omega^j)_{i,j=0,\ldots,q-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \omega^{q-1} \end{bmatrix},$$

with $pp' \equiv -1 (\bmod q)(0 < p' < q)$,

$$W_2 = V_0^{-p''} = \begin{bmatrix} 0 & I_{p''} \\ I_{q-p''} & 0 \end{bmatrix},$$

with $pp' \equiv -1 (\bmod q)(0 < p'' < q)$, and $I_t \in M_t$ is the $t \times t$ identity matrix.

DEFINITION 2.2.1. The rotation algebra $\mathcal{A}_\theta$ is the universal C^*-algebra generated by two unitaries u and v satisfying $vu = \rho uv$ where $\rho = e^{2\pi i\theta}$ and $0 \leqslant \theta < 1$.

THEOREM 2.2.2. *The rotation algebra $\mathcal{A}_\theta$, $\theta = p/q$, where p, q are comprime positive integers, can be described as,*

$$\mathcal{A}_\theta = \left\{ f \in C(\mathbf{R}^2, M_q) \middle| \begin{array}{l} f(x, y+n) = \alpha_1^n(f(x,y)), \ x, y \in \mathbf{R}, \\ f(x+m, y) = \alpha_2^m(f(x,y)), \ n, m \in \mathbf{Z}. \end{array} \right\}$$

with pointwise multiplication and involution.

The generators u and v correspond to the functions:

$$U(x, y) = \omega^x U_0,$$

$$V(x, y) = \omega^y V_0, \quad \text{for } (x, y) \in \mathbf{R} \times \mathbf{R},$$

where $\omega^t = e^{\frac{2\pi i t}{q}}$.

3. FIXED POINT SUBALGEBRA

3.1. Introduction

In this section we will state the main theorem concerning the fixed point algebra of the cubic automorphism.

3.2. The cubic automorphism

DEFINITION 3.2.1. The cubic automorphism ζ is the automorphism of $\mathcal{A}_\theta$ defined by $\zeta(u) = e^{-\pi i\theta} u^{-1} v$, $\zeta(v) = u^{-1}$.

LEMMA 3.2.2. *In the description of $\mathcal{A}_\theta$ given in Theorem 2.2.2 ζ corresponds to the automorphism,*

$$(\zeta f)(x, y) = \zeta_0(f(\delta - x + y, -x)),$$

where $\delta = [-1 + (-1)^q]p/4$.

Proof. Here we prove the case q odd, q even follows similarly.

$$e^{-\pi i p/q} U(x,y)^{-1} V(x,y) = e^{-\pi i p/q} \omega^{y-x} U_0^{-1} V_0 = \zeta_0(U(-p/2 - x + y, -x)),$$

$$(U(x,y))^{-1} = \omega^{-x}U_0^{-1} = \zeta_0(V(p/2-x+y,-x)). \qquad \blacksquare$$

REMARK 3.2.3. The fixed points of the transformation Ψ on $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ induced by the transformation ψ of $\mathbf{R}^2$:

$$\psi : (x,y) \longmapsto (\delta - x + y, -x) \quad \text{for all } (x,y) \in \mathbf{R}^2,$$

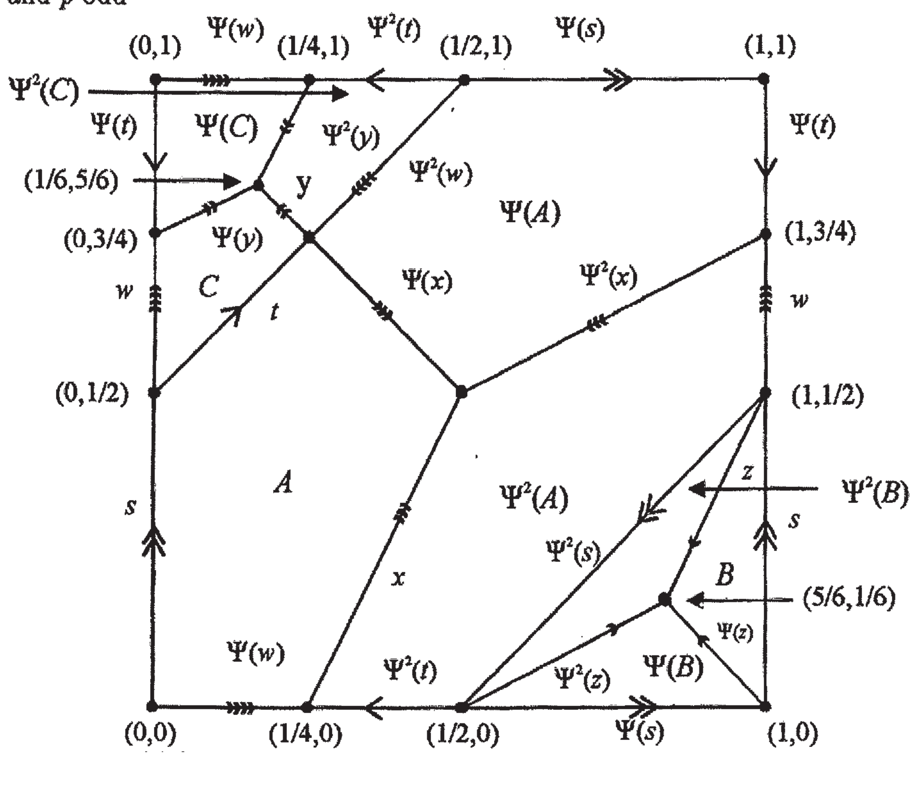
are the images in $\mathbf{T}^2$ of the following points:

1. q and p odd : $\Omega_0 = (1/6, 5/6)$, $\Omega_1 = (1/2, 1/2)$, $\Omega_2 = (-1/6, 1/6)$.
2. q or q even : $\Omega = (0,0)$, $\Omega_1 = (1/3, 2/3)$, $\Omega_2 = (2/3, 1/3)$.

3.3. Fundamental domains

Figure 1 below shows how the orientable, area preserving transformation Ψ acts on $\mathbf{T}^2$ (which is identified with $[0,1] \times [0,1]$ modulo boundary identifications) with capital letters denoting plane regions, while small letters denote line segments. It is straightforward to check how Ψ acts on the various line segments, and since Ψ is continuous and preserves the orientation, the plane regions inside get transformed accordingly.

Figure 1: The action of Ψ on the 2-torus,
q and p odd



q or p even

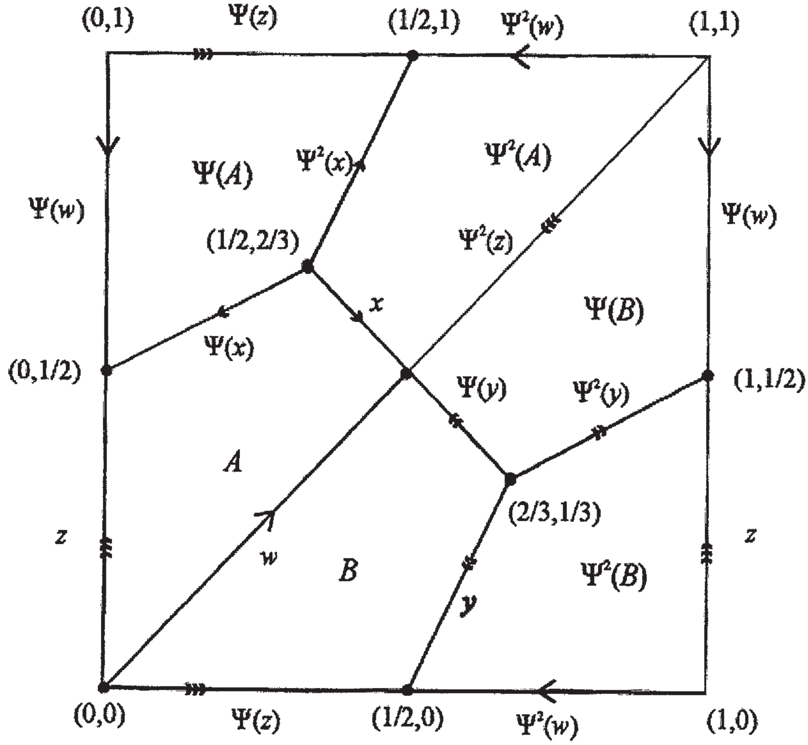
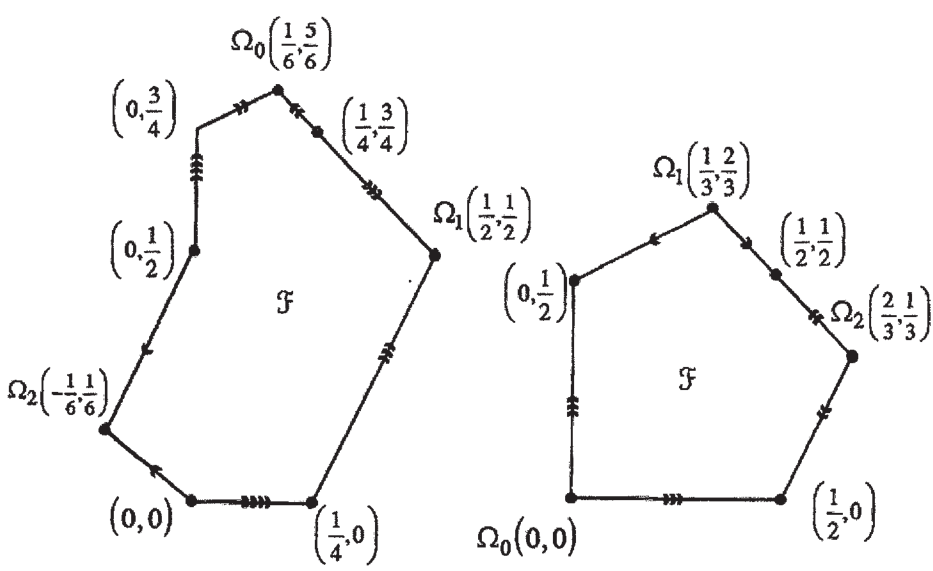


Figure 2: The fundamental domain $\mathcal{F}$,

q and p odd

q or p even



By using Figure 1 it is clear that a fundamental domain, say $\mathcal{F}$, of the transformation Ψ is given in Figure 2 above (where we now identify $\mathbb{T}^2$ with $[-1/2, 1/2] \times \times [-1/2, 1/2]$ modulo boundary identifications, when p and q are odd). The quotient orbifold in both cases is the 2-sphere $\mathbb{S}^2$ with three singular points $\Omega_0, \Omega_1, \Omega_2$.

3.4. The main theorem

We will now state our main theorem, the proof of which will be presented in Sections 4, 5, 6 and 7.

THEOREM 3.4.1. *Let $\theta = p/q$, with p, q coprime positive integers and let Ω_i, $i = 0, 1, 2$ be any three distinct points of the 2-sphere $\mathbb{S}^2$. Then the fixed point subalgebra of the cubic automorphism ζ, $\mathcal{A}_\theta^\zeta$, is isomorphic to the following subalgebra of the C^*-algebra $C(\mathbb{S}^2, M_q)$,*

$$\mathcal{A}_\theta^\zeta = \{f \in C(\mathbb{S}^2, M_q) \mid f(\Omega_i) \text{ commutes with } P_i^j,\ i = 0, 1, 2,\ j = 0, 1\},$$

where P_i^j, $i = 0, 1, 2$, $j = 0, 1$, are self adjoint orthogonal projections in M_q. The dimensions of P_i^j are given by the following table, where q is given modulo 12 and $(|)$ denotes the Jacobi symbol.

*	$q = 0, 6$	$q = 1, 5, 7, 11$	$q = 2, 4, 8, 10$	$q = 3, 9$
P_0^0	$\frac{q-3}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q}{3}$
P_0^1	$\frac{q}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q+2(q\|3)}{3}$	$\frac{q}{3}$
P_1^0	$\frac{q}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q+2(q\|3)}{3}$	$\frac{q+3}{3}$
P_1^1	$\frac{q}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q-3}{3}$
P_2^0	$\frac{q}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q}{3}$
P_2^1	$\frac{q}{3}$	$\frac{q-(q\|3)}{3}$	$\frac{q+2(q\|3)}{3}$	$\frac{q}{3}$

COROLLARY 3.4.2. *Let $\theta = p/q$, where p, q are coprime positive integers. Then the K-theory of $\mathcal{A}_\theta^\zeta$ is given by,*

*	$q = 1$	$q = 2$	$q = 3$	$q > 3$
$K_0(\mathcal{A}_\theta^\zeta)$	$\mathbb{Z}^2$	$\mathbb{Z}^5$	$\mathbb{Z}^7$	$\mathbb{Z}^8$
$K_1(\mathcal{A}_\theta^\zeta)$	0	0	0	0

Proof. See [5] and [6]. ∎

4. SCHEME OF THE PROOF OF THEOREM 3.4.1.

4.1. Introduction

This section will begin the proof of Theorem 3.4.1. After some preliminary results we will outline the scheme of proof leaving the necessary calculations to Sections 5, 6 and 7.

4.2. A characterization of the fixed point algebra

REMARK 4.2.1. As a consequence of Lemma 3.2.2 we obtain that on the boundary of $\mathcal{F}$, for a fixed point of ζ, we have,

1. q and p odd:

$$\begin{aligned} f(x,-x+1) &= \zeta_0 f(-p/2-2x+1,-x) = \zeta_0\alpha_1^{-1}\alpha_2^{-p/2+1/2}f(-2x+1/2,-x+1),\\ f(x,2x-1/2) &= \zeta_0 f(-p/2+x-1/2,-x) = \zeta_0\alpha_1^{-1}\alpha_2^{-p/2-1/2}f(x,-x+1),\\ f(0,y) &= \zeta_0 f(-p/2+y,0) = \zeta_0\alpha_2^{-p/2+1/2}f(y-1/2,0),\\ f(x,2x+1/2) &= \zeta_0 f(-p/2+x+1/2,-x) = \zeta_0\alpha_2^{-p/2+1/2}f(x,-x). \end{aligned}$$

2. q or p *even*:

$$\begin{aligned} f(0,y) &= \zeta_0 f(\delta+y,0) = \zeta_0\alpha_2^{\delta}f(y,0),\\ f(x,-x+1) &= \zeta_0 f(\delta-2x+1,-x) = \zeta_0\alpha_2^{\delta}\alpha_1^{-1}f(-2x+1,-x+1),\\ f(x,2x-1) &= \zeta_0 f(\delta+x-1,-x) = \zeta_0\alpha_2^{\delta-1}\alpha_1^{-1}f(x,-x+1). \end{aligned}$$

We claim that $\mathcal{A}_\theta^\zeta$ is isomorphic to the algebra,

$$\tilde{\mathcal{A}}_\theta^\zeta = \{f \in C(\mathcal{F}, M_q) \mid \text{the relations described in Remark 4.2.1 are satisfied}\},$$

where the identification is by restricting $f \in \mathcal{A}_\theta^\zeta$ from $\mathbf{R}^2$ to $\mathcal{F}$. In fact, given $\tilde{f} \in \tilde{\mathcal{A}}_\theta^\zeta$, define the function $f \in \mathcal{A}_\theta$ by,

$$f(x,y) = \tilde{f}(x,y), f(\psi(x,y)) = \zeta_0^{-1}\tilde{f}(x,y), f(\psi^2(x,y)) = \zeta_0^{-2}\tilde{f}(x,y)\forall(x,y) \in \mathcal{F},$$

$$f(x',y') = \alpha_1^n\alpha_2^m f(x,y),$$

where $x'-x = m \in \mathbf{Z}$, $y'-y = n \in \mathbf{Z}$, $(x,y) \in \mathcal{F}\cup\psi(\mathcal{F})\psi^2(\mathcal{F})$. It is strainghtforward to check that on the boundaries of the three regions $\mathcal{F}$, $\psi(\mathcal{F})$, $\psi^2(\mathcal{F})$ the relations in

Remark 4.2.1 actually guarantee that f is well defined and also it is clear from Figure 1 that the interiors of the three regions do not overlap and that $\mathcal{F} \cup \psi(\mathcal{F}) \cup \psi^2(\mathcal{F})$ modulo boundary identifications is homeomorphic to $\mathbf{T}^2$ so that $f \in \mathcal{A}_\theta$ (and $f \in \mathcal{A}_\theta^\zeta$ as it can be seen by using Lemma 3.2.2 and the relations in Section 2.2) . Conversely the restriction to $\mathcal{F}$ of any $f \in \mathcal{A}_\theta^\zeta$ satisfies the relations in Remark 4.2.1 and since f is a fixed point of ζ, its values on $\mathbb{R}^2$ are determined by its values on $\mathcal{F}$ (see Lemma 3.2.2).

REMARK 4.2.2. By using the definition of ζ we see that at the exceptional point Ω_i the values of $f \in \mathcal{A}_\theta^\zeta$ are restricted to a subalgebra A_i of M_q described as follows:

(i) q and p *odd*:

1. At $\Omega_0 = (1/6, 5/6) : \{A \in M_q | \zeta_0 \alpha_2^{-p/2+1/2} \alpha_1^{-1}(A) = A\}$,
2. At $\Omega_1 = (1/2, 1/2) : \{A \in M_q | \zeta_0 \alpha_2^{-p/2-1/2} \alpha_1^{-1}(A) = A\}$,
3. At $\Omega_2 = (-1/6, 1/6) : \{A \in M_q | \zeta_0 \alpha_2^{-p/2+1/2} \alpha_1^{-1}(A) = A\}$.
4. Note that at the points (0,3/4), (1/4,0) and (1/4,3/4) the restricting automorphisms would be $\zeta_0 \alpha_2^{-(p-1)/2} \zeta_0 \alpha_1^{-1} \alpha_2^{-(p+1)/2} \zeta_0 \alpha_1^{-1} \alpha_2^{-(p-1)/2}$, $\zeta_0 \alpha_1^{-1} \alpha_2^{-(p+1)/2} \cdot \zeta_0 \alpha_1^{-1} \alpha_2^{-(p-1)/2} \zeta_0 \alpha_2^{-(p-1)/2}$ and $\zeta_0 \alpha_1^{-1} \alpha_2^{-(p-1)/2} \zeta_0 \alpha_2^{-(p-1)/2} \zeta_0 \alpha_1^{-1} \alpha_2^{-(p+1)/2}$ respectively.

By using the relations between α_1, α_2 and ζ_0 given in Section 2.2 it is straightforward to show that these are in fact the identity, so there is no restriction at these points and therefore the corresponding algebras are M_q.

(ii) q or p *even*:

1. At $\Omega_0 = (0, 0) : \{A \in M_q | \zeta_0 \alpha_2^\delta(A) = A\}$,
2. At $\Omega_1 = (1/3, 2/3) : \{A \in M_q | \zeta_0 \alpha_2^\delta \alpha_1^{-1}(A) = A\}$,
3. At $\Omega_2 = (2/3, 1/3) : \{A \in M_q | \zeta_0 \alpha_2^{\delta-1} \alpha_1^{-1}(A) = A\}$.
4. As in the previous case there is no restriction at the points (1/2,1/2),(0,1/2) and (1/2,0) since the restricting automorphisms $\zeta_0 \alpha_2^\delta \alpha_1^{-1} \zeta_0 \alpha_2^\delta \zeta_0 \alpha_1^{-1} \alpha_2^{\delta-1}$, $\zeta_0 \alpha_2^\delta \zeta_0 \alpha_2^{\delta-1} \cdot \alpha_1^{-1} \zeta_0 \alpha_1^{-1} \alpha_2^\delta$ and $\zeta_0 \alpha_2^{\delta-1} \alpha_1^{-1} \zeta_0 \alpha_2^\delta \alpha_1^{-1} \zeta_0 \alpha_2^\delta$ are in fact the identity.

The algebra A_i is determined by the dimensions of the eigenspaces of any matrix implementing the isomorphism, say τ_i, of M_q associated to A_i. Such dimensions will be computed in Section 6 and this will show that the dimensions associated to the three points $\Omega_0, \Omega_1, \Omega_2$ in Theorem 3.4.1 are as stated. We will call T_i the matrix implementing τ_i obtained by composing the matrices associated to the isomorphisms ζ_0, γ_0 and α_i, $i = 1, 2$, in Section 2.2 (cf. Section 5.2). In order to end the proof of Theorem 3.4.1 we still need to show that $\tilde{\mathcal{A}}_\theta^\zeta$ is isomorphic to the algebra $\mathcal{B}$,

$$\mathcal{B} = \{f \in C(\mathbb{S}^2, M_q) | f(\Omega_i) = \tau_i(f(\Omega_i))\}.$$

To do so, we will construct a non necesarilly single-valued map $\tilde{\eta} : \mathbb{S}^2 \to \mathrm{Aut}\,(M_q)$ which is single-valued and continuous in $\mathbb{S}^2 - Y$, where Y is the image in $\mathbb{S}^2$ of the boundary of $\mathcal{F}$, and has the limits show in Figures 3 and 4 below on the edges of the tree Y. This describes how $\tilde{A}_\theta^\zeta$ wraps up to give $\mathcal{B}$. Note that we use τ_i to go (clockwise) around Ω_i. We have also used,

$$\alpha_2^{(p-1)/2}\zeta_0^{-1} = \tau_1\left(\zeta_0\alpha_2^{-(p-1)/2}\alpha_1^{-1}\right) = \left(\zeta_0\alpha_2^{-(p+1)/2}\alpha_1^{-1}\right)\left(\zeta_0\alpha_2^{-(p-1)/2}\alpha_1^{-1}\right),$$

$$\text{and } \alpha_1\alpha_2^{-\delta+1}\zeta_0^{-1} = \tau_1(\zeta_0\alpha_2^\delta) = (\zeta\alpha_2^\delta\alpha_1^{-1})(\zeta_0\alpha_2^\delta).$$

The map $\tilde{\eta}$ is given in terms of the adjoint of a unitary field in $\mathbb{S}^2$, which we will call η,

$$\eta : s \mapsto U(s) \in U_q,\ s \in \mathbb{S}^2.$$

Now we will make the construction of η precise.

REMARK 4.2.3. Let τ_i be the automorphism of M_q associated to the point Ω_i, i.e.,

$$A_i\{A \in M_q | \tau_i(A) = A\}.$$

Figure 3: The fundamental domain $\mathcal{F}$,

q and p odd $\qquad$ q or p even

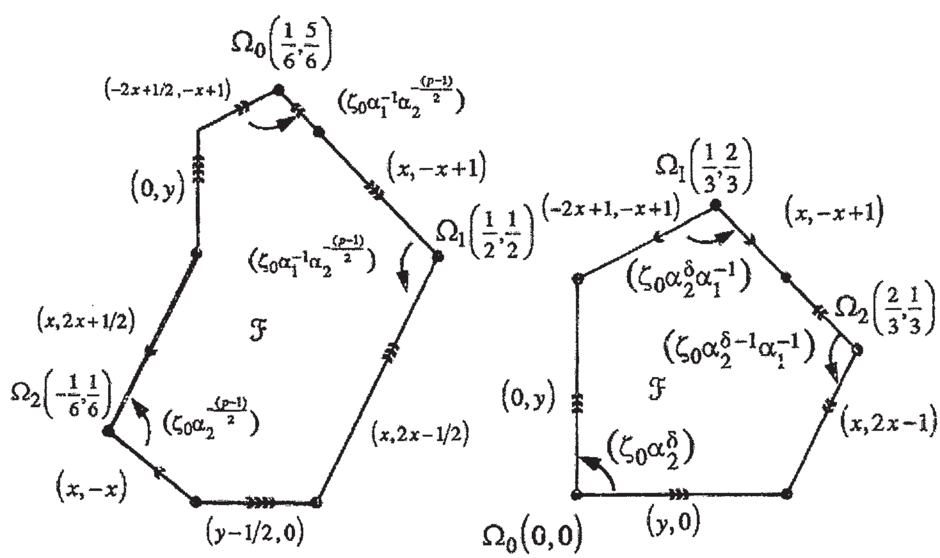
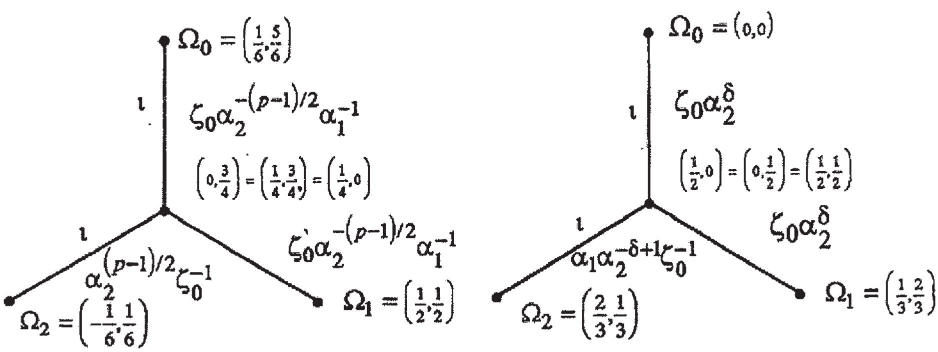


Figure 4: The Tree Y,

q and p odd q or p even



By using the definitions it is straightforward to check that $\tau_i^3 = 1$. The phase we are going to introduce makes the unitary T_i implementing τ_i into a matrix whose third power is the identity. We will call it $\tilde{T}_i$. In particular we put $T_i^3 = \mu_{2i+1}^3 I_q$, $\tilde{T}_i = \mu_{2i+1}^{-1} T_i$, $i = 0, 1$, $T_2^3 = \mu_{-1}^3 I_q$, $\tilde{T}_2 = \mu_{-1}^{-1} T_2$, for q odd and $T_i^3 = \mu_{2i}^3 I_q$, $\tilde{T}_i = \mu_{2i}^{-1} T_{2i}$, $i = 0, 1, 2$, for q even.

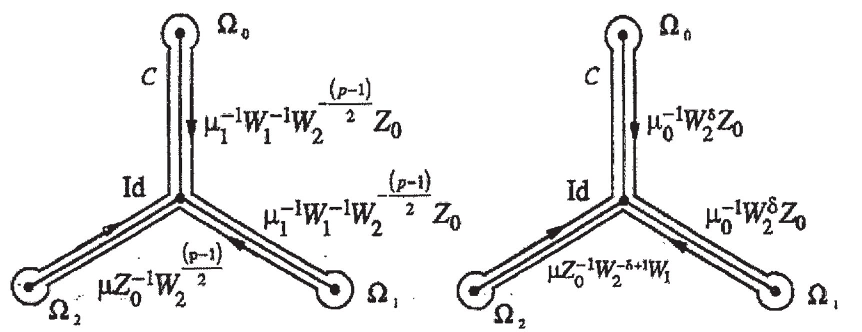
Let C be a path around Y as in Figure 5 below. On the straight parts of C, the value of $\eta : s \mapsto U(s)$ is given by the matrix implementing the automorphism $\tilde{\eta}$ as shown, with μ some phase factor. Note that we will use $\tilde{T}_i$ to go (clockwise) around Ω_i. Accordingly:

$$\mu_1^{-1}\mu_3^{-1} W_1^{-1} W_2^{-\frac{(p-1)}{2}} Z_0 (W_1^{-1} W_2^{-\frac{(p+1)}{2}} Z_0) = \mu Z_0^{-1} W_2^{\frac{(p-1)}{2}},$$

$$\mu_0^{-1}\mu_2^{-1} W_2^{\delta} Z_0 (W_1^{-1} W_2^{\delta} Z_0) = \mu Z_0^{-1} W_2^{-\delta+1} W_1.$$

Figure 5: The Tree Y,

q and p odd $\qquad$ q or p even



Let P_0 and P_1 be the spectral projections of $\tilde{T}_0$ onto the $e^{2\pi i/3}$ and the $e^{4\pi i/3}$ eigenspaces (resp.), Q_0 and Q_1 the spectral projections of $\tilde{T}_1$ onto the $e^{2\pi i/3}$ and the $e^{4\pi i/3}$ eigenspaces (resp.) and R_0 and R_1 the spectral projections of $\tilde{T}_2$ onto the $e^{2\pi i/3}$ and the $e^{4\pi i/3}$ eigenspaces (resp.).

To go (clockwise) around the circular parts of C we use the following paths:

1. At $\Omega_0 : C(t) = 1-(P_0+P_1)+e^{2\pi i(\frac{t}{3}+k_0t)}P_0+e^{2\pi i(\frac{2t}{3}+k_1t)}P_1,\ t\in[0,1]$,
2. At $\Omega_1 : C(t) = 1-(Q_0+Q_1)+e^{2\pi i(\frac{t}{3}+n_0t)}Q_0+e^{2\pi i(\frac{2t}{3}+n_1t)}Q_1,\ t\in[0,1]$,
3. At $\Omega_2 : C(t) = 1-(R_0+R_1)+e^{2\pi i(\frac{(1-t)}{3}+l_0t)}R_1+e^{2\pi i(\frac{2(1-t)}{3}+l_1t)}R_0,\ t\in[0,1]$.

This defines η (and hence $\tilde{\eta}$) on C. If we deform continously the path C to make it coincide with Y in the limit, we have also defined η in the interior of C. In order to finish the proof of Theorem 3.4.1 we need to show that the map η extends to $\mathbf{S}^2$ and for that it is enought to prove that the winding number Σ_Y of the path $\eta : s \mapsto U(s)$ around C is an integer multiple of q (cf. [4]). This shows that $[C]=0$ in $\pi_1(U_q)$. We have,

$$\Sigma_Y = \left[\left(\frac{1}{3}+k_0\right)\dim P_0+\left(\frac{2}{3}+k_1\right)\dim P_1\right]+\left[\left(\frac{1}{3}+n_0\right)\dim Q_0+\left(\frac{2}{3}+n_1\right)\dim Q_1\right]-$$

$$-\left[\left(\frac{1}{3}+l_0\right)\dim R_1+\left(\frac{2}{3}+l_1\right)\dim R_0\right]+\mathrm{UWPP}.$$

The last term UWPP in Σ_Y comes from unwinding the final phase factor $\lambda = e^{2\pi i w_0}$, $w_0 \in \mathbb{Q}$, (λ is the product of all the phase factors we introduced) by using the path,

$$C(t) = e^{2\pi i w_0(1-t)}e^{2\pi i t m_0}, \quad m_0 \in \mathbf{Z},\ t \in [0,1].$$

so that,

$$\mathrm{UWPP} = (-w_0 + m_0)q,$$

and it will be discussed in detail in Section 7. The isomorphism Φ from $\mathcal{B}$ to $\tilde{\mathcal{A}}_\theta^\zeta$ is then given by,

$$(\Phi f)(s) = \tilde{\eta}(s)f(s), \quad f \in \mathcal{B},\ s \in \mathbb{S}^2,$$

where $\tilde{\eta}(s) = \mathrm{Ad}\,\eta(s)$.

So, to finish the proof of Theorem 3.4.1 we need to compute the dimensions of the spectral projections of $\tilde{T}_i$, $i = 0,1,2$ and to show that Σ_Y is an integer multiple of q for some choice of the integer parameters k_i, n_i, l_i, $i = 0,1$ and m_0. (We choose $m_0 = 0$.)

5. GAUSSIAN SUMS

5.1. Introduction

Here we will compute the Gaussian sums which will be needed in the Section 6, but first an overiew of the general situation.

5.2. General remarks

We now describe what is to be computed. We distinguish two cases according to the parity of q and p.

(i) q and p *odd*:

$$T_i = W_1^{-1}W_2^{-(p-1+2i)/2}Z_0,\ \tilde{T}_i = \mu_{2i+1}^{-1}T_i, \quad i = 0,1,$$

$$T_2 = W_2^{-(p-1)/2}Z_0,\ \tilde{T}_2 = \mu_{-1}^{-1}T_2,$$

where μ_{2i+1}^{-1} and μ_{-1}^{-1} are determined by $T_i^3 = \mu_{2i+1}^3 I_q$, and $T_2^3 = \mu_{-1}^3 I_q$ $i = 0,1$.

(ii) p or q *even*:

$$T_0 = W_2^\delta Z_0,\ \tilde{T}_0 = \mu_0^{-1}T_0,$$

$$T_1 = W_1^{-1}W_2^\delta Z_0,\ \tilde{T}_1 = \mu_2^{-1}T_1,$$

$$T_2 = W_1^{-1}W_2^{-(-\delta+1)}Z_0,\ \tilde{T}_2 = \mu_4^{-1}T_2,$$

where μ_{2i}^{-1} is determined by $T_3 = \mu_{2i}^3 I_q$, $i = 0,1,2$.

By Remark 4.2.2 $\tilde{T}_i^3 = I_q$, $i = 0,1,2$. Therefore $\tilde{T}_i$ has only the eigenvalues $1, e^{2\pi i/3}, e^{4\pi i/3}$ and to determine the dimensions of its eingenspaces it is enough to compute the Trace of $\tilde{T}_i$. To do so we actually determine the Trace of T_i and the phase factors $\mu_{a(i)}$. These are given by Gaussian sums, as will be explained in Sections 5.3 and 5.4.

5.3. Traces

All the matrices T_i are of type,

$$W_1^{-r} W_2^{-s} Z_0, \quad r, s \in \mathbf{N}.$$

If we put,

$$\mathrm{Tr}_a(p,q) = \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \rho^{-3m^2/2} \omega^{-am/2} = \frac{1}{\sqrt{q}} \sum_{m=0}^{q-1} \mathrm{e}^{-3\pi i p m^2/q - \pi i a m/q},$$

we have,

(i) q and p *odd*: $\mathrm{Trace}(T_i) = \mathrm{Tr}_{2i+1}(p,q)$, $i = 0, 1$, $\mathrm{Trace}(T_2) = \mathrm{Tr}_{-1}(p,q)$.
(ii) q or p *even*: $\mathrm{Trace}(T_i) = \mathrm{Tr}_{2i}(p,q)$, $i = 0, 1, 2$.

5.4. Phase factors

We have,

$$W_1^{-r} W_2^{-s} Z_0 = \frac{1}{\sqrt{q}} (\omega^{-i-j(a-2)/2} \rho^{-j^2/2-ji})_{i,j=0,\cdots,q-1},$$

where $a = 2(r+s) - [1-(-1)^q]p/2$, $r, s \in \mathbf{N}$. Therefore,

$$(W_1^{-r} W_2^{-s} Z_0)^2_{0,j} = 1/q \left(\sum_{m=0}^{q-1} \omega^{-m-(a-2)(j+m)/2} \rho^{-m^2/2-j^2/2-mj} \right).$$

So,

$$\mu_a^3 = (W_1^{-r} W_2^{-s} Z_0)^3_{0,0} = \frac{1}{q\sqrt{q}} \sum_{j=0}^{q-1} \sum_{m=0}^{q-1} \omega^{-(1+(a-2)/2)(j+m)} \rho^{-(m+j)^2/2},$$

putting $j + m = d$,

$$= \frac{1}{q\sqrt{q}} \sum_{j=0}^{q-1} \sum_{d=j}^{q-1+j} \omega^{-d(1+(a-2)/2)} \rho^{-d^2/2},$$

exchanging the order of the two sums,

$$= \frac{1}{q\sqrt{q}} \sum_{d=0}^{q-1} \sum_{j=0}^{d} \omega^{-d(1+(a-2)/2)} \rho^{-d^2/2} + \frac{1}{q\sqrt{q}} \sum_{d=q}^{2q-1} \sum_{j=d-(q-1)}^{q-1} \omega^{-d(1+(a-2)/2)} \rho^{-d^2/2} =$$

$$= \frac{1}{q\sqrt{q}}\sum_{d=0}^{q-1}(d+1)\omega^{-d(1+(a-2)/2)}\rho^{-d^2/2}+\frac{1}{q\sqrt{q}}\sum_{d=q}^{2q-1}[2(q-1)-d+1]\omega^{-d(1+(a-2)/2)}\rho^{-d^2/2}.$$

If we put $d' = d - q$ in the second sum we obtain,

$$\mu_a^3 = \frac{1}{\sqrt{q}}\sum_{d=0}^{q-1}\omega^{-ad/2}\rho^{-d^2/2}.$$

5.5. Computation of Gaussian sums

As seen in Sections 5.3 and 5.4, we need to compute,

$$\mathrm{Tr}_a(p,q) = \frac{1}{\sqrt{q}}\sum_{m=0}^{q-1} e^{-3\pi i p m^2/q - \pi i a m/q},$$

$$\mu_a^3 = \frac{1}{\sqrt{q}}\sum_{d=0}^{q-1} e^{-\pi i p d^2/q} e^{-\pi i a d/q},\quad a = -1,0,1,2,3,4.$$

This will be done in Section 6, here we state the results which will make such computations possible.

DEFINITION 5.5.1. Let p, q be two non-negative integers and $a \in \mathbb{Z}$. Then we define the Gaussian sum $G(p,q,a)$ by,

$$G(p,q,a) = \sum_{d=0}^{q-1} e^{2\pi i p d^2/q} e^{2\pi i a d/q}.$$

REMARK 5.5.2. If $a = 0$, $G(p,q,0)$ is the usual Gaussian sum.

LEMMA 5.5.3. *If* $(k_1,k_2) = 1$, *then,*

$$G(p,k_1k_2,a) = G(pk_1,k_2,a)G(pk_2,k_1,a).$$

Proof. This is a generalization of the multiplicative law for Gaussian sums. The proof given in [9] works verbatim in this case too. ■

Thus if $q = 2^c r$, with r odd and $(p,q) = 1$, we have,

$$G(p,q,a) = G(p,2^c r,a) = G(p2^c,r,a)G(pr,2^c,a).$$

Therefore we only neeed to compute the two types of sums:

(I) $G(p,2^c,a)$ with p *odd*,

(II) $G(p,q,a)$ with q *odd* and $(p,q) = 1$.

LEMMA 5.5.4. *Let $c \geqslant 2$. Then,*

$$|G(p, 2^c, a)|^2 = \begin{cases} 0 & \text{if } a \text{ is odd} \\ 2^{c+1} & \text{if } a \text{ is even} \end{cases}.$$

Proof. Straightforward computation. ■

We will also make use of the following reciprocity law for generalized Gaussian sums given, for example,in [2].

THEOREM 5.5.5. *Let a, b, c be integers with $ac \neq 0$ and $ac + b$ even. Then,*

$$\sum_{m=0}^{c-1} e^{\pi i(am^2+bm)/c} = |c/a|^{1/2} e^{\pi i(|ac|-b^2)/(4ac)} \sum_{m=0}^{|a|-1} e^{-\pi i(cm^2+bm)/a}.$$

COROLLARY 5.5.6. *If p is odd, $c \geqslant 2$ and a is even, we have,*

$$G(p, 2^c, a) = \frac{2^{(c+1)/2}}{\sqrt{p}} e^{\pi i(1/4 - a^2/(p2^{c+1}))} \overline{G(2^{c-2}, p, a/2)}.$$

Proof. Straightforward computation. ■

REMARK 5.5.7. If p is odd,

$$G(p, 2, a) = 1 + (-1)^{p+a}.$$

Corollary 5.5.6 and Remark 5.5.7 give the value of the sum of type (I). We now describe the sum of type (II):

LEMMA 5.5.8. *If p and q are two positive relatively prime integers with q odd then,*

$$G(p, q, a) = \sqrt{q}(p|q) \begin{cases} e^{\pi iax(-q+1)/(2q)} & \text{if } q \equiv 1 (\text{mod}\, 4) \\ ie^{\pi iax(q+1)/(2q)} & \text{if } q \equiv 3 (\text{mod}\, 4) \end{cases},$$

where $px \equiv -a (\text{mod}\, q)$ and $(|)$ is the Jacobi symbol.

Proof. This can be proved by an adaptation of Schur's evaluation of Gaussian sums (an account of which is in [11]). See also [6] Section 5 for similar results. ■

6. DIMENSIONS

6.1. Introduction

In this section we compute the dimensions of the projections in Theorem 3.4.1, which are the dimensions of the $e^{\frac{2\pi i}{3}}$ and $e^{\frac{4\pi i}{3}}$ eigenspaces of $\tilde{T}_i$. We will devide the computation into three cases: q and p odd, q odd and p even, q even. Each of these cases is in turn divided into subcases: $q \equiv 0 (\bmod 3)$ and $(q,3)=1$. But first some introductory remarks.

LEMMA 6.1.1. *In the cases we consider we have that,*

$$\mathrm{Tr}_a(p,q) = \frac{1}{2\sqrt{q}}\overline{G(3p,2q,a)},$$

$$\mu_a^3 = \frac{1}{2\sqrt{q}}\overline{G(p,2q,a)}, \quad a=-1,0,1,2,3,4.$$

Proof. For the first sum we have,

$$G(3p,2q,a) = \sum_{m=0}^{2q-1} \mathrm{e}^{6\pi i p m^2/(2q)+2\pi i a m/(2q)} =$$

$$= \sum_{m=0}^{q-1} \mathrm{e}^{3\pi i p m^2/q+2\pi i a m/q} + \sum_{m=q}^{2q-1} \mathrm{e}^{3\pi i p m^2/q+\pi i a m/q},$$

put $m' = m-q$ into the second sum to get,

$$= \sum_{m=0}^{q-1} \mathrm{e}^{3\pi i p m^2/q+\pi i a m/q} + \sum_{m=0}^{q-1} \mathrm{e}^{3\pi i p m^2/q+\pi i a m/q}\mathrm{e}^{\pi i(3pq+a)}.$$

Now, since in the cases we consider, p and q odd $\Longrightarrow$ a odd, and p or q even $\Longrightarrow$ a even, we always have that $3pq+a$ is even so,

$$\mathrm{Tr}_a(p,q) = \frac{1}{2\sqrt{q}}\overline{G(3p,2q,a)}.$$

The other equality can be proved in an analogous way. ∎

REMARK 6.1.2. Remark 6.1.1 and Section 5 clearly allow us to compute Trace($\tilde{T}_i$) and hence the dimensions of the eigenspaces of $\tilde{T}_i$.

DEFINITION 6.1.3. Let m_i, u_i, v_i, $i=0,1,2$, be defined by,

$$\begin{aligned} m_i &= \dim +1 \ \text{ eigenspace of } \tilde{T}_i,\\ u_i &= \dim \mathrm{e}^{2\pi i/3} \ \text{ eigenspace of } \tilde{T}_i,\\ v_i &= \dim \mathrm{e}^{4\pi i/3} \ \text{ eigenspace of } \tilde{T}_i. \end{aligned}$$

Note that m_i, u_i, v_i are the solutions of the following linear system:

$$\begin{cases} m_i + u_i \mathrm{e}^{2\pi i/3} + v_i \mathrm{e}^{4\pi i/3} = \mathrm{Trace}(\tilde{T}_i)\\ m_i + u_i + v_i = q \end{cases} \quad \text{or} \quad \begin{cases} m_i - (u_i+v_i)/2 = \mathrm{Re}(\mathrm{Trace}\tilde{T}_i)\\ (u_i - v_i)\sqrt{3}/2 = \mathrm{Im}(\mathrm{Trace}\tilde{T}_i)\\ m_i + u_i + v_i = q \end{cases}.$$

6.2. q and p odd

We first consider the case when $q \equiv 0 (\mathrm{mod}\, 3)$.

Case 1: $q \equiv 0 (\mathrm{mod}\, 3)$. Let $q = 3r$. Now,

$$\mathrm{Trace}(T_0) = \mathrm{Trace}(T_2) = 0,$$

which can be seen, for example, by computing the norm square of the trace. At Ω_1 we have a more complicated situation.

LEMMA 6.2.1. *If $q = 3r$ then,*

$$G(3p, 2q, 3) = 3G(p, 2r, 1).$$

Proof. Straightforward computation using the definitions. ■

Now,

$$\mathrm{Trace}(\tilde{T}_1) = \mathrm{Trace}(\mu_3^{-1} T_1) = \frac{3}{2\sqrt{q}} \overline{G(p, 2r, 1)} \left(\frac{1}{2\sqrt{q}} G(p, 2q, 3) \right)^{1/3}.$$

By using Lemma 5.5.3 we get,

$$\mathrm{Trace}(\tilde{T}_1) = \frac{3}{2\sqrt{q}} \overline{G(2p, r, 1)}\, \overline{G(pr, 2, 1)} \left(\frac{1}{2\sqrt{q}} G(2p, q, 3) G(pq, 2, 3) \right)^{1/3}.$$

Therefore by Lemma 5.5.8 and Remark 5.5.7,

$$\mathrm{Trace}(\tilde{T}_1) = \sqrt{3}(2p|3) i^{-1}.$$

Also notice that $(2p|3) = -(p|3)$ since $(2|3) = -1$. Therefore we get:

$$\begin{array}{llll} m_i = q/3 & u_i = q/3, & v_i = q/3, & i = 0, 2, \\ m_1 = q/3, & u_1 = q/3 + (p|3), & v_i = q/3 - (p|3). & \end{array}$$

Case 2: $(q, 3) = 1$. We have that,

$$\mathrm{Trace}(\tilde{T}_k) = \frac{1}{2\sqrt{q}} \overline{G(3p, 2q, a(k))} \left(\frac{1}{2\sqrt{q}} G(p, 2q, a(k)) \right)^{1/3} =$$

$$= \frac{1}{\sqrt{q}} \overline{G(6p, q, a(k))} \left(\frac{1}{\sqrt{q}} G(2p, q, a(k)) \right)^{1/3} = (q|3),\ k = 0, 1, 2.$$

Therefore,

$$m_i = \frac{q + 2(q|3)}{3},\quad u_i = v_i = \frac{q - (q|3)}{3}, i = 0, 1, 2.$$

6.3. q odd and p even

This is the simplest case since we do not need to "double up" the sums involved.

LEMMA 6.3.1. *Let $p = 2t$. Then,*

$$\mathrm{Trace}(\tilde{T}_k) = \frac{1}{\sqrt{q}}\overline{G(3p,q,k)}\left(\frac{1}{\sqrt{q}}G(t,q,k)\right)^{1/3}, \; k = 0,1,2.$$

Proof. It is sufficient to use the formulae in 5.5 and 6.1 by noticing that $a(k) = 2k$, $p = 2t$. ■

Case 1: $q \equiv 0(\mathrm{mod}\, 3)$. Let $q = 3r$ and note $\mathrm{Trace}(\tilde{T}_k) = 0$ if $k = 1,2$, so it only remains to compute,

$$\mathrm{Trace}(\tilde{T}_0) = \frac{1}{\sqrt{q}}\overline{G(3t,q,0)}\left(\frac{1}{\sqrt{q}}G(t,q,0)\right)^{1/3} = \frac{3}{\sqrt{q}}\overline{G(t,r,0)}\left(\frac{1}{\sqrt{q}}G(t,q,0)\right)^{1/3} =$$

$$= \sqrt{3}(t|r)\,(t|q)i^{-1} = \sqrt{3}(t|3)i^{-1}.$$

Therefore:

$$\begin{array}{llll} m_0 = q/3, & u_0 = q/3 - (t|3), & v_0 = q/3 + (t|3), & \\ m_i = q/3, & u_i = q/3, & v_i = q/3, & i = 1,2, \end{array}$$

Case 2: $(q,3) = 1$. We have that,

$$\mathrm{Trace}(\tilde{T}_k) = (q|3), \quad k = 0,1,2.$$

So the dimensions of the eigenspaces are given by,

$$m_i = \frac{q + 2(q|3)}{3}, \quad u_i = v_i = \frac{q - (q|3)}{3}, \quad i = 0,1,2.$$

6.4. q even

Case 1: $q \equiv 0(\mathrm{mod}\, 3)$. Since,

$$\mathrm{Trace}(\tilde{T}_i) = 0, \quad i = 1,2,$$

we only need to compute,

$$\mathrm{Trace}(\tilde{T}_0) = \frac{3}{2\sqrt{q}}\overline{G(p,2q/3,0)}\left(\frac{1}{2\sqrt{q}}G(p,2q,0)\right)^{1/3}.$$

Thus if $q = 2^{c-1}r$, with r odd, $r = 3s$, we have,

$$\mathrm{Trace}(\tilde{T}_0) = \frac{3}{2\sqrt{q}}\overline{G(p2^c, s, 0)}\,\overline{G(ps, 2^c, 0)}\left(\frac{1}{2\sqrt{q}}G(p2^c, r, 0)G(pr, 2^c, 0)\right)^{1/3} =$$

$$= \sqrt{3}(p2^c|3)(2^{c-2}|3)i^{-1}e^{-\pi i/6}\, i = \sqrt{3}(p|3)e^{-\pi i/6}.$$

Therefore,

$$m_0 = q/3 + (p|3),\ u_0 = q/3 - (p|3),\ v_0 = q/3,$$

$$m_i = u_i = v_i = q/3, \quad i = 1, 2.$$

Case 2: $(q, 3) = 1$. Suppose that $q = 2^{c-1}r$, with r odd. Then,

$$\mathrm{Trace}(\tilde{T}_{a/2}) = \frac{1}{2\sqrt{q}}\overline{G(3p, 2q, a)}\,\overline{G(3pr, 2^c, a)}\left(\frac{1}{2\sqrt{q}}G(p, 2q, a)\right)^{1/3}.$$

By Lemma 5.5.3, Corollary 5.5.6 and Lemma 5.5.8 we have,

$$\mathrm{Trace}(\tilde{T}_{a/2}) = \frac{1}{2\sqrt{q}}\overline{G(3p2^c, r, a)}\left(\frac{1}{2\sqrt{q}}G(p2^c, r, a)G(pr, 2^c, a)\right)^{1/3} =$$

$$= \frac{2^{(c+1)/2}}{2\sqrt{3qpr}}e^{-\pi i(1/4 - a^2/(3pr2^{c+1}))}\overline{G(3p2^c, r, a)}\times$$

$$\times G(2^{c-2}, 3pr, a/2)\left(\frac{2^{(c+1)/2}}{2\sqrt{qpr}}e^{\pi i(1/4 - a^2/(pr2^{c+1}))}G(p2^c, r, a)\overline{G(2^{c-2}, pr, a/2)}\right)^{1/3},$$

$$\text{so } \mathrm{Trace}(\tilde{T}_b) = e^{\pi i/3}(2q|3)\begin{cases} e^{+2\pi iby/3} & \text{if } pr \equiv 1(\mathrm{mod}\, 4) \\ e^{-2\pi iby/3} & \text{if } pr \equiv 1(\mathrm{mod}\, 4) \end{cases},$$

where $2^{c-2}y = -b(\mathrm{mod}\, 3pr)$ and $b = a/2$. Therefore at Ω_0 we have $b = 0$ and,

$$\mathrm{Trace}(\tilde{T}_0) = e^{\pi i/3}(2q|3).$$

The congruence class of $y(\mathrm{mod}\, 3)$ is determined by the equation $2^{c-2}y \equiv -b(\mathrm{mod}\, 3pr)$ which implies $2^{c-2}y + b \equiv 0(\mathrm{mod}\, 3)$. Therefore $\mathrm{Trace}(\tilde{T}_b)$ is determined by c. By solving for the dimensions we get:

$$m_0 = [q - (q|3)]/3, u_0 = [q - (q|3)]/3, v_0 = [q + 2(q|3)]/3,$$

$$m_i = \begin{cases} \dfrac{q + 2(q|3)}{3} \\ \dfrac{q - (q|3)}{3} \end{cases}, u_i = \begin{cases} \dfrac{q - (q|3)}{3} \\ \dfrac{q + 2(q|3)}{3} \end{cases}, v_i = \begin{cases} \dfrac{q - (q|3)}{3} & 3^{c-1}pr \equiv 1(\mathrm{mod}\, 4) \\ \dfrac{q - (q|3)}{3} & 3^{c-1}pr \equiv 3(\mathrm{mod}\, 4) \end{cases},$$

$i = 1, 2.$

7. WINDING NUMBERS

7.1. Introduction

In this section we will compute the phase factors and winding numbers then show that the winding numbers are always integer multiples of q.

7.2. Phase factors

We are going to determine the phase factor μ of Section 4.2. We will do in detail only q and p odd. The other cases can be computed by using analogous techniques.

(i) q and p *odd*: μ is determined by the equation,

$$\mu_3^{-1}\mu_1^{-1}W_1^{-1}W_2^{-(p-1)/2}Z_0W_1^{-1}W_2^{-(p+1)/2}Z_0 = \mu Z_0^{-1}W_2^{\frac{(p-1)}{2}}.$$

Since $W_2^{-1}Z_0 = Z_0W_1^{-1}$ (shown by direct computation) we get,

$$\mathrm{LHS} = \mu_3^{-1}\mu_1^{-1}W_1^{-1}Z_0W_1^{-(p+1)/2}Z_0W_1^{-\frac{(p+1)}{2}}.$$

Notice that $W_1^{-1}Z_0 = \rho^{p''(p''-1)/2}Z_0W_2W_1$ and,

$$(W_2W_1)^n = \omega^{p''n(n-1)/2}W_2^nW_1^n, \quad n \in \mathbb{N},$$

so,

$$\mathrm{LHS} = \mu_3^{-1}\mu_1^{-1}\omega^{p''(p^2-1)/8}\rho^{p''(p''-1)(p+1)/4}W_1^{-1}Z_0^2W_2^{(p+1)/2}.$$

Because $W_1^{-1}Z_0^2 = Z_0^2W_2^{-1}$ we finally get,

$$\mathrm{LHS} = \mu_3^{-1}\mu_1^{-1}\omega^{p''(p^2-1)/8}\rho^{p''(p''-1)(p+1)/4}Z_0^2W_2^{(p-1)/2}.$$

Therefore,

$$\mu = \mu_3^{-1}\mu_1^{-1}\omega^{p''(p^2-1)/8}\rho^{p''(p''-1)(p+1)/4}z_0,$$

where $Z_0^3 = z_0I_q$.

(ii) q *odd* and p *even*: $\mu = \mu_0^{-1}\mu_2^{-1}\omega^{pp''(p/2-1)/4}\rho^{p''(p''-1)(p/2+1)/2}z_0$.

(iii) q *even*: $\mu = \mu_0^{-1}\mu_2^{-1}\rho^{(p'')^2/2}z_0$.

7.3. Explicit conputation of phase factors

Here we are going to find a simpler formula for the "fractional part" FP of the final phase factor λ. We call the "fractional part" of λ that part which can not be written as $e^{2\pi ik/q}$, where $k \in \mathbb{Z}$. This is the only fractional contribution to UWPP.

Note that if q and p are odd $\lambda = \mu_{-1}^{-1}\mu$ and if q or p even $\lambda = \mu_4^{-1}\mu$, where μ is given in Section 7.2.

(i) q and p *odd*: The fractional part FP of λ is in this case,

$$\mathrm{FP} = \mu_{-1}^{-1}\mu_3^{-1}\mu_1^{-1}z_0,$$

since $\omega^{p''(p^2-1)/8}\rho^{p''(p''-1)(p+1)/4}$ are exponentials whose exponent has q as denominator. By computing Z_0^3 directly we get $z_0 = \frac{1}{2\sqrt{q}}\overline{G(p,2q,-p)}$ and so,

$$\mathrm{FP} = \begin{cases} e^{\pi i(-q+1)11x/(6q)} & \text{if } q \equiv 1 (\mathrm{mod}\, 4) \\ e^{\pi i(q+1)11x/(6q)} & \text{if } q \equiv 3 (\mathrm{mod}\, 4) \end{cases},$$

where $2px \equiv -1 (\mathrm{mod}\, q)$.

(ii) q *odd* and p *even*:

$$\mathrm{FP} = \begin{cases} e^{\pi i(-q+1)(3t+5x)/(6q)} & \text{if } q \equiv 1 (\mathrm{mod}\, 4) \\ e^{\pi i(q+1)(3t+5x)/(6q)} & \text{if } q \equiv 3 (\mathrm{mod}\, 4) \end{cases},$$

where $tx \equiv -1 (\mathrm{mod}\, q)$.

(iii) q *even*:

$$\mathrm{FP} = e^{-5\pi i/(3pq)}\rho^{(p'')^2/2}\times$$
$$\times \begin{cases} e^{\pi i(-r+1)5x/(3r)} & \text{if } r \equiv 1 (\mathrm{mod}\, 4) \\ e^{\pi i(r+1)5x/(3r)} & \text{if } r \equiv 3 (\mathrm{mod}\, 4) \end{cases}, \begin{cases} e^{\pi i(-pr+1)5y/(6pr)} & \text{if } pr \equiv 1 (\mathrm{mod}\, 4) \\ e^{\pi i(pr+1)5y/(6pr)} & \text{if } pr \equiv 3 (\mathrm{mod}\, 4) \end{cases},$$

where $2^c px \equiv -2 (\mathrm{mod}\, r)$ and $2^{c-2}y \equiv -1 (\mathrm{mod}\, pr)$.

7.4. Explicit computation of winding numbers

To illustrate the method we only do the case q, p odd. The remaining two cases (q odd and p even and q even) can be shown by doing analogous computations. As in Section 6 we distinguish two subcases $q \equiv 0 (\mathrm{mod}\, 3)$ and $(q, 3) = 1$.

Case 1: $q \equiv 0 (\mathrm{mod}\, 3)$. We have,

$$\begin{aligned} \Sigma_Y = & \left[\left(\frac{1}{3}+k_0\right)q/3 + \left(\frac{2}{3}+k_1\right)q/3\right] + \\ & + \left[\left(\frac{1}{3}+n_0\right)[q/3+(p|3)] + \left(\frac{2}{3}+n_1\right)[q/3-(p|3)]\right] - \\ & - \left[\left(\frac{1}{3}+l_0\right)q/3 + \left(\frac{2}{3}+l_1\right)q/3\right] + \mathrm{UWPP}. \end{aligned}$$

Let $k_i = n_i = l_i,\ i = 0, 1$. Then,

$$\Sigma_Y = (1+n_0+n_1)q/3 + (n_0 - n_1 - 1/3)(p|3) + \mathrm{UWPP}.$$

Now from Section 7.3 we have that UWPP $= 11\alpha x/3+$(Int), where (Int) is an integer, $2px \equiv -1 (\mathrm{mod}\, q)$, and if $q = 4t+1$, $\alpha = t$, while if $q = 4t+3$, $\alpha = -(t+1)$. Therefore since $q \equiv 0 (\mathrm{mod}\, 3)$, $\alpha \equiv 2 (\mathrm{mod}\, 3)$ and $px \equiv 1 (\mathrm{mod}\, 3)$ so p and x must have the same congruence modulo 3.

For $p \equiv k (\mathrm{mod}\, 3)$, $k = -1, 1, (p|3) = k$, so by the above,

$$\mathrm{UWPP} = \frac{k}{3} + (\mathrm{Int}'),$$

and,

$$\Sigma_Y = (1 + n_0 + n_1)q/3 + k(n_0 - n_1) + (\mathrm{Int}''),$$

which can be made into an integer multiple of q by choosing $n_0 = -k(\mathrm{Int}'') + n_1$ and n_1 such that $1 + 2n_1 - k(\mathrm{Int}'') \equiv 0 (\mathrm{mod}\, 3)$.

Case 2: $(q, 3) = 1$. In this case,

$$\begin{aligned}\Sigma_Y = &\left[\left(\frac{1}{3} + k_0\right)\frac{[q-(q|3)]}{3} + \left(\frac{2}{3} + k_1\right)\frac{[q-(q|3)]}{3}\right] + \\ &+\left[\left(\frac{1}{3} + n_0\right)\frac{[q-(q|3)]}{3} + \left(\frac{2}{3} + n_1\right)\frac{[q-(q|3)]}{3}\right] - \\ &-\left[\left(\frac{1}{3} + l_0\right)\frac{[q-(q|3)]}{3} + \left(\frac{2}{3} + l_1\right)\frac{[q-(q|3)]}{3}\right] + \mathrm{UWPP}.\end{aligned}$$

We have,

$$\mathrm{UWPP} = (-1)^{(q-1)/2}\frac{[q-(-1)^{(q-1)/2}]}{12}11x + (\mathrm{Int}),$$

where (Int) is an integer. By choosing $k_i = n_i = l_i$, $i = 0, 1$, we get,

$$\Sigma_Y = (1 + n_0 + n_1)\frac{[q-(q|3)]}{3} + (-1)^{(q-1)/2}\frac{[q-(-1)^{(q-1)/2}]}{12}11x + (\mathrm{Int}).$$

We can choose $x \equiv 0 (\mathrm{mod}\, 3)$ since $(q, 3) = 1$ thus,

$$\Sigma_Y = (1 + n_0 + n_1)\frac{[q-(q|3)]}{3} + (\mathrm{Int}').$$

Since q and $\frac{q-(q|3)}{3}$ are coprime, $(\mathrm{Int}') = \alpha q + \beta\frac{[q-(q|3)]}{3}$, for some integers α, β. Thus if we choose $1 + n_0 + n_1 = -\beta$,

$$\Sigma_Y = \alpha q.$$

This completes the proof of theorem 3.4.1. ■

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CARLA FARSI
Department of Mathematics,
Unversity of Colorado,
Boulder, CO 80309-0395.

NEIL WATLING
Department of Mathematics,
SUNY at Buffalo,
Buffalo, NY 14214-3093.

current address:
Science Division,
Widener University,
Chester, PA,
U.S.A.

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