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ON THE REFLEXIVE ALGEBRA WITH TWO INVARIANT SUBSPACES

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INTRODUCTION AND PRELIMINARIES

This paper is devoted to the study of the algebra $\mathcal{A}$ of all bounded operators on a Hilbert space H which leave invariant two closed subspaces L, M of H, with $L \cap M = 0$, $\overline{L + M} = H$. In symbols,

$$\mathcal{A} = \{A \in \mathcal{B}(H): A(L) \subseteq L, A(M) \subseteq M\}.$$

When L and M are orthogonal, $\mathcal{A}$ is simply the algebra of all block-diagonal operators with respect to the decomposition $H = L \oplus M$. We are interested in the case where L and M are not orthogonal, especially when the "angle" between L and M is zero (by which we mean that the sum $L + M$ is not closed). In this case, $\mathcal{A}$ is the simplest instance of a reflexive algebra which is not a CSL algebra (a reflexive algebra whose invariant projection lattice is not commutative). Results related to this algebra can be found in [2], [17], [20] and [24].

In Section 1, we determine when the set $\mathcal{A} + \mathcal{A}^*$ is ultraweakly dense in $\mathcal{B}(H)$. This property (called $*$-density by Gilfeather and Larson) is always valid for nest algebras; in fact, it characterizes them among CSL algebras [11]. Our result shows that non-CSL algebras, such as the ones we examine, may be $*$-dense, thus answering a question of [11] in the negative.

In Section 3, we examine the set $\mathcal{A} + \mathcal{S}^*$, where $\mathcal{S}$ is the $\mathcal{A}$-module

$$\mathcal{S} = \{S \in \mathcal{B}(H): S(L) \subseteq M, S(M) \subseteq L\}$$

(when L is orthogonal to M, $\mathcal{S}$ consists of all off-diagonal operators).

We show that $\mathcal{A}+\mathcal{S}^*$ is always ultraweakly dense in $\mathcal{B}(H)$, but is only equal to $\mathcal{B}(H)$ when the angle between L and M is positive. We also investigate the validity of the corresponding equality in the Von Neumann-Schatten classes.

For CSL algebras, Katsoulis [15] has also studied "approximate complements"; more precisely, he has shown that $\mathcal{A}+\mathcal{T}^*$ is always ultraweakly dense in $\mathcal{B}(H)$ (where $\mathcal{T}^*$ is the annihilator of the rank one subalgebra of the CSL algebra $\mathcal{A}$). For example, when $\mathcal{A}$ is the algebra of all upper triangular operators with respect to some basis, $\mathcal{T}$ is the set of all strictly upper triangular operators. Similar decompositions with respect to a nest in a Von Neumann algebra are studied in [11].

In our case, the annihilator of the rank one subalgera of $\mathcal{A}$ is precisely the adjoint of the $\mathcal{A}$-module $\mathcal{S}$.

In Section 4, we study compact perturbations of the algebra $\mathcal{A}$. The algebra $\mathcal{A}+\mathcal{K}$ is always norm-closed; we show, however, that it is not equal to $\mathcal{B}=\{X\in \mathcal{B}(H): P^{\perp}XP$ is compact for all invariant projections $P\}$, unless a "distance estimate" holds for $\mathcal{A}$.

We also investigate whether $\mathcal{A}+\mathcal{K}$ remains invariant under compact perturbations of the operator expressing the "angle" between the subspaces L and M. We prove that this only happens when the "angle" is positive. However, both algebras $\mathcal{B}$ and $\mathcal{D}+\mathcal{K}$ (where $\mathcal{D}$ is the ideal generated by operators in $\mathcal{A}$ which annihilate one of the subspaces L, M) always remain invariant.

Finally, in Section 5, we show that the essential commutant of our algebra $\mathcal{A}$ is the sum of its commutant and the compacts. By contrast, the essential commutant of the ideal $\mathcal{D}$ is always larger, when the "angle" between L and M is zero.

TERMINOLOGY AND NOTATION

Let H be a (complex, separable, infinite dimensional) Hilbert space. The symbols $\mathcal{R}$, $\mathcal{F}$, $\mathcal{K}$, and $\mathcal{B}(H)$ denote the sets of rank one, finite rank, compact, and bounded operators on H respectively; $\mathcal{C}_p$ denotes the Von Neumann-Schatten class (we identify $\mathcal{C}_\infty$ with $\mathcal{K}$). If $\mathcal{X}\subseteq\mathcal{B}(H)$, we write $\mathcal{R}(\mathcal{X})=\mathcal{R}\cap\mathcal{X}$, $\mathcal{F}(\mathcal{X})=\mathcal{F}\cap\mathcal{X}$, $\mathcal{C}_p(\mathcal{X})=\mathcal{C}_p\cap\mathcal{X}$, $\mathcal{K}(\mathcal{X})=\mathcal{K}\cap\mathcal{X}$. The commutant of $\mathcal{X}$ is denoted by $\mathcal{X}'$.

For $1<p<+\infty$ and $1/p+1/q=1$ (or $q=1$ if $p=+\infty$), we identify the Banach space dual $(\mathcal{C}_p)^*$ of $\mathcal{C}_p$ (antilinearly) with $\mathcal{C}_q$, using the sesquilinear mapping

$$\mathcal{C}_p\times\mathcal{C}_q\to\mathbb{C}$$
$$(X,T)\to\operatorname{tr}(XT^*)$$

where tr(.) is the usual trace on $\mathcal{B}(H)$. Thus, if $\mathcal{X}\subseteq\mathcal{C}_p$ $(1<p\leqslant\infty)$, we write $\mathcal{X}^{\perp p}$

or $\mathcal{X}^{\perp}$ for its annihilator in $\mathcal{C}_q$, namely

$$\mathcal{X}^{\perp_p} = \mathcal{X}^{\perp} = \{T \in \mathcal{C}_q : \operatorname{tr}(XT^*) = 0 \ \forall X \in \mathcal{X}\}.$$

When $p = 1$, this duality identifies $(\mathcal{C}_1)^*$ with $\mathcal{B}(H)$, and so, if $\mathcal{X} \subseteq \mathcal{C}_1$,

$$\mathcal{X}^{\perp_1} = \mathcal{X}^{\perp} = \{T \in \mathcal{B}(H) : \operatorname{tr}(XT^*) = 0 \ \forall X \in \mathcal{X}\}$$

is the annihilator of $\mathcal{X}$ in $\mathcal{B}(H)$. For $\mathcal{Y} \subseteq \mathcal{B}(H)$, we write ${}^{\perp}\mathcal{Y}$ for the preannihilator of $\mathcal{Y}$ in $\mathcal{C}_1$, namely

$${}^{\perp}\mathcal{Y} = \{T \in \mathcal{C}_1 : \operatorname{tr}(YT^*) = 0 \ \forall Y \in \mathcal{Y}\}.$$

Also, $\overline{\mathcal{X}}^{p}$ will denote the closure of a subset $\mathcal{X} \subseteq \mathcal{C}_p$ in the norm $\| \ \|_p$ of $\mathcal{C}_p$, while $\overline{\mathcal{Y}}^{uw}$ will be the closure of $\mathcal{Y} \subseteq \mathcal{B}(H)$ in the ultraweak (w^*) topology.

If $\mathcal{X}$ is a subset of a vector space, we will denote the linear span of $\mathcal{X}$ by $[\mathcal{X}]$.

The (orthogonal) projection onto a subspace N of H will be denoted by $P(N)$. If M and N are subspaces of H, $M \vee N$ denotes the closure of $M + N$. If $\mathcal{L}$ is a set of (closed) subspaces of H, we write Alg$\mathcal{L}$ for the algebra of all bounded operators leaving each element of $\mathcal{L}$ invariant. If e and f are two vectors in H, the operator $e \otimes f^*$ is defined by $(e \otimes f^*) = \langle x, f \rangle e$. We generally follow the, by now standard, notation of [6] and [26] for matters concerning invariant subspace theory.

PRELIMINARY RESULTS

Given two subspaces L and M of a Hilbert space H with $L \cap M = 0$ and $\overline{L + M} = H$, we study the algebra

$$\mathcal{A} = \operatorname{Alg}\{L, M\} = \{A \in \mathcal{B}(H) : A(L) \subseteq L, A(M) \subseteq M\}$$

and the $\mathcal{A}$-module

$$\mathcal{S} = \{S \in \mathcal{B}(H) : S(L) \subseteq M, S(M) \subseteq L\}.$$

We will use the following facts about $\mathcal{A}$ and $\mathcal{S}$:

LEMMA 0.1 (see [22], [17]). *Every finite rank operator in $\mathcal{A}$ (resp. $\mathcal{S}$) is a (finite) sum of rank one operators in $\mathcal{A}$ (resp. $\mathcal{S}$). In fact,*

$$\mathcal{F}(\mathcal{A}) = [P(L)RP(M^{\perp}), P(M)RP(L^{\perp}) : R \in \mathcal{R}]$$
$$\mathcal{F}(\mathcal{S}) = [P(L)RP(L^{\perp}), P(M)RP(M^{\perp}) : R \in \mathcal{R}].$$

THEOREM 0.2 (see [24]). *The finite rank subalgebra $\mathcal{F}(\mathcal{A})$ of $\mathcal{A}$ is ultraweakly dense in $\mathcal{A}$.*

Note that Lemma 0.1 implies that $\mathcal{A}$ and $\mathcal{S}^*$ annihilate each other's finite rank operators. It follows that $\mathcal{C}_p(\mathcal{A})^\perp \subseteq \mathcal{C}_q(\mathcal{S}^*)$. Conversely, if T is in $\mathcal{C}_q(\mathcal{S}^*)$, then for each A in $\mathcal{C}_p(\mathcal{A})$ the operator T^*A is trace class; by Theorem 0.2 there is a net $\{R_i\}$ in $\mathcal{F}(\mathcal{A})$ such that $w^* - \lim R_i = I$, hence $\lim(\operatorname{tr}(T^*AR_i)) = \operatorname{tr}(T^*A)$. Since AR_i is in $\mathcal{F}(\mathcal{A})$, by Theorem 0.1 we have $\operatorname{tr}(T^*AR_i) = 0$ for each i, hence $T \in \mathcal{C}_p(\mathcal{A})^\perp$. It also follows that $\overline{\mathcal{F}(\mathcal{A})}^p = \mathcal{C}_p(\mathcal{A})$. Similar arguments complete the proof of

COROLLARY 0.3.

(i) $\quad {}^\perp\mathcal{A} = \mathcal{C}_1(\mathcal{S}^*)$, $\mathcal{C}_p(\mathcal{A})^\perp = \mathcal{C}_q(\mathcal{S}^*)$, $\overline{\mathcal{F}(\mathcal{A})}^p = \mathcal{C}_p(\mathcal{A})$, $\overline{\mathcal{F}(\mathcal{A})}^\infty = \mathcal{K}(\mathcal{A})$.

(ii) ${}^\perp\mathcal{S} = \mathcal{C}_1(\mathcal{A}^*)$, $\mathcal{C}_p(\mathcal{S})^\perp = \mathcal{C}_q(\mathcal{A}^*)$, $\overline{\mathcal{F}(\mathcal{S})}^{uw} = \mathcal{S}$, $\overline{\mathcal{F}(\mathcal{S})}^p = \mathcal{C}_p(\mathcal{S})$, $\overline{\mathcal{F}(\mathcal{S})}^\infty = \mathcal{K}(\mathcal{S})$.

1. THE SPACES $\mathcal{A}+\mathcal{A}^*$ AND $\mathcal{S}+\mathcal{S}^*$

Suppose that L and M are orthogonal. Then $\mathcal{A}$ is the (von Neumann) algebra of all block-diagonal operators with respect to the decomposition $H = L \oplus M$, and $\mathcal{S}$ is the (selfadjoint) subspace of all off-diagonal operators. Thus $\mathcal{A}+\mathcal{S} = \mathcal{A}+\mathcal{S}^* = \mathcal{B}(H)$ while $\mathcal{A}+\mathcal{A}^* = \mathcal{A}$ is "small".

When L is not orthogonal to M, the sum $\mathcal{A}+\mathcal{A}^*$ may be "large" in the sense that $(\mathcal{A}+\mathcal{A}^*)^{-uw} = \mathcal{B}(H)$ (Theorem 1.3). This allows us to answer the question of Gilfeather and Larson [11] mentioned in the introduction (Corollary 1.4). We also give an unexpected operator-theoretic characterisation of Halmos' "generic position" in terms of the "largeness" of $\mathcal{S}+\mathcal{S}^*$ in a weaker sense (Proposition 1.6).

As long as the sum $L+M$ remains closed (even though it may not be an orthogonal sum), the equalities $\mathcal{A}+\mathcal{S} = \mathcal{A}+\mathcal{S}^* = \mathcal{B}(H)$ remain valid. Indeed, in this case there is an equivalent inner product on H with respect to which L and M become orthogonal, hence $\mathcal{A}+\mathcal{S} = \mathcal{B}(H)$. For the other equality, see Theorem 3.3.

Things change dramatically as soon as $L+M \neq H$.

As we show in Section 2, while $\mathcal{A}+\mathcal{S}$ remains ultraweakly dense in $\mathcal{B}(H)$, it is never equal to it (Theorem 2.1).

In Section 3, we show that the same is true for $\mathcal{A}+\mathcal{S}^*$ (Theorem 3.1); moreover, we can construct a compact operator which is not in $\mathcal{A}+\mathcal{S}^*$ (Theorem 3.3). However, any Hilbert-Schmidt operator decomposes as an orthogonal sum of two Hilbert-Schmidt operators in $\mathcal{A}$ and $\mathcal{S}^*$ respectively (Theorem 3.2). This decomposability always fails for some operator in $\mathcal{C}_1$ (Theorem 3.3); however, there are situations in which $\mathcal{C}_p(\mathcal{A}) \oplus \mathcal{C}_p(\mathcal{S}^*) = \mathcal{C}_p$ for all $p \in (1, +\infty)$ (Proposition 3.8).

Now for the details:

1.1. Given the subspaces L and M of H with $L \cap M = 0$ and $\overline{L+M} = H$, following Halmos [12] we call L and M in *generic position* if $M^{\perp} \cap L = L^{\perp} \cap M = 0$. A fundamental result [12] in the study of two subspaces in generic position is that, up to unitary equivalence, we may write H as an orthogonal direct sum $H = H_0 \oplus H_0$ and L, M in the form $L = \mathrm{Gr}(B) = \{(x, Bx) : x \in H_0\}$ and $M = \mathrm{Gr}(-B)$, where $B \in \mathcal{B}(H_0)$ satisfies $0 \leqslant B \leqslant I$ and $\mathrm{Ker}(B) = \mathrm{Ker}(I - B) = 0$. Furthermore, the sum $L + M$ is closed if and only if B is invertible.

When L and M are of this form, the sets $\mathcal{A}$ an $\mathcal{S}$ are easily described: indeed, a calculation shows that

$$\mathcal{A} = \left\{ \begin{pmatrix} P & Q \\ BQB & R \end{pmatrix} : P, Q, R \in \mathcal{B}(H_0) \text{ and } BP = RB \right\}$$

and

$$\mathcal{S} = \left\{ \begin{pmatrix} X & -Y \\ BYB & -Z \end{pmatrix} : X, Y, Z \in \mathcal{B}(H_0) \text{ and } BX = ZB \right\}.$$

In the general case, we may decompose H as an orthogonal direct sum

$$H = H_1 \oplus H_2 = (L_1 \vee M_1) \oplus ((M^{\perp} \cap L) \oplus (M \cap L^{\perp}))$$

where $L_1 \equiv L \cap H_1 = L \ominus (M^{\perp} \cap L)$ and $M_1 \equiv M \cap H_1 = M \ominus (M \cap L^{\perp})$. It is easy to see that L_1 and M_1 are in generic position as subspaces of H_1, and the above applies. We call H_1 the *generic part* and H_2 the *non-generic part* of H.

The subset $\mathcal{A} \cap \mathcal{A}^*$ of $\mathcal{A}$ is the (von Neumann) algebra of all operators leaving both L, M and their orthogonal complements invariant; in other words, it is the commutant of the projections onto L and M.

Observe that both $\mathcal{A} \cap \mathcal{A}^*$ and $\mathcal{S} \cap \mathcal{S}^*$ reduce the non-generic part H_2 and that the compressions of $\mathcal{A}$ and $\mathcal{S}$ to that space are easy to describe:

$$(\mathcal{A} \cap \mathcal{A}^*)|H_2 = \left\{ \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \right\} \quad (\mathcal{S} \cap \mathcal{S}^*)|H_2 = \left\{ \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \right\}$$

where P, Q, X, Y are arbitrary operators on the appropriate spaces.

Restricting attention to the generic part H_1, we may take $L_1 = \mathrm{Gr}(B)$, $M_1 = \mathrm{Gr}(-B)$. Then, as can be deduced from [12], we have

$$(\mathcal{A} \cap \mathcal{A}^*)|H_1 = \left\{ \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} : P \in \{B\}' \right\}.$$

Observe that the operator $\Theta \in \mathcal{B}(H_1)$ given by

$$\Theta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

is a selfadjoint unitary in $(\mathcal{S}\cap\mathcal{S}^*)|H_1$. Hence $(\mathcal{S}\cap\mathcal{S}^*)|H_1 = \Theta(\mathcal{A}\cap\mathcal{A}^*)|H_1$, and therefore

$$(\mathcal{S}\cap\mathcal{S}^*)|H_1 = \left\{ \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} : X \in \{B\}' \right\}.$$

PROPOSITION 1.2. *If L and M are in generic position, the subspaces $\mathcal{A}\cap\mathcal{A}^*$ and $\mathcal{S}\cap\mathcal{S}^*$ contain no rank one operators. They contain rank two operators if and only if they contain compact operators, if and only if B has eigenvalues.*

Proof. The first statement follows trivially from the matrix forms just given.

If B has eigenvalues, let P denote the rank one projection onto an eigenvector. By the above, the operator

$$A = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

is a rank two operator in $\mathcal{A}\cap\mathcal{A}^*$, while the operator $S = \Theta A$ is a rank two operator in $\mathcal{S}\cap\mathcal{S}^*$.

Conversely, if $\mathcal{A}\cap\mathcal{A}^*$ or $\mathcal{S}\cap\mathcal{S}^*$ contain a (non-zero) compact operator, then there is a non-zero compact selfadjoint operator $P \in \{B\}'$. But then, by the spectral theorem, there is a compact non-zero projection commuting with B, so B has finite dimensional non-trivial reducing subspaces, hence also eigenvalues. ∎

THEOREM 1.3. *If L and M are in generic position, the following are equivalent:*

(a) *No finite rank proper projection commutes with both $P(L)$ and $P(M)$.*

(b) *B has no eigenvalues.*

(c) *$(\mathcal{A}+\mathcal{A}^*)^{-uw} = \mathcal{B}(H)$ (resp. $(\mathcal{S}+\mathcal{S}^*)^{-uw} = \mathcal{B}(H)$).*

(d) *$(\mathcal{C}_p(\mathcal{A})+\mathcal{C}_p(\mathcal{A}^*))^{-p} = \mathcal{C}_p$ (resp. $(\mathcal{C}_p(\mathcal{S})+\mathcal{C}_p(\mathcal{S}^*))^{-p} = \mathcal{C}_p$) for some $p \in (1,\infty]$.*

(e) *$(\mathcal{C}_p(\mathcal{A})+\mathcal{C}_p(\mathcal{A}^*))^{-p} = \mathcal{C}_p$ (resp. $(\mathcal{C}_p(\mathcal{S})+\mathcal{C}_p(\mathcal{S}^*))^{-p} = \mathcal{C}_p$) for all $p \in (1,\infty]$.*

However, $\mathcal{C}_1(\mathcal{A})+\mathcal{C}_1(\mathcal{A}^*)$ and $\mathcal{C}_1(\mathcal{S})+\mathcal{C}_1(\mathcal{S}^*)$ are never dense in $\mathcal{C}_1$.

Proof. The equivalence of (a) and (b) is obvious from Proposition 1.2.

Since, by Corollary 0.3,

$$^\perp(\mathcal{A}+\mathcal{A}^*) = ^\perp\mathcal{A}\cap^\perp(\mathcal{A}^*) = \mathcal{C}_1(\mathcal{S}^*)\cap\mathcal{C}_1(\mathcal{S}) = \mathcal{C}_1(\mathcal{S}\cap\mathcal{S}^*),$$

and

$$(\mathcal{C}_p(\mathcal{A})+\mathcal{C}_p(\mathcal{A}^*))^\perp = \mathcal{C}_p(\mathcal{A})^\perp\cap\mathcal{C}_p(\mathcal{A}^*)^\perp = \mathcal{C}_q(\mathcal{S}^*)\cap\mathcal{C}_q(\mathcal{S}) = \mathcal{C}_q(\mathcal{S}\cap\mathcal{S}^*),$$

for $1 < p \leqslant +\infty$, the remaining equivalences follow by Proposition 1.2. Finally,

$$(\mathcal{C}_1(\mathcal{A})+\mathcal{C}_1(\mathcal{A}^*))^\perp = \mathcal{S}\cap\mathcal{S}^*$$

which is never trivial. The statements about $\mathcal{S}$ are proved in the same way. ∎

REMARK. The set $\mathcal{A}+\mathcal{A}^*$ is never equal to $\mathcal{B}(H)$. It will be convenient to prove this fact using techniques of Section 3 (see Proposition 3.10).

There are several ways in which a subset $\mathcal{Y}$ of $\mathcal{B}(H)$ can be "large": it can be dense in $\mathcal{B}(H)$ for some operator topology, or it can have the weaker property of acting (topologically) transitively on H. These notions were studied by Gilfeather and Larson [11], to whom the following definition is essentially due.

DEFINITION. A linear subset $\mathcal{X}$ of $\mathcal{B}(H)$ is said to be *$*$-dense* in case $\mathcal{X}+\mathcal{X}^*$ is ultraweakly dense in $\mathcal{B}(H)$. $\mathcal{X}$ is said to be *$*$-transitive* if, for each nonzero $x \in H$, $(\mathcal{X}+\mathcal{X}^*)x$ is dense in H.

In [11], the authors ask whether $*$-density of an algebra implies that its invariant projection lattice is totally ordered (they show that this is true when the lattice is commutative). It follows from Theorem 1.3 above that, not only is the answer negative in general, but the opposite extreme might occur.

COROLLARY 1.4. *There exists a subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ which is $*$-dense but whose invariant subspace lattice has no pair of (non-trivial) comparable elemants. Moreover, there exists such an $\mathcal{A}$ which is similar to a (proper) von Neumann subalgebra of $\mathcal{B}(H)$.*

Proof. Apply Theorem 1.3 to $\mathcal{A} = \mathrm{Alg}(\mathrm{Gr}(B), \mathrm{Gr}(-B))$ where B has no eigenvalues. For the second statement, take B to be, in addition, invertible. ∎

As shown in [11], a unital algebra $\mathcal{A}$ has the (weaker) property of $*$-transitivity if and only if $N_1 \cap N_2^{\perp} = 0$ or $N_1^{\perp} \cap N_2 = 0$ for every pair of $\mathcal{A}$-invariant subspaces N_1 and N_2. Thus, in our case, when L and M are in generic position, $\mathcal{A} = \mathrm{Alg}\{L, M\}$ is always $*$-transitive. On the other hand, when both $L \cap M^{\perp}$ and $L^{\perp} \cap M$ are nonzero, $\mathcal{A}$ cannot be $*$-dense. If, however, one of $L^{\perp} \cap M$, $L \cap M^{\perp}$ is zero, then the $*$-density of $\mathcal{A}$ is governed by its generic part.

[For an example in which this situation occurs, let A be a non-injective operator on H_0 with dense range (for instance, the adjoint of the unilateral shift). Set $L = 0 \oplus H_0$, $M = \mathrm{Gr}(A)$. Then $L^{\perp} \cap M \neq 0$ while $L \cap M^{\perp} = 0$. Thus, by the result of [11], $\mathcal{A}$ is $*$-transitive.]

PROPOSITION 1.5. *Suppose that $\mathcal{A} = \mathrm{Alg}\{L, M\}$ is $*$-transitive. Then $\mathcal{A}$ is $*$-dense if and only if its compression $\mathcal{A}_1$ to the generic part H_1 of the space is $*$-dense (in $\mathcal{B}(H_1)$), if and only if any finite rank projection in $\{P(L), P(M)\}'$ is orthogonal to $P(H_1)$.*

Proof. If $\mathcal{A}$ is $*$-dense, then clearly so is $\mathcal{A}_1$.

Conversely, suppose $\mathcal{A}_1$ is $*$-dense. To show that $\mathcal{A}+\mathcal{A}^*$ is ultraweakly dense in $\mathcal{B}(H)$, we show that ${}^{\perp}(\mathcal{A}+\mathcal{A}^*)$ is trivial. As noted in Theorem 1.3, ${}^{\perp}(\mathcal{A}+\mathcal{A}^*) =$

$= \mathcal{C}_1(\mathcal{S} \cap \mathcal{S}^*)$. But an operator in $\mathcal{S} \cap \mathcal{S}^*$ must map $M \cap L^\perp$ into $L \cap M^\perp$, and one of these subspaces is zero by assumption. Hence $\mathcal{S} \cap \mathcal{S}^*$ lives in the generic part of the space. But since $\mathcal{A}_1 + \mathcal{A}_1^*$ is ultraweakly dense in $\mathcal{B}(H_1)$, the set $(\mathcal{S} \cap \mathcal{S}^*)|H_1$ contains no trace class operators. Therefore $\mathcal{C}_1(\mathcal{S} \cap \mathcal{S}^*) = 0$.

For the second equivalence, take a finite rank-projection E in the commutant of $\{P(L), P(M)\}$ and decompose it as $E = E_1 + E_2$ with respect to $H = H_1 \oplus H_2$. Clearly, E_1 commutes with $P(L_1)$ and $P(M_1)$. By Theorem 1.3, $\mathcal{A}_1$ is $*$-dense if and only if E_1 is zero.

This completes the proof. ∎

We conclude this section with an operator-theoretic characterisation of generic position.

PROPOSITION 1.6. *$\mathcal{S}$ is $*$-transitive if and only if L and M are in generic position.*

Proof. If L and M are in generic position, then $\mathcal{A}$ is $*$-transitive by the result of [11] quoted above. As we have seen in 1.1, there exists then a symmetry Θ such that $\mathcal{S} = \Theta\mathcal{A}$. Since Θ is invertible, it follows that $\mathcal{S}$ is also $*$-transitive.

Conversely, if, for instance, $L \cap M^\perp$ contains a nonzero vector x, then $\mathcal{S}x$ is in M hence $x \perp \mathcal{S}x$; thus also $x \perp \mathcal{S}^*x$ and therefore $(\mathcal{S} + \mathcal{S}^*)x$ cannot be dense in H. ∎

2. THE SPACE $\mathcal{A} + \mathcal{S}$

We now show that $\mathcal{A} + \mathcal{S}$ is always ultraweakly dense in $\mathcal{B}(H)$, but is equal to it precisely when the sum $L + M$ is closed. The situation is similar in the various operator ideals.

THEOREM 2.1.
(i) $\overline{\mathcal{A} + \mathcal{S}}^{uw} = \mathcal{B}(H)$ *and* $\overline{\mathcal{C}_p(\mathcal{A}) + \mathcal{C}_p(\mathcal{S})}^p = \mathcal{C}_p$ *for* $1 \leqslant p \leqslant +\infty$.
(ii) $\mathcal{A} + \mathcal{S} = \mathcal{B}(H) \Leftrightarrow \mathcal{C}_p(\mathcal{A}) + \mathcal{C}_p(\mathcal{S}) = \mathcal{C}_p \Leftrightarrow \mathcal{F}(\mathcal{A}) + \mathcal{F}(\mathcal{S}) = \mathcal{F} \Leftrightarrow L + M = H$.

Proof. (i) Since $L \cap M = 0$, clearly $\mathcal{A} \cap \mathcal{S} = 0$. But

$$^\perp(\mathcal{A} + \mathcal{S}) = {}^\perp\mathcal{A} \cap {}^\perp\mathcal{S} = \mathcal{C}_1(\mathcal{S}^*) \cap \mathcal{C}_1(\mathcal{A}^*)$$

(by Corollary 0.3) hence $^\perp(\mathcal{A} + \mathcal{S}) = 0$. The second equality is analogous.

(ii) First notice that, by Lemma 0.1, $\mathcal{F}(\mathcal{A}) + \mathcal{F}(\mathcal{S})$ contains all finite rank operators mapping $L^\perp + M^\perp$ into $L + M$. Since $L + M = H$ if and only if $L^\perp + M^\perp = H$, this proves the last equivalence. Now let

$$E: L + M \to L + M$$

$$x + y \to x$$

be the (skew) projection. It is bounded if and only if $L+M=H$. If $T(L+M)\subseteq$ $\subseteq L+M$, define

$$\Phi_E(T)=ETE+(I-E)T(I-E).$$

This is an idempotent, well-defined on $\mathcal{A}+\mathcal{S}$, with range $\mathcal{A}$ and kernel $\mathcal{S}$.
If E is bounded, then Φ_E is defined and bounded on $\mathcal{B}(H)$. But then, for every $X\in\mathcal{B}(H)$,

$$X=\Phi_E(X)+(X-\Phi_E(X)),$$

showing that $\mathcal{A}+\mathcal{S}=\mathcal{B}(H)$. The proof for $\mathcal{C}_p$ is identical.
If E is not bounded, for each $n\in\mathbb{N}$ there are $x_n\in L$, $y_n\in M$ with $\|x_n\|=\|y_n\|=1$ and $\|x_n-y_n\|\leqslant 1/n$. Choose $z\in M^{\perp}$, $z\neq 0$ and consider $T_n=(x_n-y_n)\otimes z^*$.

Observe that $\|T_n\|\leqslant\|z\|/n$ while $\|\Phi_E(T_n)\|=\|x_n\otimes z^*\|=\|z\|$ (because $x_n\otimes$ $\otimes z^*\in\mathcal{A}$ and $y_n\otimes z^*\in\mathcal{S}$). Thus Φ_E is unbounded and so $\mathcal{A}+\mathcal{S}$ cannot be closed. Similarly $\mathcal{C}_p(\mathcal{A})+\mathcal{C}_p(\mathcal{S})\neq\mathcal{C}_p$ for all $p\in[1,+\infty]$. ∎

3. THE SPACE $\mathcal{A}+\mathcal{S}^*$

It is shown in [17] that $\mathcal{A}\cap\mathcal{S}^*=0$; hence the sum $\mathcal{A}+\mathcal{S}^*$ is direct. How "large" is this space?

THEOREM 3.1. *The space $\mathcal{F}(\mathcal{A})+\mathcal{F}(\mathcal{S}^*)$ is ultraweakly dense in $\mathcal{B}(H)$, and $(\mathcal{F}(\mathcal{A})+\mathcal{F}(\mathcal{S}^*))^{-p}=\mathcal{C}_p$ for all p. A fortiori, $(\mathcal{A}+\mathcal{S}^*)^{-uw}=\mathcal{B}(H)$ and $(\mathcal{C}_p(\mathcal{A})+$ $+\mathcal{C}_p(\mathcal{S}^*))^{-p}=\mathcal{C}_p$ for all p, and the sums are direct.*

Proof. We have

$$^{\perp}(\mathcal{F}(\mathcal{A})+\mathcal{F}(\mathcal{S}^*))=\mathcal{C}_1(\mathcal{S}^*)\cap\mathcal{C}_1(\mathcal{A})$$

and

$$(\mathcal{F}(\mathcal{A})+\mathcal{F}(\mathcal{S}^*))^{\perp p}=\mathcal{C}_q(\mathcal{S}^*)\cap\mathcal{C}_q(\mathcal{A})$$

by Corollary 0.3. The result follows from the Theorem of [17] just quoted. ∎

For the case $p=2$ we can actually say more:

THEOREM 3.2. *In the Hilbert space $\mathcal{C}_2$, the subspaces $\mathcal{C}_2(\mathcal{A})$ and $\mathcal{C}_2(\mathcal{S}^*)$ are each other's orthogonal complements:*

$$\mathcal{C}_2(\mathcal{A})\oplus\mathcal{C}_2(\mathcal{S}^*)=\mathcal{C}_2.$$

Proof. By Corollary 0.3, $(\mathcal{C}_2(\mathcal{S}^*))^{\perp}=\mathcal{C}_2(\mathcal{A})$. ∎

The situation is altogether different for $p=1$ or $p=\infty$:

THEOREM 3.3. *The following are equivalent:*
(a) $\mathcal{C}_1(\mathcal{S}^*) + \mathcal{C}_1(\mathcal{A}) = \mathcal{C}_1$
(b) $\mathcal{S}^* + \mathcal{A} = \mathcal{B}(H)$
(c) $\mathcal{K}(\mathcal{S}^*) + \mathcal{K}(\mathcal{A}) = \mathcal{K}$
(d) $L + M = H$.

For the proof, we will need some preliminary results:

LEMMA 3.4. *Suppose that $L = \mathrm{Gr}(B)$ and $M = \mathrm{Gr}(-B)$ are as in 1.1. Then an operator $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ belongs to $\mathcal{A} + \mathcal{S}^*$ if and only if there exist (necessarily unique) bounded operators P, Q such that P^* and $T_1 - P$ leave the range of B invariant and*

$$B^2P + PB^2 = T_1B^2 + BT_4B \tag{1}$$

$$Q + B^2QB^2 = T_2 + BT_3B. \tag{2}$$

Proof. If $A \in \mathcal{A}$ and $S \in \mathcal{S}$, then $A + S^*$ has the form

$$\begin{pmatrix} P & Q \\ BQB & R \end{pmatrix} + \begin{pmatrix} X & BYB \\ -Y & -Z \end{pmatrix}$$

where P, Q, R, X, Y, Z are bounded operators such that $BP = RB$ and $XB = BZ$ (see 1.1). In particular, the relations $P^*B = BR^*$ and $XB = BZ$ imply that P^* and X both leave the range of B invariant.

If T is as in the statement, then $T = A + S^*$ if and only if

$$T_1 = P + X$$
$$T_2 = Q + BYB$$
$$T_3 = BQB - Y$$
$$T_4 = R - Z$$

The equations for T_2 and T_3 imply

$$Q + B^2QB^2 = T_2 + BT_3B$$

and

$$Y = BQB - T_3.$$

Moreover,

$$R - Z = T_4 \Rightarrow BRB - BZB = BT_4B \Rightarrow B^2P - XB^2 = BT_4B \Rightarrow$$
$$\Rightarrow B^2P + PB^2 = BT_4B + T_1B^2.$$

Conversely, if P satisfies this last equation and both P^* and $T_1 - P$ leave the range of B invariant, then defining $X = T_1 - P$, $R = BPB^{-1}$ and $Z = B^{-1}XB$ (note that R and Z are bounded operators by Douglas' range inclusion Theorem [8]) we have $T_1 = P + X$ and

$$BRB = B^2P = BT_4B + T_1B^2 - PB^2 = BT_4B + XB^2 = BT_4B + BZB$$

which, by the fact that B is injective and has dense range, gives the required relation $T_4 = R - Z$. Finally if Q satisfies (2), we set $Y = BQB - T_3$ and we are done.

Uniqueness follows from the fact that $\mathcal{A} \cap \mathcal{S}^* = 0$ [17]. ∎

Lemma 3.4 shows that the proof of Theorem 3.3 relies on the solvability of the operator equations (1) and (2) above. The next proposition disposes of the easier cases (with A in place of B^2).

PROPOSITION 3.5. *Let $A \in \mathcal{B}(H)$ be a positive operator.*

(i) *Given T in $\mathcal{B}(H)$ (resp. $\mathcal{K}$, $\mathcal{C}_p$), the equation*

$$X + AXA = T \tag{2'}$$

has a unique solution X in $\mathcal{B}(H)$ (resp. $\mathcal{K}$, $\mathcal{C}_p$), which is a continuous function of T.

(ii) *If the spectrum $\sigma(A)$ of A has limit points in $[0,1)$ (for example, if A is injective but not invertible), then there is a rank one operator T such that the solution X of (2′) is not in $\mathcal{F}$.*

(iii) *If A is invertible, for each T in $\mathcal{B}(H)$ (resp. $\mathcal{K}$, $\mathcal{C}_p$) the equation*

$$AY + YA = T \tag{1'}$$

has a unique solution Y in $\mathcal{B}(H)$ (resp. $\mathcal{K}$, $\mathcal{C}_p$), which is a continuous function of T.

Proof. (i) Denote by $\mathcal{X}$ the $\mathcal{B}(H)$-Banach module $\mathcal{B}(H)$ (resp. $\mathcal{K}$, $\mathcal{C}_p$). If $L\colon \mathcal{X} \to \mathcal{X}$ denotes the operator of left multiplication by A, namely $L(T) = AT$, and similarly $R(T) = TA$, then clearly $\sigma(L) = \sigma(R) \subseteq \sigma(A) \subseteq [0,+\infty)$. Since L and R commute, we have $\sigma(LR) \subseteq \sigma(L)\cdot\sigma(R) \subseteq [0,+\infty)$ (see for example [26], Lemma 0.11). Thus $0 \notin \sigma(I + LR)$, i.e. $I + LR$ has a bounded inverse on $\mathcal{X}$. Therefore,

$$X + AXA = T \Leftrightarrow (I + LR)(X) = T \Leftrightarrow X = (I + LR)^{-1}(T).$$

giving the solution as an operator in $\mathcal{X}$, depending continuously on T.

(ii) Suppose first that $\|A\| < 1$. Let x be a unit separating vector for A, and let $T = x \otimes x^*$. If $x_k = A^k x$ then $\|x_k\| \leqslant \|A\|^k\|x\|$, so $\|x_k\| \to 0$.

Therefore, if we define

$$X = \sum_k (-1)^k x_k \otimes x_k^*$$

then X is a well-defined (compact) operator, and direct substitution shows that $X+$ $+AXA = T$. However, we claim that X has infinite rank. Indeed, suppose y is in the kernel of X. Then

$$0 = Xy = \sum_k (-1)^k \langle y, x_k \rangle A^k x = f(A)x$$

where

$$f(z) = \sum_k (-1)^k \langle y, x_k \rangle z^k$$

is analytic in the unit disc. Since x is separating for A, it follows that $f(A) = 0$. The spectral mapping theorem ([5], VIII.2.7) now shows that f vanishes on $\sigma(A)$. Since $\sigma(A)$ has limit points in the unit disc, the analytic function f must vanish identically. Thus $\langle y, x_k \rangle = 0$ for all k. Hence each x_k is in the orthogonal complement of the kernel of X, that is, in the closure of the range of X. But $\{x_k\}$ is an infinite linearly independent set. Indeed, if μ_k $(k = 1, \ldots, n)$ are scalars such that $\sum_{k=0}^{n} \mu_k x_k = 0$, then $\sum_{k=0}^{n} \mu_k A^k = 0$, since x is separating. Hence $\sum_{k=0}^{n} \mu_k \lambda^k = 0$ for each λ in $\sigma(A)$, and thus all μ_k are zero.

For the general case, let $0 < a < 1$ be such that $\sigma(A)$ has limit points in $[0, a)$ and let $A_1 = AE$, where E is the spectral projection of A corresponding to the interval $[0, a]$. If $x \in E(H)$ is a unit separating vector for A_1 and $T = x \otimes x^*$, then, since $\sigma(A_1)$ is infinite, the previous paragraph shows that the unique solution Z of $Z + A_1 Z A_1 = T$ has infinite rank. But, since $Ax = A_1 x$, the definition of Z shows that $Z + AZA = Z + A_1 Z A_1 = T$ and we are done.

(iii) If A is invertible, then $(1')$ is equivalent to

$$Y + A^{-1} Y A = A^{-1} T$$

or

$$(I + L_1 R)(Y) = A^{-1} T$$

where R denotes, as in (i), right multiplication by A, and L_1 now denotes left multiplication by A^{-1}. As in (i), we find that the operator $I + L_1 R$ is invertible. The result follows. ∎

The solvability of equation (1) of Lemma 3.4 requires a finer argument. We need the following

NOTATION 3.6. Let A be a positive non-invertible injective contraction on a Hilbert space H. For $\lambda \in (0,1)$, let $E_n = E((\lambda^n, 1])$, $n = 1, 2, \ldots$ where $E(\cdot)$ is the spectral measure for A. Let $\Delta_0 = E((\lambda, 1])$, $\Delta_n = E((\lambda^{n+1}, \lambda^n]) = E_{n+1} - E_n$ $(n = 1, 2, \ldots)$, so that $\sum \Delta_n = I$. For $T \in \mathcal{B}(H)$, let $T_{nm} = \Delta_n T \Delta_m$. By the upper triangular part of T we mean the operator X defined on the linear span of $\{\Delta_n(H): n = 0, 1, \ldots\}$ by $X_{nm} = T_{nm}$ if $m \geqslant n$ and $X_{nm} = 0$ otherwise.

LEMMA 3.7. *If A is a positive non-invertible injective operator, there exists a compact operator T such that the equation $AX + XA = TA$ has no bounded solution X.*

Proof. We may clearly assume that A is a contraction. Form the nest $\{0, E_1, E_2, \ldots, H\}$ as in Notation 3.6. Denote by $\mathcal{T}$ the algebra of all bounded operators that are upper triangular with respect to the decomposition $H = \oplus \Delta_n(H)$, i.e. all operators that leave the nest invariant. A theorem in [23] states that a bounded operator Z leaves the range of A invariant if and only if Z belongs to $\mathcal{T} + (A^{-1}\mathcal{T}A)^*$.

The equality $AX = (T - X)A$ gives $(T - X)(\mathrm{Ran}(A)) \subseteq \mathrm{Ran}(A)$ and therefore, by the theorem just stated, there exist bounded operators P, Q in $\mathcal{T}$ with $T - X =$ $= P + (A^{-1}QA)^*$. Since both terms have bounded upper triangular parts, so does $T - X$. The equality $A(T - X)^* = X^*A$ similarly shows that X^*, and hence also X, has a bounded upper triangular part (since the diagonal part of a bounded operator is bounded). It therefore suffices to produce a compact operator T with unbounded upper triangular part.

Let T_n be the $n \times n$ matrix (t_{ij}) where $t_{ij} = (\sqrt{\log n}(i - j))^{-1}$ for $i \neq j$ and $t_{ii} = 0$ $(n \geqslant 2)$. It is known [6, Chapter 4] that $\|T_n\| \leqslant \pi(\log n)^{-1/2}$ while its upper triangular part, say S_n, has norm at least $4(\log n)^{1/2}/5$.

Since A is injective but not invertible, we may find a subsequence $\{\Delta_{m(n)}\}$ of nonzero Δ_n's. For each n, pick a unit vector e_n in $\Delta_{m(n)}(H)$. Let T be the operator which is zero on $[e_n: n \geqslant 2]^\perp$ and is T_2 on $[e_2, e_3]$, T_3 on $[e_4, e_5, e_6]$, etc. This operator is compact, being a norm-convergent sum of finite rank operators (since $\|T_n\| \to 0$). But its upper trianguler part, which is well-defined on the (non-closed) linear span of the $\Delta_n(H)$ and equal on $[e_n: n \in \mathbb{N}]$ to the direct sum of the S_n, is unbounded. ∎

Proof of Theorem 3.3. (i) Suppose first that $L + M = H$.

(a) If L and M are in generic position, then, up to a unitary equivalence, we may write $L = \mathrm{Gr}(B)$, $M = \mathrm{Gr}(-B)$, where B is positive and invertible (see 1.1). Using Proposition 3.5, we see that equations (1) and (2) of Lemma 3.4 are uniquely solvable in $\mathcal{B}(H)$ (resp. in $\mathcal{K}$, $\mathcal{C}_p$) whenever their right hand sides are any operators in $\mathcal{B}(H)$ (resp. in $\mathcal{K}$, $\mathcal{C}_p$). Since the range of B is the whole space, Lemma 3.4 now shows that $\mathcal{A} + \mathcal{S}^* = \mathcal{B}(H)$ and $\mathcal{C}_p(\mathcal{A}) + \mathcal{C}_p(\mathcal{S}^*) = \mathcal{C}_p$ for any p in $[1, +\infty]$.

(b) If L and M are in general position, as in 1.1 we decompose H as a direct sum

$$H = (L_1 \vee M_1) \oplus ((M^{\perp} \cap L) \oplus (M \cap L^{\perp}))$$

where $L_1 \equiv L \ominus (M^{\perp} \cap L)$ and $M_1 \equiv M \ominus (M \cap L^{\perp})$ are in generic position as subspaces of $L_1 \vee M_1$.

Let T be an arbitrary operator in $\mathcal{B}(H)$. Let T_1 be $TP(L \cap M^{\perp})$. Then $T_1 = P(L)T_1 + P(L^{\perp})T_1$, and $P(L)T_1$ is in $\mathcal{A}$, because it maps L into L and M to 0, while $P(L^{\perp})T_1$ is in $\mathcal{S}^*$, because it maps $L^{\perp}$ to 0 and $M^{\perp}$ into $L^{\perp}$. Thus T_1 is in $\mathcal{A} + \mathcal{S}^*$. We may similarly decompose $T_2 = TP(M \cap L^{\perp})$, and we deal with $P(L \cap M^{\perp})T$ and $P(M \cap L^{\perp})T$ by taking adjoints. Since T is arbitrary, it follows that $P(L \cap M^{\perp})TP(L_1 \vee M_1)$ and $P(M \cap L^{\perp})TP(L_1 \vee M_1)$ are also in $\mathcal{A} + \mathcal{S}^*$. But

$$T = T_1 + T_2 + TP(L_1 \vee M_1) =$$

$$= T_1 + T_2 + P(L \cap M^{\perp})TP(L_1 \vee M_1) + P(M \cap L^{\perp})TP(L_1 \vee M_1) + P(L_1 \vee M_1)TP(L_1 \vee M_1)$$

and the last term is in $\mathcal{A} + \mathcal{S}^*$, by part (a). Thus T is in $\mathcal{A} + \mathcal{S}^*$. By the same argument, we also have $\mathcal{C}_p(\mathcal{A}) + \mathcal{C}_p(\mathcal{S}^*) = \mathcal{C}_p$ for any p in $[1, +\infty]$.

(ii) Suppose now that $L + M \neq H$. We will construct a compact operator T living in the generic part of the space such that $T \notin \mathcal{A} + \mathcal{S}^*$. Thus we may assume that $L = \mathrm{Gr}(B)$, $M = \mathrm{Gr}(-B)$, where B is a positive, injective, non-invertible contraction and $I - B$ is also injective.

By Lemma 3.7 (applied to $A = B^2$), there is a compact operator T_1 such that the equation

$$B^2P + PB^2 = T_1B^2$$

has no bounded (let alone compact) solution P. In view of 3.4, this shows that the compact operator

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

is not in $\mathcal{S}^* + \mathcal{A}$ (let alone in $\mathcal{K}(\mathcal{S}^*) + \mathcal{K}(\mathcal{A})$). This deals with the bounded, as well as the compact case.

If $\mathcal{C}_1(\mathcal{S}^*) + \mathcal{C}_1(\mathcal{A}) = \mathcal{C}_1$, there would be a bounded idempotent P acting on the Banach space $\mathcal{C}_1$ with $\mathrm{Im}(P) = \mathcal{C}_1(\mathcal{A})$ and $\mathrm{Ker}(P) = \mathcal{C}_1(\mathcal{S}^*)$. Then the (Banach space) adjoint operator P^* would be a bounded idempotent acting on $\mathcal{B}(H)$ with $\mathrm{Ker}(P^*) = (\mathrm{Im}(P))^{\perp} = (\mathcal{C}_1(\mathcal{A}))^{\perp} = \mathcal{S}^*$ and $\mathrm{Im}(P^*)^{-uw} = (\mathrm{Ker}(P))^{\perp} = (\mathcal{C}_1(\mathcal{S}^*))^{\perp} = \mathcal{A}$ (by Corollary 0.3). Note that P^* is a $w^* - w^*$ continuous idempotent, hence $\mathrm{Im}(P^*)$ is w^*-closed; thus $\mathrm{Im}(P^*) = \mathcal{A}$. This would imply that $\mathcal{A} + \mathcal{S}^*$ is closed, which we have just shown not to be the case. ∎

Note that the proof of Theorem 3.3 relies on the fact that compact operators do not necessarily have bounded upper triangular parts. However, for $1 < p < +\infty$, operators in $\mathcal{C}_p$ do have bounded upper triangular parts, which in fact lie in $\mathcal{C}_p$. Hence it is natural to ask whether the equality $\mathcal{C}_p(\mathcal{A}) \oplus \mathcal{C}_p(\mathcal{S}^*) = \mathcal{C}_p$ (valid for $p = 2$ by Theorem 3.2) holds for all p in $(1, +\infty)$. We have been able to answer this affirmatively in a special case:

PROPOSITION 3.8. *If the generic parts of L and M are of the form* $\mathrm{Gr}(B)$, $\mathrm{Gr}(-B)$ *where* $B = \mathrm{diag}(b^n)$ *for some b with $0 < b < 1$, then*

$$\mathcal{C}_p(\mathcal{A}) \oplus \mathcal{C}_p(\mathcal{S}^*) = \mathcal{C}_p$$

for all $p \in (1, +\infty)$.

Proof. Arguing as in part (i.b) of the proof of Theorem 3.3, we see that it is enough to assume that L and M are in generic position. We now apply Lemma 3.4.

(a) By Proposition 3.5, we know that equation (2) of Lemma 3.4 is uniquely solvable. To solve (1), we solve the equations

$$B^2X + XB^2 = TB^2 \tag{3}$$

and

$$B^2X + XB^2 = BTB \tag{4}$$

separately for all $T \in \mathcal{C}_p$, and then obtain a solution of (1) by addition. We have

(i) With respect to the orthonormal basis $\{e_n\}$ which diagonalizes B, (4) can be written

$$b^{2n}x_{nm} + x_{nm}b^{2m} = b^n t_{nm} b^m$$

or

$$x_{nm} = b^{m-n}(1 + b^{2(m-n)})^{-1} t_{nm}.$$

For each $k \in \mathbf{N}$, let X_k be the operator whose matrix with respect to the basis has k'th diagonal equal to that of X, and the remaining diagonals equal to zero. Since $X_k = c_k T_k$ where $c_k = b^k(1 + b^{2k})^{-1}$, it is clear that each X_k is in $\mathcal{C}_p$ and that $\|X_k\|_p = c_k\|T_k\|_p \leqslant b^k \cdot \|T\|_p$.

Hence the series

$$Y = \sum_{k=0}^{\infty} X_k$$

converges absolutely in p-norm, because

$$\sum_{k=0}^{\infty} \|X_k\|_p \leqslant \sum_{k=0}^{\infty} b^k \cdot \|T\|_p \leqslant (1 - b)^{-1} \cdot \|T\|_p.$$

Therefore Y is in $\mathcal{C}_p$ and $\|Y\|_p \leqslant (1-b)^{-1} \cdot \|T\|_p$. Clearly Y is the upper triangular part of X. But

$$B^2X + XB^2 = BTB \Rightarrow B^2X^* + X^*B^2 = BT^*B$$

therefore, applying the same reasoning to the latter equation, we find that the lower triangular part of X is also in $\mathcal{C}_p$. Finally, since $x_{nn} = t_{nn}/2$ for all n, the diagonal of X is clearly in $\mathcal{C}_p$ and has p-norm at most 1/2 that of T. Hence X itself is in $\mathcal{C}_p$ and in fact

$$\|X\|_p \leqslant C \cdot \|T\|_p$$

where C is a suitable constant (independent of T). This completes the solution of (4).

(ii) For (3), we argue as follows:

The equation gives

$$t_{nm} - x_{nm} = \frac{b^{2n}}{b^{2n} + b^{2m}} t_{nm} = \frac{b^{2(n-m)}}{b^{2(n-m)} + 1} t_{nm}.$$

As above, this shows that the lower triangular part of $T - X$ is a (bounded) operator in $\mathcal{C}_p$ with p-norm at most $(1-b^2)^{-1}\|T\|_p$. Since $T \in \mathcal{C}_p$, its lower triangular part is in $\mathcal{C}_p$, hence the same is true for X.

For the upper triangular part of X, we argue in the same way, using

$$x_{nm} = \frac{b^{2m}}{b^{2n} + b^{2m}} t_{nm} = \frac{b^{2(m-n)}}{b^{2(m-n)} + 1} t_{nm}.$$

Again the preceeding equality shows that the diagonal of X is in $\mathcal{C}_p$. Thus we have shown that $X \in \mathcal{C}_p$ and $\|X\|_p \leqslant M \cdot \|T\|_p$ for some constant M.

(b) We have now proved that, for any T_1, T_2, T_3, T_4 in $\mathcal{C}_p$, equations (1) and (2) of Lemma 3.4 are (uniquely) solvable and the solutions lie in $\mathcal{C}_p$. It remains to prove that if P solves (1), then P^* and $T_1 - P$ leave the range of B invariant.

Observe that, if X is a solution of (3), then, since $(T-X)B^2 = B^2X$, both $T-X$ and X^* leave the range of B^2 invariant. Thus, by Proposition II.5 of [10] or Corollary 2 of [23], both leave the range of B invariant. Next, we claim that any solution of (4) and its adjoint leave the range of B invariant.

For this, let Y be the solution of $B^2Y + YB^2 = B^2T^*$ (which exists if $T \in \mathcal{C}_p$, as we have just proved). By the result just quoted, Y leaves the range of B invariant; thus (by Douglas' range inclusion theorem) there is a bounded operator Z such that $YB = BZ^*$. But

$$B(B^2Y + YB^2)B = BB^2T^*B$$

so

$$B^3(BZ^*) + B(BZ^*)B^2 = B^3T^*B$$

hence, since B^2 is injective,

$$B^2Z^* + Z^*B^2 = BT^*B$$

therefore, taking adjoints,

$$B^2Z + ZB^2 = BTB.$$

Hence Z is the unique solution of (4), and by construction $ZB = BY^*$, so it leaves the range of B invariant. Since Z^* solves an equation of the same type, it also leaves the range of B invariant, as claimed.

Summarising, given T_1, T_4 in $\mathcal{C}_p$, we find P_1, P_4 in $\mathcal{C}_p$ such that

$$B^2P_1 + P_1B^2 = T_1B^2$$

$$B^2P_4 + P_4B^2 = BT_4B$$

and P_1^*, $T_1 - P_1$, P_4^* and P_4 all leave the range of B invariant. Hence if we define $P = P_1 + P_4$, then P solves the equation

$$B^2P + PB^2 = T_1B^2 + BT_4B$$

and P^*, $T_1 - P$ leave the range of B invariant.

The proof is complete. ■

REMARK. When A is not invertible, we do not know whether the equation $AX + XA = TA$ is solvable in $\mathcal{C}_p$ $(1 < p < +\infty)$ for any $T \in \mathcal{C}_p$, except when A is of a special form (Proposition 3.8). Remark, however, that Theroem 3.2 (combined with Lemma 3.4) gives an indirect proof that this equation is always solvable in $\mathcal{C}_2$ if $T \in \mathcal{C}_2$.

There is a, perhaps unexpected, difference between $\mathcal{A} + \mathcal{S}$ and $\mathcal{A} + \mathcal{S}^*$ regarding decomposability of finite rank operators: Recall (Theorem 2.1) that $\mathcal{F}(\mathcal{A}) + \mathcal{F}(\mathcal{S}) = \mathcal{F}$ if and only if $L + M = H$. Only one direction of this equivalence is valid for $\mathcal{A} + \mathcal{S}^*$.

PROPOSITION 3.9. *Whenever $L + M \neq H$, there exists a rank one operator which is not in $\mathcal{F}(\mathcal{A}) + \mathcal{F}(\mathcal{S}^*)$. The converse is, however, false.*

Proof. We will construct a rank one operator living in the generic part of the space. Thus we may assume that L, M are in generic position, and take $L = \mathrm{Gr}(B)$, $M = \mathrm{Gr}(-B)$ with B positive, injective and non-invertible. Then Proposition 3.5 (ii) shows that there is a rank one operator T such that the equation $Q + B^2QB^2 = T$ has no finite rank solution. But then the rank one operator

$$\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$$

does not decompose as a sum of finite rank operators in $\mathcal{A}$ and $\mathcal{S}^*$, for if it did, then by Lemma 3.4 the equation $Q + B^2QB^2 = T$ would have a finite rank solution.

For the second assertion, we repeat the same argument, this time using an invertible B whose spectrum has limit points in $[0, 1)$. Now $L + M = H$, but Proposition 3.5 (ii) still applies. ■

We conclude this paragraph with the proof of a fact already noted after Theorem 1.3:

PROPOSITION 3.10. *The set $\mathcal{A} + \mathcal{A}^*$ (respectively $\mathcal{S} + \mathcal{S}^*$) never equals $\mathcal{B}(H)$.*

Proof. As usual, it is enough to consider the case $H = H_0 \oplus H_0$, $L = \mathrm{Gr}(B)$, $M = \mathrm{Gr}(-B)$. A short calculation (as in Lemma 3.4) shows that if an operator of the form $T \oplus 0$ is in $\mathcal{A} + \mathcal{A}^*$ then T satisfies

$$TB^2 = PB^2 - B^2P$$

for a suitable operator P on H_0. If B is invertible, the fact that the identity cannot be a commutator ([13], Problem 230) shows that $B^{-2} \oplus 0$ cannot be in $\mathcal{A} + \mathcal{A}^*$. If B is not invertible, then arguing exactly as in Lemma 3.7 we find that the above equation implies that T must have bounded upper triangular part in the sense of Notation 3.6. So if T is the operator constructed in Lemma 3.7, then $T \oplus 0$ is not in $\mathcal{A} + \mathcal{A}^*$. This concludes the proof for $\mathcal{A}$, and the proof for $\mathcal{S}$ is similar. ■

4. COMPACT PERTURBATIONS

Let $\mathcal{B} = \{T \in \mathcal{B}(H): P(L)^\perp TP(L) \in \mathcal{K}, P(M)^\perp TP(M) \in \mathcal{K}\}$. It is clear that $\mathcal{B}$ is a $\|\ \|$-closed subalgebra of $\mathcal{B}(H)$ and $\mathcal{B} \supseteq \mathcal{A} + \mathcal{K}$. Note that $\mathcal{A} + \mathcal{K}$ is also $\|\ \|$-closed. This follows, by a result of [9] (see also [25]), from the existence in $\mathcal{A}$ of a bounded approximate identity (for the strong operator topology) consisting of finite rank operators (see [2] or [20]). If $L + M = H$, then, orthogonalizing L and M with a similarity, it is easy to see that $\mathcal{B} = \mathcal{A} + \mathcal{K}$.

Recall that, for a nest algebra $\mathcal{U}$, the equality $\mathcal{B} = \mathcal{U} + \mathcal{K}$ (with the appropriate modification (see 6.3) in the definition of $\mathcal{B}$) is a consequence of rank one density and the validity of a distance estimate [9] for $\mathcal{U}$. The link between distance estimates and the equality $\mathcal{B} = \mathcal{A} + \mathcal{K}$ again appears in our situation. Indeed, in [24] (see also [17]) it was shown that the validity of a distance estimate is equivalent to $L + M = H$. Here, by a direct argument, we show the following:

THEOREM 4.1. *The equality* $\mathcal{B} = \mathcal{A} + \mathcal{K}$ *is equivalent to* $L + M = H$.

Proof. As remarked above, if suffices to suppose that $L + M \neq H$ and construct an operator in $\mathcal{B}$ which is not in $\mathcal{A} + \mathcal{K}$. Hence we may assume that $L = \mathrm{Gr}(B)$, $M = \mathrm{Gr}(-B)$ where B is not invertible. A short calculation then shows that

$$P(L) = \begin{pmatrix} (I+B^2)^{-1} & B(I+B^2)^{-1} \\ B(I+B^2)^{-1} & B^2(I+B^2)^{-1} \end{pmatrix} = \begin{pmatrix} (I+B^2)^{-1} & 0 \\ 0 & (I+B^2)^{-1} \end{pmatrix} \begin{pmatrix} I & B \\ B & B^2 \end{pmatrix}$$

and

$$P(M) = \begin{pmatrix} (I+B^2)^{-1} & 0 \\ 0 & (I+B^2)^{-1} \end{pmatrix} \begin{pmatrix} I & -B \\ -B & B^2 \end{pmatrix}.$$

Let $\{\Delta_n\}$ be the sequence of spectral projections of $A = B^2$ defined in 3.6. As B is not invertible, there is a subsequence $\{\Delta_{m(n)}\}$ of non-zero Δ_n's. For each $n \in \mathbb{N}$, choose a unit vector x_n in $\Delta_{m(n)}(H)$ such that $\|Bx_n\| > \|B\Delta_{m(n)}\|/2$. Let F be the orthogonal projection on $[x_n : n \in \mathbb{N}]$. Observe that, since $\|Bx_n\| \to 0$, $BF = \sum (Bx_n) \otimes x_n^*$ is compact. Using this, an easy calculation shows that the operator

$$T = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}$$

is in $\mathcal{B}$. If it were the case that $T \in \mathcal{A} + \mathcal{K}$, we would have

$$\begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P & Q \\ BQB & R \end{pmatrix} + \begin{pmatrix} A & C \\ D & E \end{pmatrix}$$

where $BP = RB$ (see 1.1) and A, C, D, E are compact. Thus $B(F - A) = -EB$ and so $BF = BA - EB$. Denoting by X_n the compression of an operator X to $\Delta_{m(n)}$, we obtain

$$B_n F_n = B_n A_n - E_n B_n.$$

Since A and E are compact, we must have $8\|A_n\| < 1$ and $8\|E_n\| < 1$ for large enough n. Hence

$$4\|B_n F_n\| < \|B_n\|.$$

On the other hand, $4\|B_n F_n\| = 4\|Bx_n\| \geqslant 2\|B_n\|$ by construction. This contradiction completes the proof. ■

Given two pairs of subspace (L, M) and (L_1, M_1), it is natural to ask for conditions under which the compact perturbations of the corresponding algebras $\mathcal{A}$ and $\mathcal{A}_1$ are equal.

In the special case where $L = \mathrm{Gr}(B)$, $M = \mathrm{Gr}(-B)$ and $L_1 = \mathrm{Gr}(D)$, $M_1 = \mathrm{Gr}(-D)$ where B, D are in $\mathcal{B}(H_0)$, a necessary condition is that the corresponding projections differ by a compact (Proposition 4.3.i). However, this condition is in general far from sufficient: indeed, in this special case, if $\mathcal{A} + \mathcal{K} = \mathcal{A}_1 + \mathcal{K}$ whenever the corresponding projections differ by a compact, then $L + M$ must be a closed sum (Proposition 4.3.ii).

Perhaps surprisingly, the above condition turns out to be sufficient both for the equality of the larger algebras $\mathcal{B}$ and $\mathcal{B}_1$ and for the smaller algebras $\mathcal{D} + \mathcal{K}$ and $\mathcal{D}_1 + \mathcal{K}$, where $\mathcal{D}$ (resp. $\mathcal{D}_1$) denotes the algebra of "decomposable" operators of $\mathcal{A}$ (resp. $\mathcal{A}_1$) introduced in [17] (Proposition 4.4).

These results mean that, when L, M are in generic position, the algebras $\mathcal{B}$ and $\mathcal{D} + \mathcal{K}$ are invariant under compact perturbations of B, whereas the intermediate algebra $\mathcal{A}+\mathcal{K}$ is not, unless of course B is invertible, in which case the three algebras are equal. (Note that the operator B may be thought of as expressing the "angle" between L and M; indeed, when the Hilbert space is $\mathbb{R}^2$, B is just (multiplication by) the tangent of the half-angle between L and M).

We will use these results to show that, whenever $L + M \neq H$, the inclusions

$$\mathcal{E} + \mathcal{K} \subseteq \mathcal{A} + \mathcal{K} \subseteq \mathcal{B}$$

(where $\mathcal{E}$ is the norm closure of $\mathcal{D}$) are strict inclusions of Banach algebras (Proposition 4.5).

Definition. If $\mathcal{A} = \mathrm{Alg}\{L, M\}$, the algebra $\mathcal{D}$ of decomposable operators is

$$\mathcal{D} = \{T \in \mathcal{A}: T(L) = 0\} + \{T \in \mathcal{A}: T(M) = 0\}.$$

Notation. For the rest of this section, let H_0 be a Hilbert space and $H = H_0 \oplus H_0$. Whenever B is a positive injective contraction on H_0 with $I - B$ injective, we will denote by P_B (resp. P_{-B}) the projection on $\mathrm{Gr}(B)$ (resp. $\mathrm{Gr}(-B)$), by $\mathcal{A}_B$ the algebra on H leaving $\mathrm{Gr}(B)$ and $\mathrm{Gr}(-B)$ invariant, by $\mathcal{D}_B$ the set of decomposable operators in $\mathcal{A}_B$, and by $\mathcal{B}_B$ the algebra of all operators T such that $P_B^\perp T P_B$ and $P_{-B}^\perp T P_{-B}$ are compact.

We will need the following

Lemma 4.2. *With the above notation,*

$$P_B - P_D \in \mathcal{K} \Leftrightarrow B - D \in \mathcal{K} \Leftrightarrow P_{-B} - P_{-D} \in \mathcal{K}.$$

Proof. As remarked in the proof of 4.1,

$$P_B = \begin{pmatrix} (I+B^2)^{-1} & 0 \\ 0 & (I+B^2)^{-1} \end{pmatrix} \begin{pmatrix} I & B \\ B & B^2 \end{pmatrix}.$$

Let $C(B) = (I + B^2)^{-1}$. If $P_B - P_D \in \mathcal{K}$ or $P_{-B} - P_{-D} \in \mathcal{K}$, the $(1,1)$ entry of the equivalent matrix condition gives $C(B) - C(D) \in \mathcal{K}$. Since

$$(*) \qquad C(B) - C(D) = C(D)((I + D^2) - (I + B^2))C(B)$$

it follows that $D^2 - B^2 \in \mathcal{K}$, i.e. that the images of D^2 and B^2 in the Calkin algebra $\mathcal{B}(H)/\mathcal{K}$ are equal. Since D and B are positive, so are their images in the Calkin algebra. But square roots in a C^*-algebra are unique; hence these images must be equal, and so $B - D \in \mathcal{K}$.

If, conversely, $B - D \in \mathcal{K}$, then $(*)$ and similar equalities show that the corresponding matrix entries of $P_B - P_D$ and of $P_{-B} - P_{-D}$ are all in $\mathcal{K}$. ∎

PROPOSITION 4.3. (i) *Let B, D be positive, injective contractions on a Hilbert space H_0 with $I - B$, $I - D$ also injective. If*

$$\mathcal{A}_B + \mathcal{K} = \mathcal{A}_D + \mathcal{K},$$

then $B - D$ is a compact operator.
(ii) *The converse is false. In fact, if B is not invertible, there exists a compact perturbation D of B such that $\mathcal{A}_B + \mathcal{K} \neq \mathcal{A}_D + \mathcal{K}$.*

Proof. (i) Suppose that $\mathcal{A}_B + \mathcal{K} = \mathcal{A}_D + \mathcal{K}$.

This means (by 1.1) that, given P, Q, R in $\mathcal{B}(H_0)$ with $BP = RB$ there must exist X, Y, Z in $\mathcal{B}(H_0)$ with $DX = ZD$ such that

$$\begin{pmatrix} P & Q \\ BQB & R \end{pmatrix} - \begin{pmatrix} X & Y \\ DYD & Z \end{pmatrix} \in \mathcal{K}.$$

In particular, setting $Q = I$, we see that $B^2 - D^2 \in \mathcal{K}$. As in the proof of 4.2, this implies that $B - D \in \mathcal{K}$.
(ii) We assume that $\mathcal{A}_B + \mathcal{K} = \mathcal{A}_D + \mathcal{K}$ whenever $B - D \in \mathcal{K}$ and show that B must be invertible. Suppose, to the contrary, that B is not invertible.

By the Weyl-Von Neumann Theorem [16], there is a diagonalizable operator C and a compact operator K of small norm such that $B = C + K$. Adding to C the diagonal of K (with respect to the basis diagonalizing C), we may assume that B and C have the same diagonals. It is then clear that C is positive, and one may check that the injectivity of B and $I - B$ implies that of C and $I - C$. Since B is not invertible, neither is C, because their spectra can only differ by eigenvalues (Weyl's Theorem, [26] Theorem 0.10). Hence there is an orthonormal sequence $\{e_n\}$ and numbers b_n with $4^{-n-1} \leqslant b_n \leqslant 4^{-n}$ such that $Ce_n = b_n e_n$. Define $D \in \mathcal{B}(H_0)$ by $De_n = \sqrt{b_n} e_n$ and $Dx = Cx$ for $x \in [e_n : n \in \mathbb{N}]^{\perp}$. Then D is positive, non-invertible, and D, $I - D$

are injective. Since $D-C$ is compact, both D and C are equal to B modulo $\mathcal{K}$; hence by assumption we must have

$$\mathcal{A}_C + \mathcal{K} = \mathcal{A}_B + \mathcal{K} = \mathcal{A}_D + \mathcal{K}.$$

Define operators P and R on H_0 by $Pe_n = e_{n+2}$, $Re_n = (b_{n+2}/b_n)e_{n+2}$ and $Px = 0 = Rx$ for $x \in [e_n : n \in \mathbb{N}]^\perp$. Then P and R are bounded and $CP = RC$.

Thus the operator

$$\begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}$$

belongs to $\mathcal{A}_C$, hence it must be a compact perturbation of an operator in $\mathcal{A}_D$. Therefore there must exist compact operators S, T such that $D(P+S) = (R+T)D$, that is $RD - DP = DS - TD$. This gives

$$\langle (DS - TD)e_n, e_{n+2}\rangle = \langle (RD - DP)e_n, e_{n+2}\rangle = (b_{n+2}d_n/b_n) - d_{n+2}$$

where $d_n = \sqrt{b_n}$. On the other hand,

$$\langle (DS - TD)e_n, e_{n+2}\rangle = d_{n+2}\langle Se_n, e_{n+2}\rangle - d_n\langle Te_n, e_{n+2}\rangle$$

hence

$$d_{n+2}\langle Se_n, e_{n+2}\rangle - d_n\langle Te_n, e_{n+2}\rangle = (d_{n+2}^2/d_n) - d_{n+2}$$

or

$$\langle Se_n, e_{n+2}\rangle = c_n^{-1}\langle Te_n, e_{n+2}\rangle + c_n - 1$$

where $c_n = (d_{n+2}/d_n)$, hence $1/8 \leqslant c_n \leqslant 1/2$ for each $n \in \mathbb{N}$. This contradicts the fact that, since S and T are compact operators,

$$\lim_n \langle Se_n, e_{n+2}\rangle = \lim_n \langle Te_n, e_{n+2}\rangle = 0.$$

This contradiction completes the proof. ∎

Proposition 4.4. *Let B, D be positive, injective contractions on a Hilbert space H_0 with $I-B$, $I-D$ also injective. Then the following are equivalent:*

(a) $$\mathcal{B}_B = \mathcal{B}_D$$

(b) $$\mathcal{D}_B + \mathcal{K} = \mathcal{D}_D + \mathcal{K}$$

(c) $$B - D \in \mathcal{K}.$$

Proof. (a)⇔(c): First check (using the form of P_B stated in the proof of 4.1) that an operator

$$T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

is in $\mathcal{B}_B$ if and only if the operators

$$\begin{pmatrix} B^2 & -B \\ -B & I \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} I & B \\ B & B^2 \end{pmatrix}$$

and

$$\begin{pmatrix} B^2 & B \\ B & I \end{pmatrix}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}\begin{pmatrix} I & -B \\ -B & B^2 \end{pmatrix}$$

are both compact. This is easily seen to be equivalent to

$$(*) \qquad \left.\begin{matrix} BX - WB \in \mathcal{K} \\ Z - BYB \in \mathcal{K} \end{matrix}\right\}.$$

Now suppose that $B - D \in \mathcal{K}$. Then

$$DX - WD = (D-B)X - W(D-B) + BX - WB \in \mathcal{K}$$

and

$$Z - DYD = Z - BYB - (D-B)YD - BY(D-B) \in \mathcal{K}$$

so that $T \in \mathcal{B}_D$ (by $(*)$ applied to $\mathcal{B}_D$).

Conversely, if $\mathcal{B}_B = \mathcal{B}_D$, observe that the operator

$$\begin{pmatrix} 0 & I \\ B^2 & 0 \end{pmatrix}$$

belongs to $\mathcal{B}_B$ (in fact to $\mathcal{A}_B$), hence to $\mathcal{B}_D$, and thus, again by $(*)$,

$$B^2 - D^2 = B^2 - DID \in \mathcal{K}.$$

This implies, as in 4.2, that $B - D \in \mathcal{K}$.

(b)$\Leftrightarrow$(c): From the definition of $\mathcal{D}$ it is not hard to see that $T \in \mathcal{D}_B$ if and only if

$$T = \begin{pmatrix} PB & Q \\ BQB & BP \end{pmatrix}$$

where P, Q are arbitrary. Thus the proof of 4.3.(i) already shows that $\mathcal{D}_B + \mathcal{K} = \mathcal{D}_D + \mathcal{K}$ implies $B - D \in \mathcal{K}$.

For the converse, suppose that $B = D + K$ for some $K \in \mathcal{K}$.

Given P, Q in $\mathcal{B}(H_0)$ we must find X, Y in $\mathcal{B}(H_0)$ such that

$$\begin{pmatrix} PB & Q \\ BQB & BP \end{pmatrix} - \begin{pmatrix} XD & Y \\ DYD & DX \end{pmatrix} \in \mathcal{K}.$$

It is enough to set $X = P$ and $Y = Q$, since $PB - PD = PK$, $BP - DP = KP$ and $BQB - DQD = (B-D)QB + DQ(B-D)$ are all compact. ■

Returning to the general situation, if $\mathcal{A} = \mathrm{Alg}\{L, M\}$ it is clear that the decomposable operators form an ideal $\mathcal{D}$ in $\mathcal{A}$. It is known (see [17] and [20]) that $\mathcal{D}$ contains all finite rank operators of $\mathcal{A}$, that its unit ball is strong-operator dense in the unit ball of $\mathcal{A}$, and that $\mathcal{D} = \mathcal{A}$ if and only if $L + M = H$. Observe that, when $L + M \neq H$, $\mathcal{D}$ is not $\|\cdot\|$-closed. Indeed, there exists (see [17]) a compact operator $K \in \mathcal{A}$ which is not decomposable, but is the $\|\cdot\|$-limit of decomposable (in fact, finite rank) operators. Combining 4.3 and 4.4, we can show that, whenever $L + M \neq H$, $\mathcal{D}$ is not $\|\cdot\|$-dense in $\mathcal{A}$, although it is ultraweakly dense (by unit ball density). In fact, we can show more:

PROPOSITION 4.5. *Suppose $L + M \neq H$. Then*

(i) *The $\|\cdot\|$-closure, say $\mathcal{E}$, of $\mathcal{D}$ is a proper ideal of $\mathcal{A}$ containing all compact operators of $\mathcal{A}$.*

(ii) *The algebra $\mathcal{E} + \mathcal{K}$ is $\|\cdot\|$-closed.*

(iii) *The algebra $\mathcal{D} + \mathcal{K}$ is not $\|\cdot\|$-dense in $\mathcal{A} + \mathcal{K}$.*

Proof. Observe first that $\mathcal{E} \supseteq \mathcal{A} \cap \mathcal{K}$. Indeed, as remarked above, $\mathcal{A}$ contains a bounded approximate identity $\{R_i\}$ (for the strong operator topology) consisting of finite rank, hence decomposable, operators. Therefore, if $K \in \mathcal{A} \cap \mathcal{K}$, $\|KR_i - K\| \to 0$ (since $\{R_i\}$ is a bounded net strongly converging to I) and $KR_i \in \mathcal{D}$, hence $K \in \mathcal{E}$.

Now let $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(H)$ be defined by $\Phi_i(T) = R_i T$. Observe that $\sup\|\Phi_i\|$ is finite, that $\Phi_i(\mathcal{B}(H)) \subseteq \mathcal{K}$ and that $\Phi_i(\mathcal{E}) \subseteq \mathcal{D} \subseteq \mathcal{E}$; also, for each $K \in \mathcal{K}$, $\|\Phi_i(K) - K\| \to 0$ as we just observed. A theorem of Rudin ([27]; also see [25]) states that, in this situation, the sum $\mathcal{E} + \mathcal{K}$ is $\|\cdot\|$-closed. This proves (ii).

To show that $\mathcal{D} + \mathcal{K}$ is not $\|\cdot\|$-dense in $\mathcal{A} + \mathcal{K}$, it clearly suffices to assume generic position, hence to let $\mathcal{A} = \mathcal{A}_B$ where B is not invertible. Suppose then that $\mathcal{D}_B + \mathcal{K}$ is $\|\cdot\|$-dense in $\mathcal{A}_B + \mathcal{K}$. By (ii), this means that $\mathcal{A}_B + \mathcal{K} = \mathcal{E}_B + \mathcal{K}$, hence $I \in \mathcal{E}_B + \mathcal{K}$. But for every compact perturbation D of B, we have $\mathcal{D}_B + \mathcal{K} = \mathcal{D}_D + \mathcal{K}$ by 4.4, and thus $\mathcal{E}_B + \mathcal{K} = \mathcal{E}_D + \mathcal{K}$. Hence $I \in \mathcal{E}_D + \mathcal{K}$, which shows, since $\mathcal{E}_D$ is an ideal of $\mathcal{A}_D$, that $\mathcal{E}_D + \mathcal{K} = \mathcal{A}_D + \mathcal{K}$. We conclude that $\mathcal{A}_B + \mathcal{K} = \mathcal{A}_D + \mathcal{K}$ for every compact perturbation D of B. Since B is not invertible, this contradicts 4.3.ii.

The same argument proves that $\mathcal{D}$ is not $\|\cdot\|$-dense in $\mathcal{A}$. ∎

REMARK. Clearly, ultraweak density of the rank one subalgebra of any algebra implies density in the strong operator topology. In fact, it implies that the unit ball of the rank one subalgebra is strongly dense in the unit ball of the algebra. This fact may be known by now to many people; in fact, $(1+\varepsilon)$-ball density (for any $\varepsilon > 0$) may be proved from results in [7]. For the reader's convenience, we include a direct proof that in fact one may achieve $\varepsilon = 0$ for subspaces (not only subalgebras) of $\mathcal{B}(H)$:

PROPOSITION 4.6. *Let $\mathcal{X}$ be an ultraweakly closed subspace of $\mathcal{B}(H)$ whose rank-one subspace $[\mathcal{R}(\mathcal{X})]$ (namely, the linear span of the rank one operators in $\mathcal{X}$) is ultraweakly dense in $\mathcal{X}$. Then $\mathcal{X}$ is (isometrically isomorphic to) the second dual $\mathcal{K}(\mathcal{X})^{**}$ of the compact operators in $\mathcal{X}$. Moreover, the unit ball $[\mathcal{R}(\mathcal{X})]_1$ of the finite rank subspace $[\mathcal{R}(\mathcal{X})]$ is dense in the unit ball $\mathcal{X}_1$ in any of the strong, ultrastrong, weak or ultraweak operator topologies.*

Proof. The restriction of the ultraweak topology $\sigma(\mathcal{B}(H),\mathcal{C}_1)$ to $\mathcal{K}$ is $\sigma(\mathcal{K},\mathcal{C}_1)$, namely the weak topology of the Banach space $\mathcal{K}$. By assumption, then, $[\mathcal{R}(\mathcal{X})]$ is weakly dense in $\mathcal{K}(\mathcal{X})$. Hence, by the Hahn-Banach theorem, $[\mathcal{R}(\mathcal{X})]$ is norm-dense in $\mathcal{K}(\mathcal{X})$. It follows that the annihilator $\mathcal{K}(\mathcal{X})^{\perp}$ of $\mathcal{K}(\mathcal{X})$ in $\mathcal{C}_1$ coincides with the annihilator of $[\mathcal{R}(\mathcal{X})]$, namely

$$\mathcal{K}(\mathcal{X})^{\perp} = \{T \in \mathcal{C}_1 : \mathrm{Tr}(KT^*) = 0\ \forall K \in \mathcal{K}(\mathcal{X})\} = \{T \in \mathcal{C}_1 : \mathrm{Tr}(RT^*) = 0\ \forall R \in \mathcal{R}(\mathcal{X})\}.$$

But this last set is ${}^{\perp}\mathcal{X}$, by the assumed ultraweak density of $[\mathcal{R}(\mathcal{X})]$ in $\mathcal{X}$. Now the Banach space dual $\mathcal{K}(\mathcal{X})^*$ of $\mathcal{K}(\mathcal{X})$ is isometrically isomorphic to $\mathcal{C}_1/(\mathcal{K}(\mathcal{X})^{\perp})$, thus to $\mathcal{C}_1/{}^{\perp}\mathcal{X}$. Therefore the second dual of $\mathcal{K}(\mathcal{X})$ is isometrically isomorphic to $(\mathcal{C}_1/{}^{\perp}\mathcal{X})^*$, which in turn is isometrically isomorphic to $({}^{\perp}\mathcal{X})^{\perp} = \mathcal{X}$. This proves the first assertion.

Using Goldstine's Theorem [5], one may now check that the unit ball of $\mathcal{K}(\mathcal{X})$ is ultraweakly dense in the unit ball of $\mathcal{X}$. But the strong, weak, ultrastrong and ultraweak closures of a convex bounded subset of $\mathcal{B}(H)$ coincide, so $\mathcal{K}(\mathcal{X})_1$ is strong-operator dense in $\mathcal{X}_1$.

Thus, for any $A \in \mathcal{X}$ with $\|A\| \leqslant 1$, any $\varepsilon > 0$ and any finite number of unit vectors $x_1, x_2, \ldots, x_n$ in H, there is a K in $\mathcal{K}(\mathcal{X})$ with $\|K\| \leqslant 1$ such that $\|(K-A)x_i\| < \varepsilon$ for $i = 1, \ldots, n$. By the norm density of $[\mathcal{R}(\mathcal{X})]$ in $\mathcal{K}(\mathcal{X})$, we may choose R in $[\mathcal{R}(\mathcal{X})]$ with $\|R - K\| < \varepsilon$. But then, if $R_1 = R/(1+\varepsilon)$, we have $\|R_1\| \leqslant 1$ and $\|R_1 - K\| < 2\varepsilon$. It follows that

$$\|(R_1 - A)x_i\| \leqslant \|(K-A)x_i\| + \|(R_1 - K)x_i\| \leqslant \|(K-A)x_i\| + \|R_1 - K\| \cdot \|x_i\| < 3\varepsilon$$

for $i = 1, \ldots, n$, which shows that $[\mathcal{R}(\mathcal{X})]_1$, is dense in $\mathcal{X}_1$ in the strong operator topology, hence also in the other three topologies. ∎

REMARK. The results of this paragraph have direct analogues for the subspace $\mathcal{S}$ and its compact perturbations.

5. THE ESSENTIAL COMMUTANT

The essential commutant of a subset $\mathcal{X}$ of $\mathcal{B}(H)$ is

$$\mathcal{C}(\mathcal{X}) = \{T \in \mathcal{B}(H) \colon AT - TA \in \mathcal{K}\ \forall A \in \mathcal{X}\}.$$

Observe that $\mathcal{X}' + \mathcal{K} \subseteq \mathcal{C}(\mathcal{X})$. When $\mathcal{A}$ is a type I von Neumann algebra, the Johnson-Parrott Theorem [14] implies that $\mathcal{C}(\mathcal{A}) = \mathcal{A}' + \mathcal{K}$. In this section, we calculate the essential commutant of $\mathcal{A} = \mathrm{Alg}\{L, M\}$ and of the ideal $\mathcal{D}$ of the decomposable operators of $\mathcal{A}$. When $L + M = H$, $\mathcal{A}$ is similar to a type I von Neumann algebra, hence $\mathcal{C}(\mathcal{A}) = \mathcal{A}' + \mathcal{K}$. We will show that this situation persists when $L + M \neq H$. By contrast, the essential commutant of $\mathcal{D}$ is then strictly larger than $\mathcal{D}' + \mathcal{K}$. This gives another proof of the fact that $\mathcal{D} + \mathcal{K}$ cannot be $\|\cdot\|$-dense in $\mathcal{A} + \mathcal{K}$ unless $\mathcal{D} = \mathcal{A}$, that is, unless $L + M = H$.

A more general result of [19] implies that when $L + M \neq H$ then $\mathcal{A}' = \mathbb{C}I$. In fact, the proof shows more: $\mathcal{D}' = \mathbb{C}I$.

We will show that $\mathcal{C}(\mathcal{D}) = \mathbb{C}I + \mathcal{G}$, where $\mathcal{G}$ is the set defined below. We will need a characterization of $\mathcal{G}$ in terms of graph subspaces. Recall (1.1) that, up to a unitary, we may decompose H as a direct sum

$$H = H_0 \oplus H_0 \oplus (M^\perp \cap L) \oplus (M \cap L^\perp)$$

and write $L = \mathrm{Gr}(B) \oplus (M^\perp \cap L)$ and $M = \mathrm{Gr}(-B) \oplus (M \cap L^\perp)$, where $B \in \mathcal{B}(H_0)$ is a suitable positive injective operator.

PROPOSITION 5.1. *Let*

$$\mathcal{G} = \{X \in \mathcal{B}(H) \colon X|M, X|L, X^*|M^\perp, X^*|L^\perp \text{ are compact}\}.$$

Then an operator X is in $\mathcal{G}$ if and only if there is a K in $\mathcal{K}(H)$ such that, with respect to the above decomposition,

$$X = \begin{pmatrix} 0 & Y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + K$$

where YB and BY are compact operators on H_0.

Proof. It is not difficult to see that an operator of the above form is in $\mathcal{G}$. Indeed, it clearly suffices to consider its compression to the generic part of the space. Then, since $(\mathrm{Gr}(B))^\perp = \{(-Bx, x) \colon x \in H_0\}$ and $(\mathrm{Gr}(-B))^\perp = \{(Bx, x) \colon x \in H_0\}$, one only needs to observe that the mappings

$$(x, \pm Bx) \to (\pm YBx, 0)$$

and

$$(\pm Bx, x) \to (\pm Y^* Bx, 0)$$

are compact.

Conversely, consider an operator X in $\mathcal{G}$ as a 4×4 operator matrix (X_{ij}) with respect to the above decomposition. Since $XP(M^\perp \cap L)$ and $P((M^\perp \cap L)^\perp)X$ are compact operators, it follows that X_{i3} and X_{3i} are compact for $i = 1, 2, 3, 4$. Similarly, the compactness of all X_{i4} and X_{4i} follows from the fact that $XP(L^\perp \cap M)$ and $P((L^\perp \cap M)^\perp)X$ are compact.

Consider finally the compression, say T, of X to the generic part $H_0 \oplus H_0$ of the space, that is the 2×2 upper left-hand corner of X. The fact that T acts compactly on $\mathrm{Gr}(B)$ shows that the mapping

$$(x, Bx) \to (X_{11}x + X_{12}Bx, X_{21}x + X_{22}Bx)$$

is compact, hence so are the mappings

$$x \to X_{11}x + X_{12}Bx$$

and

$$x \to X_{21}x + X_{22}Bx.$$

Similarly, the fact that T acts compactly on $\mathrm{Gr}(-B)$ shows that the mappings

$$x \to X_{11}x - X_{12}Bx$$

and

$$x \to X_{21}x - X_{22}Bx$$

are compact. Combining the above, we see that X_{11}, X_{21}, $X_{12}B$ and $X_{22}B$ are compact operators. Finally, since T^* acts compactly on $(\mathrm{Gr}(B))^\perp$ and $(\mathrm{Gr}(-B))^\perp$ we similarly find that X_{11}^*B, X_{21}^*, X_{12}^*B and X_{22}^* are compact. We conclude that X_{11}, X_{21}, X_{22}, $X_{12}B$ and BX_{12} are compact operators, as required. ∎

REMARK. Observe that the set $\mathcal{G}^2$ consists of compact operators. However, whenever $L + M \neq H$, $\mathcal{G}$ always contains non-compact operators (thus justifying (in view of 5.2 below) our earlier assertion that the essential commutant of $\mathcal{D}$ is strictly larger than $\mathcal{D}' + \mathcal{K}$).

Indeed, if B is not invertible, there is an infinite orthonormal sequence $\{y_n\}$ in H_0 such that $\|By_n\| \to 0$. If Y denotes the orthogonal projection onto $[y_n : n \in \mathbb{N}]$, then Y is not compact but both YB and BY are. (Incidentally, if B happens to be compact, then any bounded operator Y on H_0 will give an element of $\mathcal{G}$.)

THEOREM 5.2. *If $L + M \neq H$, then the essential commutant of the ideal $\mathcal{D}$ of decomposable operators is $\mathcal{G} + \mathbb{C}I$.*

Proof. (i) Suppose that T is in $\mathcal{G}$. If $A \in \mathcal{D}$ satisfies $A(L) = 0$, then $TA = (T|M)A$ is compact. Also, since $A^*(H) \subseteq L^*$, $T^*A^* = (T^*|L^\perp)A^*$ is also compact, hence so is AT. Thus $AT - TA$ is compact. The same is true when $A \in \mathcal{D}$ satisfies $A(M) = 0$. Since any element of $\mathcal{D}$ decomposes as a sum of two operators of the above forms, we have shown that T essentially commutes with $\mathcal{D}$. It follows that $\mathcal{G} + \mathbb{C}I \subseteq \mathcal{C}(\mathcal{D})$.

(ii) For the converse, we first assume that L and M are in generic position. As mentioned earlier, an operator A of the form

$$A = \begin{pmatrix} CB & Q \\ BQB & BC \end{pmatrix}$$

is in $\mathcal{D}$ for all bounded operators Q and C. If

$$T = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

is in $\mathcal{C}(\mathcal{D})$, we must have

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \cdot \begin{pmatrix} CB & Q \\ BQB & BC \end{pmatrix} - \begin{pmatrix} CB & Q \\ BQB & BC \end{pmatrix} \cdot \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in \mathcal{K}$$

which implies

$$(XCB - CBX) + (YBQB - QZ) \in \mathcal{K} \tag{1}$$

$$(XQ - QW) + (YBC - CBY) \in \mathcal{K} \tag{2}$$

for all bounded operators Q and C.

Setting $C = 0$ in (2) gives $(XQ - QW) \in \mathcal{K}$ for all Q and this implies (as first proved by Calkin [4]) that there exists $a \in \mathbb{C}$ such that $X = aI + K_1$ and $W = aI + K_2$ where K_1 and K_2 are compact.

Similarly, setting $Q = 0$ in (2) gives $(YB)C - C(BY) \in \mathcal{K}$ for all C so that $YB = bI + K_3$ and $BY = bI + K_4$ for some scalar b and compact operators K_3, K_4. If $b \neq 0$, this means that B must be Fredholm, hence by Atkinson's Theorem ([5], XI.2.10) Ran(B) must be closed; since it is also dense, B would be onto, hence invertible, contradicting the fact that $L + M \neq H$. Thus $b = 0$, hence YB and BY are compact.

On the other hand, setting $C = 0$ and $Q = I$ in (1) gives $YB^2 - Z \in \mathcal{K}$ which now shows that Z must also be compact.

We have shown that

$$T = \begin{pmatrix} a & Y \\ 0 & a \end{pmatrix} + K$$

where K, YB and BY are compact operators. By Proposition 5.1, T is therefore in $\mathcal{G} + \mathbb{C}I$.

(iii) For the general case, decompose the space $H = L \vee M$ in an orthogonal direct sum

$$H = (L_1 \vee M_1) \oplus (M^{\perp} \cap L) \oplus (M \cap L^{\perp})$$

where $L_1 = L \ominus (M^{\perp} \cap L)$ and $M_1 = M \ominus (M \cap L^{\perp})$ are the generic parts of L and M. Since $L_1 + M_1 \neq L_1 \vee M_1$, the space $L_1 \vee M_1$ must be infinite dimensional. We assume that $M^{\perp} \cap L$ and $M \cap L^{\perp}$ are also infinite dimensional, the other cases being similar and in fact simpler.

With respect to this decomposition, any $A \in \mathcal{D}$ corresponds to a 3×3 operator matrix $A = (A_{mn})$, where the A_{mn} are operators between the appropriate spaces, A_{11} is decomposable in $\mathrm{Alg}\{L_1, M_1\}$ and $A_{23} = A_{32} = 0$. Let $T = (T_{mn})$ be the corresponding decomposition of an operator $T \in \mathcal{C}(\mathcal{D})$.

(a) Since T must essentially commute with all diagonal operators with scalar diagonal entries, it follows that the off-diagonal entries of T must be compact. We may therefore assume that T is block-diagonal.

(b) Since the $(2,2)$ and $(3,3)$ entries of A are arbitrary operators on the appropriate spaces, the corresponding entries of T must be scalar plus compact, and we may therefore assume that they are (possibly different) scalars.

(c) The $(1,1)$ entry T_{11} of T is an operator in the essential commutant of the decomposable operators in $\mathrm{Alg}(L_1, M_1)$. Since L_1 and M_1 are in generic position in their span, part (ii) above implies that T_{11} can be taken to be of the form $aI + W$, where a is a scalar and W is the compression of an element of $\mathcal{G}$.

(d) Thus T can be taken to be of the form

$$T = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} + G,$$

where a, b, $c \in \mathbb{C}$ and $G \in \mathcal{G}$. It remains to prove that $a = b = c$. Since we have shown in part (i) that $\mathcal{G} \subseteq \mathcal{C}(\mathcal{D})$, it follows that

$$S = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

is also in $\mathcal{C}(\mathcal{D})$. But if A is any non-compact operator of the form

$$A = \begin{pmatrix} 0 & X & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $X(M^\perp \cap L) \subseteq L_1$ (for example, X might be a unitary mapping $M^\perp \cap L$ onto L_1), then $A \in \mathcal{D}$ and the fact that $AS - SA$ must be compact forces $a = b$. Similarly, if we choose $A' \in \mathcal{A}$ of the form

$$A' = \begin{pmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for a suitable Y, we obtain $a = c$. Therefore $T \in \mathbb{C}I + \mathcal{G}$ and the proof is complete.■

THEOREM 5.3. *The essential commutant of $\mathcal{A}$ is $\mathcal{A}' + \mathcal{K}$.*

Proof. As pointed out in the introductory remarks, if $L + M = H$ the result follows from the Johnson-Parrott Theorem. Assume therefore that $L + M \neq H$, in which case, as observed earlier, $\mathcal{A}' = \mathbb{C}I$. If T is in $\mathcal{C}(\mathcal{A})$, then a fortiori T is in $\mathcal{C}(\mathcal{D}) = \mathcal{G} + \mathbb{C}I$. Hence, by Proposition 5.1, we may assume that $L = \mathrm{Gr}(B)$, $M = \mathrm{Gr}(-B)$ and T is of the form

$$T = \begin{pmatrix} a & Y \\ 0 & a \end{pmatrix}$$

where $a \in \mathbb{C}$ and YB, BY are compact. It remains to prove that Y itself must be compact. Since

$$\begin{pmatrix} P & 0 \\ 0 & R \end{pmatrix}$$

is in $\mathcal{A}$ whenever $BP = RB$, we must have $YR - PY \in \mathcal{K}$ whenever $BP = RB$.

In particular, $YR - RY \in \mathcal{K}$ for all $R \in \{B\}'$ which implies (by the Johnson-Parrott Theorem, since $\{B\}''$ is abelian) that $Y \in \{B\}'' + \mathcal{K}$. We may thus take $Y \in \{B\}''$. Then YB equals BY and is a compact operator in $\{B\}''$.

If $\{\Delta_n\} \subseteq \{B\}''$ is the sequence of spectral projections of B defined in 3.6 and $Y_n = Y\Delta_n$, then since each B_n is invertible (if nonzero) and $Y_n B_n$ is compact, each Y_n must be compact.

Suppose that Y is not compact. Then the (strong-operator convergent) sum $Y = \sum_n Y_n$ cannot converge in norm, so there exists $\delta > 0$ such that $\|Y_n\| \geqslant 2\delta$ for infinitely many $n \in \mathbf{N}$. Thus for each $n \in \mathbf{N}$ there is an $m(n) \in \mathbf{N}$ such that $m(n) \geqslant n$ and a unit vector x_n in $\Delta_{m(n)}$ with $\|Yx_n\| \geqslant \delta$. Define

$$R = \sum_n x_n \otimes x_{2n}^*.$$

Since

$$\|(B^{-1}x_n)\otimes(Bx_{2n})^*\| \leqslant \|B^{-1}\Delta_{m(n)}\|\cdot\|B\Delta_{2m(n)}\| \leqslant \lambda^{2m(n)-m(n)-1}$$

where $\lambda\in(0,1)$, the sum

$$P\equiv\sum_n(B^{-1}x_n)\otimes(Bx_{2n})^*$$

converges in norm. Therefore P is a compact operator, and clearly $BP = RB$. However $YR-PY$ is not compact, since PY is compact and

$$YR=\sum_n Yx_n\otimes x_{2n}^*$$

is not compact because $\|Yx_n\|\geqslant\delta$ for all $n\in\mathbb{N}$. This contradiction proves that Y must be compact.

Thus $T\in\mathbb{C}I+\mathcal{K}$ as required. ■

6. CONCLUDING REMARKS

REMARK 6.1. If $\mathcal{A}$ is a CSL algebra, the behaviour of $\mathcal{C}_p(\mathcal{A})$ is examined in [1]. It is shown that there exists such an $\mathcal{A}$ whose rank one subalgebra is ultraweakly dense ($\mathcal{A}$ is a "CDCSL algebra"), but $\mathcal{C}_p(\mathcal{A})$ is not a complemented subspace of $\mathcal{C}_p$ for $p\neq 2$.

If $\mathcal{A}=\mathrm{Alg}\{L,M\}$ and $1<p<+\infty$, we have exhibited, in some special cases, an explicit complement of $\mathcal{C}_p(\mathcal{A})$, namely $\mathcal{C}_p(\mathcal{S}^*)$ (Proposition 3.8). Is $\mathcal{C}_p(\mathcal{S}^*)$ a complement of $\mathcal{C}_p(\mathcal{A})$ in general?

Using Lemma 3.4 we may reformulate this question as follows:

Given T and S in $\mathcal{C}_p$ and a positive injective contraction B, does the operator equation

$$B^2X+XB^2=TB^2+BSB$$

have a solution X in $\mathcal{C}_p$? By duality, it is enough to consider the case $1<p<2$ (for $p=2$ the answer is affirmative–see the Remark after Proposition 3.8). Since $\mathcal{C}_p\subseteq\mathcal{C}_2$, the equation then has a unique solution in $\mathcal{C}_2$. Is it in $\mathcal{C}_p$?

More generally, is $\mathcal{A}$ (resp. $\mathcal{K}(\mathcal{A})$, $\mathcal{C}_p(\mathcal{A})$) a complemented subspace of $\mathcal{B}(H)$ (resp. $\mathcal{K}$, $\mathcal{C}_p$)? Note that Theorem 3.3 shows that $\mathcal{S}^*$ (resp. $\mathcal{K}(\mathcal{S}^*)$, $\mathcal{C}_1(\mathcal{S}^*)$) is not a complement.

REMARK 6.2. Given two pairs of subspaces (L,M) and (L_1,M_1), when are the compact perturbations of the corresponding algebras $\mathcal{A}$, $\mathcal{A}_1$ equal? Note that we have been able to answer this question only in special cases (Proposition 4.3).

REMARK 6.3. If $\mathcal{A} \subseteq \mathcal{B}(H)$ is a nest algebra, the quasitriangular algebra $\mathcal{Q}$ is, by definition, the set of all $T \in \mathcal{B}(H)$ for which the function $P \to P^{\perp}TP$ defined on the set $\mathcal{N}$ of invariant projections is compactly valued and (strong operator-norm) continuous. It follows from the existence of a bounded approximate identity in the rank one subalgebra of $\mathcal{A}$ that the algebra $\mathcal{A}+\mathcal{K}$ is norm-closed; this, combined with the validity of a distance estimate for $\mathcal{A}$, shows that $\mathcal{Q} = \mathcal{A}+\mathcal{K}$ [9].

In the case of the two-atom algebra $\mathcal{A} = \mathrm{Alg}\{L, M\}$, our $\mathcal{B}$ precisely corresponds to the quasitriangular algebra, and our Theorem 4.1 shows that the situation is analogous (although $\mathcal{A}$ is in no sense "triangular").

More generally, let $\mathcal{L}$ be a complete atomic Boolean lattice of subspaces of H, let $\mathcal{A} = \mathrm{Alg}\mathcal{L}$ and define $\mathcal{Q}$ as above. The algebra $\mathcal{A}$ contains an abundance of rank one operators, but there are examples ([21], [18]) when the rank one subalgebra is not strong-operator dense. Moreover, distance estimates are not always valid [24]. Thus the investigation of the validity of the equality $\mathcal{Q} = \mathcal{A}+\mathcal{K}$ leads to two distinct questions:

(a) When is $\mathcal{A}+\mathcal{K}$ $\|\cdot\|$-closed?

(b) If $\mathcal{A}+\mathcal{K}$ is $\|\cdot\|$-closed and a distance estimate is valid for $\mathcal{A}$, is it the case that $\mathcal{Q} = \mathcal{A}+\mathcal{K}$?

The interesting case arises when there are L, M in $\mathcal{L}$ such that $L \vee M \neq L + M$.

REFERENCES

1. ANOUSSIS, M.; KATSOULIS, E., Complemented subspaces of $\mathcal{C}_p$ spaces and CSL algebras, *J. London Math. Soc. (2)*, **45**(1992).
2. ARGYROS, S.; LAMBROU, M.S.; LONGSTAFF, W., Atomic boolean subspace lattices and applications to the theory of bases, *Mem. Amer. Math. Soc.*, May 1991.
3. ARVESON, W., Interpolation problems in nest algebras, *J. Functional Analysis*, **20**(1975).
4. CALKIN, J., Two-sided ideals and congruences in the ring of bounded operators in Hilbert space, *Ann. of Math.*, **42**(1941).
5. CONWAY, J., *A course in Functional Analysis*, Springer-Verlag, New York (1985).
6. DAVIDSON, K., Nest Algebras, Pitman Research Notes in *Mathematics Series*, No. 191, England (1988).
7. DAVIDSON, K.; POWER, S., Best approximation in C^*-algebras, *J. Reine Angew. Math.*, **368**(1986).
8. DOUGLAS, R., On majorization, factorisation and range inclusion of operators on Hilbert space, *Proc. Amer. Math. Soc.*, **17**(1966).
9. FALL, T.; ARVESON, W.; MUHLY, P., Perturbations of nest algebras, *J. Operator Theory*, **1**(1979).
10. FOIAŞ, C., Invariant para-closed subspaces, *Indiana Univ. Math. J.*, **21**(1972).
11. GILFEATHER, F.; LARSON, D., Structure in reflexive subspace lattices, *J. London Math. Soc. (2)*, **26**(1982).
12. HALMOS, P., Two Subspaces, *Trans. Amer. Math. Soc.*, **144**(1969).

13. HALMOS, P., *A Hilbert space problem book*, Second Edition, Springer-Verlag, New York (1985).
14. JOHNSON, B.; PARROTT, S., Operators commuting with a von Neumann algebra modulo the set of compact operators, *J. Functional Analysis*, **11**(1972).
15. KATSOULIS, E., *Semisimplicity and triangular integration in operator algebras*, PhD Thesis (in Greek), Athens Univ. (1989).
16. KATO, T., *Perturbation Theory for Linear Operators*, Grund. Math. Wiss. 132, Springer-Verlag, Berlin (1966).
17. KATAVOLOS, A.; LAMBROU, M.S.; LONGSTAFF, W., *The decomposability of operators relative to two subspaces*, Studia math., **105**(1993).
18. KATAVOLOS, A.; LAMBROU, M.S.; PAPADAKIS, M., On some algebras diagonalized by M-bases of l^2, *Integral Equations Operator Theory*, **17**(1993).
19. LAMBROU, M.S., Approximants, commutants and double commutants in normed algebras, *J. London Math. Soc.*, (2) **25**(1982).
20. LAMBROU, M.S.; LONGSTAFF, W., Unit ball density and the operator equation $AX = YB$, *J. Operator Theory*, **25**(1991).
21. LARSON, D.; WOGEN, W., Reflexivity properties of $T \oplus 0$, *J. Functional Analysis*, **92**(1990).
22. LONGSTAFF, W., Operators of rank one in reflexive algebras, *Canad. J. Math.*, **28**(1976).
23. NORDGREN, E.; RADJABALIPOUR, M.; RADJAVI, H.; ROSENTHAL, P., On invariant operator ranges, *Trans. Amer. Math. Soc.*, **251**(1979).
24. PAPADAKIS, M., On hyperreflexivity and rank one density for non-CSL algebras, *Studia Math.*, (1) **98**(1991).
25. POWER, S., Analysis in nest algebras, in: CONWAY, J.; MORREL, B. (eds.), *Surveys of some recent results in Operator Theory*, Vol. II, Pitman Research Notes in Mathematics Series, no. 192, England (1988).
26. RADJAVI, H.; ROSENTHAL, P., *Invariant Subspaces*, Ergebnisse 77, Springer-Verlag, Berlin (1973).
27. RUDIN, W., Spaces of type $H^\infty + C$, *Ann. Inst. Fourier (Grenoble)*, **25**(1975).

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Added in proof. V. S. Shul'man recently informed us that he has independently obtained a result related to Corollary 1.4.