J. OPERATOR THEORY
31(1994), 207–218

OPERATOR SPACE STRUCTURES AND THE SPLIT PROPERTY

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1. INTRODUCTION

Recently the split property for inclusions of von Neumann algebras and its relations with the nuclearity condition of suitable maps was studied and relevant applications was given (see [1–8]).

Given an inclusion $A \subset B$ of factors with B acting standardly on the Hilbert space $\mathcal{H}$ with cyclic and separating vector, the nuclearity of the maps

$$\Phi_1 : a \in A \mapsto (\Omega, Ja\Omega) \in B_*; \quad \Phi_2 : a \in A \mapsto \Delta_B^{\frac{1}{4}} a\Omega \in \mathcal{H}$$

assures the split property for $A \subset B$. Conversely, if $A \subset B$ is a split inclusion, the above maps are nuclear for a dense set of cyclic separating vectors for B, see [1]. Hence it can be of interest to characterize the split property for $A \subset B$ in terms of the properties of the above maps constructed via a fixed cyclic separating vector for B. Moreover, in several papers [9–13], the general theory of the operator spaces was developed and, in [10], an explicit description of the predual of a W^*-tensor product was given.

In this paper we characterize a class of linear maps between operator spaces, the matrix-decomposable operators, that seem to be the natural substitutes, in the operator space context, of the nuclear operators in the Banach space context. This characterization furnishes an equivalent condition for the split property in terms of the above L^1-embedding associated with a fixed cyclic separating vector for B.

For the reader's convenience we collect some preliminary results about the operator spaces contained principally in [10] of which we need in the following. Details and proofs can be found in [9–12].

For an arbitrary normed space X, X_1 denotes its (closed) unit ball. We consider a normed space V together with a sequence of norms $\| \ \|_n$ on $\mathbf{M}_n(V)$, the space of $n \times n$ matrices with entries in V. For $a, b \in \mathbf{M}_n$ these norms verify:

$$(1.1)\quad \begin{array}{ll} \text{i)} & \|avb\|_n \leqslant \|a\|\,\|b\|\,\|v\|_n; \\ \text{ii)} & \|v_1 \oplus v_2\|_{n+m} = \max\{\|v_1\|_n, \|v_2\|_m\} \end{array}$$

(the above products are the usual row-column ones and make $\mathbf{M}_n(V)$ a bimodule on $\mathbf{M}_n$). This space with the above norms is called an (abstract) operator space.

If $T : V \mapsto W$, $T_n : \mathbf{M}_n(V) \mapsto \mathbf{M}_n(W)$ are defined by $T_n(a)_{ij} = T(a_{ij})$. T is said to be completely bounded if $\sup \|T_n\| = \|T\|_{CB} < +\infty$; $\mathcal{M}(V, W)$ denotes the set of such maps between V, W. Complete contractions and complete isometries have an obvious meaning. It is an important fact (see [12]) that a linear space V with norms on each $\mathbf{M}_n(V)$ has a realization as a concrete operator space i.e. a subspace of a C^*-algebra, if and only if these norms verify the properties in (1.1).

Given an operator space V and $f = \mathbf{M}_n(V^*)$, the norms

$$(1.2)\quad \begin{array}{c} \|f\|_n = \sup\{\|(f(v))_{(i,k)(j,l)}\| : v \in \mathbf{M}_m(V)_1,\ m \in \mathbb{N}\}; \\ (f(v))_{(i,k)(j,l)} = f_{ij}(v_{kl}) \in \mathbf{M}_{mn} \end{array}$$

determines an operator structure on V^* that becomes itself an operator space.

Given operator spaces V, W, $V \otimes_\wedge W$ and $V \otimes_\vee W$ denote respectively the algebraic tensor product with norms for $u \in \mathbf{M}_n(V \otimes W)$ given by

$$\|u\|_\wedge = \inf\{\|\alpha\|\,\|v\|\,\|w\|\,\|\beta\|\}$$

where the infimum is taken on all the decompositions

$$u = \sum \alpha_{ik} v_{ij} \otimes w_{kl} \beta_{jl}$$

with $\alpha \in \mathbf{M}_{n,pq}$, $v \in \mathbf{M}_p(V)$, $w \in \mathbf{M}_q(W)$, $\beta \in \mathbf{M}_{pq,n}$; and $\|u\|_\vee$ the norms determined by the inclusion $V \otimes W \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ if $V \subset \mathcal{B}(\mathcal{H})$, $W \subset \mathcal{B}(\mathcal{K})$; the last caracterization does not depend on the specific realization of V, W as concrete operator spaces. The completions of these tensor products are denoted respectively by $V \widehat{\otimes} W$, $V \check{\otimes} W$ and are referred as projective and spatial tensor product; these tensors are themselves operator spaces (see [9–11]). Moreover one can completely and isometrically identify $V^* \otimes_\vee W$ with the finite rank maps in $\mathcal{M}(V, W)$ and, as W is complete, we have the inclusion $V^* \check{\otimes} W \subset \mathcal{M}(V, W)$ ($\mathcal{M}(V, W)$ is itself an operator space; see [10]).

We now consider the linear space $\mathbf{M}_I(V)$ for any index I as the $I \times I$ matrices with entries in V such that

$$\|v\| \equiv \sup_{\Delta} \|v^{\Delta}\| < +\infty$$

(Δ denotes an arbitrary finite subset of I). For every index set I, $\mathbf{M}_I(V)$ is in a natural manner an operator space via inclusion $\mathbf{M}_I(V) \subset \mathcal{B}(\mathcal{H} \otimes \ell^2(I))$ if V is realized as a subspace of $\mathcal{B}(\mathcal{H})$. Of interest is also the definition of $\mathbf{K}_I(V)$ as those elements $v \in \mathbf{M}_I(V)$ such that $v = \lim_{\Delta} v^{\Delta}$. Obviously $\mathbf{M}_I(\mathbb{C}) \equiv \mathbf{M}_I = \mathcal{B}(\ell^2(I))$ and $\mathbf{K}_I(\mathbb{C}) \equiv \mathbf{K}_I = \mathcal{K}(\ell^2(I))$, the set of all the compact operators on $\ell^2(I)$. For V complete we remark the bimodule property of $\mathbf{M}_I(V)$ over $\mathbf{K}_I$ because, for $\alpha \in \mathbf{K}_I$, $\alpha^{\Delta} v$, $v\alpha^{\Delta}$ are Cauchy nets in $\mathbf{M}_I(V)$ and its limits define unique elements αv, $v\alpha$ that can be calculated via the usual row-column product.

We conclude these preliminaires with two results for complete operator spaces contained in [10, sect.3]. We indicate with ∞ any index set with denumerable cardinality.

PROPOSITION 1.1. *$u \in \mathbf{M}_n(V\widehat{\otimes}W)$ satisfies $\|u\|_\wedge < 1$ iff*

$$u = \alpha(v \otimes w)\beta \tag{1.3}$$

where α, β are $n \times \infty^2$, $\infty^2 \times n$ complex matrices; $v \in \mathbf{M}_\infty(V)$, $w \in \mathbf{M}_\infty(V)$ and $\|\alpha\|\,\|v\|\,\|w\|\,\|\beta\| < 1$. One can furthermore choose $v \in \mathbf{K}_\infty(V)$, $w \in \mathbf{K}_\infty(W)$.

The simbol $v \otimes w$ above indicates the $\infty^2 \times \infty^2$ bounded matrix with entries in $V \otimes_\wedge W$ defined by

$$(v \otimes w)_{(i,k)(j,l)} = v_{ij} \otimes w_{kl}. \tag{1.4}$$

Because of the above discussion about the bimodule property of $\mathbf{M}_I(V)$ and the fact that $\alpha, \beta \in \mathbf{K}_\infty$, one can write

$$u_{rs} = \sum \alpha_{r,ik} v_{ij} \otimes w_{kl} \beta_{jl,s} \tag{1.5}$$

where the sum is unconditionally convergent in $\mathbf{M}_n(V\widehat{\otimes}W)$.

The following important result concern the description of the predual of a W^*-tensor product in terms of the preduals of its individual factors.

THEOREM 1.2. *let A, B be von Neumann algebras. The predual $(A\overline{\otimes}B)_*$, with its operator space structure, is completely isomorphic to $A_*\widehat{\otimes}B_*$.*

The detailed proof of the above results can be found in [10, sect. 3].

2. THE MATRIX-DECOMPOSABLE OPERATORS

We characterize a class of completely bounded operators which seems to play an analogous role to that of the nuclear maps in the Banach space context. To simplify we only deal with complete operator spaces.

DEFINITION 2.1. A linear map $T : V \mapsto W$ is called *matrix-decomposable* if we can find matrices $\alpha \in \mathbf{M}_{1,\infty^2}$, $\beta \in \mathbf{M}_{\infty^2,1}$, $f \in \mathbf{M}_\infty(V^*)$, $w \in \mathbf{M}_\infty(W)$ such that

$$Tv = \alpha(f(v) \otimes w)\beta \tag{2.1}$$

We indicate by $\mathcal{D}(V,W)$ the set of such maps and define on $\mathcal{D}(V,W)$ a norm given by

$$\|T\|_\delta = \inf\{\|\alpha\| \, \|f\| \, \|w\| \, \|\beta\|\}$$

where the infimum is taken on all the decompositions (2.1) for T.

Following the discussion contained in the introduction, one has for T a summation analogous to that in (1.5). It is easy to see that such summation is unconditionally convergent in the norm topology of $\mathcal{B}(V,W)$ (Indeed one can easily see that such sum converge unconditionally in $(\mathcal{D}(V,W), \| \; \|_\delta)$. Moreover, if T has the form (2.1), there exists $\omega = \alpha(f \otimes w)\beta \in V^*\hat{\otimes}W$ such that $T = I(\omega) \in V^*\hat{\otimes}W \subset \mathcal{M}(V,W)$ where I is the mapping given in [10, (4.3)] and T is completely bounded.

We have the following

PROPOSITION 2.2. *$(\mathcal{D}(V,W)), \| \; \|_\delta)$ is a Banach space.*

Proof. If $T \in \mathcal{D}(V,W)$, one has $\|T\| \leqslant \|\alpha\| \, \|f\| \, \|w\| \, \|\beta\|$ and, taking the infimum on the right, one obtains $\|T\| \leqslant \|T\|_\delta$ and $\|T\|_\delta$ is nondegenerate. We now prove that an absolutely summable sequence $\{T_n\} \subset (\mathcal{D}(V,W), \| \; \|_\delta)$ is norm summable. Let $\{T_n\}$ an absolutely summable sequence with $T_n = \alpha_{(n)}(f_{(n)} \otimes w_{(n)})\beta_{(n)}$ and, for fixed $\varepsilon > 0$, $\|\alpha_{(n)}\|\|f_{(n)}\|\|w_{(n)}\|\|\beta_{(n)}\| \leqslant \|T_n\|_\delta + \frac{\varepsilon}{2^n}$; we can choose the above decomposition such that $\|\alpha_{(n)}\| = \|\beta_{(n)}\|$, $\|f_{(n)}\| = \|w_{(n)}\| = 1$; moreover $\sum_n \|T_n\|_\delta < +\infty$.

We consider the row matrix $\alpha_{ij} = \alpha_{(i)}\delta_{ij}$ with entries in $\mathbf{M}_{\infty^2}$ and, as $\mathbf{M}_{\infty^2}(\mathbf{M}_{\infty^2}) \cong \mathbf{M}_{\infty^2}$, it is itself a row-matrix in $\mathbf{M}_{\infty^2}$. Analogously the matrices $f_{ik} = f_{(i)}\delta_{ik}$, $w_{jl} = w_{(j)}\delta_{jl}$, $\beta_{kl} = \beta_{(k)}\delta_{kl}$ are respectively matrices in $\mathbf{M}_\infty(V^*)$, $\mathbf{M}_\infty(W)$ and column-matrix in $\mathbf{M}_{\infty^2}$. One easily verifies that $T = \sum_n T_n = \alpha(f \otimes w)\beta \in \mathcal{D}(V,W)$ and $\|T\|_\delta \leqslant \|\alpha\| \, \|\beta\| = \sum_{n;r,s} |(\alpha_{(n)})_{rs}|^2 \leqslant \sum_n(\|T_n\|_\delta + \frac{\varepsilon}{2^n})$. Hence we obtain, for arbitrary $\varepsilon > 0$, $\|T\|_\delta \leqslant \sum_n \|T_n\|_\delta + \varepsilon$. ∎

We now consider the linear space $\mathbf{M}_n(\mathcal{D}(V,W))$ and note that if $T \in \mathbf{M}_n(\mathcal{D}(V,W))$, T has a decomposition as

$$Tv = \alpha(f(v) \otimes w)\beta \tag{2.2}$$

where $\alpha \in \mathbf{M}_{n,\infty^2}$ and $\beta \in \mathbf{M}_{\infty^2,n}$, f, w are as above. We can define in a natural manner the norms on $\mathbf{M}_n(\mathcal{D}(V,W))$:

$$\|T\|_\delta = \inf\{\|\alpha\| \, \|f\| \, \|w\| \, \|\beta\|\}$$

where the infimum is taken on all the decompositions (2.2).

A natural mapping

$$\mathcal{X} : \mathbf{M}_n(V^*\hat{\otimes}W) \mapsto \mathbf{M}_n(\mathcal{D}(V,W))$$

arises from our definition; this mapping is a complete surjection, furthermore we have

THEOREM 2.3. *$\mathcal{D}(V,W)$ with the above norms is a complete operator space and $\mathcal{X}$ is a complete quotient mapping.*

Proof. Obviously $\|\mathcal{X}(u)\|_\delta \leqslant \|u\|_\wedge$; however the same proof of that of theorem 3.1 of [9] tell us that $\mathcal{D}(V,W)$ is a complete operator space. We now verify that the δ-norms are the quotient norms on all $\mathbf{M}_n(\mathcal{D}(V,W))$'s. Let $\varepsilon > 0$ be fixed and suppose that $\|T\|_\delta + \varepsilon < \inf\{\|u\|_\wedge : \mathcal{X}(u) = T\}$ holds for some $T \in \mathbf{M}_n(\mathcal{D}(V,W))$. Then there exists $u = \alpha(f \otimes w)\beta \in \mathbf{M}_n(V^*\hat{\otimes}W)$ such that $\mathcal{X}(u) = T$ and $\|\alpha\|\ \|f\|\ \|w\|\ \|\beta\| \leqslant \|t\|_\delta + \varepsilon < \|u\|_\wedge \leqslant \|\alpha\|\ \|f\|\ \|w\|\ \|\beta\|$ but this is a contradiction. ■

One cannot conclude nothing about the injectivity of $\mathcal{X}$ but, if W verifies the (operator) approximation property then

$$\mathcal{X} : \mathbf{M}_n(V^*\hat{\otimes}W) \mapsto \mathbf{M}_n(\mathcal{D}(V,W))$$

is injective for every operator space V (see [10, sect. 4]).

Besides one also has the so called ideal property for $\mathcal{D}$.

PROPOSITION 2.4. *Let X, Y, Z be operator spaces and $T_1 : X \mapsto Y$, $T_2 : Y \mapsto Z$ be linear mappings.*

i) *If $T_1 \in \mathcal{M}(X,Y)$, $T_2 \in \mathcal{D}(Y,Z)$ then $T_2T_1 \in \mathcal{D}(X,Z)$ and*

$$\|T_2T_1\|_\delta \leqslant \|T_2\|_\delta \|T_1\|_{CB};$$

ii) *If $T_1 \in \mathcal{D}(X,Y)$, $T_2 \in \mathcal{M}(Y,Z)$ then $T_2T_1 \in \mathcal{D}(X,Z)$ and*

$$\|T_2T_1\|_\delta \leqslant \|T_2\|_{CB} \|T_1\|_\delta.$$

Proof. a). There exists scalar matrices α, β and $f \in \mathbf{M}_\infty(Y^*)$, $z \in \mathbf{M}_\infty(Z)$ such that $T_2y = \alpha(f(y) \otimes z)\beta$ and $\|f\| \leqslant \|T_2\|_\delta + \frac{\varepsilon}{\|T_1\|_{CB}}$ (ε fixed) but $T_2T_1x = \alpha(f(T_1x) \otimes z)\beta$ where $f \circ T_1 \in \mathbf{M}_\infty(X^*)$ with $\|f \circ T_1\| \leqslant \|f\|\ \|T_1\|_{CB}$ (see [10, (3.2)]). We obtain $T_2T_1 \in \mathcal{D}(X,Z)$ and $\|T_2T_1\|_\delta \leqslant \|f \circ T_1\| \leqslant \|f\|\ \|T_1\|_{CB} \leqslant \|T_2\|_\delta \|T_1\|_{CB} + \varepsilon$. The proof of ii) is similar if we note that $T_2y \in \mathbf{M}_\infty(Z)$ for $y \in \mathbf{M}_\infty(Y)$ and $\|T_2y\| \leqslant \|T_2\|_{CB}\|y\|$. ■

In the light of the above considerations, $\mathcal{D}(V,W)$ seems to play a similar role to that of the nuclear operators in the Banach space context and one can prove the following

PROPOSITION 2.5. *Let V, W be operator spaces and $T : V \mapsto W$ a nuclear map with $\|T\|_\nu$ its nuclear norm (as an operator between Banach spaces, see [14]).*

Then $T \in \mathcal{D}(V,W)$ and $\|T\|_\delta \leqslant \|T\|_\nu$.

Proof. Let $\varepsilon > 0$ be fixed. Then there exists nonzero functionals $\{f_n\} \subset (V^*)_1$, nonzero vectors $\{w_n\} \subset W_1$ and a ℓ^1-positive sequence $\{\lambda_n\}$ such that $Tv = \sum_n \lambda_n f_n(v) w_n$ and $\sum_n \lambda_n \leqslant \|T\|_\nu + \varepsilon$. We define $\alpha_{ij} = \lambda_i^{\frac{1}{2}}\delta_{ij}$, $f_{ik} = f_i\delta_{ik}$, $w_{jl} = w_j\delta_{jl}$, $\beta_{kl} = \lambda_k^{\frac{1}{2}}\delta_{kl}$. One can easily verify that the above matrices give a decomposition for T as $Tv = \alpha(f(v)\otimes w)\beta$ and $\|T\|_\delta \leqslant \|\alpha\|\,\|\beta\| = \sum_n \lambda_n \leqslant \|T\|_\nu + \varepsilon$.

∎

Finally, analogously to the nuclear operator setting, we give a suitable geometrical description for the range of a matrix-decomposable injective operator. We start with an absolutely convex set Q in an operator space V and we indicate with E its algebric span. We consider a sequence $\mathfrak{q} \equiv \{Q_n\}$ of sets such that

i) $Q_1 \equiv Q$ and every Q_n is an absolutely convex absorbing set of $\mathbf{M}_n(E)$ with $Q_n \subset \mathbf{M}_n(Q)$;

ii) $Q_{m+n} \cap (\mathbf{M}_m(E) \oplus \mathbf{M}_n(E)) = Q_m \oplus Q_n$;

iii) for $x \in Q_n$ then $x \in \lambda Q_n$ implies $bx \in \lambda Q_n, xb \in \lambda Q_n$ where $b \in (\mathbf{M}_n)_1$.

We say that a (possibly) infinite matrix F with entries in the algebric dual of E has finite $\mathfrak{q}$-norm if

$$\|F\|_{\mathfrak{q}} \equiv \sup\{\|F^\Delta(q)\| : q \in Q_n;\ n \in \mathbf{N};\ \Delta\} < +\infty \tag{2.3}$$

where F^Δ indicates an arbitrary finite truncation corresponding to Δ; the numerical matrix $F^\Delta(q)$ has entries as those in (1.2).

DEFINITION 2.6. An absolutely convex set $Q \subset V$ is said to be $\mathfrak{q}$-*decomposable* (where $\mathfrak{q}$ is a fixed sequence as above) if there exists matrices α, β as in definition 2.1, an infinite matrix F of linear functionals as above with $\|F\|_{\mathfrak{q}} < +\infty$ and $v \in \mathbf{M}_\infty(V)$ such that every $q \in Q$ can be written as

$$q = \alpha(F(q)\otimes v)\beta. \tag{2.4}$$

We note that the sum in (2.4) is unconditionally convergent in V.

One can easily see that a q-decomposable set is relatively compact, hence bounded in the norm topology of V and therefore E, together the Minkowski norms determinated by the Q_n's on $\mathsf{M}_n(E)$, is a (not necessarily complete) operator space and the canonical immersion $E \mapsto V$ is matrix-decomposable. We now consider an injective completely bounded operator $T : V \mapsto W$ and the sequence $\mathfrak{q}_T$ given by

$$\mathfrak{q}_T = \{T_n(\mathsf{M}_n(V)_1)\};$$

for such sequences the properties i)–ii) are automatically verified and if $T(V_1)$ is $\mathfrak{q}_T$-decomposable we call it simply T-decomposable and indicate the $\mathfrak{q}_T$-norm of F by $\|F\|_T$.

THEOREM 2.7. *Let V, W operator spaces and $T :\mapsto W$ a completely bounded injective operator. $T \in \mathcal{D}(V, W)$ iff $T(V_1)$ is a T-decomposable set in W.*

Proof: It is easy to verify that, if $T \in \mathcal{D}(V, W)$ is injective, one can write for T a decomposition

$$Tv = \alpha(f(v) \otimes w)\beta$$

and α, $f \circ T^{-1}$, w, β furnish a decomposition for $T(V_1)$. Conversely, if $T(V_1)$ is a T-decomposable set in W, with

$$q = \alpha(F(q) \otimes w)\beta;\ q \in T(V_1),$$

then α, $F \circ T$, w, β give, as usual, a decomposition for T and $T \in \mathcal{D}(V, W)$. ∎

The definition of a decomposable set may appear rather involved; this is due to the fact that the inclusion $\mathsf{M}_n(V)_1 \subset \mathsf{M}_n(V_1)$ is strict in general; but, for an injective completely bounded operator T as above, the T-decomposable set $T(V_1)$ is intrinsecally defined directly via T.

3. AN EQUIVALENT CONDITION FOR THE SPLIT PROPERTY

We consider an inclusion $A \subset B$ of factors with separable preduals where B acts standardly on $\mathcal{H}$ with cyclic separating vector Ω with Δ, J its modular operator and conjugation relative to Ω. Together with B we consider the opposite algebra B^0 of B (the appearence of the opposite algebra B^0 is well explained in [1, sect. 1] in the correspondence setting) which is $*$-isomorphic to B' via

$$x \in B^0 \mapsto j(x) \equiv Jx^*J \in B'; \tag{3.1}$$

the inverse of (3.1) is given in the same way where B^0 is identified with B as a linear space.

Following [1], we give an equivalent condition for the split property for the inclusion $A \subset B$ in terms of the natural embedding

$$\Phi_1 : b \in B \equiv L^\infty(B) \mapsto (\Omega, Jb\Omega) \in L^1(B) \tag{3.2}$$

where we identify in a natural manner $L^1(B)$ with $(B^0)_*$. The following other embedding is also of particular interest (see [2])

$$\Phi_2 : b \in B \mapsto \Delta^{\frac{1}{4}} b\Omega \in L^2(B) \equiv \mathcal{H}. \tag{3.3}$$

We start with the following

PROPOSITION 3.1. *The mapping Φ_1 in (3.2) is completely bounded as a map between B and $(B^0)_*$.*

Proof. With the identification (3.1) the map becomes $\Phi_1(a)(j(b)) = (ab\Omega, \Omega)$ and $\|\Phi_1(a)(j(b))\| = \|\omega_{pq}(c)\|$ with $a \in \mathbf{M}_p(B)$, $b \in \mathbf{M}_q(B')$, and c has entries $c_{(i,k)(j,l)} = a_{ij} b_{kl}$. As $\omega = (\cdot\Omega, \Omega)$ is completely positive hence completely bounded with $\|\omega\|_{CB} = 1$ (see [13, prop. 3.5]), we obtain

$$\|\Phi_1(a)(j(b))\| \leqslant \|a\| \, \|b\|.$$

Taking the supremum on the unit balls of $\mathbf{M}_p(B)$, $\mathbf{M}_q(B')$ and on $p, q \in \mathbf{N}$ we have done. ∎

Let M be a von Neumann algebra, X a normed space and $F \subset X^*$ a separating set of functionals. A continuous linear map $T : M \mapsto X$ is said to be normal (singular) according to the functionals $f \circ T$ on M are normal (singular) for every $f \in F$ (for the definition of singular functional on a von Neumann algebra see [15]). It is easy to see that the unique map which is at the same time normal and singular is the zero mapping. In the above discussion, if X is also a von Neumann algebra and F is its predual, the normal maps are just those ultraweakly continuous and we have for arbitrary bounded map T the decomposition given in [15, page 128] in its normal and singular parts

$$T = T^n + T^s$$

and

$$\|T^n\|, \ \|T^s\| \leqslant \|T\| \leqslant \|T^n\| + \|T^s\|. \tag{3.4}$$

As $\mathbf{M}_I(V^*)$ can be identified with $\mathcal{M}(V, \mathbf{M}_I)$ for arbitrary index set I (see [10]), we can decompose $f \in \mathbf{M}_\infty(M^*)$ (M a von Neumann algebra) in its normal and singular parts f^n, f^s given by the normal and singular parts of its entries. One has

$$(f^n)_{ij} = f^n_{ij}, \ (f^s)_{ij} = f^s_{ij} \tag{3.5}$$

and the inequalities (3.4) for f (hence $f^n \in \mathsf{M}_\infty(M_*)$).

The following lemma is analogous to the lemma 2.1 of [1].

LEMMA 3.2. *Let M be a von Neumann algebra and V an operator space. If $T \in \mathcal{D}(M,V)$ is a normal map with respect to V^*, then T has a decomposition*

$$Ta = \alpha(f(a) \otimes v)\beta$$

where α is a row matrix, β is a column matrix as in Definition 2.1; $f \in \mathsf{M}_\infty(M_*)$, $v \in \mathsf{M}_\infty(V)$ and $\|\alpha\|\,\|f\|\,\|v\|\,\|\beta\|$ is arbitrary close to the δ-norm of T.

Proof. Suppose that $Ta = \alpha(f(a) \otimes v)\beta$ where α, v, β are as above and $f \in \mathsf{M}_\infty(M^*)$ with $\|f\| \leqslant \|T\|_\delta + \varepsilon$. We can decompose f in its normal and singular parts f^n, f^s; then we have

$$Ta = \alpha(f^n(a) \otimes v)\beta + \alpha(f^s(a) \otimes v)\beta$$

where the above summands are well defined, respectively normal and singular, operators in $\mathcal{D}(M,V)$ via (3.4). We than have that $T - \alpha(f^n(\cdot) \otimes v)\beta$ is, at the same time, normal and singular and must be the zero mapping.

Therefore

$$T = \alpha(f^n(\cdot) \otimes v)\beta$$

and $\|T\|_\delta \leqslant \|f^n\| \leqslant \|f\| \leqslant \|T\|_\delta + \varepsilon$. ■

We remark that the above lemma also works for $T \in \mathsf{M}_n(\mathcal{D}(M,V))$.

We recall that the inclusion $A \subset B$ of von Neumann algebras is said to be split if there exists a type I factor N such that $A \subset N \subset B$.

We are now able to prove the announced equivalent condition for the split property.

THEOREM 3.3. *Let $A \subset B$ be an inclusion of factors with separable preduals and ω a faithful normal state of B. Let $\Phi_1 : B \mapsto (B^o)_*$ be the mapping associated to ω given in (3.2). The following statements are equivalent.*

i) *$A \subset B$ is a split inclusion.*

ii) *$\Phi_{1|A} : a \in A \mapsto (\cdot\Omega, Ja\Omega) \in (B^o)_*$ is a matrix-decomposable map i.e. $\Phi_{1|A} \in \mathcal{D}(A,(B^o)_*)$.*

Proof. i)⇒ii)). As $A \subset B$ is a split inclusion, $A \vee B'$ naturally isomorphic to $A \bar{\otimes} B'$ (see [3]) and $\Phi_1(a)(j(b)) = (ab\Omega, \Omega)$ uniquely extends to a normal state of $A \bar{\otimes} B'$. As B' is $*$-isomorphic (hence completely isometric) to B^0, by Theorem 1.2 there will exist α, β matrices as in Definition 2.1 and $f \in \mathsf{M}_\infty(A_*)$, $g \in \mathsf{M}_\infty((B^0)_*)$ such that $\Phi_1(a)(j(b)) = \alpha(f(a) \otimes g(j(b)))\beta$. Hence $\Phi_{1|A} \in \mathcal{D}(A,(B^o)_*)$. ii)⇒i).

Suppose that $\Phi_{1|A} \in \mathcal{D}(A,(B^o)_*)$. By Lemma 3.2 $\Phi_1(a)(j(b)) \equiv \omega(a \otimes b)$ defines a normal state on the W^*-algebra $A\overline{\otimes}B^0$. Let $\tilde{A} \supset A$ be a von Neumann algebra with separable predual containing A and consider a normal state $\tilde{\omega}$ of $\tilde{A}\overline{\otimes}B^0$ which extends ω. The map $\tilde{\Phi}_1 : \tilde{A} \mapsto (B^0)_*$ given by $\tilde{\Phi}_1(a) = \tilde{\omega}(a\otimes\cdot)$ is a completely positive normal map which extends $\Phi_{1|A}$ (let A be a C^*-algebra and $f \in \mathsf{M}_n(A^*)$. Here f is said to be positive if $\sum_{i,j} f_{ij}(a_{ij}) \geqslant 0$ whenever $a \in \mathsf{M}_n(A)_+$, the positive part of $\mathsf{M}_n(A)$). Hence $\Phi_{1|A}$ is extendible and the proof is now complete (see [1, sect. 1]). ∎

Finally we can consider, in an obviously manner, the following map $\widehat{\Phi}_1 : B \mapsto (B')_*$ given by

$$\widehat{\Phi}_1 : b \in B \mapsto (b \cdot \Omega, \Omega) \in (B')_* \tag{3.6}$$

or, equally well the images of the unit ball of A under $\Phi_1, \widehat{\Phi}_1$ in $(B^0)_*, (B')_*$ respectively.

Summarizing, we have the following

THEOREM 3.4. *Let $A \subset B$ be as in the previous theorem. The following statements are equivalent.*

i) *$A \subset B$ is a split inclusion.*
ii) $\Phi_{1|A} \in \mathcal{D}(A,(B^o)_*)$.
iii) *The set $\{(\cdot\Omega, Ja\Omega) : a \in A,\ \|a\| < 1\}$ is Φ_1-decomposable.*
iv) $\widehat{\Phi}_{1|A} \in \mathcal{D}(A,(B')_*)$.
v) *The set $\{(a \cdot \Omega, \Omega) : a \in A,\ \|a\| < 1\}$ is $\widehat{\Phi}_1$-decomposable.*

Proof. Immediate by the above considerations and theorem 2.7. ∎

For the non factor case the conditions ii)–v) are all "almost" equivalent to the split property, namely there will exist a normal $*$-homomorphism of $A\overline{\otimes}B'$ onto $A \vee B'$ carrying $a \otimes b$ to ab. In quantum field theory where $A = \mathcal{U}(O) \subset \mathcal{U}(\widehat{O}) = B$ is an inclusion of local algebras of observables ($O \subset int(\widehat{O})$ are suitable compact regions in Minkowski space), one can prove, under general assumptions, that the above homomorphism is in fact an isomorphism and the condition of matrix-decomposability for $\Phi_{1|A}$ turns out to be equivalent to the split property (see [1, remark 4]).

4. COMMENTS

It could be very interesting for several application in quantum field theory (see [2–7]) to provide an equivalent condition for the split property of the inclusion $A \subset B$

directly in terms of $\Phi_2 : a \in L^\infty(A) \mapsto \Delta_B^{1/4} a\Omega \in L^2(B)$ but this approach seems to be hard.

One can consider the following commutative diagram

$$\begin{array}{ccc} b \in L^\infty(B) & \xrightarrow{\Phi_1} & (\cdot\Omega, Jb\Omega) \in L^1(B) \\ \Phi_2 \searrow & & \nearrow \Psi \\ & \Delta^{\frac{1}{4}} b\Omega \in L^2(B) & \end{array} \tag{4.1}$$

where in (4.1)

$$\Psi : x \in L^2(B) \mapsto (\Delta^{1/4} \cdot \Omega, Jx) \in L^1(B). \tag{4.2}$$

Obviously $A \subset B$ is a split inclusion iff $\Psi(\Phi_2(A_1))$ is a Φ_1-decomposable set and one could indirectly characterize the split property in this way. By this characterization one might establish, in quantum field theory, relations between the split property for the inclusion $\mathcal{U}(O) \subset \mathcal{U}(\widehat{O})$ of local algebras and the properties of the maps $a \in \mathcal{U}(O) \mapsto e^{-\beta H} a\Omega; \beta > 0$ based on the Hamiltonian operator via the images of the sets $\{e^{-\beta H} a\Omega : a \in \mathcal{U}(O)_1\}; \beta > 0$ under Φ_1 in $L^1(\mathcal{U}(\widehat{O}))$. However, for this approach, one needs new ideas about the appropriate space substitute of the p-mappings (see [16]) considered in [2]. But one would have a more direct characterization of the split property in terms of the L^2-mapping Φ_2.

For a better understanding of the above problem, one could start from the commutative case treated in [1, sect. 3] where the operator structure plays no role. In this case, the problem can be expressed in the following simpler form.

Let (X, μ) be a probability measure space and $S \subset L^\infty(X, \mu)$ a nuclear set in $L^1(X, \mu)$. One has to characterize the properties of S considered as a set in $L^p(X, \mu)$; $1 < p < +\infty$, which are equivalent to the L^1-nuclearity. Unfortunately, also this problem seems to be rather difficult.

Acknowledgments. We wish to thank Prof. R. Longo for suggesting the problem and for many stimulating discussions. We are also grateful to Prof. C. D'Antoni for several useful comments.

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Received January 10, 1991.

June 28, 1991 - note added - After submitting this paper, the author received a preprint of E. Effros, Z.-J. Ruan [17] where they define and apply the "metrically nuclear maps" to relevant developments in the operator context. Their metrically nuclear maps are the same as the matrix-de