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ON THE K-THEORY OF THE NON-COMMUTATIVE CIRCLE

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In [2] we study a certain field of C^*-algebras $(\mathcal{T}_q)_{q\in\mathbf{R}}$, where $\mathcal{T}_q$ us defined as the *universal unital C^*-algebra* generated by an element ζ subject to $1-\zeta\zeta^* = q(1-\zeta^*\zeta)$. Of particular interest we considered the algebra $\mathcal{T}_{-1}$- the *Non-Commutative Circle.*

This note, which should be considered as a continuation of [2], is devoted to the computation of the K-groups of this C^*-algebra. We show that the K-theory is the same with the one of the disk, and as a consequence, the quantum disk deformation gives a KK equivalence.

1. We recall first some notations and facts from [2]. The C^*-algebra that we are dealing with is the universal unital C^*-algebra $\mathcal{T}_{-1}$ generated by two self-adjoint elements x and y subject to the relation $x^2 + y^2 = 1$.

Let us consider A to be the full C^*-algebra $C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$ of the infinite dihedral group, that is the universal unital C^*-algebra generated by two self-adjoint unitaries u and v. The basic facts that we shall need are contained in the following.

PROPOSITION. *The K-groups of the C^*-algebra A are $K_0(A) \simeq \mathbf{Z}^3$ and $K_1(A) = 0$. Moreover, the generators of $K_0(A)$ as a free abelian group are [1], [p] and [q], where $p = \frac{1}{2}(1-u)$ and $q = \frac{1}{2}(1-v)$.*

Proof. Parts of this proof are standard (see [3], [4] or [5]). However, since some notations will be used later, we have chosen to give all the details.

The statment follows essentialy from the results of Paschke (see [4]), plus the fact that one has a group isomorphism between $\mathbf{Z}_2 * \mathbf{Z}_2$ and $\mathbf{Z}\rtimes\mathbf{Z}_2$ where the action of $\mathbf{Z}_2$ on $\mathbf{Z}$ is given by the automorphism $n \mapsto -n$. Then we consider a $*$-isomorphism $\Phi : A \to C^*(\mathbf{Z})\rtimes\mathbf{Z}_2$ which is given by $\Phi(u) = ZV$, $\Phi(v) = V$, where Z is the canonical generator for $C^*(\mathbf{Z})$ and V is the unitary in $C^*(\mathbf{Z})\rtimes\mathbf{Z}_2$ that implements the above action of $\mathbf{Z}_2$ on $\mathbf{Z}$ and hence on $C^*(\mathbf{Z})$ (that is, $VZV = Z^{-1} = Z^*$).

Since $C^*(\mathbf{Z})$ is isomorphic with $C(\mathbf{T})$ - the algebra of continuous functions on the unit circle - we can view Φ as an isomorphism between A and $C(\mathbf{T})\rtimes\mathbf{Z}_2$. The action of $\mathbf{Z}_2$ on $C(\mathbf{T})$ is now implemented by the homeomorphism $z \mapsto \overline{z}$ of $\mathbf{T}$.

Following the method of Paschke, $C(\mathbf{T})\rtimes\mathbf{Z}_2$ gets identified with the subalgebra of $M_2(C(\mathbf{T}))$ consisting of those 2 by 2 matrices of the form

$$a = \begin{pmatrix} f_1 & g \\ h & f_2 \end{pmatrix}$$

with $f_k(z) = f_k(\overline{z})$ and $g(z) + g(\overline{z}) = h(z) + h(\overline{z}) = 0$ for all $z \in \mathbf{T}$, $k = 1, 2$. That is, using the $\mathbf{Z}_2$-grading of $C(\mathbf{T})$ given by the above action, $f_1, f_2 \in C(\mathbf{T})_0$ and $g, h \in C(\mathbf{T})_1$. But if we take $\mathbf{T}^+$ to be the upper semicircle, clearly all the functions from $C(\mathbf{T})_k$, $k = 0, 1$ are uniquely determined by their restriction on $\mathbf{T}^+$. This allows us to identify $C(\mathbf{T})\rtimes\mathbf{Z}_2$ with the subalgebra B of $M_2(C(\mathbf{T}^+))$ consisting of all matrices

$$a = \begin{pmatrix} f_1 & g \\ h & f_2 \end{pmatrix}$$

with $g(1) = g(-1) = h(1) = h(-1) = 0$. Taking now the homeomorphism $\varphi : [0,1] \to$ $\to \mathbf{T}^+$ given as $\varphi(s) = 2s - 1 + 2i\sqrt{s - s^2}$ we will identify A with the subalgebra D of $M_2(C([0,1]))$ consisting of all matrices

$$a = \begin{pmatrix} f_1 & g \\ h & f_2 \end{pmatrix}$$

with $g, h \in C_0((0,1))$, that is $g(0) = g(1) = h(0) = h(1) = 0$.

Let us see now what the images of u and v are, under this identification. To do this, first we explicity write down the embedding $\Psi : C(\mathbf{T})\rtimes\mathbf{Z}_2 \to M_2(C(\mathbf{T}))$. For $a = F + GV \in C(\mathbf{T})\rtimes\mathbf{Z}_2$, with $F, G \in C(\mathbf{T})$ we have

$$\Psi(a) = \begin{pmatrix} E_0(F+G) & E_1(F-G) \\ E_1(F+G) & E_0(F-G) \end{pmatrix}$$

where $E_k : C(\mathbf{T}) \to C(\mathbf{T})_k$, $k = 0, 1$ are the projections $E_0(F)(z) = \frac{1}{2}(F(z)+$ $+F(\overline{z}))$, $E_1(F)(z) = \frac{1}{2}(F(z) - F(\overline{z}))$. Take $Z \in C(\mathbf{T})$ to be the function defined by $Z(z) = z$. For the elements $Z, V \in C(\mathbf{T})\rtimes\mathbf{Z}_2$ we get

$$\Psi(Z) = \begin{pmatrix} \mathrm{Re}Z & i\mathrm{Im}Z \\ i\mathrm{Im}Z & \mathrm{Re}Z \end{pmatrix}, \ \Psi(V) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The same formulas work if we view everything in B. Finally in D these elements correspond (we keep the same notations) to

$$Z(s)=\begin{pmatrix}2s-1 & 2i\sqrt{s-s^2}\\ 2i\sqrt{s-s^2} & 2s-1\end{pmatrix},\ V(s)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

But $\Psi(u)=ZV$ and $\Psi(v)=V$, so the unitaries $u,v\in A$ correspond to

$$U(s)=\begin{pmatrix}2s-1 & -2i\sqrt{s-s^2}\\ 2i\sqrt{s-s^2} & 1-2s\end{pmatrix},\ V(s)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Finally, the projections $1,p,q\in A$ correspond in D to

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ P(s)=\begin{pmatrix}1-s & i\sqrt{s-s^2}\\ -i\sqrt{s-s^2} & s\end{pmatrix},\ Q(s)=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

From now on we will work excusively with the algebra D, and we will find its K-groups using the exact sequence

$$M_2(C_0((0,1)))\overset{i}{\hookrightarrow}D\overset{\pi}{\twoheadrightarrow}\mathbf{C}^4$$

given by

$$\begin{pmatrix}f_1&g\\h&f_2\end{pmatrix}\overset{\pi}{\longmapsto}(f_1(0),f_1(1),f_2(0),f_2(1)).$$

The exact sequence of K-theory

$$\begin{array}{ccccc}K_0(M_2(C_0((0,1)))) & \xrightarrow{i_*} & K_0(D) & \xrightarrow{\pi_*} & K_0(\mathbf{C}^4)\\ \partial\uparrow & & & & \downarrow\delta\\ K_1(\mathbf{C}^4) & \underset{\pi_*}{\longleftarrow} & K_1(D) & \underset{i_*}{\longleftarrow} & K_1(M_2(C_0((0,1))))\end{array}$$

becomes

$$\begin{array}{ccccc}0 & \longrightarrow & K_0(D) & \xrightarrow{\pi_*} & \mathbf{Z}^4\\ \uparrow & & & & \downarrow\delta\\ 0 & \longleftarrow & K_1(D) & \underset{i_*}{\longleftarrow} & \mathbf{Z}\end{array}$$

Let us first examine the map $i_*:K_1(M_2(C_0(0,1)))\to K_1(D)$ (the bottom right arrow in the above diagrams). Of course we can view $M_2(C_0(0,1))$ as the *suspension* of $M_2(\mathbf{C})$, so the generator for $K_1(M_2(C_0(0,1)))$ is the unitary matrix

$$W(s)=\begin{pmatrix}1&0\\0&e^{2\pi is}\end{pmatrix}$$

But if we take this element in D, clearly we have a path $(W_t)_{t\in[0,1]}$ of unitaries given by

$$W_t(s) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi its} \end{pmatrix}$$

that connects W with I. This shows that the map $i_* : K_1(M_2(C_0(0,1))) \to K_1(D)$ is the null homomorphism. In particular we get $K_1(D) = 0$. On the other hand we obtain a short exact sequence of groups

$$K_0(D) \overset{\pi_*}{\hookrightarrow} \mathbf{Z}^4 \overset{\delta}{\twoheadrightarrow} \mathbf{Z}.$$

Take now in $\mathbf{Z}^4 = K_0(\mathbf{C}^4)$ the subgroup L generated by $\alpha = \pi_*([I])$, $\beta = \pi_*([P])$, $\gamma = \pi_*([Q])$. Note that $\pi(I) = (1,1,1,1)$, $\pi(P) = (1,0,0,1)$, $\pi(Q) = (0,0,1,1)$ and this clearly proves that $\mathbf{Z}^4/L \simeq \mathbf{Z}$. But, of course $L \subset \operatorname{im}\pi_* = \ker\delta$ and, on the other hand $\mathbf{Z}^4/\operatorname{Ker}\delta \simeq \operatorname{Im}\delta = \mathbf{Z}$. This clearly enforces $L = \operatorname{im}\pi_*$ and, since π_* is injective this gives the desired characterization of $K_0(D)$. ■

2. We shall use the above informations about A making now the link with the algebra $\mathcal{T}_{-1}$.

PROPOSITION. *Let $\Lambda : \mathcal{T}_{-1} \to C([0,1]) \otimes A$ be the injective (cf [2]) $*$-homomorphism defined by $\Lambda(x) = \sqrt{t} \otimes u$, $\Lambda(y) = \sqrt{1-t} \otimes v$. Let $J \subset \mathcal{T}_{-1}$ be the smallest closed two-sided ideal containing xy. Then:*
(a) $\Lambda(J) = C_0((0,1)) \otimes A$.
(b) $\mathcal{T}_{-1}/J \simeq \mathbf{C}^4$, this isomorphism being given as $x \longmapsto (1,-1,0,0)$, $y \longmapsto (0,0,1,-1)$.

Proof. To prove *(a)* take ρ any representation of $C([0,1]) \otimes A$ which has $\operatorname{Ker}\rho = C_0((0,1)) \otimes A$. Then, clearly the representation $\rho \circ \Lambda$ of $\mathcal{T}_{-1}$ vanishes at xy, simply because $\Lambda(xy) = \sqrt{t-t^2} \otimes uv \in C_0((0,1)) \otimes A$. This gives $\operatorname{Ker}\rho\circ\Lambda \supset J$, which proves the "$\subset$" part of *(a)*. To prove the other inculsion, take a total set in $C_0((0,1)) \otimes A$ consisting of elements of the form $t(1-t)(\sqrt{t})^k(\sqrt{1-t})^l \otimes a$ where $a \in A$ is a product of u's and v's with $k =$ the number of u's and $l =$ the number of v's. (To prove that this forms a total set simply use the Stone-Weierstrass theorem.). But then it is clear that any such element is of the form $\Lambda(x^2y^2b)$ where b is a product of x's and y's written in exactly the same order and number as the u's and v's appear in a. It is now obvious that $x^2y^2b \in J$, so our total set is contained in $\Lambda(J)$ which concludes the proof of part *(a)*.

To prove *(b)*, let M be the quotient $\mathcal{T}_{-1}/J$. Denote by X and Y the images of x and y in M. Since in M we have $XY = 0$ we obtain that M is commutative. On the other hand, in M we have $X^3 = X(1 - Y^2) = X$ which proves the existence of two orthogonal projections $P_X^{\pm}$ such that $X = P_X^+ - P_X^-$. Similarly $Y = P_Y^+ - P_Y^-$ and the four projections that we have are mutually orthogonal. Because of $X^2 + Y^2 = 1$ we get $P_X^+ + P_X^- + P_Y^+ + P_Y^- = 1$ we get that M is generated by these four projections, hence its dimension is at most 4. But if we take $\omega : \mathcal{T}_{-1} \to \mathbf{C}^4$ the $*$-homomorphism indicated in the statement, note that $\omega(xy) = 0$ so we get $J \subset \operatorname{Ker}\omega$. So $\operatorname{Ran}\omega = \mathbf{C}^4$ is a quatient of M which leaves only the possibility that $\operatorname{Ker}\omega = J$. ■

3. We are ready to prove now the main result

THEOREM. *The K-groups of the algebra $\mathcal{T}_{-1}$ are $K_1(\mathcal{T}_{-1}) = 0$ and $K_0(\mathcal{T}_{-1}) = \mathbf{Z}$.*

Proof. Let us use the exact sequence $J \overset{j}{\hookrightarrow} \mathcal{T}_{-1} \overset{\omega}{\twoheadrightarrow} \mathbf{C}^4$. The exact sequence of K-theory

$$
\begin{array}{ccccc}
K_0(J) & \xrightarrow{j_*} & K_0(\mathcal{T}_{-1}) & \xrightarrow{\omega_*} & K_0(\mathbf{C}^4) \\
\partial \uparrow & & & & \downarrow \delta \\
K_1(\mathbf{C}^4) & \xleftarrow[\omega_*]{} & K_1(\mathcal{T}_{-1}) & \xleftarrow[j_*]{} & K_1(J)
\end{array}
$$

reads

$$
\begin{array}{ccccc}
0 & \longrightarrow & K_0(\mathcal{T}_{-1}) & \xrightarrow{\omega_*} & \mathbf{Z}^4 \\
\uparrow & & & & \downarrow \delta \\
0 & \longleftarrow & K_1(\mathcal{T}_1) & \xleftarrow[j_*]{} & \mathbf{Z}^3
\end{array}
$$

What we have used here was the isomorphism $J \simeq C_0((0,1)) \otimes A$ which gives $K_*(J) = = K_{1-*}(A)$. We examine now the map $j_* : K_1(J) \to K_1(\mathcal{T}_{-1})$. For this purpose we use the above two Propositions. On one hand we know that $K_1(J) = K_0(A) = \mathbf{Z}^3$, but we also know the generators of $K_0(A)$. They are [1], [p] and [q]. If we replace the generator [q] by -[q], the generators for $K_1(C_0((0,1)) \otimes A$ will be the (classes of the) unitary loops $e^{2\pi i t\otimes 1}$, $e^{2\pi i t\otimes(1-u)} = e^{\pi i t\otimes p}$, $e^{2\pi i(1-t)\otimes q} = e^{\pi i(1-t)\otimes(1-v)}$. But if we look at the preimages of these elements through Λ we get that the generators for $K_1(J)$ are the unitaries $e^{2\pi i x^2}$, $e^{\pi i x(x-|x|)}$ and $e^{\pi i y(y-|y|)}$. But it is then obvious that if we view these elements in $\mathcal{T}_{-1}$ (that is when we apply j) those elements are all of the form $e^{\pi i h}$ with $h = h^* \in \mathcal{T}_{-1}$ so all of them represent the zero element in $K_1(\mathcal{T}_{-1})$. This proves exactly that $j_* : K_1(J) \to K_1(\mathcal{T}_{-1})$ is the null homomorphism. This gives $K_1(\mathcal{T}_{-1}) = 0$, plus a short exact sequence $K_0(\mathcal{T}_{-1}) \to \mathbf{Z}^4 \to \mathbf{Z}^3$. This gives us of course the isomorphism $K_0(\mathcal{T}_{-1}) \simeq \mathbf{Z}$. ■

4. We shall conclude these considerations with the description for the generator of the K_0-group. This will be done by examining the comutator ideal $\mathcal{C}$ of $\mathcal{T}_{-1}$, that is the closed two sided ideal generated by $xy - yx$.

PROPOSITION. *(a) The quotient $\mathcal{T}_{-1}/\mathcal{C}$ is isomorphic to $C(\mathbf{T})$, the isomorphism being the one sending x any y to the functions* $\operatorname{Re} z$ *and* $\operatorname{Im} z$.
(b) The group $K_0(\mathcal{T}_{-1})$ is generated by [1].
(c) The K-groups of $\mathcal{C}$ are $K_0(\mathcal{C}) = \mathbf{Z}$, $K_1(\mathcal{C}) = 0$.

Proof. *(a)* Take $\eta : \mathcal{T}_{-1} \to C(\mathbf{T})$ the $*$-homomorphism defined as in the statement. Clearly η is surjective and $\operatorname{Ker}\eta \supset \mathcal{C}$. But on the other hand if we take X and Y the images of x and y in $\mathcal{T}_{-1}/\mathcal{C}$ we get a unitary element $Z = X + iY$ which obviously generates the quotient algebra. This makes the algebra $\mathcal{T}_{-1}/\mathcal{C}$ a quotient of $C(\mathbf{T})$ in an obvious way ($Z \mapsto Z$). The combination of those two facts gives clearly the desired identification.

(b)(c) We shall consider again the sequence of K-groups, associated to the extension $\mathcal{C} \to \mathcal{T}_{-1} \xrightarrow{\eta} C(\mathbf{T})$. The corresponding exact sequence is

$$\begin{array}{ccccc} K_0(\mathcal{C}) & \longrightarrow & \mathbf{Z} & \xrightarrow{\eta_*} & \mathbf{Z} \\ \partial \uparrow & & & & \downarrow \delta \\ \mathbf{Z} & \longleftarrow & 0 & \longleftarrow & K_1(\mathcal{C}) \end{array}$$

Using the fact that $\mathbf{Z} = K_0(C(\mathbf{T}))$ is generated by [1] it follows that the map $\eta_* : \mathbf{Z} = K_0(\mathcal{T}_{-1}) \to K_0(C(\mathbf{T})) = \mathbf{Z}$ is surjective. But this enforces η_* to be an isomorphism, and since $\eta_*[1] = [1]$ we get statement *(b)*. Finally this shows that ∂ and δ are isomorphisms, which proves statement *(c)*. ■

REMARK. One can write a unitary matrix $W \in M_2(\mathcal{T}_{-1})$ which is a lifting for the unitary $w = \begin{pmatrix} Z & 0 \\ 0 & \overline{Z} \end{pmatrix} \in M_2(C(\mathbf{T}))$ (this can be used, for instance, to describe the generator of $K_0(\mathcal{C})$ by means of the isomorphism ∂). This unitary is

$$W = \begin{pmatrix} 0 & (1 - z^*z + (z^*z)^2)^{-1/2} \\ (1 - zz^* + (zz^*)^2)^{-1/2} & 0 \end{pmatrix} \cdot \begin{pmatrix} z & 1 - z^*z \\ 1 - z^*z & z^* \end{pmatrix},$$

where $z = x + iy$. Actually the unitary matrix W can be obtained, in a "section-wise" fashion in all the algebras $\mathcal{T}_q$ in the following manner. Take

$$U_q = \begin{pmatrix} \zeta & -q(1 - \zeta\zeta^*) \\ 1 - \zeta\zeta^* & \zeta^* \end{pmatrix}$$

and note that U_q is invertible. Then we simply take $W_q = U_q(U_q^*U_q)^{-1/2} = (U_qU_q^*)^{-1/2}U_q$.

5. COMMENT. Since, for $|q| < 1$ we have $\mathcal{T}_q \simeq \mathcal{T}$ - the Toeplitz algebra - using E-theory (see [1]), the quantum disk deformation ([2]) produces an element $\alpha \in KK(\mathcal{T}_{-1}, \mathcal{T})$. All the C^*-algebras involved in our discussion are *nuclear*. Hence, by the *Universal Coefficient Theorem* (cf [6]), the element α is a KK equivalence. Since the ideal $\mathcal{K}$ of compact operators in $\mathcal{T}$ is also described as the commutator ideal, the element α gives, after "restriction", an element $\beta \in KK(\mathcal{C}, \mathcal{K})$. Using the commutative diagram

$$\begin{array}{ccccc} K_*(\mathcal{C}) & \longrightarrow & K_*(\mathcal{T}_{-1}) & \longrightarrow & K_*(\mathcal{C}(\mathbf{T})) \\ \beta \big\downarrow & & \alpha \big\downarrow & & \big\downarrow \mathrm{Id} \\ K_*(\mathcal{K}) & \longrightarrow & K_*(\mathcal{T}) & \longrightarrow & K_*(\mathcal{C}(\mathbf{T})) \end{array}$$

we get that β is also a KK equivalence.

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