J. OPERATOR THEORY
31(1994), 319–326

LIE SUBALGEBRA OF NORMAL ELEMENTS IN A LIE ALGEBRA WITH INVOLUTION

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1. INTRODUCTION

In [3], I. E. Segal has proposed the reduction of study of the representations of a complex semisimple Lie group to the study of holomorphic representations by normal operators. The peculiarity of such a representation (the product of two normal operators is usually not normal) is discussed by T. Sherman in Lie algebra terms. He obtains a decomposition of a representation of a semisimple Lie algebra by normal operators into the sum of two representations which commute with each other; one of this consists of skew-adjoint operators, the other consists of normal operators and commutes with its contragradient.

In [6–10], some results concerning operatorially or topologically irreducible representations of a class of Lie agebras (which may be of infinite dimension) by bounded operators in a complex Banach space are obtained. In this context professor P. de la Harpe asked me if it possible to give an analogue of Sherman's result "for representations by ad hoc operators on some Banach space".

In the following we give an answer when the operators of the representations are generalized scalar operators [1]. At the same time we extend Sherman's result to an infinite dimensional Lie algebra which is ideally finite and semisimple.

1. Let $\mathcal{L}$ be a Lie algebra over a field of characteristic different from 2 and $^* : \mathcal{L} \to \mathcal{L}$ an anti-automorphism of order 2. An element x of $\mathcal{L}$ is normal if $[x, x^*] = 0$, self adjoint if $x = x^*$ and skew if $x = -x^*$.

The main result of Sherman [4] in Lie algebra terms is the following.

THEOREM 1. *Let $\mathcal{G} \subset \mathcal{L}$ be a Lie subalgebra consisting of normal elements. Then*

$$\mathcal{G}^+ = \mathcal{G} + \mathcal{G}^* + [\mathcal{G}, \mathcal{G}^*]$$

is a Lie algebra and $\mathcal{I} = [\mathcal{G}, \mathcal{G}^*]$ is an ideal in $\mathcal{G}^+$ consisting of skew elements.

If $\mathcal{G}$ is a semisimple finite dimensional Lie algebra and the characteristic of the field is 0, then

$$\mathcal{G}^+ = \mathcal{G}_0 \oplus \mathcal{I}$$

where $\mathcal{G}_0$ is the centralizer of the ideal $\mathcal{I}$ in $\mathcal{G}^+$. Hence, for $x, y \in \mathcal{G}$ we can write $x = x_0 + x_1$, $y = y_0 + y_1$; x_0, y_0 are normal elements, x_1, y_1 are skew elements, $[x_0, y_1] = [x_1, y_0] = 0$, $[x_0, y_0^*] = 0$.

This theorem can be applied for representations on a Hilbert space H, because there is the natural operation * (i.e., if x is an operator on H, x^* is the adjoint operator on H).

In the case of an arbitrary Banach space such a *-operation must be defined. We shall prove that one can define such a *-operation for some Lie algebras of generalized scalar operators.

SECTION 2.

We begin by the transcription of the existence of *-antiautomorphism of order two in Lie algebra terms.

LEMMA 2. *Let $\mathcal{L}$ be a complex Lie algebra. The following assertions are equivalent:*

(j) *There is a mapping $^* : \mathcal{L} \to \mathcal{L}$ such that*

$$(A+B)^* = A^* + B^*,\ (\lambda A)^* = \overline{\lambda} A^*,\ (A^*)^* = A \text{ and } [A,B]^* = -[A^*, B^*]$$

(jj) *There is $\mathcal{A} \subset \mathcal{L}$ a real vector space such that:*

$$\mathcal{L} = \mathcal{A} \oplus \mathrm{i}\mathcal{A} \text{ and } \mathrm{i}[A,B] \in \mathcal{A} \text{ for every } A, B \in \mathcal{A}.$$

Proof. If (j) holds, $\mathcal{A} = \{A \in \mathcal{L} | A = A^*\}$ and for $A \in \mathcal{L}$ we may uniquely write

$$A = \frac{A + A^*}{2} + \mathrm{i}\frac{A - A^*}{2\mathrm{i}} \quad \text{for every } A \in \mathcal{L}.$$

Obviously, $\mathcal{A}$ is a real vector space, c is skew iff $ic \in \mathcal{A}$ and $i[A, B] \in \mathcal{A}$ for every $A, B \in \mathcal{A}$ because $[A, B]^* = -[A, B]$ for every $A, B \in \mathcal{A}$. Hence (j)⇒(jj).

If (jj) holds we may uniquely write for $A \in \mathcal{L}$

$$A = A_1 + iA_2$$

We define in a natural way,

$$A^* = A_1 - iA_2$$

Clearly, $A = A^*$ iff $A \in \mathcal{A}$ and the properties of * are obvious. So, (jj)⇒(j) ■

REMARKS. If $\mathcal{L}$ satisfies (j) or (jj) and $A \in \mathcal{L}$, we have:

1. $[A, A^*]^* = -[A^*, A] = [A, A^*]$
2. A is normal iff $[A, A^*]$ is skew
3. $A = A_1 + iA_2$, $A_1, A_2 \in \mathcal{A}, A$ is normal iff $[A_1, A_2] = 0$, A is self-adjoint iff $A = A_1 \in \mathcal{A}$, A is skew iff $A = iA_2 \in i\mathcal{A}$.

LEMMA 2. *If $\mathcal{L}$ is a Lie algebra over some field K of characteristic other than two then the following assertions are equivalent:*

(α) there is a K-linear mapping $^* : \mathcal{L} \mapsto \mathcal{L}$ such that $(A^*)^* = A$, $[A, B]^* = -[A^*, B^*]$ for every $A, B \in \mathcal{L}$

(β) there exists $\mathcal{A}$ a subspace of $\mathcal{L}$, $\mathcal{S}$ a Lie subalgebra of $\mathcal{L}$ such that,

$$\mathcal{L} = \mathcal{A} \oplus \mathcal{S}, \ [\mathcal{A}, \mathcal{A}] \subset \mathcal{S}, \ [\mathcal{A}, \mathcal{S}] \subset \mathcal{A}.$$

Proof. If we have (α) we put

$$\mathcal{A} = \{A \in \mathcal{L} | A = A^*\}, \ \mathcal{S} = \{A | A^* = -A\}.$$

If we have (β), we may uniquely write for $A \in \mathcal{L}$,

$$A = A_0 + A_1, \quad A_0 \in \mathcal{A}, \ A_1 \in \mathcal{S}$$

and we can define

$$A^* = A_0 - A_1.$$

■

DEFINITION 1. A complex Lie algebra which satisfies (j) or (jj) of Lemma 1 is called a complex Lie algebra with involution or a *-complex Lie algebra.

Let $\mathcal{X}$ be a complex Banach space, $\mathcal{B}(\mathcal{X})$ the Lie algebra of all bounded operators on $\mathcal{X}$; $[A, B] = AB - BA$. We can obtain a Lie algebra with involution in the following way:

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ be a real subspace such that,

(*I*)

$$\begin{cases} [A, B] \in \mathrm{i}\mathcal{A} & \text{for every } A, B \in \mathcal{A} \\ 0 \neq \sigma(A) \subset \mathbf{R} & \text{for every } A \in \mathcal{A},\ A \neq 0 \\ & \text{i.e., every nonzero operator of } \mathcal{A} \text{ has a nonzero real spectrum.} \end{cases}$$

Then $\mathcal{A} \cap \mathrm{i}\mathcal{A} = \{0\}$ and $\mathcal{L} = \mathcal{A} \oplus \mathrm{i}\mathcal{A}$ is a complex Lie algebra with involution.

DEFINITION 2. $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ is called a space of self-adjoint operators if $\mathcal{A}$ is a real subspace which satisfies (j) and every operator of $\mathcal{A}$ is a generalized scalar operator.

The following proposition is a direct consequence of the results of [1] (Chapter 4, Theorem 1.11, Theorem 3.3 and Corollary 3.4; Chapter 5, Theorem 4.5 (jj)⇒(j)) because the spectrum of any operator of $\mathcal{A}$ is thin.

PROPOSITION 1. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$ be a space of self adjoint operators and let $\mathcal{L} = \mathcal{A} \oplus \mathrm{i}\mathcal{A}$ the corresponding *-complex Lie algebra. The normal elements in $\mathcal{L}$ are generalized scalar operators $g \in \mathcal{L}$ having a spectral distribution U^g with $U^g_{\mathrm{Re}\,\lambda} \in$ $\in \mathcal{A}$, $U^g_{\mathrm{Im}\,\lambda} \in \mathcal{A}$.*

Proof. If g has a spectral distribution U^g and $U^g_{\mathrm{Re}\,\lambda} \in \mathcal{A}$, $U^g_{\mathrm{Im}\,\lambda} \in \mathcal{A}$ we have $g = U^g_{\mathrm{Re}\,\lambda} + \mathrm{i}U^g_{\mathrm{Im}\,\lambda} = U^g_\lambda$, $g^* = U^g_{\bar\lambda}$ and $[g, g^*] = 0$ because $U^g_\lambda U^g_{\bar\lambda} = U^g_{\bar\lambda} U^g_\lambda = U^g_{|\lambda|^2}$ (multiplicativity of spectral distribution $\varphi \mapsto U^g_\varphi$ see [1]).

Conversely, we have $g = a + ib$, $a, b \in \mathcal{A}$, a and b being generalized scalar operators with spectrum on real line and $[g, g^*] = 0$ implies $[a, b] = 0$. Then g is a generalized scalar operator because a and b are commuting generalized scalar operators with thin spectrum ([1], Chapter 4, Theorems 1.11, 3.3, 3.4).

A spectral distribution of g may be obtained

$$U^g_\varphi = (U^a \otimes U^b)_{\varphi\circ(\zeta\otimes 1 + 1\otimes \mathrm{i}\mu)}, \quad \varphi \in C^\infty(R^2),\ U^g_{\mathrm{Re}\,\lambda} = U^a_{\mathrm{Re}\,\zeta} + U^b_{-\mathrm{Im}\,\mu}$$

where R is the field of real numbers, R^2 is identified with C and ζ and μ denotes the identity function of C. By [1] (Chapter 5, Theorem 4.5 (jj)⇒(j)) we have

$$U^a_{\mathrm{Re}\,\zeta} = U^a_{\zeta|R} = a, \quad U^b_{-\mathrm{Im}\,\mu} = U^b_{-\mathrm{Im}\mu|R} = 0.$$

Hence $U^g_{\mathrm{Re}\,\lambda} \in \mathcal{A}$. I an analogous manner we have $U^g_{\mathrm{Im}\,\lambda} \in \mathcal{A}$. ■

REMARK. When $\mathcal{X}$ is a Hilbert space, any normal operator on $\mathcal{X}$ has a spectral distribution (given by its spectral measure) which verifies the above mentioned property, where $\mathcal{A}$ is the space of all self-adjoint operators on $\mathcal{X}$.

The following corollaries are similar to Corollaries 1, 2 of [4].

COROLLARY 1. *Let $\mathcal{L} = \mathcal{A} \oplus \mathrm{i}\mathcal{A}$ be a *-complex Lie subalgebra in $\mathcal{B}(\mathcal{X})$. If $\mathcal{G}$ is a complex Lie subalgebra of $\mathcal{B}(\mathcal{X})$ consisting of generalized scalar operators g*

which have spectral distributions U^g ($g = U^g_\lambda$) such that $U^g_{\mathrm{Re}\,\lambda} \in \mathcal{A}$, $U^g_{\mathrm{Im}\,\lambda} \in \mathcal{A}$, then $[U^{g_1}_\lambda, U^{g_2}_{\overline{\lambda}}] = 0$ for every $g_1, g_2 \in \mathcal{G}$. If $[g_1, g_2] = 0$, then $[U^{g_1}_\varphi, U^{g_2}_\varphi] = 0$ for any $\varphi \in C^\infty(R^2)$.

Proof. We have $g^* = (U^g_\lambda)^* = U^g_{\overline{\lambda}}$ and by Sherman's theorem $[g_1, g_2^*]$, $[ig_1, g_2^*]$ are skew. Then it follows (as in [4]) that $[g_1, g_2^*] = 0$ for any $g_1, g_2^* \in \mathcal{G}$.

COROLLARY 2. *Let $\mathcal{G}$ be a semisimple finite dimensional Lie subalgebra of a *-complex Lie algebra $\mathcal{A} \oplus i\mathcal{A} \subset \mathcal{B}(\mathcal{X})$, where $\mathcal{A}$ is a space of self-adjoint operators and $\mathcal{G}$ consists of generalized scalar operators g having a spectral distribution U^g with $U^g_{\mathrm{Re}\,\lambda} \in \mathcal{A}, U^g_{\mathrm{Im}\,\lambda} \in \mathcal{A}$. Then for every $g \in \mathcal{G}$ we may uniquely write $g = g_0 + g_1$; g_0 has the same property as g and $ig_1 \in \mathcal{A}$. Moreover, for any $h \in \mathcal{G}$, if we similarely write $h = h_0 + h_1$, then $[h_0, g_1] = [h_1, g_0] = 0$ and $[g_0, h_0^*] = 0$ (i.e., $[U^{g_0}_\lambda, U^{h_0}_{\overline{\lambda}}] = 0$).*

Finally we observe that a structure of *-Lie algebra may be implicitely contained in an ordinary Lie algebra structure. A significant example is given by $\mathrm{sl}(2, C)$ with basis $\{\tau, X_+, X_-\}$ and relations $[\tau, X_+] = 2X_+, [\tau, X_-] = -2X_-$, $[X_+, X_-] = \tau$. We have $\mathrm{sl}(2, C) = \mathcal{A} \oplus i\mathcal{A}$, $\mathcal{A} = \mathrm{sp}_R\{i\tau, X_+, X_-\}$ and $\mathcal{A}$ satisfies (jj) of Lemma 1. The *-operation given by $\mathcal{A}$ is the following:

$$\begin{pmatrix} \lambda & \mu \\ \nu & -\lambda \end{pmatrix}^* = \begin{pmatrix} \tilde{\lambda} & \overline{\mu} \\ \overline{\nu} & -\tilde{\lambda} \end{pmatrix}; \begin{cases} \tilde{\lambda} = \widetilde{\lambda_1 + i\lambda_2} = -\lambda_1 + i\lambda_2, \\ \overline{\mu} = \overline{\mu_1 + i\mu_2} = \mu_1 - i\mu_2. \end{cases}$$

A normal element is given by

$$\begin{pmatrix} \lambda & \mu \\ \nu & -\lambda \end{pmatrix} \quad \text{with } \frac{\overline{\mu}}{\mu} = \frac{\tilde{\lambda}}{\lambda} = \frac{\overline{\nu}}{\nu}.$$

A skew element is given by

$$\begin{pmatrix} \alpha & i\gamma \\ i\beta & -\alpha \end{pmatrix} \quad \text{with } \alpha, \beta, \gamma \in \mathbf{R},$$

and in this case the transcription of the Theorem 1 is obvious.

SECTION 3.

In this section we give an extension of the main result of [4], Theorem 1, to an (infinite dimensional) ideally-finite semisimple Lie algebra.

An ideally finite semisimple Lie algebra is a direct sum of finite dimensional simple ideals [5]. Now let $\mathcal{L}$ be a *-Lie algebra and $\mathcal{G} \subset \mathcal{L}$ ideally finite semisimple Lie subalgebra consisting of normal elements. Hence we may write $\mathcal{G} = \bigoplus \mathcal{G}_i$, $\mathcal{G}_i$ simple ideal of $\mathcal{G}$, $[\mathcal{G}_i, \mathcal{G}_j] = 0$ for $i \neq j$, $[\mathcal{G}_i, \mathcal{G}_i] = \mathcal{G}_i$ and $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$.

We consider as in [4] $\mathcal{G}^+ = \mathcal{G} + \mathcal{G}^* + \mathcal{I}$, where $\mathcal{I} = [\mathcal{G}, \mathcal{G}^*]$ is an ideal in $\mathcal{G}^+$ and $\mathcal{G}_0 = \{x \in \mathcal{G}^+ | [x, \mathcal{I}] = 0\}$ is the centralizer of $\mathcal{I}$ in $\mathcal{G}^+$, which is an ideal in $\mathcal{G}^+$. Obviously we have, $\mathcal{G}^+ = \bigoplus \mathcal{G}_i + \bigoplus \mathcal{G}_i^* + \sum_{i,j} [\mathcal{G}_i, \mathcal{G}_j^*]$. We have $x^* = -x$ for every $x \in \mathcal{I} = [\mathcal{G}, \mathcal{G}^*] = \sum_{i,j} [\mathcal{G}_i, \mathcal{G}_j^*]$ and $\mathcal{I}^* = \mathcal{I}$.

If $x \in \mathcal{G}_0$ then the following equalities hold: $[x^*, \mathcal{I}] = [x, \mathcal{I}^*]^* = [x, \mathcal{I}]^* = \{0\}$. Hence $\mathcal{G}_0^* = \mathcal{G}_0$ because $(x^*)^* = x$. For a self-adjoint $x \in \mathcal{G}^+$ and $y \in \mathcal{I}$ we may write,

$$-[x, y] = [x, y]^* = [y^*, x^*] = -[y, x] = [x, y].$$

This proves that $[x, y] = 0$. Hence, $x = x^* \in \mathcal{G}^+$ implies $x \in \mathcal{G}_0$.

Let z, y be arbitrary elements of $\mathcal{G}$. We have $z + z^* \in \mathcal{G}_0$ because $z + z^*$ is self adjoint element of $\mathcal{G}^+$. But $\mathcal{G}_0$ is an ideal in $\mathcal{G}^+$. Hence,

$$\mathcal{G}_0 \ni [y, z + z^*] = [y, z] + [y, z^*]$$

and

$$[y, z] \in \mathcal{G}_0 + \mathcal{I} \quad \text{for every } y, z \in \mathcal{G}$$

because $[y, z^*] \in \mathcal{I}$. So we have

$$\mathcal{G} = [\mathcal{G}, \mathcal{G}] \subset \mathcal{G}_0 + \mathcal{I}$$

and

$$\mathcal{G}^+ \subset \mathcal{G}_0 + \mathcal{I}$$

since $(\mathcal{G}_0 + \mathcal{I})^* = \mathcal{G}_0^* + \mathcal{I}^* = \mathcal{G}_0 + \mathcal{I}$.

We denote

$$\mathcal{I}_0 = \mathcal{G}_0 \cap \mathcal{I} = \sum_{j,k} \mathcal{G}_0 \cap [\mathcal{G}_j, \mathcal{G}_k^*].$$

Obviously we have

$$[\mathcal{G}_0, \mathcal{I}_0] \subset [\mathcal{G}_0, \mathcal{I}] = \{0\}, \quad [\mathcal{I}, \mathcal{I}_0] \subset [\mathcal{I}, \mathcal{G}_0] = \{0\}, \quad [\mathcal{G}, \mathcal{I}_0] \subset [\mathcal{G}_0 + \mathcal{I}, \mathcal{I}_0] = \{0\}.$$

We will describe the action of $\operatorname{ad} \mathcal{G}$ on $\mathcal{I}$. Let $x, y, z \in \mathcal{G}$ (normal elements in $\mathcal{L}$) and

$$[[x+z, y], (x+z)^*] = [x+z, [y, (x+z)^*]] = [x, [y, x^*]] + [x, [y, z^*]] + [z, [y, x^*]] + [z, [y, z^*]] =$$

$$= [[x, y], x^*] + [x, [y, z^*]] + [z, [y, x^*]] + [[z, y], z^*].$$

We have

$$[z, [y, x^*]] = [z, [y^*, x]] = [[z, y^*], x] + [y^*, [z, x]]$$

because $[y, x^*]$ is skew (i.e. $[y, x^*] = [y^*, x]$). Hence we may write,

$$[[x+z,y],(x+z)^*] = [[x,y],x^*] + 2[x,[y,z^*]] + [y^*,[z,x]] + [[z,y],z^*]$$

or equivalent,

$$(1) \qquad 2[[x,[y,z^*]] = [[x+z,y],(x+z)^*] - [[x,y],x^*] - [y^*,[z,x]] - [[z,y],z^*].$$

If we take $x \in \mathcal{G}_i,\ y \in \mathcal{G}_j,\ z \in \mathcal{G}_k$, by (1) there are the following possibilities

$$[\mathcal{G}_i,[\mathcal{G}_j,\mathcal{G}_k^*]] = 0, \quad \text{for } i \neq j \neq k \neq i;$$

$$[\mathcal{G}_k,[\mathcal{G}_j,\mathcal{G}_k^*]] \subset [\mathcal{G}_j,\mathcal{G}_k^*] = [\mathcal{G}_k,\mathcal{G}_j^*].$$

Also we have,

$$[\mathcal{G}_j,[\mathcal{G}_j,\mathcal{G}_k^*]] = [\mathcal{G}_j,[\mathcal{G}_k^*,\mathcal{G}_j]] \subset [\mathcal{G}_k,\mathcal{G}_j^*] = [\mathcal{G}_j,\mathcal{G}_k^*]$$

$$[\mathcal{G}_i,[\mathcal{G}_k,\mathcal{G}_k^*]] \subset [\mathcal{G}_k,\mathcal{G}_i^*], \quad [\mathcal{G}_i,[\mathcal{G}_i,\mathcal{G}_i^*]] \subset [\mathcal{G}_i,\mathcal{G}_i^*].$$

We denote $\mathcal{I}_{jk} = [\mathcal{G}_j,\mathcal{G}_k^*]$ $(= [\mathcal{G}_k,\mathcal{G}_j^*] = [\mathcal{G}_j^*,\mathcal{G}_k] = [\mathcal{G}_k^*,\mathcal{G}_j])$. Then the above inclusions give the following:

$$[\mathcal{G},\mathcal{I}_{jk}] \subset \mathcal{I}_{jk}, \quad \text{for } j \neq k,\ [\mathcal{G},\mathcal{I}_{jj}] \subset \sum_k \mathcal{I}_{kj},\ [\mathcal{G},\sum_k \mathcal{I}_{kj}] \subset \sum_k \mathcal{I}_{kj}.$$

If we suppose that the ground field is of characteristic zero then for $j \neq k$, $\operatorname{ad}\mathcal{G}|\mathcal{I}_{jk}$ are completely reducible because $\mathcal{G}$ is semisimple and $\mathcal{I}_{jk}$ are finite dimensional ideals. Let $\mathcal{I}_1$ be a complement of $\mathcal{I}_0 \cap \mathcal{I}_{jk}$ in $\mathcal{I}_{jk}$. We have

$$[\mathcal{G},\mathcal{I}_{jk}] = [\mathcal{G},\mathcal{I}_0 \cap \mathcal{I}_{jk} + \mathcal{I}_1] = [\mathcal{G},\mathcal{I}_1] \subset \mathcal{I}_1.$$

But $[\mathcal{G}_j,\mathcal{G}_k^* + \mathcal{I}_{jk}] \subset \mathcal{I}_{jk}$. Hence, by the same reason we may find a complement $\mathcal{I}_{jk}^{\perp}$ (of $\mathcal{I}_{jk}$ in $\mathcal{G}_k^* + \mathcal{I}_{jk}$) invariant to $\operatorname{ad}\mathcal{G}_j$. We have,

$$[\mathcal{G}_j,\mathcal{I}_{jk}^{\perp}] \subset \mathcal{I}_{jk}^{\perp} \cap \mathcal{I}_{jk} = \{0\},$$

$$\mathcal{I}_{jk} = [\mathcal{G}_j,\mathcal{G}_k^*] \subset [\mathcal{G}_j,\mathcal{G}_k^* + \mathcal{I}_{jk}] = [\mathcal{G}_j,\mathcal{I}_{jk} + \mathcal{I}_{jk}^{\perp}] = [\mathcal{G}_j,\mathcal{I}_{jk}] \subset \mathcal{I}_{jk}.$$

Hence $\mathcal{I}_{jk} = [\mathcal{G}_j,\mathcal{I}_{jk}]$ and $[\mathcal{G},\mathcal{I}_{jk}] \supset \mathcal{I}_{jk}$. We obtain,

$$\mathcal{I}_{jk} \subset [\mathcal{G},\mathcal{I}_{jk}] \subset [\mathcal{G},\mathcal{I}_1] \subset \mathcal{I}_1.$$

It follows that $\mathcal{I}_{jk} = \mathcal{I}_1$, hence $\mathcal{I}_0 \cap \mathcal{I}_{jk} = \{0\}$ for $j \neq k$ and $\mathcal{I}_0 = \sum_{jk} \mathcal{G}_0 \cap \mathcal{I}_{jk} =$ $= \sum_j \mathcal{G}_0 \cap \mathcal{I}_{jj}$. On the other hand, by theorem of [4] we obtain

$$\mathcal{G}_0 \cap \mathcal{I}_{jj} = \{z \in \mathcal{G}_j^{+} \,|\, [z,\mathcal{I}_{jj}] = 0\} = \{0\}$$

because $\mathcal{I}_{jj} = [\mathcal{G}_j, \mathcal{G}_j^*] \subset \mathcal{G}_j^+ = \mathcal{G}_j + \mathcal{G}_j^* + [\mathcal{G}_j, \mathcal{G}_j^*]$.

Hence, $\mathcal{I}_0 = \mathcal{G}_0 \cap \mathcal{I} = \{0\}$ and $\mathcal{G}^+ = \mathcal{G}_0 \oplus \mathcal{I}$. So we have proved the extension of the main result of Sherman's Theorem 1 stated in introduction, for $\mathcal{G}$ an ideally finite semisimple Lie algebra.

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Received June 20, 1993.