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STRONG MORITA EQUIVALENCE FOR THE QUASI-ROTATION C^*-ALGEBRAS

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INTRODUCTION.

Let $\varphi(z) = aA(z)$, $z \in \mathbf{T}^2$, where $A \in \mathrm{GL}(2,\mathbf{Z})$ and $a \in \mathbf{T}^2$, be an affine transformation of the 2-torus $\mathbf{T}^2$. It is called a *quasi-rotation* if $A \neq I_2$ and φ has an eigenvalue $\lambda \neq 1$ and an invertible function $f \in C(\mathbf{T}^2)$ such that $f \circ \varphi = \lambda f$. Such a transformation has a unique (up to conjugacy) primitive eigenvalue which is denoted by $X_A(a)$. Denote by $\mathcal{B}(a, A)$ the associated crossed product C^*-algebra of the continuous functions on $\mathbf{T}^2$, $C(\mathbf{T}^2)$, by φ. These algebras were classified up to isomorphism for two cases:

(1) when the primitive eigenvalue (or "rotation" angle) $X_A(a) = e^{2\pi i\theta}$ of φ has irrational angle θ, and

(2) when θ is rational, in the orientation reversing case ($\det(A) = -1$).

A complete invariant for $\mathcal{B}(a, A)$ consists of the primitive eigenvalue $X_A(a)$ of φ (which comes from the range trace on its K_0-group), the determinant of $A(= \pm 1)$, and the order $m(A)$ of the torsion part of its K_1-group (see [18] and [19]). It turns out that $m(A)$ (along with $\det(A)$) classifies the conjugacy classes of all such A's in $\mathrm{GL}(2,\mathbf{Z})$.

In the present paper the author's aim is to study the strong Morita equivalence of these algebras. Recall that Rieffel studied this problem for the irrational rotation algebras $\mathcal{A}_\theta$ in [13]. He proved that their Morita equivalence is determined by the action of the group $\mathrm{GL}(2,\mathbf{Z})$. For irrational numbers $\theta, \theta', \mathcal{A}_\theta$ and $\mathcal{A}_{\theta'}$ are strongly Morita equivalent if and only if

$$\theta' = \frac{a\theta + b}{c\theta + d} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbf{Z}).$$

Here, GL(2, $\mathbf{Z}$) "acts" on the reals by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \theta = \frac{a\theta + b}{c\theta + d}$$

whenever it is defined. For rational angles θ, Rieffel showed that A_θ is strongly Morita equivalent to the C^*-algebra of continuous functions on the 2-torus [16, Theorem 3.1]. Two essential ingredients which he used to show one direction of the former result is the computation of the tracial range on the K_0-group and the fact that the tracial ranges of strongly Morita equivalent algebras are scalar multiplies of each other (this holds when all tracial states on the algebra induce the same map on the K_0-group). Implicit use of these two facts will be made in the following treatment as well as the computation of the tracial range or the algebras $\mathcal{B}(a, A)$ done in [18] and [19].

The determination of the strong Morita equivalence of the algebras $\mathcal{B}(a, A)$ for the orientation preserving case ($\det(A) = 1$) has been done by the (independent) work of Ji [7, Theorem 4.11] and Packer [10, Theorem 4.1] on the classification of the crossed products of $C(\mathbf{T}^2)$ by the Anzai transformations $\varphi_{\theta,k}(x, y) = (e^{2\pi i\theta}x, x^k y)$, $k \neq 0$, and using the elementary fact that an affine orientation preserving quasi-rotation $\varphi(z) = aA(z)$ is topologically conjugate to $\varphi_{\theta,k}$, where $e^{2\pi i\theta} = X_A(a)$ and $k = m(A)$ [19, Lemma 2.1]. Letting $H_{\theta,k} = C(\mathbf{T}^2) \times_{\varphi^*_{\theta,k}} \mathbf{Z}$, and letting $\sim$ denote strong Morita equivalence, Ji and Packer showed, for irrational numbers θ, θ', that $H_{\theta,k} \sim H_{\theta',k'}$ if and only if $|k| = |k'|$ and θ, θ' are in the same GL(2, $\mathbf{Z}$)-orbit. Further, Ji[7, Theorem 4.12] showed that for all rational numbers θ one has $H_{\theta,k} \sim H_{0,k}$. Thus one obtains:

(1) For the orientation preserving irrational affine quasi-rotations, $\mathcal{B}(a, A) \sim$ $\sim \mathcal{B}(a', A')$ if and only if $m(A) = m(A')$ and θ, θ' are in the same GL(2, $\mathbf{Z}$)-orbit, where $e^{2\pi i\theta} = X_A(a)$, $e^{2\pi i\theta'} = X_{A'}(a')$; and

(2) for the orientation preserving rational affine quasi-rotations, $\mathcal{B}(a, A) \sim$ $\sim H_{0,m(A)}$.

The author sought to investigate what would happen in the orientation reversing case, whether the orientation-reversing feature showed itself somehow in the classification of the strong Morita equivalence classes. In [19, Lemma 2.4] it was shown that there are only two orientation reversing affine quasi-rotations of $\mathbf{T}^2$ up to topological conjugacy. The basic ones being given by $\varphi(x, y) = (e^{2\pi i\theta}x, \overline{y})$ and $\psi(x, y) =$ $= (e^{2\pi i\theta}x, x\overline{y})$. Throughout, we shall let $\mathcal{B}_\theta = C(\mathbf{T}^2) \times_{\varphi^*} \mathbf{Z}$ and $\mathcal{C}_\theta = C(\mathbf{T}^2) \times_{\psi^*} \mathbf{Z}$ denote their respective crossed product algebras, and our attention in this paper will focus entirely upon the determination of their strong Morita equivalence classes for rational θ (Sections 2 and 3) and non-quadratic θ (Section 4). Some partial results in the quadratic case are later given for the algebras $\mathcal{B}_\theta$.

Recall that a real number θ is said to be non-quadratic if it does not satisfy a (non-trivial) quadratic equation with integer coefficients.

The tracial range on the K_0-group of $\mathcal{B}_\theta$ (and $\mathcal{C}_\theta$) is $\mathbf{Z}+\theta\mathbf{Z}$. So if θ,θ' are irrational and if $\mathcal{B}_\theta$ and $\mathcal{B}_{\theta'}$ are strongly Morita equivalent then $\mathbf{Z}+\theta'\mathbf{Z}$ is a multiple of $\mathbf{Z}+\theta\mathbf{Z}$, and hence θ' in the orbit of θ under the action of the group $\mathrm{GL}(2,\mathbf{Z})$ (as alluded to above). However, as will be shown, the converse does not hold. In fact, the Morita equivalence will be shown to be determined by the subgroup $\mathrm{GL}_e(2,\mathbf{Z})$ of $\mathrm{GL}(2,\mathbf{Z})$ consisting of the matrices with even $(2,1)$-entry. More precisely, the aim of this paper will be to prove the following.

THEOREM.
(1) *$\mathcal{B}_\theta \sim \mathcal{B}_{X\theta}$ for all $X\in\mathrm{GL}_e(2,\mathbf{Z})$ and all $\theta\in\mathbf{R}$, whenever $X\theta$ is defined.*
(2) *$\mathcal{C}_\theta \sim \mathcal{C}_{X\theta}$ for all $X\in\mathrm{GL}_e(2,\mathbf{Z})$ and all $\theta\in\mathbf{R}$, whenever $X\theta$ is defined.*
(3) *For $\theta=p/q$, where p, q are positive relatively prime integers, one has:*

$$\mathcal{B}_\theta\sim\begin{cases}\mathcal{B}_0 & \text{if } q \text{ is odd}\\ \mathcal{B}_{1/2} & \text{if } q \text{ is even}\end{cases}\qquad \mathcal{C}_\theta\sim\begin{cases}\mathcal{C}_0 & \text{if } q \text{ is odd}\\ \mathcal{C}_{1/2} & \text{if } q \text{ is even.}\end{cases}$$

(4) *$\mathcal{B}_{1/2}$ and $\mathcal{B}_0$ are not strongly Morita equivalent; ditto $\mathcal{C}_{1/2}$ and $\mathcal{C}_0$.*
(5) *For the orientation reversing rational affine quasi-rotations with primitive eigenvalue $X_A(a)=e^{2\pi i(p/q)}$, where p, q are relatively prime positive integers, one has*

$$\mathcal{B}(a,A)\sim\begin{cases}\mathcal{B}_0 & \text{if } q \text{ is odd and } m(A)=2,\\ \mathcal{B}_{1/2} & \text{if } q \text{ is even and } m(A)=2,\\ \mathcal{C}_0 & \text{if } q \text{ is odd and } m(A)=1,\\ \mathcal{C}_{1/2} & \text{if } q \text{ is even and } m(A)=1.\end{cases}$$

(6) *For the non-quadratic numbers θ,θ' one has: $\mathcal{B}_\theta\sim\mathcal{B}_{\theta'}$ if and only if $\theta'=X\theta$ for some $X\in\mathrm{GL}_e(2,\mathbf{Z})$.*
(7) *For the non-quadratic numbers θ,θ', one has: $\mathcal{C}_\theta\sim\mathcal{C}_{\theta'}$ if and only if $\theta'=X\theta$ for some $X\in\mathrm{GL}_e(2,\mathbf{Z})$.*
(8) *Partial result:* (6) *holds for all quadratic irrationals θ and θ' satisfying $\alpha\theta^2+\beta\theta+$ $+\gamma=0$, where α, β and γ are integers with $gcd(\alpha,\beta,\gamma)=1$ and β is even.*

The orientation reversing nature of the underlying transformations reveals itself in two ways. Through their associated *rational* quasi-rotation algebras $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ which are not all strongly Morita equivalent (as in the case for the rational rotation C^*-algebras $\mathcal{A}_\theta$ and the rational Anzai C^*-algebras $H_{\theta,k}$). Furthermore, unlike the rotation C^*-algebras $\mathcal{A}_\theta$ for which $\mathcal{A}_\theta\sim\mathcal{A}_{1/\theta}$ (for all $\theta\neq 0$), this does not hold for $\theta=2$, as $\mathcal{B}_{1/2}$ and $\mathcal{B}_2=\mathcal{B}_0$ are not strongly Morita equivalent by (4). It is also revealed through the fact that the strong Morita equivalence classes of $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$, for (at least) non-quadratic θ, is governed by the action of the subgroup $\mathrm{GL}_e(2,\mathbf{Z})$.

On the slightly more abstract level, use is made of the well-known fact from vector bundle theory which says that from every vector bundle of dimension $\geqslant 2$ over a (compact and connected) CW-complex X of dimension $\leqslant 3$, one can split off a trivial line subbundle as a direct summand. From this it will follow that $C(X)$ is the only unital integral C^*-algebra which is strongly Morita equivalent to $C(X)$ (Theorem 2.7). This, in turn, will imply that no rotation algebra (rational or irrational) is strongly Morita equivalent to any non-commutative unital integral C^*-algebra (which answers a question raised in [18, Remark p. 60]), and also that $\mathcal{B}_0$ and $\mathcal{B}_{1/2}$ are not strongly Morita equivalent. Another, less *ad hoc*, proof of the latter is given in Section 3 which works for $\mathcal{C}_0$ and $\mathcal{C}_{1/2}$.

From the above theorem it follows that for the orientation reversing non-quadratic affine quasi-rotations on the 2-torus, one has: $\mathcal{B}(a, A) \sim \mathcal{B}(a', A')$ if and only if θ, θ' are in the same $\mathrm{GL}_\epsilon(2, \mathbf{Z})$-orbit and $m(A) = m(A')$ (Section 4).

A summary of the classifications in the various cases is shown by the following table:

Classifications for $\mathcal{B}(a, A)$		
	Isomorphism	Strong Morita Equivalence
Orientation Preserving	Rational: unknown Irrational: [18] or [19]	Rational: [7] Irrational: [7] or [10]
Orientation Reversing	Rational: [18] or [19] Irrational: [18] or [19]	Rational: present paper Irrational: present paper for non-quadratics

Assertions (1), (2), and (3) of the above theorem are proved (in Section 2) by applying Phil Green's Theorem [14] and a certain lemma on $\mathrm{GL}(2, \mathbf{Z})$ proved in Section 1 below. In fact, application of Green's result shows that crossed products of certain rotations on the Klein bottle arise naturally from consideration of the orientation reversing quasi-rotations on the torus, which in itself yields some information above the former (cf. Corollaries 2.4 and 4.6 below). Assertion (5) will then follow from these and the fact that there are only two orientation reversing affine quasi-rotations up to topological conjugacy. To prove (4) we use the fact that every irreducible representation of the algebras $\mathcal{B}_{1/2}$, $\mathcal{B}_0$, $\mathcal{C}_{1/2}$ and $\mathcal{C}_0$ is finite-dimensional, and thereby compute their primitive spectra using [8]. Once they are found one invokes the result of Rieffel in [15] that the primitive spectrum is topologically invariant under strong Morita equivalence. For instance, it is shown that $\mathrm{Prim}(\mathcal{B}_0)$, as a topological manifold, has boundary, whereas $\mathrm{Prim}(\mathcal{B}_{1/2})$ is the direct product of the Klein bottle and a circle, which has no boundary.

In proving (6) and (7) one considers the elementary invariant, denoted by $\natural(\mathcal{A})$, consisting of the strong Morita equivalence classes of the *simple* quotients of $\mathcal{A}$. Since $\mathcal{B}_\theta$ and $\mathcal{C}_\theta$ are non-simple C^*-algebras, this invariant is particularly convenient for their study and shows that their simple quotients consist of certain irrational rotation algebras, the classification of which is used to obtain (6) and (7). For instance, it is shown, for θ non-quadratic, that $\mathcal{B}_\theta$ has exactly two simple quotients, $\mathcal{A}_\theta$ and $M_2(\mathcal{A}_{2\theta})$, so that $\sharp(\mathcal{B}_\theta)$ has cardinality 2 (this number is invariant under strong Morita equivalence); cf. Section 4. Also, $\sharp(\mathcal{C}_\theta)$ has cardinality 3.

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1. LEMMAS RELATED TO GL(2, Z) AND GL$_e$(2, Z).

Let

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = UVU.$$

It is known that $\mathrm{GL}(2,\mathbf{Z}) = \langle U, V\rangle$, where $\langle U, V\rangle$ denotes the subgroup generated by U, V (see Kurosh [9], Appendix B). Recall that $\mathrm{SL} = \mathrm{SL}(2,\mathbf{Z})$ is the subgroup of $\mathrm{GL}(2,\mathbf{Z})$ of matrices of determinant 1. We shall let SL_e denote the subgroup of SL consisting of matrices with even $(2,1)$-entry.

LEMMA 1.1 $\mathrm{SL}(2,\mathbf{Z}) = \langle R, V\rangle$.

Proof. Any $A \in \mathrm{SL}$ can be written as

$$A = U^{n_1}V^{m_1}\dots U^{n_k}V^{m_k},$$

where $n_i = 0, 1$. As the determinant of A is 1, there is an even number of U's in this expression for A. One argues inductively as follows. Let us require that the expression for A is written in reduced form. If $n_1 = 1$ then $n_2 = 1$, and since $UV^{m_1}U = R^{m_1}$ one obtains

$$A = R^{m_1}V^{m_2}U^{n_3}V^{m_3}\dots U^{n_k}V^{m_k}.$$

If $n_1 = 0$ the same argument applies to the remainder $U^{n_2}V^{m_2}\dots U^{n_k}V^{m_k}$. In this way we can express A in terms of powers of R and V. ■

Let $\mathcal{H} = \langle R^2, V\rangle \subseteq \mathrm{SL}_e(2,\mathbf{Z})$. We claim that $\mathcal{H} = \mathrm{SL}_e$ (the author is indebted to George Skandalis for suggesting an index argument to achieve this). To do this we need the following.

LEMMA 1.2 *We have the (disjoint) coset decomposition*

$$\mathrm{SL}(2,\mathbb{Z}) = \mathcal{H} + \mathcal{H}R + \mathcal{H}RV.$$

Hence the index is $[\mathrm{SL}(2,\mathbb{Z}) : \mathcal{H}] = 3$, *and* $\mathcal{H} = \mathrm{SL}_e(2,\mathbb{Z})$.

Proof. It is easy to verify the formula

$$RVR^{-1} = VR^{-1}V^{-1}$$

from which we obtain

$$RV^m = (VR^{-m}V^{-1})R.$$

It is also easy to check that $-I \in \mathcal{H}$, where I is the identity matrix. Thus,

$$-I = UV^{-2}UVUV^{-2}UV = R^{-2}VR^{-2}V \in \mathcal{H}.$$

From this we also have the formula

$$-V^{-1}R^2 = R^{-2}V.$$

Fix

$$A = R^{n_1}V^{m_1}\ldots R^{n_{k-1}}V^{m_{k-1}}R^{n_k}V^{m_k},$$

and write it as $A = XR^{n_k}V^{m_k}$ where $X = R^{n_1}V^{m_1}\ldots R^{n_{k-1}}V^{m_{k-1}}$. We now use induction and assume that X is in the coset decomposition in the statment of the lemma and then show that A is also in the coset decomposition. For simplicity write $A = XR^nV^m$.

Case 1: $X \in \mathcal{H}$. If n is even then $A \in \mathcal{H}$, done. If n is odd, then $A = XR^nV^m \in \mathcal{H}R^nV^m = \mathcal{H}RV^m$. If m is even, $A \in \mathcal{H}(VR^{-m}V^{-1})R = \mathcal{H}R$, by above formula. If m is odd, say $m = 2s+1$, then

$$A \in \mathcal{H}RV^{2s}V = \mathcal{H}(VR^{-2s}V^{-1}R)V = \mathcal{H}RV.$$

Case 2: $X \in \mathcal{H}R$. Here, $A \in \mathcal{H}R^{n+1}V^m$ and this reduces to case 1 again.

Case 3: $X \in \mathcal{H}RV$. Suppose first that n is even, say $n = 2s$. Then $R^n = R^{2s} = (-I)^s(VR^{-2}V)^s$, hence using the above formulas and that $(-I)^s \in \mathcal{H}$ we have

$$\begin{aligned} A \in \mathcal{H}RVR^nV^m &= \mathcal{H}RV(-I)^s(VR^{-2}V)^sV^m = \mathcal{H}RV(VR^{-2}V)(VR^{-2}V)^{s-1}V^m = \\ &= \mathcal{H}RV^2R^{-2}V(VR^{-2}V)^{s-1}V^m = \mathcal{H}(VR^{-2}V^{-1})RR^{-2}V(VR^{-2}V)^{s-1}V^m = \\ &= \mathcal{H}RV(VR^{-2}V)^{s-1}V^m = \ldots = \mathcal{H}RV^{m+1}. \end{aligned}$$

and this reduces to case 1.

Now suppose that n is odd, $n = 2s+1$. Then

$$A \in \mathcal{H}RVR^nV^m = \mathcal{H}R(VR^{2s})RV^m = \mathcal{H}R(RV^{-2s}R^{-1}V)RV^m =$$
$$= \mathcal{H}R^{-1}VRV^m = \mathcal{H}R^{-1}(RV^{-1}R^{-1}V)V^m = \mathcal{H}R^{-1}V^{m+1} = \mathcal{H}RV^{m+1}$$

and again this reduces to case 1.

To see that the decomposition is disjoint we observe that R, RV, $RVR^{-1} \notin \mathcal{H}$ since they have odd $(2,1)$-entry.

For the final assertion note that as we have $\mathcal{H} \subseteq \mathrm{SL}_e(2,\mathbf{Z}) \subset \mathrm{SL}(2,\mathbf{Z})$ and the index $[\mathrm{SL}(2,\mathbf{Z}) : \mathcal{H}] = 3$ is prime, thus $\mathcal{H} = \mathrm{SL}_e(2,\mathbf{Z})$. ∎

Let $S = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $U = SRV^{-1}$ we get $\mathrm{GL}(2,\mathbf{Z}) = \langle R, S, V\rangle$. Let $\widetilde{\mathcal{H}} = \langle R^2, S, V\rangle$, so that $\mathcal{H} \subset \widetilde{\mathcal{H}} \subseteq \mathrm{GL}_e(2,\mathbf{Z})$. The previous lemma gives us the following extension:

Lemma 1.3. *We have the (disjoint) coset decomposition*

$$\mathrm{GL}(2,\mathbf{Z}) = \widetilde{\mathcal{H}} + \widetilde{\mathcal{H}}R + \widetilde{\mathcal{H}}RV.$$

Hence $[\mathrm{GL}(2,\mathbf{Z}) : \widetilde{\mathcal{H}}] = 3$ *and* $\widetilde{\mathcal{H}} = \mathrm{GL}_e(2,\mathbf{Z})$.

Proof. As we argued in the proof of the previous lemma, the coset decompostion is disjoint and that $\widetilde{\mathcal{H}} = \mathrm{GL}_e(2,\mathbf{Z})$ follows in the same way from the fact that the index is 3.

Pick $A \in \mathrm{GL}(2,\mathbf{Z})$. If $\det(A) = 1$, then $A \in \mathrm{SL}(2,\mathbf{Z})$ and by Lemma 1.2 we're done. If $\det(A) = -1$, then $SA \in \mathrm{SL}(2,\mathbf{Z})$ and we're done again. ∎

Lemma 1.4. *Let* $\theta = p/2q < 1$, *where* p, $2q$ *are positive relatively prime integers. Then* $\exists X_1, \ldots, X_n \in \{S, R^{-2}\} \subset \mathrm{GL}_e(2,\mathbf{Z})$ *such that* $X_i \ldots X_n\theta$ *is defined for* $1 \leqslant i \leqslant n$ *and* $X_1 \ldots X_n\theta = 1/2$.

Proof. If $p = q$, then $p = q = 1$ so $\theta = 1/2$ and there is nothing to do. Assume therefore that $p \neq q$.

First observe that θ can be reduced so that $p < q$. For suppose $q < p$, then $q < p < 2q$. Then $S\theta = 1 - (p/2q) = p'/2q$ where $p' = 2q - p$, $0 < p' < q$, as desired.

Thus assume $p < q$. Now we show that if $p > 1$ then $\exists X_1, \ldots, X_m \in \{S, R^{-2}\}$ such that

$$X_1 \ldots X_m \left(\frac{p}{2q}\right) = \frac{p_1}{2q_1},$$

where $0 < p_1 < p$, $0 < q_1 < q$ and $p_1 < q_1$, where each successive operation is defined.

To see this let R^{-2} act r times on $\theta = p/2q$, where r is the positive integer such that $rp < q < (r+1)p$. Thus

$$R^{-2r}\left(\frac{p}{2q}\right) = \begin{pmatrix} 1 & 0 \\ -2r & 1 \end{pmatrix} \frac{p}{2q} = \frac{p}{2q_1},$$

where $0 < q_1 = q - rp < p$. Now since $q_1 < p$ we can apply S as in above observation to reduce $p/2q_1$ as follows:

$$SR^{-2r}\left(\frac{p}{2q}\right) = S\left(\frac{p}{2q_1}\right) = \frac{p_1}{2q_1},$$

where $p_1 = 2q_1 - p < q_1$, $0 < p_1 < p$ and $0 < q_1 < q$, as desired.

Proceeding in this way, inductively, we eventually get

$$SR^{-2r_m} \dots SR^{-2r_1}\left(\frac{p}{2q}\right) = \frac{p_m}{2q_m} = \frac{1}{2q_m},$$

where $p_m = 1 < q_m$. Now letting R^{-2} act $q_m - 1$ times on $1/2q_m$ we obtain

$$R^{-2(q_m-1)}\left(\frac{1}{2q_m}\right) = \frac{1}{2}.$$

■

The reason of course we should be careful to ensure that each of the operations is defined is because, for instance, R^{-2} is not defined at $1/2$. By, contrast, for $\theta = p/q$, where q is odd, the operation $X\theta$ is always defined for $X \in \mathrm{GL}_e(2, \mathbb{Z})$. This case is easier and we have:

LEMMA 1.5. *Let $\theta = p/q$, where p, q are positive relatively prime integers and q is odd. Then $\exists X \in \mathrm{GL}_e(2, \mathbb{Z})$ such that $X\theta = 0$.*

Proof. Since q is odd and q, $2p$ are relatively prime, there are integers m, n such that $qm + 2pn = 1$. Then

$$X = \begin{pmatrix} q & -p \\ 2n & m \end{pmatrix}$$

is in $\mathrm{GL}_e(2, \mathbb{Z}) = \widetilde{\mathcal{H}}$ and $X\theta = 0$. ■

2. STRONG MORITA EQUIVALENCE

The original definition of strong Morita equivalence of two C^*-algebras $\mathcal{A}$, $\mathcal{B}$ is given by the existance of an imprimitivity $\mathcal{A}$, $\mathcal{B}$-bimodule Ξ [17, Definition 6.10]. We shall abbreviate this by writing $\mathcal{A} \sim \mathcal{B}$. From this definition one easily checks that if e

is a projection in a C^*-algebra $\mathcal{A}$ such that $e\mathcal{A}e$ is a full corner of $\mathcal{A}$ (i.e., not contained in any proper 2-sided ideal of $\mathcal{A}$), then $\Xi = \mathcal{A}e$ is an imprimitivity $\mathcal{A}$, $e\mathcal{A}e$-bimodule. Thus, $\mathcal{A} \sim e\mathcal{A}e$. Combining this together with Rieffel's result [13, Propostion 2.1] we may use the following as an equivalent definition of strong Morita equivalence: For $\mathcal{A}$, $\mathcal{B}$ unital C^*-algebras, $\mathcal{A} \sim \mathcal{B}$ if and only if either one of them is isomorphic to a full corner of some matrix algebra over the other. Thus, $\mathcal{B} \cong eM_n(\mathcal{A})e$ is a full corner of $M_n(\mathcal{A})$, where n is some positive integer and e is a projection in $M_n(\mathcal{A})$.

Let us now restate a result due to Phil Green as described in Rieffel's "Situation 10"[14].

Let H be a locally compact group acting on a locally compact Hausdorff space X. The action of H on X is said to be *free* if whenever $hx = x$ for some $h \in H$ and $x \in X$, then $h = 1_H$ = identity of H. The action is said to be *wandering* if for any compact subset Q of X the set $\{h \in H : hQ \cap Q \neq \emptyset\}$ has compact closure in H. These two conditions together ensure that the orbit space X/H is a locally compact Hausdorff space.

THEOREM 2.1. *(Phil Green, cf. [14]) Let G, H be two locally compact groups which act on a locally compact Hausdorff space X so that the actions commute, are free and wandering. Then*

$$C_0(X/H) \times G \sim C_0(X/G) \times H.$$

Now we can apply this result in our case with $G = H = \mathbf{Z}$.

THEOREM 2.2. *Let θ be real number.*
(1) $\mathcal{B}_\theta \sim \mathcal{B}_{A\theta}$ for all $A \in \mathrm{GL}_e(2, \mathbf{Z})$, whenever $A\theta$ is defined.
(2) For $\theta = m/n$, where m, n are positive relatively prime integers, one has:

$$\mathcal{B}_\theta \sim \begin{cases} \mathcal{B}_0 & \text{if } n \text{ is odd} \\ \mathcal{B}_{1/2} & \text{if } n \text{ is even.} \end{cases}$$

(3) $\mathcal{B}_{1/2}$ and $\mathcal{B}_0$ are not strongly Morita equivalent.

Proof. If $\theta = 0$, the result is trivial. Suppose then that $\theta \neq 0$. The map $\beta_\theta(t, y) = (t + \theta, \overline{y})$ induces the map $\varphi(x, y) = (e^{2\pi i\theta}x, \overline{y})$ on the orbit space of $X = \mathbf{R} \times \mathbf{T}$ by $\alpha(t, y) = (t + 1, y)$. Now the actions of $\mathbf{Z}$ induced by β_θ and α clearly commute, are free (as $\theta \neq 0$) and wandering. By Green's theorem we have

$$\mathcal{B}_\theta \cong C\left(\frac{\mathbf{R} \times \mathbf{T}}{\alpha}\right) \times_{\beta_\theta^*} \mathbf{Z} \sim C\left(\frac{\mathbf{R} \times \mathbf{T}}{\beta_\theta}\right) \times_{\alpha^*} \mathbf{Z}.$$

The latter algebra can be suitably scaled as follows. Consider the homeomorphism $H : (\mathbf{R} \times \mathbf{T})/\beta_1 \to (\mathbf{R} \times \mathbf{T})/\beta_\theta$ given by $H[t, y] = [\theta t, y]_\theta$, where $[t, y]$ and $[t, y]_\theta$

denote the orbit elements under β_1 and β_θ, respectively. It is clearly well-defined, continuous, onto, one-to-one, and hence a homeomorphism. Conjugating the action of α on $(\mathbf{R}\times\mathbf{T})/\beta_\theta$ by H gives one the homeomorphism $\alpha_{1/\theta}$ on $(\mathbf{R}\times\mathbf{T})/\beta_1$ defined by

$$\alpha_{1/\theta}[t,y] = \left[t+\frac{1}{\theta}, y\right].$$

It is not hard to see that the orbit space $K = (\mathbf{R}\times\mathbf{T})/\beta_1$ is the Klein bottle and that the second variable (corresponding to $\overline{y}$) is the "twisted" variable. So $\alpha_{1/\theta}$ is a rotation by angle $1/\theta$ in the "untwisted" variable of the Klein bottle. Thus for all $\theta \neq 0$

$$\mathcal{B}_\theta \sim C(K)\times_{\alpha^*_{1/\theta}}\mathbf{Z} = \mathcal{K}_{1/\theta}.$$

Now as $\alpha_{1/\theta} = \alpha_{(1/\theta)+2}$ one obtains

$$\mathcal{B}_\theta \sim \mathcal{K}_{\frac{1}{\theta}} \cong \mathcal{K}_{\frac{1}{\theta}+2} = \mathcal{K}_{\frac{2\theta+1}{\theta}} \sim \mathcal{B}_{\frac{\theta}{2\theta+1}}.$$

Therefore, $\mathcal{B}_\theta \sim \mathcal{B}_{R^2\theta}$, if $R^2\theta$ is defined. This holds whenever θ is irrational or $\theta = m/n$ where n is odd. Since already $\mathcal{B}_\theta \cong \mathcal{B}_{V\theta} \cong \mathcal{B}_{S\theta}$, we see that $\mathcal{B}_{A\theta} \sim \mathcal{B}_\theta$ for all $A \in \widetilde{\mathcal{H}} = \mathrm{GL}_e(2,\mathbf{Z})$. If $\theta = m/2n$, where $gcd(m,2n) = 1$, then Lemma 1.4 shows that the same result holds in this case. In fact, in this case we have $\mathcal{B}_{A\theta} \sim \mathcal{B}_{1/2}$ for all $A \in \mathrm{GL}_e(2,\mathbf{Z})$. If $\theta = m/n$ where n is odd, then by Lemma 1.5 one has $\mathcal{B}_{A\theta} \sim \mathcal{B}_0$ for all $A \in \mathrm{GL}_e(2,\mathbf{Z})$.

For (3), we shall offer two proofs that $\mathcal{B}_0$ and $\mathcal{B}_{1/2}$ are not strongly Morita equivalent (see Corollary 2.8 and Section 3 below).

THEOREM 2.3.

(1) $\mathcal{C}_\theta \sim \mathcal{C}_{A\theta}$ for all $A \in GL_e(2,\mathbf{Z})$ and $\theta \in \mathbf{R}$, whenever $A\theta$ is defined.

(2) For $\theta = m/n$, where m, n are positive relatively prime integers, one has:

$$\mathcal{C}_\theta \sim \begin{cases} \mathcal{C}_0 & \text{if } n \text{ is odd} \\ \mathcal{C}_{1/2} & \text{if } n \text{ is even.} \end{cases}$$

(3) $\mathcal{C}_{1/2}$ and $\mathcal{C}_0$ are not strongly Morita equivalent.

Proof. We proceed similarly as in the above proof. Suppose $\theta \neq 0$. The map $\gamma_\theta(t,y) = (t+\theta,\ e^{2\pi it}\overline{y})$ induces $\psi(x,y) = (e^{2\pi i\theta}x, x\overline{y})$ on the orbit space of $X = \mathbf{R}\times\mathbf{T}$ under $\alpha(t,y) = (t+1,y)$. The actions of $\mathbf{Z}$ induced by γ_θ and α clearly commute, are free and wandering. Therefore, by Green's theorem we have

$$\mathcal{C}_\theta \cong C\left(\frac{\mathbf{R}\times\mathbf{T}}{\alpha}\right)\times_{\gamma_\theta^*}\mathbf{Z} \sim C\left(\frac{\mathbf{R}\times\mathbf{T}}{\gamma_\theta}\right)\times_{\alpha^*}\mathbf{Z}.$$

The latter may be properly scaled as follows. We use the homeomorphism

$$\Psi : \frac{\mathbf{R}\times\mathbf{T}}{\beta_1} \to \frac{\mathbf{R}\times\mathbf{T}}{\gamma_\theta},$$

given by

$$\Psi[t,y] = [\theta t, e^{2\pi i\theta(2t-1)/4}\overline{y}]_{\gamma_\theta},$$

where β_1 is as in the previous proof, to conjugate the action of α on $(\mathbf{R}\times\mathbf{T})/\gamma_\theta$ to yield the transformation $\alpha'_{1/\theta}$ on the Klein bottle K defined by

$$\alpha'_{1/\theta}[t,y] = [t+\frac{1}{\theta}, -y].$$

Thus for all $\theta \neq 0$,

$$\mathcal{C}_\theta \sim C(K)\times_{\alpha'_{1/\theta}} \mathbf{Z} = \mathcal{K}^-_{\frac{1}{\theta}}.$$

Now as $\alpha'_{1/\theta} = \alpha'_{(1/\theta)+2}$ we get

$$\mathcal{C}_\theta \sim \mathcal{K}^-_{\frac{1}{\theta}} \cong \mathcal{K}^-_{\frac{1}{\theta}+2} \sim \mathcal{C}_{\frac{\theta}{2\theta+1}} = \mathcal{C}_{R^2\theta},$$

if $R^2\theta$ is defined. Since also $\mathcal{C}_\theta \cong \mathcal{C}_{V\theta} \cong \mathcal{C}_{S\theta}$, we may use Lemmas 1.4 and 1.5 and argue as in the end of the preceding proof to obtain (1) and (2). The proof of (3) is deferred to Section 3. ∎

The proofs of the preceding two theorems contain the following result for crossed products of the Klein bottle by rational rotations along the "untwisted" component.

COROLLARY 2.4. *For $\theta = m/n$, where m, n are relatively prime positive integers, one has*

$$\mathcal{K}_\theta \sim \begin{cases} \mathcal{K}_0 & \text{if } m \text{ is even,} \\ \mathcal{K}_1 & \text{if } m \text{ is odd.} \end{cases}$$

Furthermore, $\mathcal{K}_0$ and $\mathcal{K}_1$ are not strongly Morita equivalent. The same result holds for the algebras $\mathcal{K}^-_\theta$.

Let us briefly describe how we shall show conditions (3) in the preceding two theorems. First, one shows that all the irreducible representations of the algebras $\mathcal{A} = \mathcal{B}_0, \mathcal{B}_{1/2}, \mathcal{C}_0, \mathcal{C}_{1/2}$ are finite dimensional. Secondly, we shall apply the computations for "finite" spectra given by Kawamura *et al* [8] to compute the spectrum $\widehat{\mathcal{A}}$, which coincides with the primitive spectrum Prim($\mathcal{A}$). Thirdly, we employ the result of Rieffel [15, Corollary 3.3] that Prim($\mathcal{A}$) is topologically invariant under strong Morita equivalence. Finally, one shows that Prim($\mathcal{B}_0$) and Prim($\mathcal{B}_{1/2}$) are not homeomorphic; ditto Prim($\mathcal{C}_0$) and Prim($\mathcal{C}_{1/2}$). The computation of Prim($\mathcal{A}$) could also be made by a (difficult and deep) result of Williams [21, Theorem 5.3].

For the remainder of this section we wish to prove that strong Morita equivalence of integral unital C^*-algebras with a CW-complex (cf. [2, page 89]) of low dimension reduces to mere isomorphism (Theorem 2.7). This will have as its consequence that $\mathcal{B}_0$ and $\mathcal{B}_{1/2}$ are not strongly Morita equivalent (cf. Corollary 2.8). However, the author is not aware whether there is a similar result for $\mathcal{C}_0$ and $\mathcal{C}_{1/2}$. In addition, this will answer a question raised in [18, Remark p.60] whether any rational rotation algebra $\mathcal{A}_\theta$ is strongly Morita equivalent to any crossed product of the form $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$ (Corollary 2.9). This is then subsequently generalized (Corollary 2.10).

LEMMA 2.5. *Let X be a compact and connected CW-complex of dimension $\leqslant 3$. If e is a non-zero projection in $M_n(C(X))$, then there exists a rank 1 projection $e' \in M_n(C(X))$ such that $e' \leqslant e$.*

Proof. For this we shall exploit the fact that for CW-complexes X of dimension $\leqslant 3$ one can always split off line bundles from every vector bundle over X of dimension $\geqslant 2$. Specifically, if X is an m-dimensional CW-complex and E is a vector bundle of dimension k such that $2k-1 \geqslant m$, then $E \cong E' \oplus$ (trivial line bundle) is a Whitney sum of subbundles (cf. Husemoller [6], Proposition 1.1, p. 99, or Theorem 7.1, p. 21).

Without loss of generality suppose $k = \operatorname{rank}(e) \geqslant 2$ (note that $\operatorname{rank}(e) = \operatorname{trace}(e)$ is continuous and hence constant on X, being connected.) In our case the condition $2k-1 \geqslant m$ is clearly met. Now $e \in C(X, \mathcal{L}(\mathbf{C}^n))$ defines a vector bundle E of dimension k with fibre $E_x = \operatorname{Im}(e(x))$. The above result gives us a nowhere-vanishing continuous cross-section ξ of E (and hence can be assumed to be normalized) and thus yields a rank 1 projection $e' \in C(X, \mathcal{L}(\mathbf{C}^n))$ given as orthogonal projections on $\xi(x)$:

$$e'(x)v = \langle v, \xi(x)\rangle \xi(x), \quad v \in \mathbf{C}^n,$$

i.e. $e' = \xi \otimes \xi$. Clearly, e' is continuous and $e' \leqslant e$. ■

LEMMA 2.6. *Let X be a compact and connected space and let $e \in M_n(C(X))$ be a projection of rank 1. Then*

$$eM_n(C(X))e \cong C(X).$$

Proof. An isomorphism may be given by $\Phi : C(X) \to eM_n(C(X))e$, $\Phi(f) = fe$. This is clearly an injective $*$-homomorphism. To see surjectivity fix $F \in M_n(C(X))$. Since $e(x)$ has rank 1, for each $x \in X$ there is a unique scalar $f(x)$ such that $e(x)F(x)e(x) = f(x)e(x)$. Evidently, f is continuous (as e never vanishes) and $eFe = \Phi(f)$. ■

REMARK. Note that Lemma 2.6 does not extend to projections e of rank $k \geqslant 2$, i.e. we cannot expect that $eM_n(C(X))e \cong M_k(C(X))$. As is well-known, we may take X to be the 6-sphere S^6 and e the projection associated with the tangent bundle of S^6 with its natural almost complex structure, the latter gives it the structure of a complex vector bundle E of dimension 3. It is known that this bundle has no line subbundles (cf. [20, Remark p. 274]). So e contains no projections of rank 1, and therefore the assumption that X has dimension at most 3 in Lemma 2.5 cannot be weakened. Another example may be provided by letting X be the complex projective 2-space $P^2(\mathbf{C})$ (a CW-complex of dimension 4) where its tangent space is viewed as a 2-dimensional complex vector bundle.

REMARK. Using Lemma 2.6 one can give an elementary argument to show that strong Morita equivalence of two $C(X)$'s, where X is a compact connected space, reduces to isomorphism and hence to homeomorphism of the underlying spaces.

Let us call a C^*-algebra $\mathcal{A}$ *integral* if for some tracial state τ one has $\tau_* K_0(\mathcal{A}) \subseteq \mathbf{Z}$. This definition differs slightly from Exel's [4, p.43] who calls an algebra "integral" for a given tracial state. In either case, however, $C(X)$ is integral (in both senses) for any compact connected Hausdorff space X, cf. [4, p.64].

THEOREM 2.7. *Let X be a compact and connected CW-complex of dimension $\leqslant 3$. Every integral unital C^*-algebra which is strongly Morita equivalent to $C(X)$ is actually isomorphic to $C(X)$.*

That is, $C(X)$ is the only integral unital C^*-algebra which is strongly Morita equivalent to $C(X)$.

Proof. Let $\mathcal{A}$ be an integral unital C^*-algebra strongly Morita equivalent to $C(X)$, say $\mathcal{A} \cong eM_n(C(X))e$ for some projection $e \in M_n(C(X))$ of rank k. We claim that $k = 1$. Lemma 2.5 gives us a rank 1 projection $e' \in M_n(C(X))$ such that $e' \leqslant e$. Let τ be a tracial state on $\mathcal{A}$ such that $\tau_* K_0(\mathcal{A}) = \mathbf{Z}$, which we shall also use to denote a tracial state on $eM_n(C(X))e$. By [11, 5.2.8] we can extended τ to a tracial positive linear functional $\tilde{\tau}$ on $M_n(C(X))$, so that $\tilde{\tau}(x) = \tau(x)$ for all $x \in eM_n(C(X))e$. Normalizing $\tilde{\tau}$ by

$$\tau'(x) = \frac{\tilde{\tau}(x)}{\tilde{\tau}(1)},$$

gives us a tracial state on $M_n(C(X))$. But such τ' must have the form

$$\tau'(x) = \frac{1}{n}\sum_{i=1}^{n} \mu(x_{ii}),$$

for $x \in M_n(C(X))$, where μ is some tracial state on $C(X)$ obtained by integration against a probability measure on X. Since e has rank k, it follows that $\tau'(e) = k/n$, and as $\tilde{\tau}(e) = \tau(e) = 1$ we have $\tilde{\tau}(1) = n/k$. Now e' has rank 1, so $\tau'(e') = 1/n$ and

$$\tau(e') = \tilde{\tau}(e') = \frac{n}{k}\tau'(e') = \frac{1}{k}.$$

Thus, $1/k \in \tau_* K_0(\mathcal{A}) = \mathbf{Z}$ and so $k = 1$. Therefore, $\mathcal{A} \cong eM_n(C(X))e$ and e has rank 1 implies $\mathcal{A} \cong C(X)$ by Lemma 2.6. ■

COROLLARY 2.8. *The algebras $\mathcal{B}_0$ and $\mathcal{B}_{1/2}$ are not strongly Morita equivalent.*

Proof. We know that $\mathcal{B}_{1/2} \sim \mathcal{K}_2 \cong C(K \times \mathbf{T})$ where $K \times \mathbf{T}$ is a CW-complex of dimension 3, compact and connected, and where K is the Klein bottle (as in the proof of Theorem 2.2). A simple application of Pimsner's computation of the tracial range [12] shows that $\mathcal{B}_0$ is an integral (for any tracial state) unital C^*-algebra. Since $\mathcal{B}_0$ is not commutative and hence is not isomorphic to $C(K \times \mathbf{T})$, by Theorem 2.7 it is not strongly Morita equivalent to $C(K \times \mathbf{T}) \sim \mathcal{B}_{1/2}$. ■

COROLLARY 2.9. *No $\mathcal{A}_\theta$ is strongly Morita equivalent to any $C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$.*

Proof. Let $\mathcal{A} = C(\mathbf{T}^2) \times_\alpha \mathbf{Z}$. Suppose $\mathcal{A} \sim \mathcal{A}_\theta$. Then $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_\theta) \cong \mathbf{Z}^2$ and from the computation of the K-groups of $\mathcal{A}$ in [18] one sees that $K_0(\mathcal{A})$ is generated by the identity and the Bott projection and both have trace 1. So $\mathcal{A}$ is integral, and being strongly Morita equivalent to $\mathcal{A}_\theta$ one sees that their tracial ranges are multiples of each other (see [13, Corollary 2.6]), so θ must be rational. However, it is known that in this case $\mathcal{A}_\theta \sim C(\mathbf{T}^2)$ (eg. see [16, Theorem 3.1]; or by applying Green's Theorem), so $\mathcal{A} \sim C(\mathbf{T}^2)$. Since $\mathbf{T}^2$ is a compact and connected CW-complex of dimension 2, Theorem 2.7 implies that $\mathcal{A} \cong C(\mathbf{T}^2)$, and being commutative, $\mathcal{A} = C(\mathbf{T}^2) \times_{\text{id}} \mathbf{Z} \cong$ $\cong C(\mathbf{T}^3)$, a contradiction. ■

COROLLARY 2.10. *No $\mathcal{A}_\theta$ is strongly Morita equivalent to any non-commutative integral unital C^*-algebra $\mathcal{A}$.*

Proof. If $\mathcal{A} \sim \mathcal{A}_\theta$ then comparing their tracial ranges we see that θ must be rational. So $\mathcal{A} \sim \mathcal{A}_\theta \sim C(\mathbf{T}^2)$. By 2.7, $\mathcal{A} \cong C(\mathbf{T}^2)$ is commutative, a contradiction. ■

3. THE RATIONAL QUASI-ROTATION ALGEBRAS

In this section we derive a simple corollary (Theorem 3.1) to a result of Kawamura, Tomiyama and Watatani on the computation of finite-dimensional irreducible representions of crossed products of a commutative C^*-algebra by the integers [8]. In fact, this corollary also follows from the generalized Effros-Hahn Conjecture [5, Corollary 3.2].

Note that if α is any automorphism of a C^*-algebra $\mathcal{A}$ and p any positive integer, then there is a unital (if $\mathcal{A}$ has a unit) embedding

$$\mathcal{A} \times_\alpha \mathbf{Z} \to M_p(\mathcal{A} \times_{\alpha^p} \mathbf{Z})$$

which is implemented by the Banach $*$-algebra (unital) injection

$$\Psi : l^1_\alpha(\mathbf{Z}, \mathcal{A}) \to M_p(l^1_{\alpha^p}(\mathbf{Z}, \mathcal{A}))$$

given by the matrix whose ij-entry is

$$\Psi(\xi)_{ij} = \alpha^i(\xi^{j-i}),$$

where $i, j = 0, \ldots, p-1$, $\xi^i_k = \xi_{kp+i}$, $\alpha(\xi)_k = \alpha(\xi_k)$, and $l^1_\alpha(\mathbf{Z}, \mathcal{A})$ has convolution twisted by α.

Let X be a compact Hausdorff space and σ a homeomorphism of X. For $n \geqslant 1$ let

$$X^n = \{x \in X : \sigma^n(x) = x\} \quad \text{and} \quad X_n = X^n \setminus \bigcup_{i=1}^{n-1} X^i.$$

Let X_n/σ denote the orbit space of X_n under σ. Let $\mathcal{A} = C(X) \times_{\sigma^*} \mathbf{Z}$ and $\widehat{\mathcal{A}}_n$ the space of equivalence classes of n-dimensional irreducible representations of $\mathcal{A}$. Kawamura, Tomiyama and Watatani proved in [8] that there is a homeomorphism

$$\widehat{\mathcal{A}}_n \cong (X_n/\sigma) \times \mathbf{T}.$$

THEOREM 3.1. *Let σ be a homeomorphism of a compact Hausdorff space X such that $\sigma^p = id$, where p is a positive integer, and let $\mathcal{A} = C(X) \times_{\sigma^*} \mathbf{Z}$. Then one has the homeomorphisms*

$$\operatorname{Prim}(\mathcal{A}) \cong \widehat{\mathcal{A}} \cong (X/\sigma) \times \mathbf{T}.$$

Proof. In view of the above embedding Ψ, $\mathcal{A}$ is a C^*-subalgebra of the $p \times p$ matrix algebra over some commutative C^*-algebra. All the latter's irreducible representations have dimension at most p. Since every irreducible representation of $\mathcal{A}$ "extends" to an irreducible representation of the $p \times p$ matrix algebra [11, 4.1.8], it follows that the irreducible representations of $\mathcal{A}$ have dimension at most p. Since, in the above notation, $X^p = X$ and so X_i is empty for $i > p$, we obtain

$$\widehat{\mathcal{A}} = \bigcup_{n=1}^{p} \widehat{\mathcal{A}}_n \cong \bigcup_{n=1}^{p} (X_n/\sigma) \times \mathbf{T} = (X/\sigma) \times \mathbf{T}.$$

Since this shows that $\widehat{\mathcal{A}}$ is Hausdorff (and hence a T_0-space), as σ has finite order, it follows that $\widehat{\mathcal{A}} \cong \mathrm{Prim}(\mathcal{A})$ (cf. [1, 3.1.6]). ■

As the underlying transformations of the algebras $\mathcal{B}_0$, $\mathcal{B}_{1/2}$, $\mathcal{C}_0$, $\mathcal{C}_{1/2}$ have finite order, application of Theorem 3.1 gives the following homeomorphisms:

$$\widehat{\mathcal{B}}_0 \cong \left(\frac{\mathbf{T}^2}{(x,y)\sim(x,\overline{y})}\right) \times \mathbf{T}, \quad \widehat{\mathcal{B}}_{1/2} \cong \left(\frac{\mathbf{T}^2}{(x,y)\sim(-x,\overline{y})}\right) \times \mathbf{T},$$

and

$$\widehat{\mathcal{C}}_0 \cong \left(\frac{\mathbf{T}^2}{(x,y)\sim(x,x\overline{y})}\right) \times \mathbf{T}, \quad \widehat{\mathcal{C}}_{1/2} \cong \left(\frac{\mathbf{T}^2}{(x,y)\sim(-x,x\overline{y})}\right) \times \mathbf{T}.$$

Now it is easy to see that $\widehat{\mathcal{B}}_0$ is homeomorphic to $[0,1] \times \mathbf{T}^2$ and hence it is a 3-dimensional topological manifold whose boundary consists of two disjoint copies of the 2-torus. However, $\widehat{\mathcal{B}}_{1/2}$ is the Klein bottle crossed with a circle which has no boundary. Consequently, $\mathrm{Prim}(\mathcal{B}_0)$ is not homeomorphic to $\mathrm{Prim}(\mathcal{B}_{1/2})$, and by Rieffel's theorem [15, Corollary 3.3] one deduces that $\mathcal{B}_0$ and $\mathcal{B}_{1/2}$ are not strongly Morita equivalent.

Now $\widehat{\mathcal{C}}_0$ is a topological manifold whose boundary is the 2-torus, whereas $\widehat{\mathcal{C}}_{1/2}$, being the Klein bottle crossed with a circle ($\cong \widehat{\mathcal{B}}_{1/2}$), has no boundary. Thus, similarly, $\mathcal{C}_0$ and $\mathcal{C}_{1/2}$ are not strongly Morita equivalent. This concludes the proof of conditions (3) of Theorems 2.2 and 2.3.

REMARK. One might wonder whether the algebra $\mathcal{A} = C(X) \times_{\sigma^*} \mathbf{Z}$, where σ has finite order, might be strongly Morita equivalent to the commutative algebra $C(\widehat{\mathcal{A}})$. Indeed, this is so for the rational rotation C^*-algebras $\mathcal{A}_\theta$ (as $\mathcal{A}_\theta \sim C(\mathbf{T}^2)$ and $\widehat{\mathcal{A}}_\theta \cong \mathbf{T}^2$), and for $\mathcal{B}_{1/2}$ (eg. from the proof of Corollary 2.7 we had $\mathcal{B}_{1/2} \sim C(K \times \mathbf{T})$ and from above $\widehat{\mathcal{B}_{1/2}} \cong K \times \mathbf{T}$). However, it is not the case for $\mathcal{B}_0$. *Proof*: From above we saw that $\widehat{\mathcal{B}_0} \cong [0,1] \times \mathbf{T}^2$, a CW-complex of dimension 3. Now if $\mathcal{B}_0 \sim C(\mathbf{T}^2 \times [0,1])$, then in view of Theorem 2.7 $\mathcal{B}_0$ is isomorphic to $C([0,1] \times \mathbf{T}^2)$, which is false. In fact, it is not the case for $\mathcal{C}_{1/2}$ either. This can be seen by noting that their K_1-groups are different: $K_1(\mathcal{C}_{1/2}) \cong \mathbf{Z}^2$ and $K_1(C(K \times \mathbf{T})) \cong$ $\cong \mathbf{Z}^2 \oplus \mathbf{Z}_2$ (see [18] or [19]). But it might be of interest to ask whether $\mathcal{A}$ is strongly Morita equivalent to any commutative C^*-algebra.

4. THE NON-QUADRATIC QUASI-ROTATION ALGEBRAS.

In this section we shall construct a simple invariant which will help us to prove the following:

THEOREM 4.1. *For non-quadratic real numbers θ, θ' we have:*

$$\mathcal{B}_\theta \sim \mathcal{B}_{\theta'} \iff \theta' = X\theta$$

for some $X \in GL_e(2, \mathbb{Z})$. *The same conclusion holds for the family of algebras* $\{\mathcal{C}_\theta\}$.

The direction ($\Leftarrow$) follows from Theorem 2.2(1) and 2.3(1). So we shall prove the direction ($\Rightarrow$) and, in doing so, we shall consider a simple strongly Morita invariant object associated with a unital C^*-algebra. Denote by $[Q]$ the strong Morita equivalence class of the C^*-algebra Q.

Let $\mathcal{A}$ be a unital C^*-algebra and look at all of its simple (non-zero) quotients, and take the strong Morita equivalence classes of these quotients, which we shall denote by

$$\sharp(\mathcal{A}) = \{[Q] : Q \text{ is a simple quotient of } \mathcal{A}\}.$$

LEMMA 4.2. *If $\mathcal{A}$ and $\mathcal{B}$ are two unital strongly Morita equivalent C^*-algebras, then*

$$\sharp(\mathcal{A}) = \sharp(\mathcal{B}).$$

Proof. Let Q be a simple quotient of $\mathcal{A}$, so that there is a C^*-surjection $\rho : \mathcal{A} \to Q$. By assumption, $\mathcal{B} \cong pM_n(\mathcal{A})p$ for some n and projection $p \in M_n(\mathcal{A})$ so that there is an induced surjection $\mathcal{B} \cong pM_n(\mathcal{A})p \to \rho(p)M_n(Q)\rho(p)$. The algebra $Q' = \rho(p)M_n(Q)\rho(p)$ is full in $M_n(Q)$ since $pM_n(\mathcal{A})p$ is full in $M_n(\mathcal{A})$, hence Q' is strongly Morita equivalent to Q. As Q is simple, so also is Q' and we have $[Q] = [Q'] \in \sharp(\mathcal{B})$. We therefore get the inclusion $\sharp(\mathcal{A}) \subseteq \sharp(\mathcal{B})$. The other containment is obtained symmetrically by envisaging $\mathcal{A}$ as a full corner of a matrix algebra over $\mathcal{B}$. ■

In particular, the cardinality of $\sharp(\mathcal{A})$ is invariant under strong Morita equivalence. In the next two lemmas we calculate $\sharp(\mathcal{B}_\theta)$ and $\sharp(\mathcal{C}_\theta)$.

LEMMA 4.3. *For any irrational number θ : $\mathcal{B}_\theta$ has $\mathcal{A}_\theta$ and $M_2(\mathcal{A}_{2\theta})$ as the only simple quotients, up to isomorphism. In particular,*

$$\sharp(\mathcal{B}_\theta) = \{[\mathcal{A}_\theta],\ [M_2(\mathcal{A}_{2\theta})]\}.$$

Proof. First, it is easy to see that $\mathcal{A}_\theta$ is a quotient of $\mathcal{B}_\theta$. In fact, for $n = \pm 1$ the surjection $\rho_n : C(\mathbf{T}^2) \to C(\mathbf{T})$ given by $\rho_n(f)(x) = f(x, n)$, is equivariant under the action of φ^* on $C(\mathbf{T}^2)$ and the rotation λ^* on $C(\mathbf{T})$ given by $\lambda^*(f)(x) = f(\overline{\lambda}x)$, where $\lambda = e^{2\pi i\theta}$. So ρ_n induces a surjection $\rho_{n*} : \mathcal{B}_\theta \to \mathcal{A}_\theta$, and $\mathcal{A}_\theta$ is a simple quotient of $\mathcal{B}_\theta$.

Now suppose $z \in \mathbf{T}$ is such that $z \neq \overline{z}$. Then we have the surjection

$$\Phi_z = \Phi : C(\mathbf{T}^2) \to C(\mathbf{T}) \oplus C(\mathbf{T}),$$

given by $\Phi(f) = (f_+, f_-)$ where

$$f_+(x) = f(x, z) \quad \text{and} \quad f_-(x) = f(x, \overline{z}).$$

As $z \neq \bar{z}$, the Tietze extension theorem shows that Φ is surjective.

The action on $C(\mathbf{T}) \oplus C(\mathbf{T}) \cong C(\mathbf{T}) \otimes C(\mathbf{Z}_2)$ is given by $\lambda^* \otimes \tau^*$, where τ is translation by 1 on $\mathbf{Z}_2 : \tau(a,b) = (b,a)$. Thus, $\lambda^* \otimes \tau^*(g,h) = (\lambda^*(h), \lambda^*(g))$ for $g,h \in C(\mathbf{T})$. It is easy to show that Φ is equivariant and hence induces the surjection

$$\Phi_* : \mathcal{B}_\theta \to (C(\mathbf{T}) \otimes C(\mathbf{Z}_2)) \times_{\lambda^* \otimes \tau^*} \mathbf{Z}.$$

It is not difficult to see that the latter algebra is isomorphic to $M_2(C(\mathbf{T}) \times_{\lambda^*} 2\mathbf{Z}) \cong$ $\cong M_2(\mathcal{A}_{2\theta})$ (see [16, Proposition 1.2]). Thus $M_2(\mathcal{A}_{2\theta})$ is also a simple quotient of $\mathcal{B}_\theta$.

It remains to show that these are the only two types of simple quotients. If Q is a simple quotient of $\mathcal{B}_\theta$, then $Q \cong \mathcal{B}_\theta / J$ for some maximal 2-sided ideal J. These ideals, however, can be calculated using the machinery of Effros and Hahn [3, Corollary 5.16]. The latter result implies that the primitive ideals of $\mathcal{B}_\theta$ (which are maximal here) are exactly in one-to-one correspondence with the closed φ-orbits of points of $\mathbf{T}^2$; namely, in our case they are $\mathbf{T} \times \{z, \bar{z}\}$ for $z \in \mathbf{T}$. Now in the above we found all those ideals: $\ker(\Phi_{z*})$ (for $z \neq \bar{z}$) and $\ker(\rho_{n*})$ (for $n = \pm 1$). Thus J must be one of these and so Q is isomorphic to $\mathcal{A}_\theta$ or $M_2(\mathcal{A}_{2\theta})$. ∎

Now suppose that θ is a non-quadratic number and that $\mathcal{B}_{\theta'}$ is strongly Morita equivalent to $\mathcal{B}_\theta$. In this case 2θ and θ are not in the same $\mathrm{GL}(2,\mathbf{Z})$ orbit so $\mathcal{A}_{2\theta}$ and $\mathcal{A}_\theta$ are not strongly Morita equivalent and hence $\sharp(\mathcal{B}_\theta) = \sharp(\mathcal{B}_{\theta'})$ has exactly two classes. From the above two lemmas we either have:

(1) $\mathcal{A}_{\theta'} \sim \mathcal{A}_\theta$ and $M_2(\mathcal{A}_{2\theta'}) \sim M_2(\mathcal{A}_{2\theta})$,

or

(2) $\mathcal{A}_{\theta'} \sim M_2(\mathcal{A}_{2\theta})$ and $M_2(\mathcal{A}_{2\theta'}) \sim \mathcal{A}_\theta$.

In case (2) we see that $\mathcal{A}_{2\theta} \sim \mathcal{A}_\theta$ (since $\mathcal{B}_\theta \sim \mathcal{B}_{\theta'}$ implies $\mathcal{A}_\theta \sim \mathcal{A}_{\theta'}$) so that 2θ and θ are in the same $\mathrm{GL}(2,\mathbf{Z})$ orbit, and a simple calculation shows that θ satisfies a quadratic equation over $\mathbf{Z}$, so case (2) cannot occur. Therefore, only case (1) can occur, in which case $\mathcal{A}_{2\theta} \sim \mathcal{A}_{2\theta'}$. Now as also $\mathcal{A}_\theta \sim \mathcal{A}_{\theta'}$ we have that θ, θ' are in the same $\mathrm{GL}(2,\mathbf{Z})$ orbit and that $2\theta, 2\theta'$ are in the same $\mathrm{GL}(2,\mathbf{Z})$ orbit, and these imply that $\theta' = X\theta$ for some $X \in \mathrm{GL}_e(2,\mathbf{Z})$, as a simple calculation shows.

Therefore, for θ non-quadratic $\mathcal{B}_{1/\theta}$ and $\mathcal{B}_\theta$ are not strongly Morita equivalent as in the case for the irrational rotation algebras. In fact, one can show that this is still the case for some quadratics, eg. $\sqrt{2}$ (see Proposition 4.7 below). Another interesting feature related to the Klein bottle is that $\mathcal{K}_\theta$ (in the notation of Theorem 2.2) and $\mathcal{B}_\theta$ are much alike in having homeomorphic primitive spectra, isomorphic centers, isomorphic K-groups and same range of trace (as can be shown), yet the above shows that they are not even strongly Morita equivalent (at least) for non-quadratic θ's: Since we showed that $\mathcal{K}_\theta \sim \mathcal{B}_{1/\theta}$, if $\mathcal{B}_\theta \sim \mathcal{K}_\theta$, then $\mathcal{B}_{1/\theta} \sim \mathcal{B}_\theta$ and so θ must be quadratic (actually a quadratic of a special sort).

LEMMA 4.4. *For any irrational number θ, $\mathcal{C}_\theta$ has $M_2(\mathcal{A}_\theta)$, $\mathcal{A}_{\theta/2}$ and $\mathcal{A}_{(\theta+1)/2}$ as the only simple quotients, up to isomorphism. In particular,*

$$\natural(\mathcal{C}_\theta) = \left\{[M_2(\mathcal{A}_\theta)],\ \left[\mathcal{A}_{\frac{\theta}{2}}\right],\ \left[\mathcal{A}_{\frac{\theta+1}{2}}\right]\right\}.$$

Proof. Let $\lambda = e^{2\pi i\theta}$ and $\mu = e^{-\pi i\theta}$ so that $\mu^2 = \overline{\lambda}$. The transformation underlying $\mathcal{C}_\theta$ here is $\psi(x,y) = (\lambda x, x\overline{y})$ and its inverse is $\psi^{-1}(x,y) = (\overline{\lambda}x, \overline{\lambda}x\overline{y})$. When we look at the closed ψ-orbits of points (x,y) in $\mathbf{T}^2$ we find that there are two kinds: the "degenerate" points for which the circles

$$T_1 = \{(t^2x, ty) : t \in \mathbf{T}\} \quad \text{and} \quad T_2 = \{(\lambda t^2 x, tx\overline{y}) : t \in \mathbf{T}\}$$

are equal, and the "non-degenerate" ones for which these circles are disjoint. Here, T_1 is generated by the even powers of ψ and T_2 by the odd powers. The closed orbit of (x,y) is then the union $T_1 \cup T_2$.

First, let's look at a non-degenerate point (x,y). It implements the surjection

$$\Psi : C(\mathbf{T}^2) \to C(\mathbf{T}) \oplus C(\mathbf{T})$$

given by $\Psi(f) = (f_+, f_-)$ where

$$f_+(t) = f(t^2x, ty) \quad \text{and} \quad f_-(t) = f(t^2x, \mu t x\overline{y}).$$

Note that $(t^2x, \mu t x\overline{y}) = (\lambda(\mu t)^2 x, (\mu t)x\overline{y}) \in T_2$ as $\mu^2\lambda = 1$. As T_1 and T_2 are disjoint, Ψ is surjective. Now we shall see that Ψ is equivariant under the action of $\overline{\mu}^* \otimes \tau^*$, where τ is as in the preceding proof. We have

$$\Psi \circ \psi^*(f) = \left((f\circ\psi^{-1})_+, (f\circ\psi^{-1})_-\right) \quad \text{and} \quad (\overline{\mu}^* \otimes \tau^*)\circ\Psi(f) = (\overline{\mu}^*(f_-), \overline{\mu}^*(f_+)).$$

For the first coordinates we have

$$\begin{aligned}(f\circ\psi^{-1})_+(t) &= f\circ\psi^{-1}(t^2x, ty) = f(\overline{\lambda}t^2x,\ \overline{\lambda}t^2x\overline{t}\overline{y}) = f(\mu^2t^2x,\ \mu^2tx\overline{y}) = \\ &= f((\mu t)^2x,\ \mu(\mu t)x\overline{y}) = f_-(\mu t) = \overline{\mu}^*(f_-)(t)\end{aligned}$$

and a similar calculation verifies the second coordinates. Hence we get the surjection

$$\mathcal{C}_\theta \to (C(\mathbf{T}) \otimes C(\mathbf{Z}_2)) \times_{\overline{\mu}^*\otimes\tau^*} \mathbf{Z},$$

and as before the latter algebra is isomorphic to $M_2(C(\mathbf{T})\times_{\overline{\mu}^{2*}}\mathbf{Z}) \cong M_2(\mathcal{A}_\theta)$, a simple quotient of $\mathcal{C}_\theta$.

Now suppose that (x, y) is degenerate. A simple calculation shows that it has the form $(x, y) = (\pm\overline{\mu}y^2, y)$. In this case we look at the surjection

$$\Phi_y = \Phi : C(\mathbf{T}^2) \to C(\mathbf{T})$$

given by $\Phi(f)(t) = f(\pm\overline{\mu}t^2, t)$. Give $C(\mathbf{T})$ the action of the rotation $(\pm\overline{\mu})^*$, so that the equivariance of Φ follows:

$$\begin{aligned}(\Phi \circ \psi^*)(f)(t) &= (f \circ \psi^{-1})(\pm\overline{\mu}t^2, t) = f(\overline{\lambda}(\pm\overline{\mu}t^2), \overline{\lambda}(\pm\overline{\mu}t^2)\overline{t}) = f(\pm\mu t^2, \pm\mu t) = \\ &= f(\pm\overline{\mu}(\pm\mu t)^2, \pm\mu t) = \Phi(f)(\pm\mu t) = ((\pm\overline{\mu})^* \circ \Phi)(f)(t).\end{aligned}$$

Thus, Φ induces the surjection

$$C_\theta \to C(\mathbf{T}) \times_{(\pm\overline{\mu})^*} \mathbf{Z},$$

and the latter algebra is (depending on the $\pm$ sign) $\mathcal{A}_{\theta/2}$ or $\mathcal{A}_{(\theta+1)/2}$. Arguing as in the preceding proof we see that these and $M_2(\mathcal{A}_\theta)$ are the only simple quotients of C_θ up to isomorphism. ■

Now again suppose that θ is a non-quadratic number and that $C_{\theta'}$ is strongly Morita equivalent to C_θ. Then $\natural(C_\theta) = \natural(C_{\theta'})$ has exactly 3 classes and the fact that θ and θ' are in the same $\mathrm{GL}(2, \mathbf{Z})$ orbit shows $M_2(\mathcal{A}_\theta) \sim M_2(\mathcal{A}_{\theta'})$ so that we have only the two possibilities:

(i) $\mathcal{A}_{\frac{\theta}{2}} \sim \mathcal{A}_{\frac{\theta'}{2}}$ and $\mathcal{A}_{\frac{\theta+1}{2}} \sim \mathcal{A}_{\frac{\theta'+1}{2}}$

or

(ii) $\mathcal{A}_{\frac{\theta}{2}} \sim \mathcal{A}_{\frac{\theta'+1}{2}}$ and $\mathcal{A}_{\frac{\theta+1}{2}} \sim \mathcal{A}_{\frac{\theta'}{2}}$.

Note that (ii) may be obtained from (i) by replacing θ' in (i) by $\theta' + 1$, which does not affect the equivalence $C_{\theta'} \sim C_\theta$. So it is sufficient to show from (i) that θ, θ' are in the same $\mathrm{GL}_e(2, \mathbf{Z})$ orbit. As these are in the same $\mathrm{GL}(2, \mathbf{Z})$ orbit we have the relations:

(1) $\theta' = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \theta \implies \theta' = \dfrac{m\theta + n}{p\theta + q}$, where $mq - np = \pm 1$,

(2) $\dfrac{\theta'}{2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \dfrac{\theta}{2} \implies \theta' = \dfrac{2a\theta + 4b}{c\theta + 2d}$, where $ad - bc = \pm 1$,

(3) $\dfrac{\theta' + 1}{2} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \dfrac{\theta' + 1}{2} \implies \theta + 1 = \dfrac{2a'(\theta + 1) + 4b'}{c'(\theta + 1) + 2d'}$, where $a'd' - b'c' = \pm 1$.

Claim: p is even.

Substituing θ' from (1) into (2) we get from the fact that θ is non-quadratic:

$$(*) \qquad 2pa = mc, \quad 4pb + 2qa = 2md + nc, \quad 4qb = 2nd.$$

The first of these shows that $2|mc$ and the second shows that $2|nc$, and as m, n are relatively prime, we get $2|c$ so that c is even. Hence a and d are odd. From the last equation in $(*)$ we have $2|nd$, so n is even. Therefore, m and q are odd.

Now substituting θ' from (1) into (3) we get the relations

$$(**) \qquad 2a'p = (m+p)c' \ \text{ and } \ 2|(m+n+p+q)c'.$$

Assume that p is odd. By the above, $m+n+p+q$ is also odd, and hance c' is even and so a' is odd. Since $2|(m+p)$ we have that $4|(m+p)c'$, and from the first equation in $(**)$ we obtain $2|a'p$, a contradiction as a' and p are both odd. Therefore, p is even and (1) shows that $\theta' = X\theta$ for some $X \in \mathrm{GL}_e(2,\mathbf{Z})$. This completes the proof of Theorem 4.1.

COROLLARY 4.5. *For the non-quadratic orientation reversing quasi-rotation algebras $\mathcal{B}(a, A)$, the following conditions are equivalent:*

1. $\mathcal{B}(a, A) \sim \mathcal{B}(a', A')$,

2. $m(A) = m(A')$ and $\theta' = X\theta$ for some $X \in \mathrm{GL}_e(2,\mathbf{Z})$, where $X_A(a) = e^{2\pi i\theta}$ and $X_{A'}(a') = e^{2\pi i\theta'}$ are their respective primitive eigenvalues.

COROLLARY 4.6. *For non-quadratic θ, θ' one has:*

$$\mathcal{K}_\theta \sim \mathcal{K}_{\theta'} \iff \frac{1}{\theta'} = X\frac{1}{\theta} \ \text{ for some } X \in \mathrm{GL}_e(2,\mathbf{Z}).$$

The same conclusion holds for the family of algebras $\{\mathcal{K}_\theta^-\}$.

Let's now make some final comments regarding the quadratic case, where we shall only address the algebras $\mathcal{B}_\theta$. Curiously, Theorem 4.1 can be shown to hold without difficulty for certain quadratic irrationals as the following shows.

PROPOSITION 4.7. *Let θ be a quadratic irrational number satisfying $\alpha\theta^2 + \beta\theta + \gamma = 0$, where α, β, γ are integers with $gcd(\alpha,\beta,\gamma) = 1$. Assume that β is even. Then:*

$$\mathcal{B}_\theta \sim \mathcal{B}_{\theta'} \iff \theta' = X\theta$$

for some $X \in \mathrm{GL}_e(2,\mathbf{Z})$.

Proof. Suppose $\mathcal{B}_\theta \sim \mathcal{B}_{\theta'}$ so that θ', θ are in the same $\mathrm{GL}(2,\mathbf{Z})$ orbit and $\natural(\mathcal{B}_\theta) = \natural(\mathcal{B}_{\theta'})$. The latter shows that $\mathcal{A}_{2\theta} \sim \mathcal{A}_{2\theta'}$ and so we have

$$\text{(i)} \ \ \theta' = \frac{a\theta + b}{c\theta + d}, \ \text{ and } \ \text{(ii)} \ \ 2\theta' = \frac{2m\theta + n}{2p\theta + q},$$

where $ad - bc = \pm 1$ and $mq - np = \pm 1$. Substituing θ' from (i) into (ii) and multiplying out the expression gives us a quadratic equation in θ in which the coefficient of θ is

$2aq+4bp-2dm-cn$. As this must be a multiple of β, which is even, we see that cn is even. If c is even, (i) shows the result. If n is even, then (ii) may be re-written as

$$\theta' = \frac{m\theta + \left(\frac{n}{2}\right)}{2p\theta + q} = \begin{pmatrix} m & \frac{n}{2} \\ 2p & q \end{pmatrix} \theta,$$

hence the result. ∎

Our difficulty thus lies in the case when β is odd. At this point we note that the conditions: θ, θ' are in the same $\mathrm{GL}(2,\mathbf{Z})$ orbit and $\sharp(B_\theta) = \sharp(B_{\theta'})$ in themselves are not sufficient to ensure that θ, θ' are in the same $\mathrm{GL}_e(2,\mathbf{Z})$ orbit.

EXAMPLE. Let θ be a quadratic root of $2\theta^2 - \theta - 4 = 0$, and let $\theta' = \theta^{-1}$. Then

$$2\theta = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix} \theta \quad\text{and}\quad 2\theta' = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \theta',$$

are in the same $\mathrm{GL}(2,\mathbf{Z})$ orbit; yet θ' is not in the $\mathrm{GL}_e(2,\mathbf{Z})$ orbit of θ as a simple algebra shows.

Thus it seems that something recondite is needed to show that the conclusion of Propostion 4.7 still holds for quadratics θ with β odd.

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