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BOUNDEDNESS OF SOME SINGULAR INTEGRAL OPERATORS IN WEIGHTED L^2 SPACES

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ABSTRACT. In the previous paper, when measurable functions α, β and a weight function W satisfy some conditions, we gave the necessary and sufficient condition of the boundedness of singular integral operators $\alpha P_+ + \beta P_-$ in the weighted norm of $L^2(W)$, where P_+ is an analytic projection and P_- is a co-analytic projection. In this paper, we give it completely in general.

KEYWORDS: *Hardy spaces, singular integral operators, analytic projections, weighted norm inequalities.*

AMS SUBJECT CLASSIFICATION: Primary 45E10; Secondary 46J15, 47B35.

1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $T = \{\zeta; |\zeta| = 1\}$. Let A be the disc algebra, that is, A is the algebra of all continuous function f on T whose negative Fourier coefficients are zero. For $0 < p < \infty$, the Hardy space $H^p = H^p(T)$ is the closure of A in $L^p = L^p(T)$, and $H^\infty = H^\infty(T)$ is the weak-$*$ closure of A in $L^\infty = L^\infty(T)$. A function Q in H^∞ is an inner function if $|Q| = 1$. A function h is an outer function if there exists a real function t in L^1 and a real constant c such that $h = \mathrm{e}^{t+\mathrm{i}\tilde{t}+\mathrm{i}c}$, where $\tilde{t}$ denotes the harmonic conjugate function of t. Let A_0 be the subspace of all functions f in A whose mean value is zero, and let $\bar{A}_0$ be the subspace of all complex conjugates $\bar{f}$ of functions f in A_0. Let S be the singular integral operator defined by

$$Sf(\zeta) = \frac{1}{\pi \mathrm{i}} \int_T \frac{f(\eta)}{\eta - \zeta} \, \mathrm{d}\eta$$

(cf. [4], p. 12), the integral being a Cauchy principal value. If f is in L^1, then $Sf(\zeta)$ exists for almost everywhere ζ on T, and $Sf(\zeta)$ is a m-measurable function. We shall define the analytic projection P_+ and co-analytic projection P_- by

$$P_+ = \frac{I+S}{2}, \qquad P_- = \frac{I-S}{2},$$

where I denotes the identity operator. Then P_+ maps $A + \overline{A}_0$ to A, P_- maps $A + \overline{A}_0$ to $\overline{A}_0$, and

$$(P_+ - P_-)f(\zeta) = Sf(\zeta) = i\tilde{f}(\zeta) + \int_T f\, dm.$$

For a non-negative function W in L^1, $L^2(W)$ is a Hilbert space of m-measurable functions equipped with the norm

$$\|f\|_W = \left\{ \int_T |f|^2 W\, dm \right\}^{\frac{1}{2}}.$$

Arocena, Cotlar and Sadosky [1] gave a refinement of the Helson-Szegö theorem (cf. [5]) and the Koosis theorem (cf. [6]). They characterized those weights W for which $S = P_+ - P_-$ is bounded in terms of the norm of S. For functions α and β in L^∞, singular integral operators $\alpha P_+ + \beta P_-$ have been studied (cf. [4]). Nakazi and the author [7] gave the necessary and sufficient condition of α, β and W satisfying the weighted norm inequality

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant \|f\|_W \qquad (f \in A + \overline{A}_0),$$

when $W|1 - \alpha\overline{\beta}|e^{\tilde{s}}$ is in L^1, where s is the argument of $1 - \alpha\overline{\beta}$. In this paper, we shall make a further development of the results in [7]. We shall not distinguish between an operator's being bounded and being densely defined and extendable by continuity to a bounded operator. The above inequality implies that $\alpha P_+ + \beta P_-$ is bounded in $L^2(W)$ with norm one.

The main theorem is Theorem 1 in Section 2 which gives the necessary and sufficient condition of α, β and W satisfying the above inequality using an inner function Q, even when $W|1 - \alpha\overline{\beta}|e^{\tilde{s}}$ is not in L^1. In Section 3, we shall obtain several corollaries of Theorem 1. We shall give another proofs of the Helson-Szegö theorem and the Koosis theorem using Theorem 1. Feldman, Krupnik and Markus [2] obtained the connection between the norms of the operators $\alpha P_+ + \beta P_-$ and P_+. In Corollary 6, we shall give the another proof. Theorem 2 is the main theorem in [7] which gives the necessary and sufficient condition of the boundedness of $\alpha P_+ + \beta P_-$ in $L^2(W)$ with norm one which does not use an inner function Q when $W|1 - \alpha\beta|e^{\tilde{s}}$ is in L^1. We shall give the another proof of Theorem 2 using Theorem 1.

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2. THE MAIN THEOREM

The proof of Lemma 1 requires the Cotlar-Sadosky theorem (cf. [1], [9]). The proof of Theorem 1 requires Lemma 1 and the inner-outer factorization theorem (cf. [3], p. 74).

DEFINITION 1. For given functions α and β in L^∞, put

$$E = \{\zeta \in T\ ;\ \alpha(\zeta) \neq \beta(\zeta)\}.$$

LEMMA 1. (cf. [7]) *Suppose α and β are in L^∞, and W is a non-negative function in L^1 satisfying $\int_E W\,dm > 0$. Then α, β and W satisfy the weighted norm inequality*

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant \|f\| \qquad (f \in A + \overline{A}_0)$$

if and only if $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $\log(|1-\alpha\overline{\beta}|W)$ is in L^1 and there exists a k in H^1 such that

$$|(1-\alpha\overline{\beta})W - k|^2 \leqslant (1-|\alpha|^2)(1-|\beta|^2)W^2.$$

Proof. We shall prove the "only if" part. The proof is reversible. Put $W_{11} = (1-|\alpha|^2)W$, $W_{22} = (1-|\beta|^2)W$ and $W_{12} = \overline{W_{21}} = (1-\alpha\overline{\beta})W$. Then $|W_{12}|^2 - W_{11}W_{22} = |\alpha-\beta|^2W^2$, and

$$\sum_{j,k=1,2} \int_T f_j \overline{f}_k W_{jk}\,dm \geqslant 0 \qquad (f_1 \in A, f_2 \in \overline{A}_0).$$

By the Cotlar-Sadosky theorem, $W_{11} \geqslant 0, W_{22} \geqslant 0$ and there exists a k in H^1 such that $|W_{12}-k|^2 \leqslant W_{11}W_{22}$. Since $\int_E W\,dm > 0$, this implies k is non-zero. Since $|k| \leqslant (W_{11}W_{22})^{1/2} + |W_{12}| \leqslant 2|W_{12}|, \log|W_{12}|$ is in L^1. Hence $W > 0$, $|\alpha| \leqslant 1$ and $|\beta| \leqslant 1$. This completes the proof. ∎

DEFINITION 2. For given functions α and β in L^∞ satisfying $|1-\alpha\overline{\beta}| > 0$, put

$$r = \frac{|\alpha-\beta|}{|1-\alpha\overline{\beta}|}.$$

THEOREM 1. *Suppose α and β are functions in L^∞ satisfying $m(E) > 0$. Suppose W is a positive function in L^1. Then α, β and W satisfy the weighted norm inequality*

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant \|f\|_W \qquad (f \in A + \overline{A}_0)$$

if and only if $|\alpha| \leqslant 1, |\beta| \leqslant 1, |1-\alpha\overline{\beta}| > 0$ *and there exists an inner function* Q *and real functions* t, u, v *in* L^1 *such that*

$$\frac{1-\alpha\overline{\beta}}{|1-\alpha\overline{\beta}|} = Qe^{i\tilde{t}}, \qquad |1-\alpha\overline{\beta}|W = e^{t+u+\tilde{v}},$$

$$|v| \leqslant \cos^{-1} r, \qquad r^2e^u + e^{-u} \leqslant 2(\cos v).$$

Proof. We shall prove the "if" part. Put $k = (1-\alpha\overline{\beta})We^{-u-iv}$, then $k = Qe^{i\tilde{t}+t+\tilde{v}-iv}$, and

$$\begin{aligned}|(1-\alpha\overline{\beta})W - k|^2 &- (1-|\alpha|^2)(1-|\beta|^2)W^2 \\ &= |1-\alpha\overline{\beta}|^2W^2\{|1-e^{-u-iv}|^2 - (1-r^2)\} \\ &= |1-\alpha\overline{\beta}|^2W^2e^{-u}\{r^2e^u + e^{-u} - 2(\cos v)\} \leqslant 0.\end{aligned}$$

Since $|\alpha| \leqslant 1$ and $|\beta| \leqslant 1, |k| \leqslant 3W$. Since W is in L^1, k is in H^1 (cf. [3], p. 74). By Lemma 1, α, β and W satisfy the weighted norm inequality. We shall prove the "only if" part. By Lemma 1, $\log(|1-\alpha\overline{\beta}|W)$ is in L^1 and there exists a k in H^1 such that

$$|(1-\alpha\overline{\beta})W - k|^2 \leqslant (1-|\alpha|^2)(1-|\beta|^2)W^2.$$

Hence

$$\left|1 - \frac{k}{(1-\alpha\overline{\beta})W}\right|^2 \leqslant 1 - r^2.$$

Since $m(E) > 0$ and $r > 0$ on E, k is non-zero. By the inner-outer factorization theorem, there exists an inner function Q such that

$$k = Qe^{\log|k| + i(\log|k|)^{\sim}}.$$

Put $u = \log|1-\alpha\overline{\beta}|W - \log|k|$, then u is in L^1. Put $v = \operatorname{Arg}\{(1-\alpha\overline{\beta})/k\}$, where $-\pi \leqslant \operatorname{Arg} z < \pi$. Then

$$|1-e^{-u-iv}|^2 \leqslant 1-r^2.$$

Hence $|v| \leqslant \cos^{-1} r$ (let the reader make a diagram, cf. [7]). Put $t = \log|k| - \tilde{v}$, then t is in L^1, and $\tilde{t} = (\log|k|)^{\sim} + v - c$, for some real constant c. Hence

$$\begin{aligned}(1-\alpha\overline{\beta})W &= e^{iv}\left(\frac{k}{|k|}\right)|1-\alpha\overline{\beta}|W \\ &= Qe^{i(\tilde{t}+c)}|1-\alpha\overline{\beta}|W = Qe^{i(\tilde{t}+c)+t+u+\tilde{v}}.\end{aligned}$$

Hence $(Qe^{ic})e^{i\tilde{t}} = (1-\alpha\overline{\beta})/|1-\alpha\overline{\beta}|$ and $|1-\alpha\overline{\beta}|W = e^{t+u+\tilde{v}}$. Then

$$r^2e^u + e^{-u} - 2(\cos v) = e^u\{|1-e^{-u-iv}|^2 - (1-r^2)\} \leqslant 0.$$

This completes the proof. ∎

3. COROLLARIES OF THEOREM 1

We shall prove Corollary 2 and Theorem 2 using Theorem 1 and the Neuwirth-Newman theorem (cf. [8]). In [7], we proved the Helson-Szegö theorem (cf. [5]) and the Koosis theorem (cf. [6]) using Theorem 2. In Corollary 3 we shall give an another proof of the Helson-Szegö theorem using Corollary 2. In Corollary 5 we shall give an another proof of the Koosis theorem using Corollaries 2 and 4. In Corollary 6, we shall give an another proof of the Feldman-Krupnik-Marcus theorem (cf.[2]) using Corollary 2.

COROLLARY 1. *In Theorem* 1, $e^t/\{|1-\alpha\overline{\beta}|W\}$ *is in* L^1.

Proof. In Theorem 1, $|1-\alpha\overline{\beta}|W = e^{t+u+\tilde{v}}$, and $e^{-u} \leqslant 2(\cos v)$. Hence

$$\frac{e^t}{|1-\alpha\overline{\beta}|W} = e^{-u-\tilde{v}} \leqslant 2(\cos v)e^{-\tilde{v}}.$$

Since $|v| \leqslant \cos^{-1} r \leqslant \pi/2$, $(\cos v)e^{-\tilde{v}}$ is in L^1 (cf. [3], p. 161). This completes the proof. ∎

COROLLARY 2. *Suppose* α *and* β *are functions in* L^∞ *satisfying* $m(E) > 0$, *and* $\alpha\overline{\beta}$ *is a complex constant. Suppose* W *is a positive function in* L^1. *Then* α, β *and* W *satisfy the weighted norm inequality*

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant \|f\|_W \qquad (f \in A + \overline{A}_0)$$

if and only if $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, $\alpha\overline{\beta}$ *is not equal to one, and there exists a constant* C *and real functions* u, v *in* L^1 *such that*

$$W = Ce^{u+\tilde{v}}, \qquad |v| \leqslant \cos^{-1} r,$$

$$r^2 e^u + e^{-u} \leqslant 2(\cos v).$$

Proof. We shall prove the "if" part. Since $1 - \alpha\overline{\beta}$ is a non-zero constant, there exists a real constant γ such that $(1-\alpha\overline{\beta})/|1-\alpha\overline{\beta}| = e^{i\gamma}$. Put $t = \log C - \log|1-\alpha\overline{\beta}|$, then

$$W = Ce^{u+\tilde{v}} = |1-\alpha\overline{\beta}|e^{t+u+\tilde{v}}.$$

By Theorem 1 with $Q = e^{i\gamma}$, this implies the weighted norm inequality. We shall prove the "only if" part. By Theorem 1, there exists an inner function Q, and real functions t, u, v in L^1 satisfying the condition in Theorem 1. Since $\alpha\overline{\beta}$ is a constant, by Lemma 1, $1-\alpha\overline{\beta}$ is a non-zero constant. Hence there exists a real constant γ such that

$$e^{i\gamma}=\frac{1-\alpha\overline{\beta}}{|1-\alpha\overline{\beta}|}=Qe^{i\tilde{t}}.$$

Hence $Qe^{t+i\tilde{t}-i\gamma}=e^{t}\geqslant 0$. By Corollary 1, e^{t}/W is in L^{1}. Since W is in L^{1}, e^{t} is in $L^{1/2}$. Hence $Qe^{t+i\tilde{t}-i\gamma}$ is a non-negative function in $H^{1/2}$. By the Neuwirth-Newman theorem, $Q=e^{i\gamma}$ and there exists a constant C such that $e^{t}=C$. This completes the proof. ∎

COROLLARY 3. (Helson-Szegö) *For a positive function W in L^{1}, the following conditions are mutually equivalent:*

(i) *There exist α and β in L^{∞} satisfying* $\operatorname{ess\,inf}|\alpha-\beta|>0$, *and*

$$\|(\alpha P_{+}+\beta P_{-})f\|_{W}\leqslant\|f\|_{W}\qquad(f\in A+\overline{A}_{0}).$$

(ii) *There exists a constant M such that*

$$\|P_{+}f\|_{W}\leqslant M\|f\|_{W}\qquad(f\in A+\overline{A}_{0}).$$

(iii) *There exist real functions u and v in L^{∞} such that*

$$W=e^{u+\tilde{v}},\qquad\|v\|_{\infty}<\frac{\pi}{2}.$$

Proof. We shall show that (i) implies (ii). Put $\delta=\operatorname{ess\,inf}|\alpha-\beta|$. Since $(\alpha-\beta)P_{+}=(\alpha P_{+}+\beta P_{-})-\beta I$, (i) implies

$$\delta\|P_{+}f\|_{W}\leqslant\|(\alpha-\beta)P_{+}f\|_{W}\leqslant(1+\|\beta\|_{\infty})\|f\|_{W}.$$

This implies (ii). The converse is clear. We shall show that (ii) implies (iii). Put $\alpha=M^{-1}$ and $\beta=0$. Then $m(E)=1, r=\alpha=M^{-1}$ and $\alpha\overline{\beta}=0$. By Corollary 2, there exists a constant C and real functions u,v in L^{1} such that

$$W=Ce^{u+\tilde{v}},\qquad|v|\leqslant\cos^{-1}(M^{-1})<\frac{\pi}{2},$$

$$M^{-2}e^{u}+e^{-u}\leqslant 2(\cos v).$$

Hence u is in L^{∞}. This implies (iii). We shall show that (iii) implies (ii). Since u is in L^{∞} and $\|v\|_{i}nfty<\pi/2$, there exists a constant M such that $M\geqslant 1$ and

$$e^{u}+e^{-u}\leqslant 2M(\cos v).$$

Put $u'=u+\log M$, then

$$MW=Me^{u+\tilde{v}}=e^{u'+\tilde{v}},$$

$$2M^{-1}\leqslant M^{-2}e^{u'}+e^{-u'}\leqslant 2(\cos v).$$

Hence $|v|\leqslant\cos^{-1}(M^{-1})$. By Corollary 2 with $\alpha=M^{-1},\beta=0$ and $r=M^{-1}$, we have

$$\|M^{-1}P_{+}f\|_{MW}\leqslant\|f\|_{MW}\qquad(f\in A+\overline{A}_{0}).$$

This implies (ii). This completes the proof. ∎

COROLLARY 4. *Suppose α and β are functions in L^∞ satisfying $m(E) > 0$. Suppose W is a positive function in L^1. Then,*

$$\|(\alpha P_+ + \beta P_-)f\|_W = \|f\|_W \qquad (f \in A + \overline{A}_0)$$

if and only if $|\alpha| = |\beta| = 1$, $|\alpha - \beta| = |1 - \alpha\overline{\beta}| > 0$ and there exists an inner function Q and a real function t in L^1 such that

$$\frac{1-\alpha\overline{\beta}}{|1-\alpha\overline{\beta}|} = Q\mathrm{e}^{i\tilde{t}}, \qquad |1-\alpha\overline{\beta}|W = \mathrm{e}^{t}.$$

Proof. We shall prove the "if" part. Since $(1-\alpha\overline{\beta})W = Q\mathrm{e}^{t+i\tilde{t}}$, $(1-\alpha\overline{\beta})W$ is in H^1. Since $|\alpha| = |\beta| = 1$,

$$\|f\|_W^2 - \|(\alpha P_+ + \beta P_-)f\|_W^2 = 2\mathrm{Re}\int_T (P_+f)(\overline{P_-f})(1-\alpha\overline{\beta})W\,\mathrm{d}m = 0.$$

We shall prove the "only if" part. Since

$$\int_T |P_+f|^2(1-|\alpha|^2)W\,\mathrm{d}m = \int_T |P_-f|^2(1-|\beta|^2)W\,\mathrm{d}m = 0,$$

$|\alpha| = |\beta| = 1$. By Theorem 1, $|\alpha - \beta| = |1-\alpha\overline{\beta}| > 0$ and there exists an inner function Q, and real functions t, u, v in L^1 satisfying the condition in Theorem 1. Hence $r = 1$ and $v = 0$. Since

$$2 \leqslant \mathrm{e}^u + \mathrm{e}^{-u} = r^2\mathrm{e}^u + \mathrm{e}^{-u} \leqslant 2(\cos v) \leqslant 2,$$

$u = 0$. This completes the proof. ∎

COROLLARY 5. (Koosis) *For a non-negative function W in L^1, the following conditions are mutually equivalent:*

(i) *There exist α and β in L^∞ satisfying $\int_E W\,\mathrm{d}m > 0$ and*

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant \|f\|_W \qquad (f \in A + \overline{A}_0).$$

(ii) *There exists a non-zero function U such that*

$$\|P_+f\|_U \leqslant \|f\|_W \qquad (f \in A + \overline{A}_0).$$

(iii) *W^{-1} is in L^1.*

(iv) *There exist* α *and* β *in* L^∞ *satisfying* $\int_E W\,dm > 0$, *and*

$$\|(\alpha P_+ + \beta P_-)f\|_W = \|f\|_W \qquad (f \in A + \overline{A}_0).$$

Proof. We shall show that (i) implies (ii). Since $(\alpha - \beta)P_+ = (\alpha P_+ + \beta P_-) - \beta I$, (i) implies

$$\|(\alpha - \beta)P_+ f\|_W \leqslant (1 + \|\beta\|_\infty)\|f\|_W.$$

This implies (ii) with $U = |\alpha - \beta|^2 W/(1 + \|\beta\|_\infty)^2$. We shall show that (ii) implies (iii). By (ii),

$$\|f_1\|_U \leqslant \|f_1\|_W \qquad (f_1 \in A).$$

This implies $U \leqslant W$. Put $G = \{\zeta; U(\zeta) > 0\}$, then $m(G) > 0$ and $W > 0$ on G. Put $\alpha = (U/W)^{1/2}\chi_G$ and $\beta = 0$. Then $E = G$ and hence $m(E) > 0$. By Lemma 1, $\log W$ is in L^1. By Corollary 2, there exist real functions u and v such that $|v| \leqslant \pi/2$ and

$$W^{-1} = C^{-1}e^{-u-\tilde{v}} \leqslant 2C^{-1}(\cos v)e^{-\tilde{v}}.$$

Since $(\cos v)e^{-\tilde{v}}$ is in L^1 (cf. [3], p. 161), W^{-1} is in L^1. We shall show that (iii) implies (iv). Since W^{-1} is a positive function in L^1, $W^{-1} + i(W^{-1})^{\sim}$ is an outer function (cf. [3], p. 68). Put $h = 2/\{W^{-1} + i(W^{-1})^{\sim}\}$. Since $|h| \leqslant 2W$, h is an outer function in H^1. Hence there exists a real function t in L^1 and a real constant c such that $h = e^{t + i\tilde{t} + ic}$. Put $\alpha = h/\overline{h}$ and $\beta = -1$, then $|\alpha - \beta| = 2(\mathrm{Re}\, h)/|h| = |h|/W > 0$. Since $(1 - \alpha\overline{\beta})W = h$, we have

$$e^{i\tilde{t} + ic} = \frac{h}{|h|} = \frac{1 - \alpha\overline{\beta}}{|1 - \alpha\overline{\beta}|},$$

$$|1 - \alpha\overline{\beta}|W = |h| = e^t.$$

By Corollary 4 with $Q = e^{ic}$, this implies (iv). It is clear that (iv) implies (i). This completes the proof. ∎

DEFINITION 3. For given functions α, β in L^∞ and a positive function W in L^1, $\|\alpha P_+ + \beta P_-\|_W$ denotes the infimum of the constant C satisfying

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant C\|f\|_W \qquad (f \in A + \overline{A}_0).$$

COROLLARY 6. (Feldman-Krupnik-Markus) *Suppose* α *and* β *are complex constant. Suppose* W *is a positive function in* L^1. *Put* $M = \|\alpha P_+ + \beta P_-\|_W$ *and* $N = \|P_+\|_W$. *Then*

$$2M = \left\{|\alpha - \beta|^2(N^2 - 1) + (|\alpha| + |\beta|)^2\right\}^{\frac{1}{2}} + \left\{|\alpha - \beta|^2(N^2 - 1) + (|\alpha| - |\beta|)^2\right\}^{\frac{1}{2}}.$$

Proof. We assume $\alpha \neq \beta$. Since

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant M\|f\|_W \qquad (f \in A + \overline{A}_0),$$

by Corollary 2, $\alpha\overline{\beta}$ is not equal to M^2, $M \geqslant \max\{|\alpha|, |\beta|\} \geqslant |\alpha\overline{\beta}|^{1/2}$, and there exists a constant C and real functions u, v in L^1 such that

$$W = Ce^{u+\tilde{v}}, \qquad |v| \leqslant \cos^{-1} r(M), \qquad r(M)^2 e^u + e^{-u} \leqslant 2(\cos v),$$

where $r(M) = |\alpha - \beta|M/|M^2 - \alpha\overline{\beta}|$. Since $M \geqslant \max\{|\alpha|, |\beta|\}$, $0 < r(M) \leqslant 1$. By Corollary 2, this implies

$$\|P_+ f\|_W \leqslant r(M)^{-1}\|f\|_W \qquad (f \in A + \overline{A}_0).$$

Hence $N \leqslant r(M)^{-1}$. Since

$$\|P_+ f\|_W \leqslant N\|f\|_W \qquad (f \in A + \overline{A}_0),$$

by Corollary 2, $N \geqslant 1$ and there exists a constant C and real functions u, v in L^1 such that

$$W = Ce^{u+\tilde{v}}, \qquad |v| \leqslant \cos^{-1} N^{-1}, \qquad N^{-2}e^u + e^{-u} \leqslant 2(\cos v).$$

Put $D = (2\mathrm{Re}(\alpha\overline{\beta}) + |\alpha - \beta|^2 N^2)^2 - 4|\alpha\beta|^2$. Since $N \geqslant 1$, $D \geqslant (|\alpha|^2 - |\beta|^2)^2 \geqslant 0$. Put $K = \{(2\mathrm{Re}(\alpha\overline{\beta}) + |\alpha - \beta|^2 N^2 + D^{1/2})/2\}^{1/2}$, then

$$K^4 - \{2\mathrm{Re}(\alpha\overline{\beta}) + |\alpha - \beta|^2 N^2\}K^2 + |\alpha\beta|^2 = 0.$$

This implies $N = r(K)^{-1}$. Since

$$|\alpha|^4 - \{2\mathrm{Re}(\alpha\overline{\beta}) + |\alpha - \beta|^2 N^2\}|\alpha|^2 + |\alpha\beta|^2 = |\alpha|^2|\alpha - \beta|^2(1 - N^2) \leqslant 0$$

and

$$|\beta|^4 - \{2\mathrm{Re}(\alpha\overline{\beta}) + |\alpha - \beta|^2 N^2\}|\beta|^2 + |\alpha\beta|^2 = |\beta|^2|\alpha - \beta|^2(1 - N^2) \leqslant 0,$$

we have $K \geqslant \max\{|\alpha|, |\beta|\}$. By Corollary 2,

$$\|(\alpha P_+ + \beta P_-)f\|_W \leqslant K\|f\|_W \qquad (f \in A + \overline{A}_0).$$

Hence $M \leqslant K$. By the calculation,

$$\begin{aligned} N^2 - r(M)^{-2} &= r(K)^{-2} - r(M)^{-2} \\ &= \frac{|K^2 - \alpha\overline{\beta}|^2}{|K(\alpha-\beta)|^2} - \frac{|M^2 - \alpha\overline{\beta}|^2}{|M(\alpha-\beta)|^2} \\ &= \frac{M^2|K^2 - \alpha\overline{\beta}|^2 - K^2|M^2 - \alpha\overline{\beta}|^2}{|KM(\alpha-\beta)|^2} \\ &= \frac{(K^2 - M^2)(K^2M^2 - |\alpha\beta|^2)}{|KM(\alpha-\beta)|^2} \geqslant 0. \end{aligned}$$

Hence $N = r(M)^{-1}$ when $\alpha \neq \beta$. Hence $|\alpha - \beta|MN = |M^2 - \alpha\overline{\beta}|$. This implies

$$M^4 - \{2\mathrm{Re}(\alpha\overline{\beta}) + |\alpha - \beta|^2 N^2\}M^2 + |\alpha\beta|^2 = 0.$$

Put $c = 2\mathrm{Re}(\alpha\overline{\beta}) + |\alpha - \beta|^2N^2$. Since $M \geqslant \max\{|\alpha|, |\beta|\}$, $M^2 \geqslant c/2$. Since $(2M^2 - c)^2 = c^2 - 4|\alpha\beta|^2$, we have $2M^2 = c + (c^2 - 4|\alpha\beta|^2)^{1/2}$. Hence

$$2M = \left\{2c + 2(c^2 - 4|\alpha\beta|^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}} = \left(c + 2|\alpha\beta|\right)^{\frac{1}{2}} + \left(c - 2|\alpha\beta|\right)^{\frac{1}{2}}.$$

This equality holds even when $\alpha = \beta$, since $\|\alpha I\|_W = |\alpha|$. This completes the proof. ∎

THEOREM 2. (cf. [7]) *Suppose α and β are functions in L^∞ satisfying $m(E) > 0$ and $|1 - \alpha\overline{\beta}| > 0$. Suppose W is a positive function in L^1. Suppose there exists a real function s in L^2 such that $|1 - \alpha\overline{\beta}|We^{\tilde{s}}$ is in L^1 and*

$$e^{is} = \frac{1 - \alpha\overline{\beta}}{|1 - \alpha\overline{\beta}|}\ .$$

Then the following conditions are mutually equivalent:

(i) $\|(\alpha P_+ + \beta P_-)f\|_W \leqslant \|f\|_W \qquad (f \in A + \overline{A}_0)$.

(ii) *There exists a positive constant C, and two real functions u, v such that*
$|1 - \alpha\overline{\beta}|W = Ce^{u+\tilde{v}-\tilde{s}}, \qquad |v| \leqslant \cos^{-1} r,$
$r^2e^u + e^{-u} \leqslant 2(\cos v).$

(iii) *There exists a positive constant C and real functions u', v such that*
$W = C\left\{\frac{1-\chi_E}{|1-\alpha\overline{\beta}|} + \frac{\chi_E}{|\alpha-\beta|}\right\}e^{u'+\tilde{v}-\tilde{s}},$
$|v| \leqslant \cos^{-1} r,$
$|u'| \leqslant \cosh^{-1}\{(\cos v)/r\}$ *on* E,
$-\log(2\cos v) \leqslant u'$ *on* E^c.

These conditions imply that $\{|1 - \alpha\overline{\beta}|We^{\tilde{s}}\}^{-1}$ is in L^1.

Proof. We shall show that (i) implies (ii). By (i), there exists an inner function Q and real functions t, u, v in L^1 satisfying the condition in Theorem 1. Since $(1-\alpha\overline{\beta})/|1-\alpha\overline{\beta}| = Qe^{i\tilde{t}}$, $e^{is} = Qe^{i\tilde{t}}$. Since $|1-\alpha\overline{\beta}|W = e^{t+u+\tilde{v}}$,

$$Qe^{t+\tilde{s}+i(\tilde{t}-s)} = e^{t+\tilde{s}} = |1-\alpha\overline{\beta}|We^{\tilde{s}-u-\tilde{v}}.$$

By Corollary 1, $e^{-u-\tilde{v}}$ is in L^1. Since $|1-\alpha\overline{\beta}|We^{\tilde{s}}$ is in L^1, $Qe^{t+\tilde{s}+i(\tilde{t}-s)}$ is a non-negative function in $H^{1/2}$. By the Neuwirth-Newman theorem, there exists a constant C such that $e^{t+\tilde{s}} = C$. This implies (ii) and that $\{|1-\alpha\overline{\beta}|We^{\tilde{s}}\}^{-1}$ is in L^1. By Theorem 1 with $Q = e^{ic}$ where $c = \int_T s\,dm$, (ii) implies (i). We shall show that (ii) implies (iii). Put

$$r' = 1 - \chi_E + r\chi_E \quad \text{and} \quad u' = u + \log r'.$$

Then

$$\frac{e^u}{|1-\alpha\overline{\beta}|} = \frac{e^{u'}}{r'|1-\alpha\overline{\beta}|} = \left\{\frac{1-\chi_E}{|1-\alpha\overline{\beta}|} + \frac{\chi_E}{|\alpha-\beta|}\right\}e^{u'}.$$

Since $r^2e^u + e^{-u} \leqslant 2(\cos v)$, $r(e^{u'} + e^{-u'}) \leqslant 2(\cos v)$. Hence

$$|u'| \leqslant \cosh^{-1}\{(\cos v)/r\} \text{ on } E,$$

$$-\log(2\cos v) \leqslant u = u' \text{ on } E^c.$$

This proof is reversible. This completes the proof. ∎

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