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SPECTRAL INVARIANCE AND THE HOLOMORPHIC FUNCTIONAL CALCULUS OF J.L. TAYLOR IN Ψ^*-ALGEBRAS

ROBERT LAUTER

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ABSTRACT. If X is a Hilbert space it is shown that very general subalgebras A of $\mathcal{L}(X)$ contain the holomorphic functional calculus in several variables in the sense of J.L. Taylor. In particular, Taylor's holomorphic functional calculus applies to Ψ^*-algebras (cf. [12], Definition 5.1), and so gives a useful tool for the investigation of certain algebras of pseudo-differential operators and of Fréchet operator algebras on singular spaces. Taylor's holomorphic functional calculus applies also to algebras of $n \times n$-matrices with elements in Ψ^*-algebras and even more general algebras. Furthermore, an example shows that Taylor's holomorphic functional calculus for at least three commuting operators on a Hilbert space is, in general, richer than any other multidimensional holomorphic functional calculus in commutative subalgebras of $\mathcal{L}(X)$.

KEYWORDS: *Multidimensional Holomorphic Functional Calculus, Joint Spectra, Ψ^*-Algebras.*

AMS SUBJECT CLASSIFICATION: Primary 47A13; Secondary 47A60.

1. INTRODUCTION

Taylor's joint spectrum $\sigma_T(a, X)$ of a commuting system $a = (a_1, \dots, a_n)$ of operators acting on a Banach space X was introduced in [24] by J.L. Taylor in 1970. Also in 1970 he gave in [25] a construction of the corresponding holomorphic functional calculus in several variables $\Theta_a : \mathcal{O}(\sigma_T(a, X)) \to \mathcal{L}(X)$. In this context the following question posed by B. Gramsch is interesting:

• When does a subalgebra A of $\mathcal{L}(X)$ contain its holomorphic functional calculus in the sense of J.L. Taylor, i.e. is $f(a) := \Theta_a(f) \in A$ for every $f \in \mathcal{O}(\sigma_T(a,X))$ and all commuting systems $a \in A^n$? Algebras with this property are briefly called T-algebras.

It is easy to see that a T-algebra is necessarily spectrally invariant in $\mathcal{L}(X)$,i.e. $A \cap \mathcal{L}(X)^{-1} = A^{-1}$ holds for the groups A^{-1} respectively $\mathcal{L}(X)^{-1}$ of invertible elements in A respectively $\mathcal{L}(X)$. If X is a Hilbert space, then it will be shown in Theorem 5.3 that for symmetric, sequentially complete, locally convex, and continuously embedded subalgebras A of $\mathcal{L}(X)$, the spectral invariance of A in $\mathcal{L}(X)$ is also sufficient for A to be a T-algebra.

In particular, every Ψ^*-algebra in $\mathcal{L}(X)$ is a T-algebra. This result was first obtained in joint work with B. Gramsch. Ψ^*-algebras, i.e. spectrally invariant, symmetric, continuously embedded Fréchet subalgebras of $\mathcal{L}(X)$ – X a Hilbert space – , were introduced by B. Gramsch in 1984 (cf. [12], Definition 5.1). Already in the early eighties B. Gramsch first stressed the importance of spectral invariance in Fréchet algebras of pseudo-differential operators (cf. [11]).

Since then, the concept of Ψ^*-algebras has developped into a useful tool in structural analysis of certain algebras of pseudo-differential operators, Fréchet operator algebras on singular spaces and C^∞-elements of C^*-dynamical systems. Subsequent to the work of Gramsch ([11], [12]), in this connection should be referred among many others to the work of Cordes (cf. [5]), Gramsch, Kaballo (cf. [13]), Gramsch, Ueberberg, Wagner (cf. [14]), Lorentz (cf. [16]), Schrohe (cf. [18], [19]), Ueberberg (cf. [26]) and the recent work of Bony, Chemin (cf. [2]) and Schrohe (cf. [20], [21]).

If X is a Hilbert space and A is only a symmetric, sequentially complete, locally pseudo-convex, continuously embedded subalgebra of $\mathcal{L}(X)$, the spectral invariance of A is still a sufficient condition for A to be T-algebra, if there is – in addition – a separating submultiplicative system of continuous pseudo-seminorms, generating the topology of A (cf. 5.11).

As a simple corollary of Theorem 5.3 and Theorem 5.11 one gets the following:

If X is a Hilbert space and A is as in 5.3 or 5.11, then the algebra $M_N(A)$ of $N \times N$-matrices with elements in A, is a T-algebra in $\mathcal{L}(X^N)$ which may be identified with $M_N(\mathcal{L}(X))$, in an obvious manner (cf. 5.5 and 5.12). A similar result, concerning the ordinary holomorphic functional calculus in one variable, was recently proved by L. Schweitzer in [22].

Now it is well known (cf. [24], Lemma 1.1) that for every unital subalgebra A of $\mathcal{L}(X)$ containing $a_1, \dots, a_n$ in its centre, one has

$$\sigma_T(a, X) \subseteq \sigma_A(a) := \left\{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : \sum_{j=1}^{n} (\lambda_j \mathrm{id}_X - a_j)A \neq A \right\}.$$

Further, an example of E. Albrecht in [1] shows that Taylor's holomorphic functional calculus is rich in the following sense: For every $n \geqslant 2$ there is a Banach space and a commuting system $a = (a_1, \dots, a_n) \in \mathcal{L}(X)^n$, such that the algebra $\mathcal{O}(\sigma_A(a))$ of germs of functions analytic on $\sigma_A(a)$ is strictly contained in the algebra $\mathcal{O}(\sigma_T(a, X))$ of germs of functions analytic on $\sigma_T(a, X)$, for every unital subalgebra A of $\mathcal{L}(X)$ containing $a_1, \dots, a_n$ in its centre. In particular, Taylor's holomorphic functional calculus on Banach spaces is in general richer than the holomorphic functional calculus for commutative Banach (cf. [4], Chapter I, Section 4) or Fréchet (cf. [32], VI Proposition 4) subalgebras of $\mathcal{L}(X)$ containing $a_1, \dots, a_n$.

A suitable modification of an example of J.L. Taylor in [24], Theorem 4.1 shows in 4.2, that also in the case of Hilbert spaces, there is for every $n \geqslant 3$, a separable Hilbert space X and a commuting system $a = (a_1, \dots, a_n) \in \mathcal{L}(X)^n$, such that the holomorphic functional calculus of J.L. Taylor is rich in the above mentioned sense. To the best of the author's knowledge the question, whether there is also a pair of commuting operators acting on a Hilbert space X, such that even only $\sigma_T(a, X)$ is strictly contained in $\sigma_A(a)$, where $A = (a)'$ is the commutant of the system $a = (a_1, \dots, a_n)$, remains still open (cf. [8]).

Summing up, it may be said that Taylor's holomorphic functional calculus applies to Ψ^*-algebras, and hence Ψ^*-algebras are algebras with a rich holomorphic functional calculus in several variables. This improves, for example, a result of E. Schrohe in [21], Theorem 4.4.

If X is only a Banach space the situation is completely different and almost nothing is known about T-algebras in $\mathcal{L}(X)$ in this context. Using an idea of [11], it was possible to prove that any non trivial left – or right – ideal $\mathcal{J}$, contained in the ideal of compact operators on X, generates a T-algebra A by adjoining a unit to $\mathcal{J}$, i.e. $A = \mathbb{C}id_X \oplus \mathcal{J}$ (cf. 5.13). The main difference between the Banach space and the Hilbert space setting is the existence of an integral formula in the Hilbert space setting, the so-called Martinelli-formula for Taylor's holomorphic functional calculus, proved by F.H. Vasilescu in 1978 (cf. [27]). The absence of such a formula and hence the use of Taylor's original construction in [25] or [30], Chapter III makes it extremely difficult to investigate the operators $f(a)$ in the case of arbitrary Banach spaces.

2. SPECTRAL INVARIANCE, Ψ^*-ALGEBRAS AND MATRIX ALGEBRAS

DEFINITION 2.1. Let B be an algebra with unit e. A subalgebra A of B with $e \in A$ is called *spectrally invariant* in B – sometimes also called a full subalgebra in B, if $A \cap B^{-1} = A^{-1}$ for the groups A^{-1} respectively B^{-1} of invertible elements in A respectively B holds. The pair (A, B) of algebras is called a Wiener-pair by Naimark.

The following lemma – due to B. Gramsch (cf. [12], Lemma 5.3) – gives a necessary and sufficient condition for spectral invariance, sometimes much easier to verify.

LEMMA 2.2. *Let B be a C^*-algebra with unit e, and $A \subseteq B$ be a symmetric subalgebra of B – i.e. $a \in A$ implies $a^* \in A$. Assume further that $e \in A$. Then A is spectrally invariant in B if and only if A is locally spectral invariant in B, i.e. there is an open neighbourhood U of e in B with $A \cap U \subseteq A^{-1}$.*

Proof. Since A is a symmetric subalgebra of B, the closure $R := \overline{A}^B$ of A in B is a C^*-subalgebra of B, hence one has $R \cap B^{-1} = R^{-1}$ by the well-known spectral invariance of C^*-subalgebras.

Now let be $a \in A \cap B^{-1} \subseteq R \cap B^{-1} = R^{-1}$. Consequently there is an $x \in R$, such that $ax = xa = e$. By definition there is a sequence $(x_n)_{n \in \mathbb{N}}$ in A, such that $x_n \stackrel{n \to \infty}{\longrightarrow} x$ in B, hence $ax_n \stackrel{n \to \infty}{\longrightarrow} ax = e$. Since $\emptyset \neq U \cap A \subseteq A$ is an open neighbourhood of $e \in A$, one obtains an $n \in \mathbb{N}$, such that $ax_n \in U \cap A \subseteq A^{-1}$; thus there exists $y \in A$ with $ax_n y = e$, hence $a^{-1} = x = x_n y \in A$. ∎

DEFINITION 2.3. [GRAMSCH, 1984]. Let B be a C^*-algebra with unit e. A subalgebra A of B with $e \in A$ is called a *Ψ^*-algebra in B*, if the following conditions are fulfilled:

(1) There is a topology τ_A on A which gives (A, τ_A) the structure of a Fréchet algebra.

(2) The natural inclusion $j : (A, \tau_A) \hookrightarrow (B, \|\cdot\|_B)$ is continuous.

(3) A is symmetric, i.e. $a \in A \Longrightarrow a^* \in A$.

(4) A is spectrally invariant in B.

In the sequel let $M_n(B)$ be the algebra of $n \times n$-matrices over an algebra B with unit e. If B is in addition a topological algebra, then let $M_n(B)$ always be equipped with the product topology. Recall that $M_n(B)$ is a topological algebra, with jointly continuous multiplication, an open group $M_n(B)^{-1}$ of invertible elements and continuous inversion, if B is such an algebra (cf. [3], Proposition A.1.1 or [23], Corollary 1.2).

Using, for example, the GNS-construction one obtains for every C^*-algebra B with unit and for every $n \in \mathbb{N}$ a unique norm on $M_n(B)$ which generates the product topology on $M_n(B)$ and gives $M_n(B)$ the structure of a C^*-algebra.

PROPOSITION 2.4. *Let B be a C^*-algebra with unit e and A a symmetric, spectrally invariant subalgebra of B with $e \in A$. Then for all $n \in \mathbb{N}, M_n(A)$ is a symmetric, spectrally invariant subalgebra of the C^*-algebra $M_n(B)$.*

Proof. Because of 2.2 one has to show only local spectral invariance. Using the Gauss-Jordan-elimination as in [23], Lemma 1.1, one sees that there is an open neighbourhood V in $M_n(B)^{-1}$ of the unit matrix $E \in M_n(B)$, such that for every $T \in V \cap M_n(A)$, there are at most $n(n-1)$ elementary row and column operations in $M_n(A)$ reducing T to an invertible diagonal matrix with elements in A. Now the spectral invariance of A in B gives $T \in M_n(A)^{-1}$. ∎

COROLLARY 2.5. *Let B be a C^*-algebra and A a Ψ^*-algebra in B. Then for each $n \in \mathbb{N}, M_n(A)$ is a Ψ^*-algebra in the C^*-algebra $M_n(B)$.*

In [22], Remark 2.4 the proof of Proposition 2.4 is attributed to B. Gramsch. It is not hard to see that the same argument still works if B is replaced by a topological algebra, with jointly continuous multiplication, an open group of invertible elements and continuous inversion, and A by a spectrally invariant, dense subalgebra in B. This was also noticed by R.G. Swan in [23], Lemma 2.1 and J.-B. Bost in [3], Proposition A.2.2.

3. TAYLOR'S JOINT SPECTRUM, TAYLOR'S HOLOMORPHIC FUNCTIONAL CALCULUS AND T-ALGEBRAS

3.1. THE KOSZUL-COMPLEX. In the sequel let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a system of indeterminates, $\Lambda[\sigma]$ be the exterior algebra generated by the system $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $S_j : \Lambda[\sigma] \longrightarrow \Lambda[\sigma] : \omega \longmapsto \sigma_j \wedge \omega$ be the creation operator, $j = 1, \ldots, n$.

The space $\Lambda[\sigma]$ has a natural Hilbert space structure in which the elements 1 and $\sigma_{j_1} \wedge \cdots \wedge \sigma_{j_q}$, where $1 \leqslant j_1 < \cdots < j_q \leqslant n$, $1 \leqslant q \leqslant n$, form an orthonormal basis. The adjoint S_j^* of the creation operator S_j with respect to this Hilbert space structure is given by $S_j^* : \Lambda[\sigma] \longrightarrow \Lambda[\sigma] : \omega = \omega_1 + \sigma_j \wedge \omega_2 \longmapsto \omega_2$, where ω_1 and ω_2 do not contain σ_j. Further, let X be a vector space over the complex numbers $\mathbb{C}$, and $a = (a_1, \ldots, a_n)$ be a commuting system of linear operators on X. On $\Lambda[\sigma, X] := X \otimes \Lambda[\sigma]$ one defines a coboundary operator δ_a by $\delta_a := \sum_{j=1}^{n} a_j \otimes S_j$. The commutativity of the system $a = (a_1, \ldots, a_n)$ gives then $\delta_a \circ \delta_a = 0$, therefore

$\mathcal{K}(a, X) := (\Lambda[\sigma, X], \delta_a)$ is a cochain complex, the so-called *Koszul-complex of the commuting system* $a = (a_1, \ldots, a_n)$. Further one defines for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ the commuting system $\lambda e - a$ by $\lambda e - a := (\lambda_1 \mathrm{id}_X - a_1, \ldots, \lambda_n \mathrm{id}_X - a_n)$.

Now let X be a Banach space. After fixing a basis of $\Lambda[\sigma]$ the space $\Lambda[\sigma, X]$ can be identified with 2^n copies of X, and hence the algebra $\mathcal{L}(\Lambda[\sigma, X])$ of all bounded operators on $\Lambda[\sigma, X]$ with the matrix algebra $M_{2^n}(\mathcal{L}(X))$ in an obvious manner.

If X is a Hilbert space, then clearly $\Lambda[\sigma, X]$ also has a Hilbert space structure. The adjoint of $\delta_a \in \mathcal{L}(\Lambda[\sigma, X])$ with respect to this Hilbert space structure is given by $\delta_a^* = \sum_{j=1}^{n} a_j^* \otimes S_j^*$.

3.2. THE TAYLOR SPECTRUM. Let X be a vector space and $a = (a_1, \ldots, a_n)$ a commuting system of linear operators on X. The set $\sigma_T(a, X)$ of all $\lambda \in \mathbf{C}^n$, such that the Koszul-complex $\mathcal{K}(\lambda e - a, X)$ is not exact, is called the *Taylor spectrum of the commuting system* $a = (a_1, \ldots, a_n)$.

If X is a Banach space and $a = (a_1, \ldots, a_n) \in \mathcal{L}(X)^n$ is a commuting system of bounded linear operators on X then it is well known (cf. [24], Theorem 3.1, Corollary 3.2) that $\sigma_T(a, X)$ is a nonvoid, compact subset of $\mathbf{C}^n$, which is contained in $\prod_{j=1}^{n} \sigma(a_j)$, where $\sigma(a_j)$ denotes the usual spectrum of a_j $(j = 1, \ldots, n)$. Furthermore, the Taylor spectrum satisfies the projection property, i.e.

If $m \leqslant n$ and $a = (a_1, \ldots, a_n)$ and $a' = (a_1, \ldots, a_m)$ are commuting systems of bounded operators on a Banach space X, then $\sigma_T(a', X) = \pi_m \sigma_T(a, X)$, where $\pi_m : \mathbf{C}^n \to \mathbf{C}^m$ is the projection onto the first m components (cf. [24], Theorem 3.2).

A proof for the following theorem on the holomorphic functional calculus can be found for example in [25], Corollary 4.4 or [30], Corollary III 9.10.

THEOREM 3.3. [TAYLOR, 1970]. *Let X be a Banach space and $a = (a_1, \ldots, a_n) \in \mathcal{L}(X)^n$ a commuting system. Then there exists an unital algebra homomorphism*

$$\Theta_a : \mathcal{O}(\sigma_T(a, X)) \to (a)'' \subseteq \mathcal{L}(X) \text{ with } \Theta_a(z = (z_1, \ldots, z_n) \mapsto z_j) = a_j, j = 1, \ldots, n.$$

Here $\mathcal{O}(\sigma_T(a, X))$ denotes the algebra of germs of analytic functions, in neighbourhoods of the compact set $\sigma_T(a, X)$, and $(a)'' = ((a)')'$ is the bicommutant of the commuting system $a = (a_1, \ldots, a_n)$. The homomorphism $\Theta_a : (\sigma_T(a, X)) \longrightarrow \mathcal{L}(X)$ is called the holomorphic functional calculus of J.L. Taylor with respect to the commuting system $a = (a_1, \ldots, a_n)$.

The construction of the homomorphism Θ_a is rather complicated in the general Banach space setting, but, in 1976, Vasilescu (cf. [27]) gave a significant improvement and simplification of the construction of the homomorphism in the Hilbert space setting. Vasilescu's construction uses a certain differential form, the so-called Martinelli kernel.

3.4 THE MARTINELLI KERNEL AND THE CONSTRUCTION OF THE APPLICATION Θ_a. Let X be a Hilbert space and $a = (a_1, \ldots, a_n) \in \mathcal{L}(X)^n$ a commuting system. Put $\Omega := \mathbb{C}^n \setminus \sigma_T(a, X)$. By a result of Vasilescu (cf. [28], Corollary 2.2) one has $z \in \Omega$ if and only if $(\delta_{ze-a} + \delta^*_{ze-a})^{-1}$ exists in $\mathcal{L}(\Lambda[\sigma, X])$, hence the mapping $\xi_0 : \Omega \longrightarrow \mathcal{L}(\Lambda[\sigma, X]) = \Lambda^0[d\bar{z}, \mathcal{L}(\Lambda[\sigma, X])] : z \longmapsto (\delta_{ze-a} + \delta^*_{ze-a})^{-1}$ is well defined and continuous. Using an appropriate form of the Neumann series (cf. [28], Lemma 2.8), one can see that ξ_0 is in fact real-analytic, in particular of the class $\mathcal{C}^\infty$, and so the construction of the following differential forms makes sense.

$$\eta_j : \Omega \xrightarrow{\mathcal{C}^\infty} \Lambda^{j+1}[d\bar{z}, \mathcal{L}(\Lambda[\sigma, X])] : z \longmapsto (\bar{\partial}\xi_j)(z), \quad j = 0, 1, \ldots, n-2,$$

$$\xi_{j+1} : \Omega \xrightarrow{\mathcal{C}^\infty} \Lambda^{j+1}[d\bar{z}, \mathcal{L}(\Lambda[\sigma, X])] : z \longmapsto \xi_0(z)\eta_j(z), \quad j = 0, 1, \ldots, n-2.$$

Further let $S := S_1 \ \ldots \ S_n : \Lambda[\sigma, X] \longrightarrow \Lambda[\sigma, X] : \omega \longmapsto \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \omega$. Then by [30], p.143 one has for all $z \in \Omega$:

(∗) If $\xi_{n-1}(z) = \sum_{k=1}^{n} A_k d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n$ with $A_k = A_k(z) \in \mathcal{L}(\Lambda[\sigma, X])$, then one has

$$A_k S|_{\Lambda^0[\sigma,X]}(\Lambda^0[\sigma, X]) \subseteq \Lambda^0[\sigma, X] \quad \text{for all} \quad k = 1, \ldots, n.$$

Since $\Lambda^0[\sigma, X] = X$ in a canonical way, $\xi_{n-1}S|_{\Lambda^0[\sigma,X]}$ gives a $\mathcal{C}^\infty$-mapping $M_a : \Omega \rightarrow \Lambda^{n-1}[d\bar{z}, \mathcal{L}(X)]$. The differential form M_a is called the *Martinelli kernel attached* to the commuting system $a = (a_1, \ldots, a_n)$. Fixing the canonical basis for $\Lambda[\sigma]$, one gets the following matrix representation for $A_k(z)$, using the same notation as in (∗).

$$\begin{array}{cc} & \cdots \Lambda^n[\sigma, X] \\ A_k(z) = \begin{pmatrix} & b_k(z) \\ * & 0 \\ & \vdots \\ & 0 \end{pmatrix} & \begin{array}{c} \Lambda^0[\sigma, X] \\ \Lambda^1[\sigma, X] \\ \vdots \\ \Lambda^n[\sigma, X] \end{array} \end{array}$$

with some $b_k \in \mathcal{C}^\infty(\Omega, \mathcal{L}(X))$. Hence one has

$M_a(z) = \sum_{k=1}^{n} b_k(z) d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_n$ for all $z \in \Omega$.

Now let $U \subseteq \mathbf{C}^n$ be open with $\sigma_T(a, X) \subsetneq U$, and $f \in \mathcal{O}(U)$ be a representative for the germ $[f] \in \mathcal{O}(\sigma_T(a, X))$. It is then possible to find a relatively compact open set $\Delta \supsetneq \sigma_T(a, X)$ with $\overline{\Delta} \subseteq U$, such that the boundary $\Sigma := \partial\Delta$ of Δ is a smooth surface (cf. [29], p.484). In this case Σ is said to be an admissible surface surrounding $\sigma_T(a, X)$ in U, Now let

$$\Theta_a([f]) := \Theta_a^U(f) := \frac{1}{(2\pi i)^n} \int_{\Sigma} f(z) M_a(z) \wedge dz \in \mathcal{L}(X),$$

where $M_a(z) \wedge dz := M_a(z) \wedge dz_1 \wedge \cdots \wedge dz_n$ is a continuous $(2n-1)$-form with values in $\mathcal{L}(X)$.

A proof for the following theorem can be found for example in [30], Proposition III.11.1 or [6], Theorem 5.18.

THEOREM 3.5. [VASILESCU, 1976]. *Let X be a Hilbert space and $a = (a_1, \ldots, a_n) \in \mathcal{L}(X)^n$ a commuting system. Then the application $\Theta_a : \mathcal{O}(\sigma_T(a, X)) \to \mathcal{L}(X)$ from* 3.4 *is well-defined and exactly the holomorphic functional calculus of J.L. Taylor in the sense of* 3.3. *In the following for $f \in \mathcal{O}(\sigma_T(a, X)), f(a)$ is always written instead of $\Theta_a(f)$.*

DEFINITION 3.6. Let X be a Banach space. A subalgebra $A \subseteq \mathcal{L}(X)$ is called a *T-algebra*, if one has $f(a) \in A$ for all $f \in \mathcal{O}(\sigma_T(a, X))$, and all commuting systems $a = (a_1, \ldots, a_n) \in A^n$, where $n \in \mathbf{N}$.

REMARKS AND EXAMPLES 3.7.

(1) *A* T-algebra in $\mathcal{L}(X)$ is necessarily spectrally invariant in $\mathcal{L}(X)$.

(2) An arbitrary intersection of T-algebras is also a T-algebra.

(3) Let X be a Banach space. Then $A := \{\lambda \mathrm{id}_X : \lambda \in \mathbf{C}\}$ is a T-algebra; in fact one has $f(\lambda \mathrm{id}_X) = f(\lambda)\mathrm{id}_X$ for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ and all $f \in \mathcal{O}(\sigma_T(a, X))$. Note that $\sigma_T(\lambda \mathrm{id}_X, X) = \{\lambda\}$.

(4) Let X be a Banach space and $a = (a_1, \ldots, a_n) \in \mathcal{L}(X)^n$ a commuting system. Then

$$\mathcal{A}_T(a) := \{f(a) \in \mathcal{L}(X) : f \in \mathcal{O}(\sigma_T(a, X))\}$$

is a commutative T-algebra. This follows directly from the composition formula $(f \circ g)(a) = f(g(a))$ due to M. Putinar (cf. [17], Theorem 2).

4. THE RICHNESS OF TAYLOR'S HOLOMORPHIC FUNCTIONAL CALCULUS ON HILBERT SPACES

DEFINITION 4.1. Let X be a Banach space and $a = (a_1, \ldots, a_n) \in \mathcal{L}(X)^n$ a commuting system. For every subalgebra A of $\mathcal{L}(X)$ containing $a_1, \ldots, a_n$ and id_X in its centre, the joint spectrum of $a = (a_1, \ldots, a_n)$ in A is defined by

$$\sigma_A(a) := \left\{ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n : \sum_{j=1}^{n} (\lambda_j \mathrm{id}_X - a_j)A \neq A \right\}.$$

It is well known (cf. [24], Lemma 1.1) that $\sigma_T(a, X) \subseteq \sigma_{(a)'}(a) \subseteq \sigma_A(a)$, where $(a)'$ is the commutant of $a = (a_1, \ldots, a_n)$, i.e. the norm-closed subalgebra of all $b \in \mathcal{L}(X)$ satisfying $a_j b = b a_j$ for $j = 1, \ldots, n$.

The following theorem is proved by a suitable generalization of an example of J.L. Taylor in [24], Theorem 4.1.

THEOREM 4.2. *For every $n \geqslant 3$ there exists a separable Hilbert space X and a commuting system $a = (a_1, \ldots, a_n) \in \mathcal{L}(X)^n$, such that $\sigma_T(a, X)$ is strictly contained in $\sigma_{(a)'}(a)$. Moreover, the algebra $\mathcal{O}(\sigma_{(a)'}(a))$ of germs of functions analytic on $\sigma_{(a)'}(a)$ is strictly contained in the algebra $\mathcal{O}(\sigma_T(a, X))$ of germs of functions analytic on $\sigma_T(a, X)$.*

In particular: Taylor's holomorphic functional calculus for a commuting system $a = (a_1, \ldots, a_n)$, of at least three operators acting on a Hilbert space X, is in general richer than the holomorphic functional calculus in any commutative subalgebra containing $a_1, \ldots, a_n$ and id_X in the sense of [4], Chapter I, Section 4 or [32], VI Proposition 4.

The proof of the theorem is divided into a few steps.

NOTATIONS 4.3. For $z = (z_1, z_2) \in \mathbf{C}^2$ let $\|z\|_2 := \sqrt{|z_1|^2 + |z_2|^2}$. Further put $G_1 := \{z \in \mathbf{C}^2 : \|z\|_2 < \frac{1}{4}\}$, $G_2 := \{z \in \mathbf{C}^2 : \frac{3}{4} < \|z\|_2 < 1\}$ and $D := \{z \in \mathbf{C}^2 : \|z\|_2 < 1\}$. Then $G := G_1 \cup G_2 \subseteq \mathbf{C}^2 = \mathbf{R}^4$ is an open bounded subset with a smooth boundary. Choose $s > 2$ and put

- $E_0 := H^s(G) \hookrightarrow C(\overline{G})$.
- $E_1 := H^{s+1}(G) \hookrightarrow C^1(\overline{G})$.
- $E_2 := \{f \in E_1 : \bar{\partial}_1 f = \bar{\partial}_2 f = 0\} \subseteq \mathcal{O}(G)$.

Here the spaces $H^r(G), r \in \mathbf{R}$ are the usual Sobolev–Hilbert spaces, and the imbeddings are given by the well known Sobolev imbedding theorem. Recall further that $\bar{\partial}_j := \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}(\frac{\partial}{\partial x_j} + \mathrm{i}\frac{\partial}{\partial y_j}), j = 1, 2$ as usual.

Then $X := E_1 \oplus E_0 \oplus E_0$ is a separable Hilbert space and one is now able to define for $j = 1, \ldots, n$ operators $a_j : X \to X$ by

$$a_j : X \longrightarrow X : (f, g, h) \longmapsto \begin{cases} (z_j f, z_j g, z_j h) & , \quad j = 1, 2 \\ (0, \bar{\partial}_1 f, \bar{\partial}_2 f) & , \quad j = 3 \\ 0 & , \quad j \geqslant 4 \end{cases} .$$

One easily checks that $a = (a_1, \ldots, a_n)$ is a commuting system of continuous linear operators on X.

LEMMA 4.4. *$\sigma_T(a, X) = \overline{G} \times \{0^{n-2}\} = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n : (z_1, z_2) \in \overline{G}, z_j = 0$ for all $j \geqslant 3\}$.*

Proof. An easy computation using Lemma 1.1 in [24] shows that $\sigma_T((a_1, a_2), X) = \overline{G}$. Since $a_j^2 = 0$ for all $j \geqslant 3$, one has $\sigma(a_j) = \{0\}$. If $\pi_2 : \mathbf{C}^n \to \mathbf{C}^2$ is the projection onto the first two components, the projection property of the Taylor spectrum (cf. 3.2) gives $\pi_2(\sigma_T(a, X)) = \sigma_T((a_1, a_2), X) = \overline{G}$. $(*)$

Hence $\sigma_T(a, X) \subseteq (\overline{G} \times \mathbf{C}^{n-2}) \cap \prod_{j=1}^{n} \sigma(a_j) \subseteq \overline{G} \times \{0\}$, where 3.2 is used, and a further look at $(*)$ completes the proof. ∎

LEMMA 4.5. *Let $A := (a)'$ be the commutant of the system $a = (a_1, \ldots, a_n)$. Then*

$$\overline{D} \times \{0^{n-2}\} := \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n : (z_1, z_2) \in \overline{D}, z_j = 0, j \geqslant 3\} \subseteq \sigma_A(a).$$

Proof.

- If $X_1 := N(a_3) = E_2 \oplus E_0 \oplus E_0$, then X_1 is a hyperinvariant subspace for the commuting system $a = (a_1, \ldots, a_n)$, i.e. $bX_1 \subseteq X_1$ for all $b \in A = (a)'$.
- Assume that $(\lambda, 0) := (\lambda_1, \lambda_2, 0, \ldots, 0) \in D \times \{0\}$ and $(\lambda, 0) \notin \sigma_A(a)$. Hence for $j = 1, \ldots, n$ there exist $b_j \in A$, such that $id_X = \sum_{j=1}^{n} \tilde{a}_j b_j = \sum_{j=1}^{3} \tilde{a}_j b_j$ with $\tilde{a}_j := \lambda_j id_X - a_j, j = 1, \ldots, n$. In particular, one gets for $(1, 0, 0) \in X_1 \subseteq X$

$$(1, 0, 0) = \sum_{j=1}^{2} \tilde{a}_j b_j(1, 0, 0) - b_3 a_3(1, 0, 0) = \sum_{j=1}^{2} \tilde{a}_j(r_j, s_j, t_j)$$

with $(r_j, s_j, t_j) := b_j(1, 0, 0) \in X_1$ for $j = 1, 2$.

- Thus one has $r_j \in \mathcal{O}(G), j = 1, 2$ with $1 = \sum_{j=1}^{2} (\lambda_j - z_j) r_j(z)$ for all $z \in G$.
- By Hartog's theorem (cf. [30], Theorem II 8.4) there exist (uniquely defined) $R_j \in \mathcal{O}(D)$ with $R_j|_{G_2} = r_j|_{G_2}$ for $j = 1, 2$. Therefore, $1 = \sum_{j=1}^{2} (\lambda_j - z_j) R_j(z)$ for all $z \in D$, in contradiction to $\lambda \in D$.
- Consequently, it follows that $\overline{D} \times \{0^{n-2}\} \subseteq \sigma_A(a)$, by the compactness of $\sigma_A(a)$. ∎

Proof of Theorem 4.2. The preceeding lemmata have proved that $\sigma_T(a,X)$ is strictly contained in $\sigma_A(a)$. The germ $[\varphi]$ represented by the function $\varphi : z \longmapsto \begin{cases} 1 & , \ z \in G_1 \times \{0\} \\ 0 & , \ \text{otherwise} \end{cases}$ gives an element in $\mathcal{O}(\sigma_T(a,X))$ that is not in $\mathcal{O}(\sigma_{(a)'}(a))$ and the proof is completed. ∎

5. THE MAIN THEOREMS CONCERNING T-ALGEBRAS

It is quite useful to collect some technical properties by the following:

DEFINITION 5.1. Let B be a C^*-algebra with unit e. A subalgebra A of B is called *suitable*, if the following conditions hold:

(1) $e \in A$.

(2) A is symmetric, i.e. $a \in A \Longrightarrow a^* \in A$.

(3) A is spectrally invariant in B, i.e. $A \cap B^{-1} = A^{-1}$.

(4) There is a topology τ_A on A, which makes (A, τ_A) into a locally convex algebra, with jointly continuous multiplication, an open group A^{-1} of invertible elements, and continuous inversion.

(5) The natural inclusion $j_A : (A, \tau_A) \hookrightarrow (B, \|\cdot\|_B)$ is continuous.

REMARKS 5.2.

(1) Every Ψ^*-algebra in a C^*-algebra is a suitable algebra.

(2) Is B a C^*-algebra with unit e and A a suitable subalgebra of B, then for all $N \in \mathbb{N}$, $M_N(A)$ is also a suitable subalgebra of the C^*-algebra $M_N(B)$.

Proof. (4) is shown for example in [3], Proposition A.1.1 or [23], Corollary 1.2 (cf. 2.3), (1), (2) and (5) are then obvious and (3) follows from 2.4 ∎

THEOREM 5.3. *Let X be a Hilbert space and let $A \subseteq \mathcal{L}(X)$ be a sequentially complete, suitable subalgebra of $\mathcal{L}(X)$. Then A is a T-algebra.*

COROLLARY 5.4. *Let X be a Hilbert space. Then every Ψ^*-algebra in $\mathcal{L}(X)$ is a T-algebra.*

COROLLARY 5.5. *Let X be a Hilbert space and let $A \subseteq \mathcal{L}(X)$ be a sequentially complete, suitable subalgebra of $\mathcal{L}(X)$. Then $M_N(A)$ is, for every $N \in \mathbb{N}$, a T-algebra in $\mathcal{L}(X^N)$.*

Proof. By 5.2 the algebra $M_N(A)$ is a suitable subalgebra of the C^*-algebra $M_N(\mathcal{L}(X))$, which may be identified with $\mathcal{L}(X^N)$ in an obvious manner. Since $M_N(A)$ is clearly sequentially complete, a look at Theorem 5.3 completes the proof. ∎

Before proving the theorem, there are some technical lemmata concerning the Martinelli kernel given.

NOTATION 5.6. Let X be a Hilbert space. Then let κ be the mapping

$$\kappa : \mathcal{L}(X)^n \longrightarrow \mathcal{L}(\Lambda[\sigma, X]) : a = (a_1, .., a_n) \longmapsto \delta_a + \delta_a^* = \sum_{j=1}^{n} a_j \otimes S_j + \sum_{j=1}^{n} a_j^* \otimes S_j^* .$$

In the following the algebra $\mathcal{L}(\Lambda[\sigma, X])$ is identified after fixing the system $\sigma = (\sigma_1, \ldots, \sigma_n)$ and the canonical basis of $\Lambda[\sigma]$ used in (*) of 3.4 with the matrix algebra $M_{2^n}(\mathcal{L}(X))$.

PROPOSITION 5.7. *Let X be a Hilbert space, $A \subseteq \mathcal{L}(X)$ a suitable subalgebra of $\mathcal{L}(X)$ and $a = (a_1, \ldots, a_n) \in A^n$ a commuting system. Further, let $\Omega := \mathbf{C}^n \setminus \sigma_T(a, X)$, and consider the differential forms used to construct the Martinelli kernel attached to the commuting system $a = (a_1, \ldots, a_n)$:*

$$\begin{array}{l} \xi_j : \Omega \to \Lambda^j[d\bar{z}, M_{2^n}(X)], \quad j = 0, 1, \ldots, n-1 \\ \eta_j : \Omega \to \Lambda^{j+1}[d\bar{z}, M_{2^n}(X)], j = 0, 1, \ldots, n-2 \end{array}, \quad \textit{where} \quad \begin{array}{r} \xi_0 : z \mapsto \kappa(ze - a)^{-1}. \\ \eta_j : z \mapsto (\bar{\partial}\xi_j)(z). \\ \xi_{j+1} : z \mapsto \xi_0(z)\eta_j(z). \end{array}$$

Then one has:

$$(1) \quad \begin{array}{llll} \xi_j(\Omega) \subseteq & \Lambda^j[d\bar{z}, M_{2^n}(A)] & \subseteq \Lambda^j[d\bar{z}, M_{2^n}(\mathcal{L}(X))], & j = 0, 1, \ldots, n-1. \\ \eta_j(\Omega) \subseteq & \Lambda^{j+1}[d\bar{z}, M_{2^n}(A)] & \subseteq \Lambda^{j+1}[d\bar{z}, M_{2^n}(\mathcal{L}(X))], & j = 0, 1, \ldots, n-2. \end{array}$$

(2) *The mappings $\xi_j^A : \Omega \longrightarrow \Lambda^j[d\bar{z}, M_{2^n}(A)] : z \longmapsto \xi_j(z)$, and*

$$\eta_j^A : \Omega \longrightarrow \Lambda^{j+1}[d\bar{z}, M_{2^n}(A)] : z \longmapsto \eta_j(z)$$

are of the class C^∞ for all j. Furthermore, one has $\eta_j^A = \bar{\partial}\xi_j^A$ and $\xi_{j+1}^A = \xi_0^A \eta_j^A$.

Proof. The proof is given by induction on j. First of all consider the case $j = 0$:

• Let φ be the mapping $\varphi : \Omega \longrightarrow M_{2^n}(\mathcal{L}(X)) : z \longmapsto \kappa(ze - a)$; since A is symmetric and $\mathrm{id}_X \in A$, one has $\varphi(z) \in M_{2^n}(A)$ for all $z \in \Omega$; now $z \in \Omega$ implies $z \notin \sigma_T(a, X)$, hence (cf. 3.4) one has $\varphi(z) \in M_{2^n}(A) \cap M_{2^n}(\mathcal{L}(X))^{-1} = M_{2^n}(A)^{-1}$ by the spectral invariance of $M_{2^n}(A)$ in $M_{2^n}(\mathcal{L}(X))$ [cf. 5.2].

• The mapping $\varphi^A : \Omega \longrightarrow M_{2^n}(A) : z \longmapsto \varphi(z)$ is by construction linear in $z = (z_1, \ldots, z_n)$ and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$, and is therefore of class C^∞. Since $M_{2^n}(A)$ has jointly continuous multiplication and continuous inversion, the inversion in $M_{2^n}(A)$ is also of the class C^∞, as one can easily see using the resolvent equation. Hence ξ_0^A is of class C^∞ and the basis of the induction is proved.

Assume now that (1) and (2) are shown for ξ_j for some j with $0 \leqslant j \leqslant n-2$.

• By (2) the mapping $\varphi : \Omega \longrightarrow \Lambda^{j+1}[d\bar{z}, M_{2^n}(A)] : z \longmapsto (\overline{\partial}\xi_j^A)(z)$ is of the class $\mathcal{C}^\infty$. Now the mappings $(j_A^q)_* : \Lambda^q[d\bar{z}, M_{2^n}(A)] \hookrightarrow \Lambda^q[d\bar{z}, M_{2^n}(\mathcal{L}(X))]$, induced naturally by $j_A : A \hookrightarrow \mathcal{L}(X)$, are continuous, linear and one-to-one. Hence one obtains $(j_A^{j+1})_*\varphi(z) = \overline{\partial}((j_A^j)_*\xi_j^A)(z) = (\overline{\partial}\xi_j)(z) = \eta_j(z)$, thus $\eta_j(\Omega) \subseteq \Lambda^{j+1}[d\bar{z}, M_{2^n}(A)]$ and $\eta_j^A = \varphi$, consequently (1) and (2) are proved for η_j.

• Since $\xi_0(z) \in M_{2^n}(A)$ and $\eta_j(z) \in \Lambda^{j+1}[d\bar{z}, M_{2^n}(A)]$, one has $\xi_{j+1}(z) = \xi_0(z)\eta_j(z) \in \Lambda^{j+1}[d\bar{z}, M_{2^n}(A)]$ for all $z \in \Omega$. The differential form ξ_{j+1}^A is of class $\mathcal{C}^\infty$, since $M_{2^n}(A)$ is a suitable algebra, and hence the $M_{2^n}(A)$-modul multiplication on $\Lambda^{j+1}[d\bar{z}, M_{2^n}(A)]$ is of class $\mathcal{C}^\infty$. ∎

COROLLARY 5.8. *Let X be a Hilbert space, $A \subseteq \mathcal{L}(X)$ a suitable subalgebra of $\mathcal{L}(X)$, $a = (a_1, \ldots, a_n) \in A^n$ a commuting system, and $\Omega := \mathbb{C}^n \setminus \sigma_T(a, X)$. Then the following properties, for the Martinelli kernel attached to the commuting system $a = (a_1, \ldots, a_n)$, are fulfilled:*

(1) $M_a(\Omega) \subseteq \Lambda^{n-1}[d\bar{z}, A] \subseteq \Lambda^{n-1}[d\bar{z}, \mathcal{L}(X)]$.

(2) *The mapping $M_a^A : \Omega \longrightarrow \Lambda^{n-1}[d\bar{z}, A] : z \longmapsto M_a(z)$ is of class $\mathcal{C}^\infty$, and in particular continuous.*

(3) *The following diagram is commutative:*

$$\begin{array}{ccc} \Omega & \xrightarrow{M_a^A} & \Lambda^{n-1}[d\bar{z}, A] \\ \mathrm{id}\downarrow & & \downarrow j_{A*} \\ \Omega & \xrightarrow{M_a} & \Lambda^{n-1}[d\bar{z}, \mathcal{L}(X)] \end{array}.$$

Here $j_{A*} : \Lambda^{n-1}[d\bar{z}, A] \hookrightarrow \Lambda^{n-1}[d\bar{z}, \mathcal{L}(X)]$ denotes the continuous, linear map, induced naturally by $j_A : A \hookrightarrow \mathcal{L}(X)$.

Proof. By 3.4, M_a is the element in the right corner of the first row of ξ_{n-1}. ∎

5.9. PROOF OF THE MAIN THEOREM. Let $a = (a_1, \ldots, a_n) \in A^n$ be a commuting system and $U \subseteq \mathbb{C}^n$ open, such that $\sigma_T(a, X) \subseteq U$. Choose an admissible surface Σ surrounding $\sigma_T(a, X)$ in U. Then for all $f \in \mathcal{O}(U)$, one has:

• The mapping $\Sigma \longrightarrow \Lambda^{n-1}[d\bar{z}, A] : z \longmapsto f(z)M_a^A(z)$ is continuous.

• $f(z)M_a(z) = j_{A*}f(z)M_a^A(z)$.

Hence, for the operator given by the holomorphic functional calculus (cf. 3.5), one obtains:

$$\begin{aligned} f(a) &= \frac{1}{(2\pi i)^n} \int_{\Sigma} f(z) M_a(z) \wedge dz \\ &= \frac{1}{(2\pi i)^n} \int_{\Sigma} j_{A*} M_a^A(z) \wedge dz \\ &= j_A \left[\frac{1}{(2\pi i)^n} \int_{\Sigma} f(z) M_a^A(z) \wedge dz \right] \in A \end{aligned}$$

since the last integrand is converging in A. Note that the integral fM_a^A is a continuous function with values in a sequentially complete, locally convex space. ∎

REMARK 5.10. Of course one can ask if the assumption of local convexity of the algebra A may be dropped or replaced by a weaker one. The main problem occurring in this context is the convergence of the integral in the last step of the above proof; the existence of a continuous factorization M_a^A of M_a does not suffice to prove the theorem in the non locally convex case; in fact B. Gramsch has showm in 1965 (cf. [9], Example 3.1) that there are continuous functions with values in complete locally bounded (i.e. p-normed), non locally convex spaces, which are not Riemann-integrable.

However, with the aid of an integration theory developed by B. Gramsch and others in the sixties (cf. [9], [31]), the integration of a wide class of continuous functions with values in a complete, locally bounded, or even locally pseudo-convex, sequentially complete space (cf. [10]), is still possible.

Recall that a locally pseudo-convex space E is a topological vector space E, on which the topology may be given by a separating system $(q_\gamma)_{\gamma \in \Gamma}$ of p_γ-seminorms $(0 < p_\gamma \leqslant 1)$, i.e. $q_\gamma : E \to [0, \infty]$ is a mappings, such that $q_\gamma(x + y) \leqslant q_\gamma(x) + q_\gamma(y)$ and $q_\gamma(\lambda x) = |\lambda|^{p_\gamma} q_\gamma(x)$ for all $x, y \in E$ and for all $\lambda \in \mathbb{C}$.

In the following, let A be an *m-suitable subalgebra* of $\mathcal{L}(X) - X$ a Hilbert space; i.e. A is a suitable subalgebra of $\mathcal{L}(X)$, except that the condition (4) in the definition of suitable 5.1 is substitueted by

4′. There is a locally pseudo-convex topology τ_A on A, generated by a separating system $(q_\gamma)_{\gamma \in \Gamma}$ of submultiplicative p_γ-seminorms $(0 < p_\gamma \leqslant 1)$, i.e. $q_\gamma(xy) \leqslant q_\gamma(x) q_\gamma(y)$ for all $x, y \in A$ and $q_\gamma(e) = 1$, which makes (A, τ_A) into a topological algebra with jointly continuous multiplication, an open group A^{-1} of invertible elements, and continuous inversion.

It is then possible to show that the Martinelli kernel M_a attached to the commuting system $a \in A^n$ allows a factorization $M_a^A : \Omega \to \Lambda^{n-1}[d\bar{z}, A]$ belonging to an integrable class of functions with values in A. Thus it is easy to prove the following theorem – a detailed proof is given by the author in [15], Chapter 9.

THEOREM 5.11. *Let X be a Hilbert space, $A \subseteq \mathcal{L}(X)$ a sequentially complete, m-suitable subalgebra of $\mathcal{L}(X)$. Then A is a T-algebra.*

The proof of the following corollary proceeds as the proof of 5.5, using the fact that $M_N(A)$ is an m-suitable subalgebra of $M_N(\mathcal{L}(X))$, if A is an m-suitable subalgebra of $\mathcal{L}(X)$ (cf. [15], Lemma 9.2.3.

COROLLARY 5.12. *Let X be a Hilbert space and $A \subseteq \mathcal{L}(X)$ a sequentially complete, m-suitable subalgebra of $\mathcal{L}(X)$. Then $M_N(A)$ is for every $N \in \mathbb{N}$ a T-algebra in $\mathcal{L}(X^N)$.*

The case of Banach spaces X is completely different, because there is no such integral formula like the Martinelli-formula in the case of Hilbert spaces. Thus it was only possible to give the following class of T-algebras in that case.

THEOREM 5.13. *Let X be an infinite dimensional Banach space and $\mathcal{J} \triangleleft \mathcal{L}(X)$ a non trivial left – or right – ideal in $\mathcal{L}(X)$ which is contained in the ideal $\mathcal{K}(X)$ of all compact operators on X. Then the algebra with unit generated by $\mathcal{J}$, $B := \mathbb{C}\mathrm{id}_X \oplus \mathcal{J}$, is a T-algebra.*

Note that no convexity or completeness of the ideal $\mathcal{J}$ is required.

Proof.

• Let $a = (a_1, \ldots, a_n) \in \mathcal{K}(X)^n$ a commuting system, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$, and $b := \mu e + a \in B^n$. Then the Taylor spectrum $\sigma_T(b, X)$ of the commuting system $b = (b_1, \ldots, b_n)$ is at most countable and has at most one limit point, namely μ (cf. [7], Example A.8 or [15], Corollary 4.4.6).

• Further, let $f \in \mathcal{O}(U)$ be a representative of the germ $[f] \in \mathcal{O}(\sigma_T(b, X))$.

• Using Corollaries III 8.16 and III 8.17 in [30], and an induction argument, one obtains a system $\Gamma_j \subseteq \mathbb{C}$ $(j = 1, \ldots, n)$ of admissible contours (cf. [30], III 3.3) surrounding $\sigma(b_j)$ in $\pi_j(U)$, where $\pi_j : \mathbb{C}^n \to \mathbb{C}$ is projection onto the j-th components, such that the operator $\Theta_b(f) = f(b)$ is given by the integral convergent in $\mathcal{L}(X)$

$$\frac{1}{(2\pi i)^n} \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} (z_n \mathrm{id}_X - b_n)^{-1} \cdots (z_1 \mathrm{id}_X - b_n)^{-1} f(z) dz_1 \ldots dz_n.$$

• Now let $\mathcal{J}$ be a left ideal, and assume further that $a = (a_1, \ldots, a_n) \in \mathcal{J}^n$.

• Since the identity

$$(\lambda e - x)^{-1} = \frac{1}{\lambda}e + \left(e - \frac{x}{\lambda}\right)^{-1}\frac{x}{\lambda^2}, \quad 0 \neq \lambda \in \mathbf{C}$$

is valid in every C-algebra with unit e, one obtains the following expression for the n-dimensional resolvent ϱ:

$$\begin{aligned}
\varrho(z) &= \prod_{j=1}^{n}(z_j \mathrm{id}_X - b_j)^{-1} \\
&= \prod_{j=1}^{n}\left(\frac{1}{z_j-\mu_j}\mathrm{id}_X + \left(\mathrm{id}_X - \frac{a_j}{z_j-\mu_j}\right)^{-1}\frac{a_j}{(z_j-\mu_j)^2}\right) \\
&= \left(\prod_{j=1}^{n}\frac{1}{z_j-\mu_j}\right)\mathrm{id}_X \\
&\quad + \sum_{k=0}^{n-1}\sum_{\substack{M_k \subsetneq \{1,\dots,n\} \\ |M_k|=k}}\prod_{j\in M_k}\prod_{\ell\in\{1,\dots,n\}\setminus M_k}\frac{1}{z_j-\mu_j}\left(\mathrm{id}_X - \frac{a_\ell}{z_\ell-\mu_\ell}\right)^{-1}\frac{a_\ell}{(z_\ell-\mu_\ell)^2},
\end{aligned}$$

where $z = (z_1, \dots, z_n) \in \Gamma_1 \times \cdots \times \Gamma_n$. Note that one has $z_j - \mu_j \neq 0$, because $z_j \notin \sigma(b_j)$ implies that $((z_j - \mu_j)\mathrm{id}_X - a_j)^{-1}$ exists, and a_j is a compact operator on a infinite-dimensional Banach space,

• Hence with the abbreviation

$$\oint_\Gamma \dots dz := \oint_{\Gamma_n} \cdots \oint_{\Gamma_1} \dots dz_1 \dots dz_n,$$

and since all integrals exist because of continuity, one obtains:

$$\begin{aligned}
f(b) &= \frac{1}{(2\pi \mathrm{i})^n}\oint_\Gamma \varrho(z)f(z)dz \\
&= \frac{1}{(2\pi \mathrm{i})^n}\Bigg[\left(\oint_\Gamma\left(\prod_{j=1}^{n}\frac{1}{z_j-\mu_j}\right)f(z)dz\right)\mathrm{id}_X \\
&\quad + \sum_{k=0}^{n-1}\sum_{|M_k|=k}\oint_\Gamma\left(\prod_{j\in M_k}\prod_{\ell\in\overline{M}_k}\frac{1}{z_j-\mu_j}\left(\mathrm{id}_X - \frac{a_\ell}{z_\ell-\mu_\ell}\right)^{-1}\frac{a_\ell}{(z_\ell-\mu_\ell)^2}dz\right)\Bigg] \\
&= f(\mu)\mathrm{id}_X + \sum_{k=0}^{n-1}\sum_{|M_k|=k} \\
&\quad \underbrace{\frac{1}{(2\pi \mathrm{i})^n}\left[\oint_\Gamma\left(\prod_{j\in M_k}\prod_{\ell\in\overline{M}_k}\frac{1}{z_j-\mu_j}\left(\mathrm{id}_X - \frac{a_\ell}{z_\ell-\mu_\ell}\right)^{-1}\frac{1}{(z_\ell-\mu_\ell)^2}\right)dz\right]}_{\in\mathcal{L}(X)}\underbrace{\prod_{\ell\in\overline{M}_k}a_\ell}_{\in\mathcal{J}},
\end{aligned}$$

where $\overline{M_k} := \{1, \ldots, n\} \setminus M_k$ for every subset M_k of $\{1, \ldots, n\}$ with k elements. In particular, one has $\overline{M_k} \neq \emptyset$ of every $k = 0, \ldots, n-1$, i.e. $\prod_{\ell \in \overline{M_k}} a_\ell \in \mathcal{J}$, hence $f(b) \in \mathbb{C}\,\mathrm{id}_X \oplus \mathcal{J} = B$.

• In the case of a right ideal $\mathcal{J}$ the proof proceeds analogously using the corresponding identity

$$(ze - a)^{-1} = \frac{1}{z}e + \frac{a}{z^2}\left(e - \frac{a}{z}\right)^{-1}.$$

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ROBERT LAUTER
Fachbereich 17 - Mathematik
Johannes Gutenberg-Universität Mainz
D-55099 Mainz
Germany

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Note added in proof. Using an idea of Kordula and Müller (*Vasilescu-Martinelli formula for operators in Banach spaces*, preprint 1994) it was recently possible to construct an integral formula analogous to the Martinelli formula (3.4) and to prove an analogue to Theorem 5.3 valid in the case of Banach spaces and Ψ_0-algebras in the sense of Gramsch, ([12] Definition 5.1) – (R. Lauter, *A multidimensional holomorphic functional calculus for Ψ_0-algebras with methods of J.L. Taylor*, to apppear in: Proceedings of the 15th conference on operator theory in Timişoara, June 1994, Birkhaäuser, Basel).