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CONTINUOUS AND DISCRETE SEMIGROUP APPROXIMATIONS WITH APPLICATIONS TO THE CAUCHY PROBLEM

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ABSTRACT. In this paper we carry on further study of the problem of approximation of semigroups for both the continuous and the discrete cases which were considered in ([1], [2]). These problems arise naturally when one considers the question of numerical solutions of linear Cauchy problems.

KEYWORDS: *Semigroups, Linear operators, Approximation, Cauchy problem.*

AMS SUBJECT CLASSIFICATION: Primary 47, 46; Secondary 35.

1. INTRODUCTION

In a previous study ([1], [2]) the problem of convergence and approximation of semigroups of operators was investigated for class (1,A) semigroups, for both the continuous and the discrete cases. The object of the present paper is to extend that study to more general classes, namely those of class (0,A) and (A). Specifically, we shall be concerned with the question of approximation of a continuous semigroup $T(t), t > 0$ by means of discrete semigroups (F_n^k), where $\{F_n : n = 1, 2, \ldots\}$ are bounded operators. This problem, as is well known, has its origin in the subject of numerical approximations to solutions of initial boundary value problems. It turns out that the mode and rate of convergence, as well as the information retrieved concerning properties and characteristics of the solution family of the associated Cauchy problem, are all governed to a large extent by the stability condition satisfied by the approximating systems. Thus the earlier studies of the problem were based on a uniform type of stability condition, originally due to Von Neumann. It was considered later by P.D. Lax and R.D. Richtmyer, in connection with finite difference systems, cf.[8], and introduced in operator-semigroup form by H. Trotter [12], namely

$$\|F_n^k\| \leqslant M e^{k\omega\rho_n}$$

where (ρ_n) is a null sequence of positive numbers and M, ω are independent of k and n.

More recent studies (cf. [2], [6]) show that convergence of discrete semigroups to a continuous one may still hold under weaker stability conditions. We shall resume this line of investigation in sections 4 and 5 below and show that other forms of stability conditions are also usable, in which case the limit semigroup belongs to one of the more general classes (0,A) or (A).

In Section 2 the question of convergence of continuous parameter semigoups is considered in a rather general setting for both classes (A) and (0,A). Section 3 is concerned with the question of convergence of certain types of Riemann sums to Lebesgue integrals. This will be useful in the subsequent sections 4,5 when dealing with the discrete case. These results are then applied to a Cauchy problem of parabolic type in the sense of Shilov, (cf. [11]).

Let $(X, \|\cdot\|)$ be a Banach space, and $(X_n, \|\cdot\|_n)$, a sequence of Banach spaces approximating X in the following sense: There exist bounded linear operators $P_n : X \to X_n, n \in \mathbf{N}$ such that for each $x \in X$

$$\lim_{n\to\infty} \|P_n x\|_n = \|x\|.$$

In particular there is a constant $\beta > 0$ such that:

$$\|P_n x\|_n \leqslant \beta \|x\|, \quad \forall n \in \mathbf{N}, x \in X.$$

The limit (in a generalized sense) of a sequence of vectors $(x_n), x_n \in X_n$, is an element $x \in X$ such that

$$\lim_{n\to\infty} \|P_n x - x_n\|_n = 0.$$

In this case we write:

$$\widetilde{\lim}\, x_n = x.$$

This generalized notion of limit reduces to the ordinary one if we take $X_n = X, \|\cdot\|_n = \|\cdot\|$ and $P_n = I$ for all n. Next, let $A_n : X_n \to X_n, n = 1, 2, \ldots$ be linear operators. The limit (in the generalized sense) of the sequence (A_n) is an operator A in X denoted $A = \widetilde{\lim}\, A_n$ and is defined as follows: $x \in \mathcal{D}(A)$ and $y = Ax$ if and only if, for all n, $P_n x \in \mathcal{D}(A_n)$, and $\widetilde{\lim}\, A_n P_n x = y$. The limit inferior

$\hat{A} = \liminf A_n$ is also an operator in X (possibly multi-valued) that is defined as follows: $x \in \mathcal{D}(\hat{A})$ and $y \in \hat{A}x$ provided there exists a sequence $(x_n), x_n \in \mathcal{D}(A_n)$, such that $\widetilde{\lim}\, x_n = x$ and $\widetilde{\lim}\, A_n x_n = y$. Finally, we introduce the *superior domain* $\mathcal{D}^\circ$ of the sequence (A_n) which we define as follows:

$\mathcal{D}^\circ = \{x \in X :$ there exists a sequence $(x_n), x_n \in D(A_n)$ such that $\widetilde{\lim}\, x_n = x$ and $\sup_n \|A_n x_n\|_n < \infty\}$.

We note here that $\hat{A}$ is a closed extension of A and that $\mathcal{D}(A) \subset \mathcal{D}(\hat{A}) \subset \mathcal{D}^\circ$.

Recall that a semigroup of linear operators on a Banach space X is a mapping $T(t) : (0, \infty) \to \mathcal{L}(X)$ (the space of bounded linear operators on X) satisfying $T(t+s) = T(t)T(s)$ for all $t, s > 0$. We assume in this discussion that the semigroup is a strongly continuous map on $(0, \infty)$. The infinitesimal operator of $T(t)$ is defined as usual by:

$$A_0 x = \lim_{h \to 0+} h^{-1}(T(h) - I)x$$

whenever the limit exists. The closure $\overline{A_o}$, when it exists, is called the infinitesimal generator (i.g.) of $T(t)$. The *type* of $T(t)$, denoted ω_0, is defined by:

$$\omega_0 = \liminf_{t > 0} t^{-1} \ln \|T(t)\|.$$

DEFINITION 1. A semigroup $T(t), t > 0$, is of class (A) (cf. Hille & Phillips [7]) if it satisfies the following:

(i) $\mathcal{R}[T] = \bigcup_{t>0} T(t)X$ is dense in X;

(ii) There exists an $\omega_1 > \omega_0$ such that, for each λ with $\Re(\lambda) > \omega_1$, there is an operator $R(\lambda) \in \mathcal{L}(X)$, with the properties:

(a)

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt, \quad x \in \mathcal{R}[T]$$

(b) $\|R(\lambda)\|$ is bounded in the half plane $\Re(\lambda) > \omega_1$, and (c) $\lim_{\lambda \to \infty} \lambda R(\lambda)x = x$ for each $x \in X$.

For such semigroups A_0 need not be closed. However, it has a smallest closed extension denoted by A, which is the infinitesimal generator (i.g.) of $T(t)$.

DEFINITION 2. $T(t)$ is said to be of class (0,A) if it satisfies the following:

$$\int_0^1 \|T(t)x\| \, dt < \infty, \quad x \in X,$$

$$\lim_{\lambda \to \infty} \lambda \int_0^\infty e^{-\lambda t} T(t)x \, dt = x, \quad \forall x \in X.$$

Every C_0 semigroup is of class (0,A), and (0,A) semigroups are of class (A). The converse is not true in general (cf. [7]). For further details and information on this subject we refer to ([4], [7]).

2. CONVERGENCE OF (A)-SEMIGROUPS

The main results of this section are Theorem 2.3 and 2.4. They extend the main result of [1] on $(1, A)$ semigroups and some of the results of [9] on (A) and (0,A) semigroups. We first state separately a set of conditions that, roughly speaking, describes the growth condition that would be appropriate for (A) and (0,A) semigroups, as well as some type of uniform Abel summability. This also replaces the commonly used (uniform) bound on the norms in the case of C_0 semigroups (cf. [12]). We note here that conditions $I_1 - I_4$ were first used in [9].

Let $(T_n(t), t > 0)$ be a sequence of (A)-semigroups defined, respectively, on the spaces (X_n). The i.g. of $T_n(t)$ is denoted by A_n.

CONDITIONS:

(I_1) There exists a non-negative, non-increasing function $\psi(t)$ of negative type (meaning that, for sufficiently large t, ψ is bounded by a decreasing exponential), such that

$$\sup_n \|T_n(t)\|_n \leqslant \psi(t), \quad t > 0.$$

(I_2) There are positive constants M, L, and ω such that

$$\sup_n \|R(\lambda; A_n)\|_n \leqslant M, \quad \Re(\lambda) > 0.$$

(I_3) $\sup_n \|\lambda R(\lambda; A_n)\|_n \leqslant L, \quad$ for all $\lambda > \omega, n \in \mathbf{N}$.

(I_4) There is a real number γ_0 such that

$$\sup_n \int_0^\infty e^{-\gamma_0 t} \|T_n(t) P_n x\|_n \, dt = M_x < \infty, \ \textit{for } x \in X.$$

PROPOSITION 2.1. *Let $(T_n(t), t > 0)$ be a sequence of* (A) *semigroups satisfying conditions $(I_2), (I_3)$. Further, assume that $\mathcal{D}^\circ$ and $\mathcal{R}(\lambda_0 I - \hat{A})$ are dense in X for some $\lambda_0 > \omega$. Then $\hat{A}$ is a densely defined, single valued operator on X with $\rho(\hat{A}) \supseteq (\omega, \infty)$, such that*

$$\lim_{\lambda \to \infty} \lambda R(\lambda; \hat{A}) x = x, \quad x \in X.$$

Moreover, for each $\lambda > \omega$

$$\lim_{n \to \infty} \|R(\lambda; A_n) P_n x - P_n R(\lambda; \hat{A}) x\|_n = 0, \quad x \in X.$$

Proof. See ([1], Theorem 1).

We shall use the notation:

$$H(\omega) = \{\lambda : \Re(\lambda) > \omega\},$$

$$S(\omega) = \rho(\hat{A}) \cap H(\omega).$$

PROPOSITION 2.2. *Suppose that $(T_n(t), t > 0)$ satisfy conditions I_2, I_3, and that $\mathcal{D}^\circ$ and $\mathcal{R}(\lambda_0 I - \hat{A})$ are dense in X for some $\lambda_0 > \omega$; then the following assertions hold:*

(a) *For each $\lambda \in S(\omega), x \in X$, and sequence (x_n), $x_n \in X_n$*

$$\widetilde{\lim}\, x_n = x \;\Rightarrow\; \widetilde{\lim}\, R(\lambda; A_n)x_n = R(\lambda; \hat{A})x.$$

In particular, $\|R(\lambda, \hat{A})\| \leqslant M, \lambda > \omega$

(b) $H(\omega) \subseteq \rho(\hat{A})$

(c) $R(\lambda; \hat{A})^k = \liminf R(\lambda; A_n)^k,\ k \geqslant 1, \lambda \in H(\omega)$

(d) *For each $z \in \mathcal{D}(\hat{A}^2)$ there exists a sequence $(z_n), z_n \in \mathcal{D}(A_n^2)$, such that $\widetilde{\lim}\, z_n = z, \widetilde{\lim}\, A_n z_n = \hat{A}z$ and $\widetilde{\lim}\, A_n^2 z_n = \hat{A}^2 z$; in particular,* $\liminf A_n^2 \supseteq \hat{A}^2$

Proof. The proof is similar to that of ([1], Proposition 1).

We now state the main result of this section.

THEOREM 2.3. *Let $(X, \|\cdot\|)$ be a complex Banach space and $(X_n, \|\cdot\|_n)$ a sequence of Banach spaces approximating X. For each $n \in \mathbb{N}$, let $T_n(t)$ be an* (A) *semigroup on X_n with i.g. A_n satisfying conditions $(I_1), (I_2)$, and (I_3). Then the following assertions are equivalent:*

(i) *There exists a semigroup $T(t), t > 0$ of class* (A), *defined on X such that, for $x \in X$ and $x_n \in X_n, n = 1, 2, 3, \ldots$*

$$\widetilde{\lim}\, x_n = x \;\Rightarrow\; \widetilde{\lim}\, T_n(t)x_n = T(t)x \tag{1}$$

uniformly on compact subsets of $(0, \infty)$

(ii) *$\mathcal{D}^\circ$ and $\mathcal{R}(\lambda_0 I - \hat{A})$ are dense in X for some $\lambda_0 > \omega$.*

In either case, $\hat{A}$ is the i.g. of $T(t)$.

Proof. **(i) $\Rightarrow$ (ii)**

Assume that $T(t)$ is a semigroup on X of class (A) with i.g. A such that (1) holds. It suffices to show that

$$\widetilde{\lim}\, x_n = x \;\Rightarrow\; \widetilde{\lim}\, R(\lambda; A_n)x_n = R(\lambda; A)x \tag{2}$$

for in this case one can deduce as in ([1], Theorem 2) that $\hat{A} = A$, hence (ii) follows.

Assume first that $x \in \mathcal{R}[T]$. There exists a sequence $(x_n), x_n \in \mathcal{R}[T_n]$ such that $\widetilde{\lim}\, x_n = x$. Indeed, there is an $s_0 > 0$ and an $x_0 \in X$, such that $x = T(s_0)x_0$.

Put $x_n = T_n(s_0)P_n x_0$; then $x_n \in \mathcal{R}[T_n]$ and by (1), we find that $\widetilde{\lim}\, x_n = x$. Using this and the relation between resolvent operators and semigroups, (definition (i)-(a)) we see that

$$(3) \qquad \|R(\lambda; A_n)x_n - P_n R(\lambda; A)x\|_n \leqslant \int_0^\infty e^{-\lambda t}\|T_n(t)x_n - P_n T(t)x\|_n \, dt$$

Now

$$\begin{aligned}\sup_n \|T_n(t)x_n - P_n T(t)x\|_n &\leqslant \sup_n \|T_n(t+s_0)P_n x_0\|_n + \sup_n \|P_n T(t+s_0)x_0\|_n \\ &\leqslant \psi(t+s_0)\beta\|x_0\| + \|T(t+s_0)\|\,\|x_0\|\beta \\ &\leqslant [\psi(s_0) + \nu(s_0)]\|x_0\|\beta \\ &= \alpha.\end{aligned}$$

Here $\nu(s) = \sup_{t>0} \|T(t+s)\|$ is finite since $T(t)$ is of class (A) and of negative type (cf. [7]). Hence $e^{-\lambda t}\|T_n(t)x_n - P_n T(t)x\|_n \leqslant \alpha e^{-\lambda t}$, which is integrable over $(0, \infty)$. Therefore, by making use of (i) and the dominated convergence theorem, we obtain

$$(4) \qquad \lim_{n\to\infty} \|R(\lambda; A_n)x_n - P_n R(\lambda; A)x\|_n = 0$$

for all $x \in \mathcal{R}[T]$. Now the family of operators $\{R(\lambda; A_n)P_n - P_n R(\lambda; A)\}$ is uniformly bounded. Thus, in view of I_2 and the assumption that $\mathcal{R}[T]$ is dense one concludes that in fact (4) holds for any $x \in X$ and any sequence (x_n), $x_n \in X_n$ satisfying $\widetilde{\lim}\, x_n = x$.

(ii) $\Rightarrow$ (i)

We note from Propositions 2.1 and 2.2 that $\hat{A}$ is a closed operator whose resolvent $R(\lambda; \hat{A})$ is bounded in the half plane $H(\omega)$. Therefore, there exists a $\gamma > \omega$ such that

$$(1.A) \qquad Y(t; z) = z + t\hat{A}z + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} R(\lambda; \hat{A})\hat{A}^2 z \frac{d\lambda}{\lambda^2}$$

and $Y(t; z)$ defines a continuous function on $t \geqslant 0$ for each $z \in \mathcal{D}(\hat{A}^2)$, such that $Y(0; z) = z$ (cf. [2]). Similarly, since each $T_n(t)$ is of class (A) with $H(\omega) \subseteq \rho(A_n)$,

$$(2.A) \quad T(t; A_n)w = w + tA_n w + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} R(\lambda; A_n)A_n^2 w \frac{d\lambda}{\lambda^2}, \quad w \in \mathcal{D}(A_n^2).$$

Now let $z \in \mathcal{D}(\hat{A}^2)$, and let (z_n) be a sequence as given by Proposition 2.2-(d). Replacing w by z_n in (2.A) and passing to the limit in the generalized sense as $n \to \infty$, we find that

$$(3.A) \qquad \widetilde{\lim}\, T_n(t)z_n = Y(t; z) = T(t)z,$$

uniformly on compacts, (cf. [1], [2]). Next, we note from condition (I_1) that

$$\|Y(t;z)\| \leqslant \lim_{n\to\infty} \|T_n(t)z_n\|_n \leqslant \psi(t) \lim_n \|z_n\|_n = \psi(t)\|z\|.$$

Therefore, by applying the argument of ([2], Theorem A) to the present case, one concludes that, in fact, (3.A) holds for any $x \in X$ and any sequence of vectors $(x_n), x_n \in X_n$ satisfying $\widetilde{\lim}\, x_n = x$. In particular, $T(t)$ has the semigroup property, and the resolvent operator of $\hat{A}$ is related to $T(t)$ (cf. [7], Lemma 11.5.2) via

$$R(\lambda;\hat{A})x = \int_0^\infty e^{-\lambda t} T(t)x \,dt, \quad x \in \mathcal{D}(\hat{A}^2). \tag{4.A}$$

It remains to show that $T(t)$ is of class (A). To verify (i) of Definition 1, assume to the contrary that $\mathcal{R}[T]$ is not dense in X, then there exists $x^* \in X^*, x^* \neq 0$ such that $x^*[\mathcal{R}[T]] = 0$, by (4.A) $x^*[R(\lambda;\hat{A})x] = 0$, for all $x \in \mathcal{D}(\hat{A}^2)$, i.e. $x^*[\mathcal{D}(\hat{A}^3)] = 0$. This is a contradiction, since $\mathcal{D}(\hat{A}^3)$ is a dense subspace of $\mathcal{D}(\hat{A}^2)$ and hence of X. Next, we note that (ii).(b) and (ii).(c) of the definition are immediate consequences of Propositions 2.1 and 2.2. For (ii).(a), we let $x \in \mathcal{R}[T], s > 0$ and $z \in X$ such that $x = T(s)z$, then

$$\begin{aligned}\|T(t)x\| = \|T(t+s)z\| &\leqslant \liminf_n \|T_n(t+s)P_n z\|_n \\ &\leqslant \psi(t+s)\|z\| \leqslant \psi(s)\|z\| \leqslant \psi(s)\|z\|, \quad t > 0.\end{aligned}$$

So the Laplace transform $\int_0^\infty e^{-\lambda t} T(t)x\, dt$ exists. As before, we can find a sequence $x_n \in \mathcal{R}[T_n]$ satisfying $\widetilde{\lim}\, x_n = x$. Hence, $\widetilde{\lim} T_n(t)x_n = T(t)x, t > 0$ and by a Lesbegue convergence argument we find that

$$R(\lambda;\hat{A})x = \widetilde{\lim}\, R(\lambda;A_n)x_n = \widetilde{\lim} \int_0^\infty e^{-\lambda t} T_n(t)x_n \,dt = \int_0^\infty e^{-\lambda t} T(t)x\,dt. \quad \blacksquare$$

THEOREM 2.4. *If $T_n(t)$ is a semigroup of class* (0,A) *defined on $X_n, n = 1,2,\ldots$ such that conditions I_1, I_2, I_3 and I_4 hold, then* (i) *and* (ii) *of* Theorem 2.3 *are equivalent. Moreover, $T(t)$ is of class* (0,A) *and satisfies I_4.*

Proof. Since (0,A) semigroups are also (A) semigroups, the equivalence of (i) and (ii) follows from Theorem 2.3. To complete the proof, we show that $T(t)$ is of class (0,A). By (1),

$$\|T(t)x\| \leqslant \liminf_n \|T_n(t)P_n x\|_n, \quad t > 0$$

and

$$\int_0^\infty e^{-\lambda t}\|T(t)x\|\,dt \leqslant \int_0^\infty e^{-\lambda t}\|T_n(t)P_n x\|_n\,dt = M_x, \ \forall n. \quad \blacksquare$$

3. CERTAIN TYPES OF RIEMANN SUMS

It is important for our investigation here to reconsider the question of boundedness of Riemann sums. This leads one to study also the question of approximation of *Lebesgue* integrals by means of Riemann sums. This problem was considered, e.g., in [5]. Some results in this direction, due to P. Chernoff, are also quoted in [2].

Consider the function

$$S(f;t) = \sum_{j=1}^{\infty} t f(jt)$$

where $f(t)$ is a non-negative function on $(0,\infty)$. We will be dealing here with Riemann sums of the type

$$S(f;\rho_n) = \sum_{j=1}^{\infty} \rho_n f(j\rho_n),$$

where (ρ_n) is a null sequence of positive numbers.

PROPOSITION 3.1. *Let $0 \leqslant f \in L^1(0,\infty)$ be absolutely continuous such that $(1+t)|f'| \in L^1(0,\infty)$. Then the Riemann sums $S(f;\rho_n)$ are uniformly bounded.*

Proof. Note that necessarily $f(t) \to 0$ as $t \to \infty$. It follows that for all $t \geqslant 0$

$$f(t) = -\int_t^{\infty} f'(t)\,\mathrm{d}t.$$

Hence, $0 \leqslant f(t) \leqslant g(t) = \int_t^{\infty} |f'(s)|\,\mathrm{d}s$. Note also that

$$\int_0^{\infty} \mathrm{d}t \int_t^{\infty} |f'(s)|\,\mathrm{d}s = \int_0^{\infty} |f'(s)|\,\mathrm{d}s \int_0^{s} \mathrm{d}t$$
$$= \int_0^{\infty} s|f'(s)|\,\mathrm{d}s < \infty.$$

Hence, $g \in L^1(0,\infty)$. Since this function is decreasing, we see in view of ([2], Lemma 1) that $S(f;\rho_n) \leqslant S(g;\rho_n) \leqslant K$, for all $n \in \mathbf{N}$. ∎

PROPOSITION 3.2. *Let $0 \leqslant f \in L^1(0,\infty)$. Then $S(f;t) \in L^1(0,1)$.*

Proof. We note that

$$\int_0^1 S(f;t)\,\mathrm{d}t = \sum_{j=1}^{\infty} \int_0^1 t f(jt)\,\mathrm{d}t$$
$$= \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^j t f(t)\,\mathrm{d}t$$
$$= \sum_{j=1}^{\infty} C_j \int_{j-1}^{j} t f(t)\,\mathrm{d}t$$

where $C_j = \sum_{k=j}^{\infty} \frac{1}{k^2}$. Also note that $jC_j \to 1$ as $j \to \infty$. ($C_1 = \frac{\pi^2}{6} > C_2 > C_3 > \ldots$).

Now $\int_{j-1}^{j} tf(t)\,dt \leqslant j \int_{j-1}^{j} f(t)\,dt$. So we have:

$$\int_0^1 S(f;t)\,dt \leqslant \sum_{j=1}^{\infty} jC_j \int_{j-1}^{j} f(t)\,dt$$
$$\leqslant K \int_{j-1}^{j} f(t)\,dt = K \sum_{j=1}^{\infty} \int_0^{\infty} f(t)\,dt$$

where $K = \sup_{1 \leqslant j \leqslant \infty} jC_j < \infty$. Thus, $S = S(f;\cdot) \in L^1(0,1)$, with $\|S\|_1 \leqslant K\|f\|_1$. In particular, $S(t) < \infty$ for a.e. $t \in (0,1)$. ∎

In some sense, $S(f;t)$ should approach $\int_0^{\infty} f(s)\,ds$ as $t \downarrow 0$. The following is a simple result that says that this is true in some "average" sense.

PROPOSITION 3.3. *Let $0 \leqslant f \in L^1(0,\infty)$, then*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} S(f;s)\,ds = \int_0^{\infty} f(s)\,ds.$$

Proof. For $\varepsilon > 0$,

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} S(f;t)\,dt = \frac{1}{\varepsilon}\sum_{j=1}^{\infty}\int_0^{\varepsilon} tf(jt)\,dt$$
$$= \frac{1}{\varepsilon}\sum_{j=1}^{\infty}\frac{1}{j^2}\int_0^{j\varepsilon} xf(x)\,dx$$
$$= \frac{1}{\varepsilon}\sum_{j=1}^{\infty} C_j \int_{(j-1)\varepsilon}^{j\varepsilon} xf(x)\,dx$$
$$\leqslant \sum_{j=1}^{\infty} jC_j \int_{(j-1)\varepsilon}^{j\varepsilon} f(x)\,dx \leqslant K\int_0^{\infty} f(x)\,dx.$$

Now, given $\delta > 0$, choose N so that $1 - \delta < jC_j < (1+\delta)$ for $j \geqslant N$. Then

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} S(f;t)\,dt \leqslant \sum_{j=1}^{\infty} jC_j \int_{(j-1)\varepsilon}^{j\varepsilon} f(x)\,dx$$
$$= \left(\sum_{j<N} + \sum_{j \geqslant N}\right)\left(jC_j \int_{(j-1)\varepsilon}^{j\varepsilon} f(x)\,dx\right).$$

The first sum $\to 0$ as $\varepsilon \downarrow 0$. The second sum satisfies the following inequalities:

$$(1-\delta)\int_{(N-1)\varepsilon}^{\infty} f(x)\,\mathrm{d}x \leqslant \sum_{j\geqslant N} \leqslant (1+\delta)\int_{(N-1)\varepsilon}^{\infty} f(x)\,\mathrm{d}x.$$

As $\varepsilon \downarrow 0$, the extreme terms converge to $(1\pm\delta)\int_0^\infty f(x)\,\mathrm{d}x$. Since $\delta > 0$ is arbitrary, we conclude that

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^{\varepsilon} S(f;t)\,\mathrm{d}t \leqslant \int_0^\infty f(x)\,\mathrm{d}x.$$

For the reverse inequality, we have

$$\frac{1}{\varepsilon}\int_0^{\varepsilon} S(f;t)\,\mathrm{d}t \geqslant \sum_{j=1}^{\infty}(j-1)C_j\int_{(j-1)\varepsilon}^{j\varepsilon} f(x)\,\mathrm{d}x.$$

Since $(j-1)C_j \to 1$ as $j\to\infty$, an argument precisely the same as the foregoing shows that

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^{\varepsilon} S(f;t)\,\mathrm{d}t \geqslant \int_0^\infty f(x)\,\mathrm{d}x. \quad ∎$$

COROLLARY 1. *Let $0 \leqslant f \in L^1(0,\infty)$, then for each $\varepsilon > 0$ there is a $t_\varepsilon \in (0,\varepsilon)$ such that the following holds:*

$$\lim_{\varepsilon\downarrow 0} S(f;t_\varepsilon) \leqslant \int_0^\infty f(x)\,\mathrm{d}x.$$

Proof. For each $\varepsilon > 0$, we can find a $t_\varepsilon \in (0,\varepsilon)$ such that

$$S(f;t_\varepsilon) \leqslant \frac{1}{\varepsilon}\int_0^{\varepsilon} S(f;t)\,\mathrm{d}t + \varepsilon.$$

This can be verified e.g. by a contrapositive argument. The result follows by passing to the limit as $\varepsilon \downarrow 0$. ∎

COROLLARY 2. *Let $0 \leqslant f, g \in L^1(0,\infty)$, then for each $\varepsilon > 0$ there is a $t_\varepsilon \in (0,\varepsilon)$ such that:*

$$S(f;t_\varepsilon) \leqslant 3\int_0^\infty f(s)\,\mathrm{d}s$$

$$S(g;t_\varepsilon) \leqslant 3\int_0^\infty g(s)\,\mathrm{d}s.$$

Proof. Assume $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\varepsilon} S(f;s)\,\mathrm{d}s = L > 0$.

Let

$$A_\varepsilon = \{t : 0 < t \leqslant \varepsilon, S(f;t) \geqslant 3L\}.$$

Then

$$\frac{1}{\varepsilon}\int_0^\varepsilon S(f;t)\,dt \geqslant \frac{1}{\varepsilon}\int_{A_\varepsilon} S(f;t)\,dt \geqslant \frac{3L\mu(A_\varepsilon)}{\varepsilon}.$$

Thus

$$\limsup_{\varepsilon\downarrow 0}\frac{3L\mu(A_\varepsilon)}{\varepsilon} \leqslant L$$

or

$$\limsup_{\varepsilon\downarrow 0}\frac{\mu(A_\varepsilon)}{\varepsilon} \leqslant \frac{1}{3}\,.$$

So if ε is sufficiently small, $\mu(A_\varepsilon) \leqslant 2\varepsilon/5$ and therefore $\mu(B_\varepsilon) \geqslant 3\varepsilon/5$ where

$$B_\varepsilon = \{t : 0 < t \leqslant \varepsilon, S(f;t) < 3L\}.$$

Now, suppose similarly $g \geqslant 0, \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^\varepsilon S(g;s)\,ds = M > 0$. Then $\mu(C_\varepsilon) \geqslant 3\varepsilon/5$ for ε sufficiently small, where

$$C_\varepsilon = \{t : 0 < t \leqslant \varepsilon, S(g;t) < 3M\}.$$

So $\mu(B_\varepsilon \cap C_\varepsilon) \geqslant \varepsilon/5$. In particular, $\mu(B_\varepsilon \cap C_\varepsilon) > 0$, so $B_\varepsilon \cap C_\varepsilon \neq \emptyset$. Therefore, for ε sufficiently small, $\exists t_\varepsilon, 0 < t_\varepsilon \leqslant \varepsilon$, with $S(f;t_\varepsilon) \leqslant 3L$ and $S(g;t_\varepsilon) \leqslant 3M$. ∎

Another useful result in this direction due to P. Chernoff (cf. Corollary 2 in Section 2, [2]) is the following:

PROPOSITION 3.4. *If $f \in L^1(0,\infty)$, then for a.e. $t \in \mathbf{R}$*

$$\frac{1}{2^n}\sum_{t+\frac{j}{2^n}>0} f(t+\frac{j}{2^n}) \to \int_0^\infty f(s)\,ds.$$

4. DISCRETE APPROXIMATION OF (A)-SEMIGROUPS

In this section, we consider the problem of approximation of (A)-semigroups by means of discrete semigroups. Aside from being interesting on their own, the results here have direct consequences in the case of (0,A) semigroups (see Section 5).

In what follows, F_n denotes a bounded linear operator on the Banach space X_n, A_n the operator defined by $A_n := \rho_n^{-1}(F_n - I), \rho_n > 0, n \in \mathbf{N}, \hat{A} = \liminf A_n$, and $\mathcal{D}^\circ$ the superior domain of (A_n).
Finally, to simplify notation we write:

$$Q_{nj}(t) = e^{-\frac{t}{\rho_n}}\left(\frac{t}{\rho_n}\right)^j\frac{1}{j!}$$

for $t, \rho_n > 0, n = 1,2,\ldots$ and j a non-negative integer.

THEOREM 4.1. *Let $\psi(t,x)$ be a non-negative mapping on $(0,\infty)\times\mathcal{D}^\circ$ that is continuous in x for each t, and let γ be a non-negative number such that for each $x\in\mathcal{D}^\circ$, $e^{-\gamma t}\psi(t,x)$ is non-increasing in t (this also covers the case where for each $x\in\mathcal{D}^\circ$, there is a $T_x>0$ such that $\Psi(t,x)$ is non-increasing over (T_x,∞)) and belongs to $L^1(0,\infty)\cap L^p(0,\infty)$, for some $p>1$. Further, let (ρ_n) be a null sequence of positive numbers such that the following conditions are satisfied*

(i) *$\mathcal{D}^\circ$ and $\mathcal{R}(\lambda_0 I-\hat{A})$ are dense in X for some $\lambda_0>\omega+\gamma$.*

(ii) $\sup_n\|\exp(tA_n)\|_n<\infty$ *for each $t>0$.*

(iii) *There are constants $L, M>0$, and $\omega\geqslant 0$ such that*

$$\|R(\lambda;A_n\|_n\leqslant M,\quad \Re(\lambda)\geqslant\omega, n\in\mathbb{N} \tag{5}$$

$$\|\lambda R(\lambda;A_n\|_n\leqslant L,\quad \lambda\geqslant\omega, n\in\mathbb{N}. \tag{6}$$

(iv) *For each $x\in\mathcal{D}^\circ$ and each sequence $(x_n), x_n\in\mathcal{D}(A_n)$, satisfying $\widetilde{\lim}\, x_n=x$ and $\sup_n\|A_nx_n\|_n<\infty$, the following holds:*

$$\|F_n^k x_n\|_n\leqslant\psi(\rho_n k,x),\quad k,n\in\mathbb{N}.$$

Then $\hat{A}$ generates an (A)-semigroup $T(t), t>0$ on X, such that for any $x\in\mathcal{D}^\circ$ and $x_n\in\mathcal{D}(A_n)\, n=1,2,\dots$ satisfying $\widetilde{\lim}\, x_n=x$ and $\sup_n\|A_nx_n\|_n<\infty$, we have

$$\widetilde{\lim}\, F_n^{[\frac{t}{\rho_n}]}x_n=T(t)x, \tag{7}$$

uniformly on compact t-intervals.

Proof. We shall prove the theorem in the case $\gamma=0$. The case $\gamma>0$ is treated as in ([2], Theorem 2). We omit the details for the latter case.

Let $T_n(t)=\exp(tA_n)$ denote the semigroup generated by A_n on the space X_n. First, we note that (ii) implies (I_1), for a proof of this fact we refer e.g. to [9]. Thus applying Theorem 2.3, we get that $\hat{A}$ generates an (A)-semigroup $T(t)$ on X, such that (1) holds. To establish the limit formula (7), we begin by showing that for $x'\in\mathcal{D}(\hat{A}^2)$, there exists a sequence $(x'_n), x'_n\in X_n$, satisfying $\widetilde{\lim}\, x'_n=x'$, and such that

$$\lim_{n\to\infty}\|(\exp(\rho_nk_nA_n)-F_n^{k_n})x'_n\|_n=0 \tag{8}$$

where for a given $t>0, k_n=[t/\rho_n]$. Note that $\rho_nk_n\leqslant t,\forall n$ and that $\rho_nk_n\to t$ as $n\to\infty$.

Let $x' \in \mathcal{D}(\hat{A}^2)$. By Proposition 2.2 there is a sequence $(x'_n), x'_n \in \mathcal{D}(A_n^2)$, such that $\widetilde{\lim}\, x'_n = x', \widetilde{\lim}\, A_n x'_n = x'$ and $\widetilde{\lim}\, A_n^2 x'_n = \hat{A}^2 x'$. We have the following:

$$\begin{aligned}
\|(\exp(\rho_n k_n A_n) - F_n^{k_n})x'_n\|_n &\leqslant \sum_{j=0}^{\infty} Q_{nj}(\rho_n k_n)\|(F_n^j - F_n^{k_n})x'_n\|_n \\
&\leqslant \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\|(F_n^j - F_n^{k_n})x'_n\|_n \\
&\quad + \mathrm{e}^{-k_n}\|(I - F_n^{k_n})x'_n\|_n \\
&= \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

Here we may assume that k_n is non-zero, since this can always be achieved by taking n sufficiently large. Also, note that the left side is trivially equal to 0 when $k_n = 0$. Now,

$$\begin{aligned}
\mathcal{I}_2 &= \mathrm{e}^{-k_n}\|(I - F_n^{k_n})x'_n\|_n \\
&\leqslant \mathrm{e}^{-k_n}(\|x'_n\|_n + \|F_n^{k_n}x'_n\|_n) \\
&\leqslant \mathrm{e}^{-k_n}(\beta_1\|x'\| + \psi(\rho_n k_n, x')) \\
&\leqslant \beta_1\|x'\|\mathrm{e}^{-k_n} + k_n\mathrm{e}^{-k_n} \cdot \frac{1}{\rho_n k_n}\rho_n\psi(\rho_n k_n, x') \\
&\leqslant \beta\|x'\|\mathrm{e}^{-k_n} + k_n\mathrm{e}^{-k_n} \cdot \frac{1}{\rho_n k_n}\sum_{j=1}^{\infty}\rho_n\psi(\rho_n j, x') \\
&\leqslant \beta\|x'\|\mathrm{e}^{-k_n} + k_n\mathrm{e}^{-k_n} \cdot \frac{1}{\rho_n k_n}\int_0^{\infty}\psi(t, x')\,\mathrm{d}t.
\end{aligned}$$

Since e^{-k_n} and $k_n\mathrm{e}^{-k_n} \to 0$, while $\rho_n k_n \to t$ as $n \to \infty$, we have that $\mathcal{I}_2 \to 0$.

For $\mathcal{I}_1$, we write

$$\begin{aligned}
\mathcal{I}_1 &= \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \cdot \|(F_n^j - F_n^{k_n})x'_n\|_n \\
&= \Big(\sum_1 + \sum_2\Big) Q_{nj}(\rho_n k_n) \cdot \|(F_n^j - F_n^{k_n})x'_n\|_n,
\end{aligned}$$

where

$$\sum_1 := \sum_{|j-k_n|>\varepsilon k_n}, \qquad \sum_2 := \sum_{|j-k_n|\leqslant\varepsilon k_n},$$

and where $\varepsilon \in (0,1)$. It is noted in [3], page 18, that:

$$\sum_{|j-k_n|>\delta} \frac{u^j}{j!} \leqslant \frac{u\mathrm{e}^u}{\delta^2}\,.$$

Hence we find that

$$\begin{aligned}\sum_1 Q_{nj}(\rho_n k_n)\|(F_n^j - F_n^{k_n})x_n'\|_n &\leqslant \sum_1 Q_{nj}(\rho_n k_n)(\|F_n^j x_n'\|_n + \|F_n^{k_n}x_n'\|_n)\\ &\leqslant \sum_1 Q_{nj}(\rho_n k_n)(\psi(\rho_n j, x') + \psi(\rho_n k_n, x'))\\ &\leqslant 2\psi(\rho_n, x')\mathrm{e}^{-k_n}\sum_1 \frac{k_n^j}{j!}\\ &\leqslant 2\sqrt[p]{\rho_n}\psi(\rho_n, x')\frac{1}{\varepsilon^2\sqrt[q]{k_n}\sqrt[p]{\rho_n k_n}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Because

$$\sqrt[p]{\rho_n}\psi(\rho_n, x') \leqslant \left(\sum_{j=1}^{\infty}\rho_n\psi^p(j\rho_n, x')\right)^{\frac{1}{p}} \leqslant \left(\int_0^{\infty}\psi^p(s, x')\,\mathrm{d}s\right)^{\frac{1}{p}}.$$

for all $n \in \mathbf{N}$, we see that the right side in the last sequence of inequelities (for $\sum_1$) tends to 0 as $n \to \infty$.

Next, for the sum $\sum_2$, we note that j satisfies $(1-\varepsilon)t/2 \leqslant \rho_n \min\{j, k_n\}$, for all n, and since $\psi(\cdot, x')$ is non-increasing, we find that

$$\sum_2 Q_{nj}(\rho_n k_n)\|(F_n^j - F_n^{k_n})x_n'\|_n \leqslant \rho_n \sum_2 Q_{nj}(\rho_n k_n)\sum_m \|F_n^m A_n x_n'\|_n$$

where the inner sum is taken over all m satisfying

$$\min\{j, k_n\} \leqslant m \leqslant \max\{j-1, k_n-1\}, \quad 0 \neq j \neq k_n.$$

Now we recall (iv) to see that

$$\begin{aligned}\sum_2 &\leqslant C_1\rho_n \sum_2 Q_{nj}(\rho_n k_n)\sum_m \psi(m\rho_n, \hat{A}x')\\ &\leqslant C_1\rho_n \sum_2 Q_{nj}(\rho_n k_n)\psi(\min\{j, k_n\}\rho_n, \hat{A}x')|j-k_n|\\ &\leqslant C_1\rho_n\psi((1-\varepsilon)\frac{t}{2}, \hat{A}x')\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)|j-k_n|\\ &\leqslant C_1\rho_n\sqrt{k_n}\psi((1-\varepsilon)\frac{t}{2}, \hat{A}x') \to 0 \quad as \quad n \to \infty,\end{aligned}$$

where the last inequality is justified by the Schwarz inequality and Lemma 2 of [2]. Thus, $\mathcal{I}_1 \to 0$ and (8) is established.

To finish the proof, assume that we have $x \in \mathcal{D}^\circ$ and $x_n \in \mathcal{D}(A_n), n = 1,2,\ldots$, satisfying $\widetilde{\lim} x_n = x$ and $\sup_n \|A_n x_n\|_n < \infty$. Since $\hat{A}$ is the i.g. of an (A) semigroup, $\mathcal{D}(\hat{A}^2)$ is dense in X. Thus, given $\varepsilon > 0$, then is an $x' \in \mathcal{D}(\hat{A}^2)$ such that $\|x - x'\| < \varepsilon$. Moreover, there is a sequence $(x_n'), x_n' \in \mathcal{D}(A_n^2)$, for which (8) is satisfied and such that $\widetilde{\lim} x_n' = x', \sup_n \|A_n x_n'\|_n < \infty$. Hence

$$\begin{aligned}\|P_nT(t)x - F_n^{k_n}x_n\|_n \leqslant \|P_n(T(t) - T(\rho_n k_n))x\|_n + \|P_nT(\rho_n k_n)(x - x')\|_n \\ + \|P_nT(\rho_n k_n)x' - T_n(\rho_n k_n)x_n'\|_n + \|F_n^{k_n}(x_n - x_n')\|_n \\ + \|(T_n(\rho_n k_n) - F_n^{k_n})x_n'\|_n.\end{aligned}$$

The first term on the right side goes to 0 by the strong continuity of $T(t)$ for $t > 0$. Also, $(\rho_n k_n)$ is contained in some interval $[a, b] \subset (0, \infty)$, and since $\|T(t)\|$ is bounded for all $t \geqslant a > 0$, (cf. [7]), the second term is bounded by a constant multiple of ε. The third term tends to 0 as $n \to \infty$ because of the uniform convergence on compacts guaranteed by (1). The fourth term is bounded by $\psi(t/2, x' - x)$ for all large values of n. This can be made arbitrarily small by virtue of the continuity of ψ in its second argument. The last term tends to 0 by (8). Thus (7) is established. ∎

In view of Section 3, one finds that results similar to Theorem 4.1 are also achieved by choosing functions $\psi(t, x)$ with certain properties, or by making certain choices of the sequence (ρ_n). We proceed now with this line of investigation.

THEOREM 4.2. *Let $\psi(t, x) : (0, \infty) \times \mathcal{D}^\circ \to [0, \infty)$ satisfy the following: $\psi(t, x)$ is continuous in x for each $t \in (0, \infty)$ and is absolutely continuous in t for each $x \in \mathcal{D}^\circ$, and also, for some $p > 1$, the functions $\psi, \psi^p, (1 + t)|\psi'|$ and $(1 + t)\psi^{p-1}|\psi'| \in L^1(0, \infty)$. Then the conclusion of Theorem 4.1 remains valid under the hypotheses* (i)-(iv).

Proof. We note from the proof of Proposition 3.1 that

$$\psi(t, x) \leqslant \varphi(t, x) = \int_t^\infty |\psi(s, x)|\, ds$$

where $\varphi(t, x)$ is non-increasing in t and belongs to $L^1(0, \infty)$. Likewise

$$\psi^p(t, x) \leqslant \varphi_p(t, x) = \int_t^\infty |\psi^p(s, x)|\, ds$$

and again $\varphi_p(t, x) \in L^1(0, \infty)$. The proof proceeds as in Theorem 4.1 but by replacing ψ and ψ^p by φ and φ_p respectively in the estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$, where appropriate. Finally, we use the fact (from Proposition 3.1) that $S(\varphi(\cdot, x); \rho_n) \leqslant L_x$ and $S(\varphi_p(\cdot, x); \rho_n) \leqslant K_x$ for all $n \in \mathbb{N}$ and $x \in \mathcal{D}^\circ$, where L_x, K_x are certain constants. ∎

We now consider the case where, for some $x \in \mathcal{D}^{\circ}, e^{-\gamma t}\varphi(t,x)$ may not be monotonic for any choise of $\gamma \geqslant 0$. In a sense, the following result tells us that in such a case there is always a null sequence of positive numbers (ρ_n) for which the convergence of the discrete system is established.

THEOREM 4.3. *Let $\psi(t,x) : (0,\infty) \times \mathcal{D}^{\circ} \to [0,\infty)$ be continuous in x for each $t \in (0,\infty)$ and let there exist a $\gamma \geqslant 0$ and $p > 3$, such that for each $x \in \mathcal{D}^{\circ}, e^{-\gamma t}\psi(t,x) \in L^1(0,\infty) \cap L^p(0,\infty)$. Then there exists a null sequence (ρ_n) of positive numbers for which the conclusion of Theorem 4.1 holds under the hypotheses (i) - (iv).*

Proof. Again, it suffices here to consider the case where $\gamma = 0$. Let $x \in \mathcal{D}^{\circ}$; semicolon according to Corollary 2 of Proposition 3.3, we can find a null sequence of positive numbers (ρ_n) such that for all $n \in \mathbf{N}$

$$\sum_{j=1}^{\infty} \rho_n \psi(\rho_n j, x) \leqslant 3 \int_0^{\infty} \psi(s,x)\,\mathrm{d}s := L_1(x),$$

$$\sum_{j=1}^{\infty} \rho_n \psi^p(\rho_n j, x) \leqslant 3 \int_0^{\infty} \psi^p(s,x)\,\mathrm{d}s := L_2(x)$$

where $L_1(x), L_2(x)$ are constants depending only on x. The proof goes parallel to that of Theorem 4.1. Thus, to verify (8), we let $x', (x'_n)$, and k_n have the same meaning as before, and we get

$$\|(\exp(\rho_n k_n A_n) - F_n^{k_n})x'_n\|_n \leqslant \mathcal{I}_1 + \mathcal{I}_2.$$

The proof that $\mathcal{I}_2 \to 0$ is the same as in Theorem 4.1. For $\mathcal{I}_1$, we have

$$\begin{aligned}
\mathcal{I}_1 &= \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\|(F_n^j - F_n^{k_n})x_n'\|_n \\
&\leqslant \rho_n \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \sum_m \|F_n^m A_n x_n'\|_n \\
&\leqslant C_2\rho_n \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n) \sum_m \psi(m\rho_n, \hat{A}x') \\
&\leqslant C_2\rho_n \sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\cdot \max_m \psi(m\rho_n; \hat{A}x')\cdot |j-k_n| \\
&\leqslant C_2\rho_n \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\cdot \max_m \psi(m\rho_n; \hat{A}x')\cdot |j-k_n|^4\right)^{\frac{1}{4}} \\
&\times \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\cdot \max_m \psi(m\rho_n; \hat{A}x')\right)^{\frac{3}{4}} \\
&= C_2\rho_n \cdot \mathcal{J}_1 \cdot \mathcal{J}_2,
\end{aligned}$$

where m plays the same role as before (see Theorem 4.1). Next, by noting that

$$\rho_n \max_m \psi(m\rho_n; \hat{A}x') \leqslant \sum_{j=1}^{\infty} \rho_n \psi(\rho_n j; \hat{A}x') \leqslant L_1(\hat{A}x'),$$

$$\rho_n \max_m \psi^p(m\rho_n; \hat{A}x') \leqslant \sum_{j=1}^{\infty} \rho_n \psi^p(\rho_n j; \hat{A}x') \leqslant L_2(\hat{A}x')$$

we can proceed as in ([2], Theorem 3), with appropriate modifications to establish (8). For convenience, we present these calculations here.

We find, using ([2], Lemma 2) that

$$\begin{aligned}
\mathcal{J}_1 &\leqslant \rho_n^{-\frac{1}{4}} L_1^{\frac{1}{4}}(\hat{A}x')\cdot \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)(j-k_n)^4\right)^{\frac{1}{4}} \\
&= L_1^{\frac{1}{4}}(\hat{A}x')\rho_n^{-\frac{3}{4}}(3\rho_n^2 k_n^2 + \rho_n^2 k_n)^{\frac{1}{4}} \\
&\leqslant L_1^{\frac{1}{4}}(\hat{A}x')\rho_n^{-\frac{3}{4}}(3t^2 + \rho_n t)^{\frac{1}{4}}.
\end{aligned}$$

While for $\mathcal{J}_2$ we apply Hölder's inequality, with $p > 3$ and q the conjugate of p to find that

$$\begin{aligned}\mathcal{J}_2 &\leqslant \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\right)^{\frac{3}{4q}} \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\cdot(\max_m \psi(m\rho_n;\hat{A}x'))^p\right)^{\frac{3}{4p}} \\ &\leqslant \rho_n^{-\frac{3}{4p}} \left(\sum_{j=1}^{\infty} Q_{nj}(\rho_n k_n)\cdot \max_m \rho_n \psi^p(m\rho_n;\hat{A}x')\right)^{\frac{3}{4p}} \\ &\leqslant \rho_n^{-\frac{3}{4p}} L_2^{\frac{3}{4p}}(\hat{A}x').\end{aligned}$$

Therefore,

$$\mathcal{I}_1 \leqslant C_2\rho_n \cdot \mathcal{J}_1 \cdot \mathcal{J}_2 \leqslant C_2'(x')\rho_n^{\frac{(p-3)}{4p}}(3t^2+\rho_n t)^{\frac{1}{4}},$$

which tends to 0 as $n \to \infty$, as long as $p > 3$. The remaining part of the proof goes similarly to that of Theorem 4.1. ∎

While Theorem 4.3 is interesting from the theoretical point of view, the following result, which is analogous to Theorem 3 of [2], is important for applications. The proof uses the same arguments as in 4.3, but depends now on Theorem 3.4 instead of Corollary 2 of Theorem 3.3, (see also [2], Theorem 3) .

THEOREM 4.4. *Let* $0 \leqslant \psi$, *and assume that* p *and* $\gamma \geqslant 0$ *satisfy the hypotheses of Theorem* 4.3. *Further, let* $\rho_n = 2^{-n}, n \in N$, *such that* (i)-(iii) *of Theorem* 4.1 *and* (iv') *are satisfied, where:*
(iv'). *There exists a* $t_0 > 0$ *satisfying Proposition* 3.4 *above, such that: For each* $x \in \mathcal{D}^\circ$ *and each sequence* $(x_n), x_n \in \mathcal{D}(A_n)$ *satisfying* $\widetilde{\lim}\, x_n = x$ *and* $\sup_n \|A_n x_n\|_n < \infty$,

$$\|F_n^{[2^n t]} x_n\|_n \leqslant \psi(t_0 + t, x), \quad t > 0.$$

Then the conclusion of Theorem 4.1 remains valid.

5. DISCRETE APPROXIMATION OF (0,A)-SEMIGROUPS

We now discuss the results of Section 4 in the case of (0,A) semigroups. It is worth noting that the results here cover a range of applications beyond that considered in [2].

THEOREM 5.1. *If, in the statement of Theorem 4.1, the set $\mathcal{D}^\circ$ is replaced by X, and* (iv) *holds for all $x \in X$ and $x_n \in X_n$ satisfying $\widetilde{\lim} x_n = x$, then the limit semigroups $T(t), t > 0$ is of the class* (0,A). *Moreover, the convergence of the discrete semigroups in* (7) *holds for all $x \in X$ and $x_n \in X_n, n = 1, 2, \dots$ satisfying $\widetilde{\lim} x_n = x$.*

Proof. Theorem 4.1 applies and we conclude that $\hat{A}$ generates an (A) semigroup $T(t), t > 0$. Furthermore, it is readily verified under the present conditions that (7) holds for all $x \in X$ and $x_n \in X_n$ satisfying $\widetilde{\lim} x_n = x$. It remains only to verify that $T(t)$ is of class (0,A), and that is satisfies the inequality in I_4. Let $x \in X$ and $\gamma_x > 0$; then

$$\begin{aligned}\int_0^\infty e^{-\gamma_x t}\|T_n(t)P_n x\|_n \,dt &\leqslant \int_0^\infty \|T_n(t)P_n x\|_n \,dt \\ &\leqslant \int_0^\infty e^{\frac{-t}{\rho_n}} + \sum_{k=1}^\infty \int_0^\infty Q_{nk}(t)\psi(\rho_n k, x)\,dt \\ &\leqslant \rho_n + \sum_{k=1}^\infty \rho_n \psi(\rho_n k, x) \\ &\leqslant M_1 + \int_0^\infty \psi(t,x)\,dt\end{aligned}$$

where M_1 is some constant. The conclusion now follows by Theorem 2.4. ∎

COROLLARY. *Let ψ be as in Theorem* 5.1, *and assume that hypotheses* (ii)-(iv) *and* (i)′ *are fulfilled where:*
(i)′ *Core A and $\mathcal{R}(\lambda_0 I - A)$ are dense in X for some core of A and some $\lambda_0 > \omega + \gamma$, where $A = \lim A_n$.*
Then A (or its closure) generates a (0,A) *semigroup $T(t), t > 0$ such that*

$$\widetilde{\lim} F_n^{[t/\rho_n]} P_n x = T(t)x, \quad x \in X, t > 0. \tag{9}$$

Proof. Core $A \subset \mathcal{D}(A) \subset \mathcal{D}(\hat{A})$ and $\mathcal{R}(\lambda_0 I - A) \subset \mathcal{R}(\lambda_0 I - \hat{A})$. Hence, condition (i) of the theorem is satisfied. ∎

Next we present some results analogous to 4.2-4.4.

THEOREM 5.2. *Let $\psi(t,x) : (0,\infty) \times X \to [0,\infty]$ satisfy the following: $\psi(t,x)$ is continuous in x for each $t \in (0,\infty)$ and is absolutely continuous in t for each $x \in X$, such that for some $p > 1$, the functions $\psi, \psi^p, (1+t)|\psi'|$ and $(1+t)\psi^{p-1}|\psi'|$ belong to the space $L^1(0,\infty)$. Then the conclusion of Theorem* 5.1 *remains valid.*

COROLLARY. *The theorem remains valid if we replace condition* (i) *by* (i′) *above.*

THEOREM 5.3. *If the set $\mathcal{D}^\circ$ is replaced by X in the statements of Theorems 4.3, 4.4, and also (iv) of Theorem 4.1 is assumed for all $x \in X$ and all sequences $(x_n), x_n \in X_n$ satisfying $\widetilde{\lim}\, x_n = x$, then the conclusion of Theorem 5.1 remains valid in both of these cases.*

REMARK. Theorems 5.2-5.4 remain valid if we replace condition (i) by (i′) (see corollary to Theorem 5.1).

EXAMPLE. Following the examples discussed in [2] and [11], we shall now deal with a situation in which the results of [2] do not seem to apply while those of Sections 4, 5 above are applicable. Consider the Cauchy problem:

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{P}(D)\mathbf{u}, \quad \mathbf{u}(x,0) = \mathbf{f}(x) \tag{10}$$

in the Banach space $\mathbf{L}^2(\mathbf{R}) = L^2(\mathbf{R}) \times L^2(\mathbf{R})$, with the standard norm. Here $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t))$ is a vector valued function of the real variables $x \in \mathbf{R}, t > 0, \mathbf{f}(x) = (f_1(x), f_2(x)), f_i(x) \in L^2(\mathbf{R})$ is the given initial condition, and $\mathbf{P}(D)$ is the partial differential operator with respect to x, given by:

$$\mathbf{P}(D) = \begin{pmatrix} D_x^2 + \mathrm{i}D_x^4 & \mathrm{i}D_x^4 \\ 0 & D_x^2 + \mathrm{i}D_x^4 \end{pmatrix}.$$

The method of discrete approximation corresponding to this problem is as described in [2], namely

$$\mathbf{u}(x, t+\rho) = \mathbf{F}(\rho)\mathbf{u}(x,t), \quad \mathbf{u}(x,0) = \mathbf{f}(x), \tag{11}$$

where

$$\mathbf{F}(\rho, h) = \begin{pmatrix} 1 + \rho\Delta_h^2 + \mathrm{i}\rho\Delta_h^4 & \mathrm{i}\rho\Delta_h^4 \\ 0 & 1 + \rho\Delta_h^2 + \mathrm{i}\rho\Delta_h^4 \end{pmatrix}.$$

By taking the Fourier transform in x, this takes the form:

$$\hat{\mathbf{u}}(\xi, t+\rho) = \hat{\mathbf{F}}(\rho)\hat{\mathbf{u}}(\xi,t), \quad \hat{\mathbf{u}}(\xi,0) = \hat{\mathbf{f}}(\xi),$$

where

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\pi}} \int_{\Re} \mathrm{e}^{-\mathrm{i}x\xi} u(x,t)\, \mathrm{d}x$$

while the transformed matrix of $\mathbf{F}(\rho, h)$ is:

$$\hat{\mathbf{F}}(\rho,h) = \begin{pmatrix} 1 - \rho(h^{-1}\sin\xi h)^2 + \mathrm{i}\rho(h^{-1}\sin\xi h)^4 & \rho(h^{-1}\sin\xi h)^4 \\ 0 & 1 - \rho(h^{-1}\sin\xi h)^2 + \mathrm{i}\rho(h^{-1}\sin\xi h)^4 \end{pmatrix}.$$

Let $(\rho_n), (h_n)$ be null sequences of positive numbers, and write $\mathbf{F}_n$ for $\mathbf{F}(\rho_n, h_n)$. Similarly, we write:

$$\hat{\mathbf{F}}_n^{[t/\rho_n]} = \begin{pmatrix} (1+\rho_n\alpha_n)^{[t/\rho_n]} & \rho_n[t/\rho_n](h_n^{-1}\sin\xi h_n)^4(1+\rho_n\alpha_n)^{[t/\rho_n]-1} \\ 0 & (1+\rho_n\alpha_n)^{[t/\rho_n]} \end{pmatrix}$$

where $\alpha_n = -(h_n^{-1}\sin\xi h_n)^2 + \mathrm{i}(h_n^{-1}\sin\xi h_n)^4$.
In applying the results of Sections 4, 5 to the present situation, we take $X_n = X = \mathbf{L}^2(\mathbf{R})$ and $P_n = I$ for all $n \in \mathbf{N}$. In this case, the notion of limit reduces to the ordinary one. Note that with $\mathbf{A}_n = \rho_n^{-1}(\mathbf{F}_n - I)$, we have that $\lim \mathbf{A}_n = \mathbf{P}(D) = A$ in the sense that $\mathbf{A}_n\mathbf{u} \to \mathbf{P}(D)\mathbf{u}, \mathbf{u} \in \mathcal{D}(A)$, in the $\mathbf{L}^2$ norm. Note that $\mathcal{D}(A) = \{\mathbf{u} : \mathbf{u} \in \mathbf{L}^2(\mathbf{R}), \mathbf{P}(D)\mathbf{u} \in \mathbf{L}^2\}$ and that a core for A is $\{\mathbf{u} \in \mathbf{L}^2(\Re) :$ $\hat{\mathbf{u}}$ has compact support$\}$. Observe that for initial data $\mathbf{u}(\cdot, 0) \in \mathcal{D}(A)$

$$\xi^4\hat{u}_1(\xi), \xi^4\hat{u}_2(\xi) \in L^2(\mathbf{R}).$$

Hence

$$||\mathbf{F}_n^{[t/\rho_n]}\mathbf{u}|| = ||\hat{\mathbf{F}}_n^{[t/\rho_n]}\hat{\mathbf{u}}|| \leqslant \kappa_1(||\hat{u}_1|| + ||\hat{u}_2||) + \kappa_2 t||\hat{u}_2||.$$

Thus, taking $\psi(t;\mathbf{u})$ equal to the right hand side, and $\gamma = 1$, we see that for example $\mathrm{e}^{-t}\psi \in L^1 \cap L^p$, for some $p > 3$. Now we can apply Theorem 4.4 in which case we work with $\rho_n = 2^{-n}$. Thus we have shown that condition (i′) of the theorem is satisfied. The remaining conditions follow as in [11]. Therefore, we conclude that there exists a semigroup $T(t), t > 0$ of class (A) on $\mathbf{L}^2(\mathbf{R})$ which solves the Cauchy problem (10), and furthermore, that the solution of the discrete system, namely $\mathbf{F}_n^{[t/\rho_n]}\mathbf{u}(x, 0)$, converges to the solution $T(t)\mathbf{u}(x, 0)$ of (10) as $n \to \infty$. In closing, we note from [11] that $T(t)$ is in fact of class (0,A).

REFERENCES

1. N.H. ABDELAZIZ, A note on the convergence of linear semigroups of class (1,A). *Hokkaido Math. J.* **18**(1989), 513–521.
2. N.H. ABDELAZIZ, On approximation by discrete semigroups, *J. Approx. Theory* **73**(1993), 253–269.
3. M. BECKER, Über den Satz von Trotter mit Anwendungen auf Approximationstheorie, *Forsch. Bericht des Landes NRW*, Nr. **2577**, Westdeutscher Verlag, 1976.
4. P.L. BUTZER, H. BERENS, *Semigroups of Operators and Approximation*, Springer-Verlag, New York 1967.
5. P.R. CHERNOFF, Product formulas, nonlinear semigroups and addition of unbounded operators, *Mem. Amer. Math. Soc.* **140**(1974).
6. E. GÖRLICH, D. PONTZEN, Approximation of operator semigroups of Oharu's class $(C_{(k)})$, *Tôhuku Math. J. (2)* **34**(1982), 539–552.

7. E. HILLE, R. PHILLIPS, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ., vol. **31**, Providence, R.I., 1957.
8. P.D. LAX, R.D. RICHTMYER, Survey of the stability of linear finite difference equations, *Comm. Pure Appl. Math.* **9**(1956), 267–293.
9. S. OHARU, H. SUNOUCHI, On the convergence of semigroups of linear operators, *J. Funct. Anal.* **6**(1970), 292–304.
10. R.S. PHILLIPS, A note on the abstract Cauchy problem, *Proc. Nat. Acad. Sci. U.S.A.* **40** (1954), 244–248.
11. H. SUNOUCHI, Convergence of semi-discrete difference schemes of abstract Cauchy problems, *Tôhoku Math. J.* **22** (1970), 394–408.
12. H.F. TROTTER, Approximation of semigroups of operators, *Pacific J. Math.* **8**(1958), 887–919.

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