J. OPERATOR THEORY
32(1994), 353–379

THE CONTINUITY OF THE CONSTANT OF HYPERREFLEXIVITY

ILEANA IONAŞCU

Communicated by Norberto Salinas

ABSTRACT. Starting from the question of what are the possible values of the constant of hyperreflexivity for subspaces of $\mathcal{B}(H)$, where H is a separable complex Hilbert space, the paper considers the continuity of the function κ : $\mathcal{B}(H) \to \overline{R}$, defined by $\kappa(T) = K(\mathcal{A}_w(T))$, $\mathcal{A}_w(T))$ denoting the unital weakly closed algebra generated by T. As a consequence, it is shown that any number bigger than or equal to one is a constant of hyperreflexivity of a subspace. Besides several results concerning the continuity of the function κ, the paper contains also more general results, like those determining the closures (in the norm topology) or the set of reflexive, respectively non-reflexive, operators.

KEYWORDS: *Hyperreflexivity, invariant subspace, Hilbert space.*

AMS SUBJECT CLASSIFICATION: Primary 47D25; Secondary 15A25.

Let H be a (complex) Hilbert space, $\mathcal{B}(H)$ be the algebra of all (bounded linear) operators on H, and $\mathcal{P}(H)$ be the lattice of all (orthogonal) projections in $\mathcal{B}(H)$. For a subset $\mathcal{S}$ of $\mathcal{B}(H)$, denote by Lat $(\mathcal{S})$ the lattice of all (closed linear) subspaces of H that are invariant under all operators in $\mathcal{S}$, and for a subset $\mathcal{L}$ of $\mathcal{P}(H)$ denote by Alg $(\mathcal{L})$ the (unital weakly closed) algebra of all operators in $\mathcal{B}(H)$ that leave invariant all subspaces corresponding to $\mathcal{L}$. A subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is called *reflexive* if Alg Lat $(\mathcal{A}) = \mathcal{A}$. An operator T in $\mathcal{B}(H)$ is called *reflexive* if $\mathcal{A}_w(T)$, the unital weakly closed algebra generated by T, is reflexive. A subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ is called ***hyperreflexive*** if there exists a (positive) constant K such that

$$(1)\qquad \operatorname{dist}(T,\mathcal{A}) \leqslant K \sup\left\{\|P^{\perp}TP\| : P \in \operatorname{Lat}(\mathcal{A})\right\}, \quad \forall T \in \mathcal{B}(H).$$

Similarly, an operator $T \in \mathcal{B}(H)$ is called ***hyperreflexive*** if $\mathcal{A}_w(T)$ is hyperreflexive. The smallest K such that (1) holds is called *the constant of hyperreflexivity* for $\mathcal{A}$ and it is denoted by $K(\mathcal{A})$. We write $K(T)$ for $K(\mathcal{A}_w(T))$. By convention, if there is no K such that (1) holds, we set $K(\mathcal{A}) = \infty$.

The notion of hyperreflexivity was introduced by W.A. Arveson in [2] (where he also gives an alternate definition that offers, sometimes, a more tractable way of calculating K), generalizing his result about nest algebras: the well-known distance formula ([1]).

Obviously, hyperreflexivity is a stronger condition than reflexivity (in general, strictly stronger; see [6]). But we are not interested in the relation between these concepts in this paper. What we are investigating here is the continuity of the function

$$\kappa : \mathcal{B}(H) \to [1,\infty],$$

$\kappa(T) = K(\mathcal{A}_w(T))$, where we consider on $\mathcal{B}(H)$ the norm topology.

This seemed to be a quite hard problem since the values of the function κ are not calculated at too many points, even for operators on finite-dimensional spaces. And when the values are calculated, a complicated machinery — different in each case — is used. Even for von Neumann algebras (which are all reflexive, by the von Neumann double commutant theorem, but not all known to be hyperreflexive) the constant of hyperreflexivity K has not been computed in all the cases it is known to be finite. Or even in the case when H is finite-dimensional space (in which case, reflexivity coincide with hyperreflexivity), there is no general way of calculating K, and the work of K. Davidson and M. Ordower [5] shows the difficulty of getting such a "recipe".

We have been, thus, obliged to avoid any attempt to think about the values of K — except for those T's for which $K(T) = \infty$ — in trying to solve the problem of continuity of K. A consequence of this is that we obtained some results that are interesting in their own right: the description of the (norm)-closures of reflexive, respectively, non-reflexive operators (Theorem 1.1, Remark 1.2 and Theorem 2.1). The main problem of this paper is completely solved in the finite-dimensional case (Theorem 1.6), and it is narrowed down quite a bit in the infinite-dimensional case (Theorem 2.14 and Theorem 2.13).

We should mention that in [23], a paper with a title similar to ours, another stability problem for the constant of hyperreflexivity has been discussed. In the last section, we will present a parallel between the two problems.

1. THE FINITE-DIMENSIONAL CASE

Throughout this section, H will denote a finite-dimensional Hilbert space. In this case, the notions of reflexivity and hyperreflexivity coincide, an obvious reason being that, on a finite-dimensional space, any two seminorms with the same null set are equivalent (In particular, for the two seminorms appearing in the inequality (1) defining K.) Recall that there exists a characterization of reflexive operators on finite-dimensional spaces (i.e., for reflexive matrices) obtained in [7]; namely, a matrix is reflexive if and only if in its Jordan canonical form, for each eigenvalue, the two biggest blocks are either of the same size or their sizes differ by one.

The goal of this section is to describe all the points of continuity for K in case H is a finite-dimensional Hilbert space. The main result, Theorem 1.6, states that this set is the union between the set of non-reflexive matrices and the set of matrices with distinct eigenvalues. In the process of obtaining this result, we show, also, that the closure of the set of non-reflexive matrices is the complement, in $\mathcal{B}(H)$, of the set of matrices with distinct eigenvalues (Theorem 1.1). Notice that the set of reflexive $n \times n$ matrices is dense in M_n (Remark 1.2).

THEOREM 1.1. *Let H be a finite-dimensional Hilbert space. Consider the set*

$$\mathcal{E} = \{T \in \mathcal{B}(H) : T \text{ has } \dim H \text{ distinct eigenvalues } \}.$$

Then

$$\overline{\{T \in \mathcal{B}(H) : T \text{ is non-reflexive } \}} = \mathcal{B}(H) \setminus \mathcal{E}. \tag{2}$$

Proof. Notice that all the operators in $\mathcal{E}$ are reflexive. So $\{T \in \mathcal{B}(H) : T \text{ is non-reflexive}\} \subseteq \mathcal{B}(H) \setminus \mathcal{E}$. Thus, to justify the inclusion "$\subseteq$" in (2), it is sufficient to prove that the set $\mathcal{E}$ is open. Let $T \in \mathcal{E}$. Herrero's result about the semi-continuity of the spectrum ([15], Theorem 1.1), implies that there exists an $\varepsilon > 0$ such that, if $X \in \mathcal{B}(H)$ and $\|X - T\| < \varepsilon$, then the spectrum of X, $\sigma(X)$, has also $\dim H$ distinct eigenvalues, i.e., $X \in \mathcal{E}$. Thus $\mathcal{E}$ is open.

To show the other inclusion in (2), let $X \in \mathcal{B}(H) \setminus \mathcal{E}$. If X is a non-reflexive operator, then there is nothing to prove. If X is reflexive, then, from [7], in the Jordan canonical form of X, for every eigenvalue, the biggest two blocks have either the same size or their sizes differ by one (the last case including, by convention, the case of eigenvalues of multiplicity one). Because $X \in \mathcal{B}(H) \setminus \mathcal{E}$, there exists a $\lambda \in \sigma(X)$ that has multiplicity bigger than one. Then, for every $k \geqslant 1$, define $X_k \in \mathcal{B}(H)$ to be the operator that has, with respect to the basis in which X has Jordan canonical form, the same matrix as X with the exception of the part corresponding to the eigenvalue λ, where we replace the 0's on the second diagonal with $\frac{1}{k}$'s. It is easy to notice that all X_k's ($k \geqslant 1$) are non-reflexive and that $\lim_{k\to\infty} X_k = X$. Thus $X \in \overline{\{T \in \mathcal{B}(H) : T \text{ is non-reflexive }\}}$. ∎

REMARK 1.2. The set of reflexive operators on a finite-dimensional Hilbert space H is dense in $\mathcal{B}(H)$. Even more, with the notation in Theorem 1.1, $\mathcal{E}$ is dense in $\mathcal{B}(H)$. (This last assertion is made by P. Halmos in [13].)

From Theorem 1.1 we conclude that the only possible points of continuity for the function κ are the non-reflexive operators and those in $\mathcal{E}$. In the sequel, we will show that, in fact, all of these are points of continuity for κ (Theorem 1.6).

First, we have to make use of some results that are true in an arbitrary Hilbert space, and we will state and prove them in their full generality. We recall that the notions of reflexivity and hyperreflexivity are defined in general for (closed linear) subspaces of $\mathcal{B}(H)$. A subspace $\mathcal{S}$ of $\mathcal{B}(H)$ is called *reflexive* if $T \in \mathcal{S}$ whenever $Tx \in \overline{[\mathcal{S}x]}$, for every $x \in H$. A subspace $\mathcal{S}$ is called *hyperreflexive* if there exists a constant $K(\geqslant 1)$ such that

$$\operatorname{dist}(T, \mathcal{S}) \leqslant K \sup\{\operatorname{dist}(Tx, \mathcal{S}x) : X \in H, \|x\| \leqslant 1\}, \quad \forall T \in \mathcal{B}(H).$$

The initial definition of hyperreflexivity for subspaces was given in [18]; the one we use here is due to D. Larson ([20]). We define $K(\mathcal{S})$ to be the smallest K that satisfies the above condition if $\mathcal{S}$ is hyperreflexive and to be ∞ otherwise. In case $\mathcal{S}$ is a unital algebra, the definitions above coincide with those given in the introduction.

PROPOSITION 1.3. *Let H be a separable Hilbert space, $\mathcal{S}$ be a subspace of $\mathcal{B}(H)$ and A and B be invertible operators in $\mathcal{B}(H)$. Then*

$$\frac{1}{\alpha}K(\mathcal{S}) \leqslant K(A\mathcal{S}B) \leqslant \alpha K(\mathcal{S}), \tag{3}$$

where $\alpha = \|A\|\,\|A^{-1}\|\,\|B\|\,\|B^{-1}\|$.

Proof. On one hand,

$$\begin{aligned} \operatorname{dist}(X, A\mathcal{S}B) &= \inf\{\|X - ASB\| : S \in \mathcal{S}\} \\ &= \inf\{\|A(A^{-1}XB^{-1} - S)B\| : S \in \mathcal{S}\} \\ &\leqslant \|A\|\,\|B\| \inf\{\|A^{-1}XB^{-1} - S\| : S \in \mathcal{S}\} \\ &= \|A\|\,\|B\| \operatorname{dist}(A^{-1}XB^{-1}, \mathcal{S}). \end{aligned} \tag{4}$$

On the other hand,

$$
(5)\quad
\begin{aligned}
&\sup\{\operatorname{dist}(Xx, ASBx) : x \in X,\ \|x\| \leqslant 1\}\\
&= \sup\{\inf\{\|Xx - ASBx\| : S \in \mathcal{S}\} : x \in X,\ \|x\| \leqslant 1\}\\
&\geqslant \frac{1}{\|B^{-1}\|}\sup\{\inf\{\|XB^{-1}y - ASy\| : S \in \mathcal{S}\} : y \in X,\ \|y\| \leqslant 1\}\\
&= \frac{1}{\|B^{-1}\|}\sup\{\inf\{\|A(A^{-1}XB^{-1}y - Sy)\| : S \in \mathcal{S}\} : y \in X,\ \|y\| \leqslant 1\}\\
&\geqslant \frac{1}{\|B^{-1}\|\,\|A^{-1}\|}\sup\{\operatorname{dist}(A^{-1}XB^{-1}y, \mathcal{S}y) : y \in X,\ \|y\| \leqslant 1\}.
\end{aligned}
$$

From (4), (5) and the definition of K it follows that

$$K(A\mathcal{S}B) \leqslant \|A\|\,\|A^{-1}\|\,\|B\|\,\|B^{-1}\|K(\mathcal{S}).$$

By applying the same reasoning to $A\mathcal{S}B$ instead of $\mathcal{S}$, we obtain the other inequality in (3). ∎

COROLLARY 1.4. *Let H be a separable Hilbert space, $\mathcal{S}$ be a subspace in $\mathcal{B}(H)$, and $\{A_\lambda\}_\lambda$ and $\{B_\lambda\}_\lambda$ be two nets of invertible operators in $\mathcal{B}(H)$. If $\lim_\lambda \operatorname{dist}(A_\lambda, \mathcal{U}) = 0$, and $\lim_\lambda \operatorname{dist}(B_\lambda, \mathcal{U}) = 0$, where $\mathcal{U}$ is the set of unitary operators on H, then $\lim_\lambda K(A_\lambda \mathcal{S} B_\lambda) = K(\mathcal{S})$.*

Proof. If $\{X_\lambda\}_\lambda$ is a net of invertible operators, it follows from [24], Theorem 3.4, that $\lim_\lambda \operatorname{dist}(X_\lambda, \mathcal{U}) = 0$ if and only if $\lim_\lambda \|X_\lambda\| = 1$. So the corollary follows directly from Proposition 1.3, applying (3) for $A = A_\lambda$ and $B = B_\lambda$, for all λ's and taking the limit with λ. ∎

COROLLARY 1.5. *If H is a separable Hilbert space, $\mathcal{S}$ is a subspace of $\mathcal{B}(H)$ and $\{x \mapsto A_x\}_x$ and $\{x \mapsto B_x\}_x$ are continuous functions defined on a topological space X with values in $\mathcal{I}(H)$, the set of invertible operators in $\mathcal{B}(H)$, then $\{x \mapsto K(A_x \mathcal{S} B_x)\}_x$ is a continuous function.*

Proof. For $s, t \in X$, it is obvious that

$$(6)\qquad K(A_s\mathcal{S}B_s) = K((A_sA_t^{-1})A_t\mathcal{S}B_t(B_t^{-1}B_s)).$$

From the continuity of the functions $\{x \mapsto A_x\}_x$ and $\{x \mapsto B_x\}_x$, it follows that $\lim_{s\to t}\|A_sA_t^{-1} - I\| = 0$ and $\lim_{s\to t}\|B_t^{-1}B_s - I\| = 0$, which, in turn, implies that $\lim_{s\to t}\operatorname{dist}(A_sA_t^{-1}, \mathcal{U}) = 0$ and $\lim_{s\to t}\operatorname{dist}(B_t^{-1}B_s, \mathcal{U}) = 0$. To end the proof, use (6) and apply Corollary 1.4 for $\{A_\lambda\}_\lambda = \{A_sA_t^{-1}\}_s$, $\{B_\lambda\}_\lambda = \{B_t^{-1}B_s\}_s$ and $\mathcal{S} = A_t\mathcal{S}B_t$, for an arbitrary, but fixed, $t \in X$. ∎

The following is the main result of this section.

THEOREM 1.6. *Let H be a finite-dimensional Hilbert space and let $T \in \mathcal{B}(H)$. Then κ is continuous at T if and only if $T \in \mathcal{E}$ or T is non-reflexive.*

Proof. Theorem 1.1 implies that any point of continuity for κ must either be a non-reflexive operator or belong to $\mathcal{E}$.

To prove that κ is indeed continuous at any such point, we consider separately each case. First, let $T \in \mathcal{B}(H)$ be a non-reflexive operator. In particular, $\kappa(T) = \infty$. Assume that

$$\lim_{k\to\infty} T_k = T. \tag{7}$$

We claim that $\lim\limits_{k\to\infty} \kappa(T_k) = \infty$, so κ is continuous at T. Suppose not, i.e. $\{T_k\}_{k\geqslant 1}$ has a bounded subsequence. Without loss of generality, we can assume that there exists an $M > 0$ such that

$$\kappa(T_k) \leqslant M, \quad \forall k \geqslant 1. \tag{8}$$

Since any operator on a finite-dimensional space is algebraic, from [3] it follows that $\operatorname{Alg}\operatorname{Lat}(T) \cap \{T\}' = \mathcal{A}_w(T)$. But T is not reflexive, so there exists an $A \in \mathcal{B}(H)$ such that $A \in \operatorname{Alg}\operatorname{Lat}(T)$, but $A \notin \{T\}'$. The first condition on A can be rewritten as

$$\sup\{\|P^{\perp}AP\| : P \in \operatorname{Lat}(T)\} = 0. \tag{9}$$

Since H is finite-dimensional, the closed unit ball of $\mathcal{B}(H)$ is compact. So, for each $k \geqslant 1$, there exists a $P_k \in \operatorname{Lat}(T_k)$ such that

$$\sup\{\|P^{\perp}AP\| : P \in \operatorname{Lat}(T_k)\} = \|P_k^{\perp}AP_k\|. \tag{10}$$

Still because of the compactness of the closed unit ball of $\mathcal{B}(H)$, we can assume, without loss of generality, that there exists a $P \in \mathcal{B}(H)$ such that $\lim\limits_{k\to\infty} P_k = P$. In the context of this last relation, of (7) and of the fact that $P_k \in \operatorname{Lat}(T_k)$, applying the result on the upper semi-continuity of Lat ([14]) it follows that $P \in \operatorname{Lat}(T)$. Using this together with (9) and (10), we can infer that

$$\lim_{k\to\infty} \sup\{\|P^{\perp}AP\| : P \in \operatorname{Lat}(T_k)\} = 0,$$

which, in view of (8), implies that

$$\lim_{k\to\infty} \operatorname{dist}(A, \mathcal{A}_w(T_k)) = 0.$$

Thus, there exists a sequence $\{X_k\}_{k\geqslant 1}$ in $\mathcal{B}(H)$ such that

$$X_k \in \mathcal{A}_w(T_k), \quad \forall k \geqslant 1, \tag{11}$$

and

$$\lim_{k\to\infty} \|A - X_k\| = 0. \tag{12}$$

From (11) it follows that $X_kT_k = T_kX_k$. So (12) and (7) imply that $AT = TA$, which contradicts the choice of A not in $\{T\}'$. So, $\lim_{k\to\infty} \kappa(T_k) = \infty$.

Finally, let T be an operator in $\mathcal{E}$, and let $\{T_k\}_{k\geqslant 1}$ be a sequence in $\mathcal{B}(H)$ such that $\lim_{k\to\infty} T_k = T$. From the general theory of matrices, there exists an invertible operator $S \in \mathcal{B}(H)$ such that STS^{-1} is diagonal (with distinct eigenvalues). Evidently, $\lim_{k\to\infty} ST_kS^{-1} = STS^{-1}$, and $STS^{-1} \in \mathcal{E}$. So, since $\mathcal{E}$ is open (see the proof of Theorem 1.1), it follows that $ST_kS^{-1} \in \mathcal{E}$, for k sufficiently large. Hence, there exists a sequence $\{S_k\}_{k\geqslant 1}$ in $\mathcal{I}(H)$ with the properties that

$$\lim_{k\to\infty} S_k = I, \tag{13}$$

and that there exists a $k_0 \geqslant 1$ such that

$$S_kST_kS^{-1}S_k^{-1} \quad \text{is diagonal}, \quad \forall k \geqslant k_0. \tag{14}$$

Thus

$$\mathcal{A}_w(S_kSTS^{-1}S_k^{-1}) = \mathcal{D} = \mathcal{A}_w(STS^{-1}), \quad \forall k \geqslant k_0,$$

where $\mathcal{D}$ is the algebra of all diagonal matrices of dimension $\dim H$. Equivalently, by multiplying by the appropriate operators,

$$\mathcal{A}_w(T_k) = (S^{-1}S_kS)\mathcal{A}_w(T)(S^{-1}S_kS)^{-1}, \quad \forall k \geqslant k_0.$$

In view of this last equality and of (13) we can apply Corollary 1.5 and obtain the following.

$$\begin{aligned}\lim_{k\to\infty} \kappa(T_k) = \lim_{k\to\infty} K(\mathcal{A}_w(T_k)) &= \lim_{k\to\infty} K((S^{-1}S_kS)\mathcal{A}_w(T)(S^{-1}S_kS)^{-1}) \\ &= \lim_{k\to\infty} K(\mathcal{A}_w(T)) = \kappa(T). \quad \blacksquare\end{aligned}$$

The following two consequences of Theorem 1.6 are interesting in their own right and also in connection with the homologous results in the infinite-dimensional case (Section 2).

COROLLARY 1.7. *If H is a finite-dimensional Hilbert space, then the set of continuity for the function κ is a dense G_δ set in $\mathcal{B}(H)$.*

Proof. As noted in Remark 1.2, the set $\mathcal{E}$ is dense. Also, Theorem 1.6 implies that the set of points of continuity for κ contains $\mathcal{E}$ and hence is dense. To justify the second part of the corollary, recall that the points of continuity for a real valued function form a G_δ set in the domain ([27]). ∎

COROLLARY 1.8. *Let H be a finite-dimensional Hilbert space and let A and B be invertible operators on H. Then, if $T \in \mathcal{B}(H)$ is a point of continuity for κ, then ATB is also a point of continuity for κ.*

Proof. Obvious. ∎

The remainder of this section sheds some light on the problem of what are all the possible values of the constant of hyperreflexivity.

COROLLARY 1.9. *If H is a finite-dimensional Hilbert space, then the set of points of continuity for κ is a connected set.*

Proof. Denote by $\mathcal{C}$ the set of points of continuity for κ. From Theorem 1.6 and the fact that $\mathcal{E}$ is dense, it follows immediately that $\mathcal{E} \subseteq \mathcal{C} \subseteq \overline{\mathcal{E}}$. And it is very easy to show that the set $\mathcal{E}$ is connected. Thus $\mathcal{C}$ itself is connected. ∎

COROLLARY 1.10. *If H is a Hilbert space of dimension two, then the function $\kappa : \mathcal{B}(H) \to [1, \infty]$ is surjective.*

Proof. As above, denote the set of points of continuity for κ by $\mathcal{C}$. Notice that $\mathcal{C}$ contains $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and that $\kappa\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 1$ ([17]). Also, since there are non-reflexive two-dimensional matrices, the range of κ contains ∞. But κ is continuous at non-reflexive matrices (Theorem 1.6) and $\mathcal{C}$ is connected, so $\operatorname{ran}(\kappa) = [1, \infty]$, i.e., κ is surjective. ∎

Denote by $\mathcal{D}_n$ the algebra of all diagonal matrices of dimension n. The fact that makes the proof of Corollary 1.10 work is that

$$\inf\{\kappa(T) : T \in \mathcal{E} \subseteq M_2\} \left(= \kappa\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = K(\mathcal{D}_2)\right) = 1. \tag{15}$$

For higher dimensions, (15) does not necessarily hold; actually $K(\mathcal{D}_n) > 1, \forall n \geqslant 3$ (see [5]). Thus, the natural question to ask is

QUESTION. What is the value of $\inf\{\kappa(T) : T \in \mathcal{E} \subset \mathcal{B}(H)\}$ if H is a finite-dimensional Hilbert space of dimension $\geqslant 3$?

If $\inf\{\kappa(T) : T \in \mathcal{E}\} = 1$, then $\kappa : \mathcal{B}(H) \to [1, \infty]$ is surjective. If $\inf\{\kappa(T) : T \in \mathcal{E}\} > 1$ it is interesting, especially in view of [11] (for implications in the infinite-dimensional case), to determine how does this quantity depend on the dimension of the underlying space.

2. THE INFINITE-DIMENSIONAL CASE

If H is an infinite-dimensional Hilbert space, then the notion of hyperreflexivity does not coincide in general with that of reflexivity (see [6]). And, in addition, there is no known characterization of reflexive or hyperreflexive operators (except for the class of algebraic operators - for which the characterization is similar to that in the finite-dimensional case). Still, we can show that there are "a lot" of points of continuity for κ : the set of points at which κ is continuous is a dense G_δ set in $\mathcal{B}(H)$ (Theorem 2.14). There are also many points at which κ is not continuous, for example, all the hyperreflexive operators and all non-hyperreflexive operators similar to operators whose C^*-algebras contain no non-zero compacts (Theorem 2.5 and Theorem 2.7). However, in view of Theorem 2.14, the set of points of discontinuity for κ is a set of first category in $\mathcal{B}(H)$ and is, therefore, topologically smaller than the set of points of continuity.

For any compact set $E \subseteq \mathbf{C}$, denote by $\widetilde{E}$ the polynomially convex hull of E.

THEOREM 2.1. *Let H be a separable infinite-dimensional Hilbert space. Then the following two sets are norm-dense in $\mathcal{B}(H)$.*

(i) $\{T \in \mathcal{B}(H) : T$ *is non-reflexive*$\}$.

(ii) $\{T \in \mathcal{B}(H) : T$ *is reflexive*$\}$.

Proof. (i) Let $T \in \mathcal{B}(H)$. Denote by U the unbounded component of $\mathbf{C}\backslash\sigma_e(T)$, where $\sigma_e(T)$ is the essential spectrum of T. Take $\lambda \in \partial U$. So, $\lambda \in \partial\sigma_e(T)$, which implies that $\lambda \in \sigma_{le}(T)$. Using now [8], it follows that there exists a compact operator $K \in \mathcal{B}(H)$ such that $\|K\|$ is arbitrarily small, and $T + K$ is unitarily equivalent to an operator X of the form $\begin{bmatrix} \lambda^{(\infty)} & * \\ 0 & A \end{bmatrix}$, where $\lambda^{(\infty)}$ represents the scalar operator λI on an infinite-dimensional Hilbert space. Obviously, X can also be written as

$$X = \begin{bmatrix} \lambda & 0 & 0 & * \\ 0 & \lambda & 0 & * \\ 0 & 0 & \lambda^{(\infty)} & * \\ 0 & 0 & 0 & A \end{bmatrix}, \tag{16}$$

where the two λ's represent operators on one-dimensional spaces. By its choice, λ is in fact in the outer boundary of $\sigma_e(T)$, or, since $\sigma_e(X) = \sigma_e(T)$, it is in the outer boundary of $\sigma_e(X)$. Using the relation between the spectrum and the essential spectrum of an operator, we can infer that there exists a sequence $\{\lambda_n\}_{n\geqslant 1}$ in $U \setminus \sigma(X)$ such that $\lim_{n\to\infty} \lambda_n = \lambda$. Consider the following sequence of operators

$$X_n = \begin{bmatrix} \lambda_n & 0 & 0 & * \\ 0 & \lambda_n & 0 & * \\ 0 & 0 & \lambda^{(\infty)} & * \\ 0 & 0 & 0 & A \end{bmatrix}. \tag{17}$$

From (16) and (17), and the fact that λ is the limit of the sequence $\{\lambda_n\}_{n\geqslant 1}$, it follows that

$$\lim_{n\to\infty} X_n = X. \tag{18}$$

Obviously, $\sigma(A) \subseteq \widetilde{\sigma(X)}$, and $\lambda \in \widetilde{\sigma(X)}$. But $\lambda_n \notin \widetilde{\sigma(X)}$, $\forall n \geqslant 1$. Thus, $\sigma\left(\begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{bmatrix}\right) \cap \sigma\left(\begin{bmatrix} \lambda^{(\infty)} & * \\ 0 & A \end{bmatrix}\right) = \emptyset$. So we can apply Rosenblum's Theorem ([25]) to conclude that

$$X_n \overset{\text{sim}}{\sim} \begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{bmatrix} \oplus \begin{bmatrix} \lambda^{(\infty)} & * \\ 0 & A \end{bmatrix} \overset{\text{def}}{=} Z_n. \tag{19}$$

The spectra of the two summands in (19) are disjoint, but, even more, the polynomially convex hulls of their spectra are disjoint; thus (by [4])

$$\mathcal{A}_w(Z_n) = \mathcal{A}_w\left(\begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{bmatrix}\right) \oplus \mathcal{A}_w\left(\begin{bmatrix} \lambda^{(\infty)} & * \\ 0 & A \end{bmatrix}\right). \tag{20}$$

To end the proof of (i), simply approximate each $\begin{bmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{bmatrix}$ $(n \geqslant 1)$ by non-reflexive operators $\begin{bmatrix} \lambda_n & \frac{1}{k} \\ 0 & \lambda_n \end{bmatrix}$, $k \geqslant 1$.

(ii) The key ingredient in the proof of the density of reflexive operators on a separable infinite-dimensional Hilbert space is the density of the Apostol-Morel simple models ([15], Theorem 1.6). This result asserts, mainly, that every $T \in \mathcal{B}(H)$ can be approximated by operators of the form $X = \begin{bmatrix} A & * & * \\ 0 & B & * \\ 0 & 0 & C \end{bmatrix}$, where A is a subnormal operator, B is a normal operator (with finite spectrum),

C is the adjoint of a subnormal operator, and $\sigma(A)$, $\sigma(B)$ and $\sigma(C)$ are mutually disjoint. Thus, by Rosenblum's Theorem, X is similar to $A \oplus B \oplus C$. Also,

$$\mathcal{A}_w(A \oplus B \oplus C) \subseteq \mathcal{A}_w(A) \oplus \mathcal{A}_w(B) \oplus \mathcal{A}_w(C). \tag{21}$$

Recall that subnormals, normals and adjoints of subnormals are reflexive and have property D ([22]). So, by [21] and [12], the algebra $\mathcal{A}_w(A) \oplus \mathcal{A}_w(B) \oplus \mathcal{A}_w(C)$ is reflexive and has property (D). Thus, by [21], every weakly closed algebra of $\mathcal{A}_w(A) \oplus \mathcal{A}_w(B) \oplus \mathcal{A}_w(C)$ is reflexive, in particular $\mathcal{A}_w(A \oplus B \oplus C)$ is reflexive, hence, $A \oplus B \oplus C$ is reflexive, as well as X. Thus T can be approximated with reflexive operators. ∎

REMARK 2.2. We could use the density of the Apostol-Morel simple models also to show the density of non-reflexive operators on a separable infinite-dimensional Hilbert space, in a similar way we showed (ii) in Theorem 2.1, but we still need quite a bit of spectral analysis. We prefer the above proof since it does not require the use of such a strong result when a more direct and elementary proof was available.

The following immediate consequence of Theorem 2.1 gives a first result about the points of continuity for κ.

COROLLARY 2.3. *If H is a separable infinite-dimensional Hilbert space, then the only possible points of continuity for $\kappa : \mathcal{B}(H) \to [1, \infty]$ are the non-hyperreflexive operators on H.*

DEFINITION 2.4. Two operators S and T from $\mathcal{B}(H)$ are called *approximately equivalent*, if there exists a sequence $\{U_n\}_{n\geqslant 1}$ of unitaries in $\mathcal{B}(H)$ such that

(a) $\lim_{n\to\infty} \|S - U_n T U_n^*\| = 0$, and

(b) $S - U_n T U_n^*$ is compact, $\forall n \geqslant 1$.

We denote by $S \sim_a T$ the fact that S is approximately equivalent to T.

THEOREM 2.5. *If H is a separable infinite-dimensional Hilbert space and $S \in \mathcal{B}(H)$ is non-hyperreflexive such that $C^*(S) \cap \mathcal{K}(H) = (0)$, then κ is not continuous at S.*

Proof. From [28] it follows that if $\pi : C^*(S) \to \mathcal{B}(H_\pi)$ is a $*$-representation of $C^*(S)$ onto a separable Hilbert space H_π and if $C^*(S) \cap \mathcal{K}(H) \subseteq \ker(\pi)$, then $S \sim_a S \oplus \pi(S)$. By the hypothesis of our theorem, for any $*$-representation of $C^*(S)$, $S \sim_a S \oplus \pi(S)$. So, by taking a convenient $*$-representation, i.e., $\pi(S) = S^{(\infty)}$, we can infer that $S \sim_a S \oplus S^{(\infty)}$. Since $S \oplus S^{(\infty)} \sim_a S$ (they are even unitarily equivalent), it follows that $S \sim_a S^{(\infty)}$. In other words, there exists a sequence $\{U_n\}_{n\geqslant 1}$ of unitaries such that, in particular,

$$S = \lim_{n\to\infty} U_n S^{(\infty)} U_n^*. \tag{22}$$

Obviously, $\kappa(U_n S^\infty U_n^*) = \kappa(S^\infty)$. Finally, recall that $S^{(\infty)}$ is hyperreflexive ([10]) and S, by hypothesis, is not. Thus, in view of (22), κ cannot be continuous at S. ∎

We mentioned above one of Herrero's results ([15], Theorem 1.1) which implies that the spectrum is a lower semi-continuous function of an operator. The next lemma shows that the polynomially convex hull of the spectrum has the same property.

LEMMA 2.6. *If $\mathcal{A}$ is a unital Banach algebra, the map $a \mapsto \widetilde{\sigma(a)}$ is lower semi-continuous on $\mathcal{A}$, i.e., if $\{a_n\}_{n\geqslant 1}$ is a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} a_n = a$, for $a \in \mathcal{A}$, and if $\{\lambda_n\}_{n\geqslant 1}$ is a sequence in $\widetilde{\sigma(a_n)}$ such that $\lim_{n\to\infty} \lambda_n = \lambda$, then $\lambda \in \widetilde{\sigma(a)}$. Equivalently, if $a \in \mathcal{A}$, U is an open set in $\mathbf{C}$ containing $\widetilde{\sigma(a)}$, then there exists a $\delta > 0$ such that $\|b - a\| < \delta$ implies $\widetilde{\sigma(b)} \subset U$.*

Proof. Let $\{a_n\}_{n\geqslant 1}$ be a sequence in $\mathcal{A}$ such that $\lim_{n\to\infty} a_n = a$, for $a \in \mathcal{A}$, and let $\{\lambda_n\}_{n\geqslant 1}$ be a sequence in $\widetilde{\sigma(a_n)}$ such that $\lim_{n\to\infty} \lambda_n = \lambda$. Assume, by way of contradiction, that $\lambda \notin \widetilde{\sigma(a)}$. Recall that $\widetilde{\sigma(a)}$ is the union of $\sigma(a)$ and its (bounded) holes. Therefore λ is in the unbounded component of $\mathbf{C} \setminus \sigma(a)$. Thus there exist two disjoint open sets U and V such that U contains $\sigma(a)$ and V is unbounded, path-connected and contains λ. Hence, we can choose a path Γ in V that connects λ to ∞ and does not intersect $\widetilde{\sigma(a)}$. Since $\lim_{n\to\infty} \lambda_n = \lambda$ and V is an open set containing λ, it follows that there exists an $n_1 \geqslant 1$ such that

$$\lambda_n \in V, \quad \forall n \geqslant n_1. \tag{23}$$

But, applying the lower semi-continuity of the spectrum, from $\lim_{n\to\infty} a_n = a$, we infer that there exists an $n_2 \geqslant 1$ such that

$$\sigma(a_n) \subset U, \quad \forall n \geqslant n_2. \tag{24}$$

Since $U \cap V = \emptyset$, (23) and (24) constitute a contradiction. So, $\lambda \in \widetilde{\sigma(a)}$. ∎

The following is the analogue in infinite dimensions of Corollary 1.8.

THEOREM 2.7. *If H is a separable infinite-dimensional Hilbert space, $T \in \mathcal{B}(H)$ is a point of continuity for κ and S is an invertible operator in $\mathcal{B}(H)$, then STS^{-1} is also a point of continuity for κ.*

Proof. Since T is a point of continuity for κ and since H is infinite-dimensional, it follows, from Corollary 2.3, that $\kappa(T) = \infty$. From Proposition 1.3, we infer that

$$\kappa(STS^{-1}) = \infty. \tag{25}$$

To show that κ is continuous at STS^{-1}, take a sequence $\{X_n\}_{n\geqslant 1}$ of operators in $\mathcal{B}(H)$ such that $\lim_{n\to\infty} X_n = STS^{-1}$. Then, obviously, $\lim_{n\to\infty} S^{-1}X_nS = T$. Since T is a point of continuity for κ, it follows that $\lim_{n\to\infty} \kappa(S^{-1}X_nS) = \kappa(T) = \infty$. Using again Proposition 1.3, we deduce that $\lim_{n\to\infty} \kappa(X_n) = \infty$, which ends the proof. ∎

REMARK 2.8. Two operators A and B from $\mathcal{B}(H)$ are said to be *approximately similar* if there exists a sequence $\{S_n\}_{n\geqslant 1}$ of invertible operators in $\mathcal{B}(H)$ such that

(a) $\sup\{\|S_n\|, \|S_n^{-1}\| : n \geqslant 1\} < \infty$, and

(b) $A = \lim_{n\to\infty} S_nBS_n^{-1}$.

With obvious modifications, it can be shown that Theorem 2.7, as well as Theorem 2.5, are still true if we replace "similar" by "approximately similar" in their statements.

LEMMA 2.9. *Let H be a separable (finite- or infinite-dimensional) Hilbert space, and consider $\{Q_n\}_{n\geqslant 1}$ to be a sequence of idempotents in $\mathcal{B}(H)$ that converges to an orthogonal projection Q. Then there exists a sequence $\{S_n\}_{n\geqslant 1}$ of invertible operators in $\mathcal{B}(H)$ such that*

(i) $\lim_{n\to\infty} S_n = I$, *and*

(ii) $\exists n_0$ *such that* $S_nQ_nS_n^{-1} = Q, \quad \forall n \geqslant n_0$.

Proof. The proof will be done by constructing the sequence $\{S_n\}_{n\geqslant 1}$. For every $n \geqslant 1$, define $S_n = Q_nQ + (I - Q_n)(I - Q)$. Since $\lim_{n\to\infty} Q_n = Q$, it is obvious that (i) holds. Moreover, from (i) it follows that, eventually, S_n is invertible. Finally, trivial linear algebra shows that $Q_nS_n = S_nQ(= Q_nS)$. Thus (ii) also holds. ∎

LEMMA 2.10. *Let $\mathcal{A}$ be a unital subalgebra of $\mathcal{A}_1 \oplus \mathcal{A}_2$, where $\mathcal{A}_i$ is a weak-operator closed unital subalgebra of $\mathcal{B}(H_i)$, $i = 1, 2$. If*

$$\operatorname{Lat}(\mathcal{A}) = \operatorname{Lat}(\mathcal{A}_1) \oplus \operatorname{Lat}(\mathcal{A}_2),$$

then

$$K(\mathcal{A}) \geqslant \max\{K(\mathcal{A}_1), K(\mathcal{A}_2)\}.$$

Proof. By definition of K in particular,

$$K(\mathcal{A}) \geqslant \frac{\operatorname{dist}(T \oplus 0, \mathcal{A})}{\mathrm{d}(T \oplus 0, \mathcal{A})}, \quad \forall T \in \mathcal{B}(H_1), \tag{26}$$

where $\mathrm{d}(X, \mathcal{A}) = \sup\{\|P^{\perp} X P\| : P \in \operatorname{Lat}(\mathcal{A})\}, \forall X \in \mathcal{B}(H_1 \oplus H_2)$. Even without the hypothesis about the lattices of invariant subspaces, it is easy to see that the following holds.

$$\operatorname{dist}(T \oplus 0, \mathcal{A}) \geqslant \operatorname{dist}(T \oplus 0, \mathcal{A}_1 \oplus \mathcal{A}_2) = \operatorname{dist}(T, \mathcal{A}_1). \tag{27}$$

But, since $\operatorname{Lat}(\mathcal{A}) = \operatorname{Lat}(\mathcal{A}_1) \oplus \operatorname{Lat}(\mathcal{A}_2)$, a straightforward calculation shows that

$$\mathrm{d}(T \oplus 0, \mathcal{A}) = \mathrm{d}(T \oplus 0, \mathcal{A}_1 \oplus \mathcal{A}_2) = \mathrm{d}(T, \mathcal{A}_1). \tag{28}$$

Hence, from (26), (27) and (28), $K(\mathcal{A}) \geqslant \frac{\operatorname{dist}(T, \mathcal{A}_1)}{\mathrm{d}(T, \mathcal{A}_1)}, \forall T \in \mathcal{B}(H_1) \setminus \mathcal{A}_1$. Therefore $K(\mathcal{A}) \geqslant K(\mathcal{A}_1)$. Similarly it can be proven that $K(\mathcal{A}) \geqslant K(\mathcal{A}_2)$. ∎

REMARK 2.11. One obvious case when the hypothesis, hence the conclusion, of Lemma 2.10 is true is when $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$. This will be the case in which we will apply Lemma 2.12.

LEMMA 2.12. *If $\{\mathcal{A}_n\}_{n \geqslant 1}$ is a sequence of weak-operator closed unital algebras such that $\mathcal{A}_n \subseteq \mathcal{B}(H_n), \forall n \geqslant 1$, then*

$$K\left(\bigoplus_{n \geqslant 1} \mathcal{A}_n\right) \leqslant 9 \sup_{n \geqslant 1} K(\mathcal{A}_n).$$

Proof. Using a result about relative hyperreflexivity from [10], it follows that

$$K\left(\bigoplus_{n \geqslant 1} \mathcal{A}_n, \mathcal{B}(H)^{(\infty)}\right) = \sup_{n \geqslant 1} K(\mathcal{A}_n). \tag{29}$$

Since $\mathcal{B}(H)^{(\infty)}$ is a type I von Neumann algebra, by [26], it is hyperreflexive and

$$K(\mathcal{B}(H)^{(\infty)}) \leqslant 4. \tag{30}$$

Applying Theorem 2.6. in [10], we derive now from (29) and (30) that

$$K\left(\bigoplus_{n\geqslant 1}\mathcal{A}_n\right)\leqslant\left(K\left(\mathcal{B}(H)^{(\infty)}\right)+1\right)\left(K\left(\bigoplus_{n\geqslant 1}\mathcal{A}_n,\mathcal{B}(H)^{(\infty)}\right)+1\right)-1$$
$$\leqslant 5\sup_{n\geqslant 1}K(\mathcal{A}_n)+4\leqslant 9\sup_{n\geqslant 1}K(\mathcal{A}_n),$$

which ends the proof. ∎

The following theorem gives a useful "recipe" for producing new points of continuity for κ on infinite-dimensional spaces, for example using Theorem 1.6, which describes all the points of continuity in finite dimension.

THEOREM 2.13. *Let H_1 and H_2 be two separable (finite- or infinite-dimensional) Hilbert spaces. Then non-hyperreflexive operators on $\mathcal{B}(H_1 \oplus H_2)$ of the form $T = A \oplus B$ with $\widetilde{\sigma(A)} \cap \widetilde{\sigma(B)} = \emptyset$ are points of continuity for κ if and only if at least one of A and B is a non-hyperreflexive point of continuity for the corresponding κ function.*

Proof. Assume that $T = A \oplus B \in \mathcal{B}(H_1 \oplus H_2)$ is an operator such that $\widetilde{\sigma(A)} \cap \widetilde{\sigma(B)} = \emptyset$ and such that A is a point of continuity for κ with $\kappa(A) = \infty$. From [4] it follows that $\mathcal{A}_w(T) = \mathcal{A}_w(A) \oplus \mathcal{A}_w(B)$. So, by Lemma 2.10, $\kappa(T) = \infty$. If $H_1 \oplus H_2$ is finite-dimensional, it follows from Theorem 1.6 that κ is continuous at T. If $H_1 \oplus H_2$ is infinite-dimensional, let $\{T_n\}_{n\geqslant 1}$ be a sequence in $\mathcal{B}(H_1 \oplus H_2)$ such that $\lim_{n\to\infty} T_n = T$. Choose two disjoint open sets U and V such that $\widetilde{\sigma(A)} \subset U$, and $\widetilde{\sigma(B)} \subset V$. Applying Lemma 2.6, we deduce that, eventually, $\widetilde{\sigma(T_n)} \subset U \cup V$. Let Q_n be the spectral projection of T_n corresponding to $\overline{U}$. It is clear that $\lim_{n\to\infty} Q_n = Q$, where Q is the spectral projection of T corresponding to $\overline{U}$. Because of the form of T, Q is an orthogonal projection. So, by Lemma 2.9, there exists a sequence $\{S_n\}_{n\geqslant 1}$ of invertible operators in $\mathcal{B}(H_1 \oplus H_2)$ such that

(i) $\lim_{n\to\infty} S_n = I$

and

(ii) $S_n Q_n S_n^{-1} = Q$, for n sufficiently large.

Define a sequence of operators on $H_1 \oplus H_2$ by $T_n' = S_n T_n S_n^{-1}, \forall n \geqslant 1$. This sequence has the property that $\lim_{n\to\infty} T_n' = A \oplus B$. Also, the spectral projection of T_n' is $Q_n' = S_n Q_n S_n^{-1}$, so, it is eventually equal to Q. Hence, eventually, $T_n' = A_n \oplus B_n$ relative to the decomposition $H_1 \oplus H_2$ of the underlying space, the same decomposition that gives $T = A \oplus B$. It follows that $\lim_{n\to\infty} A_n = A$ and

$\lim_{n\to\infty} B_n = B$. Recall that A is a point of continuity for κ and that $\kappa(A) = \infty$. Thus

$$\lim_{n\to\infty} \kappa(A_n) = \infty. \tag{31}$$

Since $\widetilde{\sigma(A)} \subset U$ and $\widetilde{\sigma(B)} \subset V$, it follows that, eventually, $\widetilde{\sigma(A_n)} \subset U$, and $\widetilde{\sigma(B_n)} \subset V$. Hence ([4]) $\mathcal{A}_w(A \oplus B) = \mathcal{A}_w(A) \oplus \mathcal{A}_w(B)$. Using Lemma 2.10, from (31) we conclude that $\lim_{n\to\infty} \kappa(T_n') = \infty$. Thus, it follows from Proposition 1.3 that $\lim_{n\to\infty} \kappa(T_n) = \infty$ as well.

Conversely, assume that $T = A \oplus B$ is a point of continuity for κ such that $\widetilde{\sigma(A)} \cap \widetilde{\sigma(B)} = \emptyset$ and $\kappa(T)(= \kappa(A \oplus B)) = \infty$. In the hypothesis about the spectra, $\mathcal{A}_w(A \oplus B) = \mathcal{A}_w(A) \oplus \mathcal{A}_w(B)$. So, from Lemma 2.12 we deduce that

$$\kappa(A \oplus B) = K(\mathcal{A}_w(A) \oplus \mathcal{A}_w(B)) \leqslant 9 \max\{\kappa(A), \kappa(B)\}. \tag{32}$$

Thus, at least one of $\kappa(A)$ and $\kappa(B)$ is infinite. Without loss of generality, consider that $\kappa(A) = \infty$. If the underlying space corresponding to A is finite-dimensional, since A is non-reflexive, by Theorem 1.6 it is a point of continuity for κ. If the underlying space for A is infinite-dimensional, assume, by way of contradiction, that neither A nor B is a point of continuity for κ. Then, there exist sequences $\{A_n\}_{n\geqslant 1} \subset \mathcal{B}(H_1)$ and $\{B_n\}_{n\geqslant 1} \subset \mathcal{B}(H_2)$ such that $\lim_{n\to\infty} A_n = A$ and $\lim_{n\to\infty} B_n = B$, but $\{\kappa(A_n)\}_{n\geqslant 1}$ and $\{\kappa(B_n)\}_{n\geqslant 1}$ are bounded. From the hypothesis about the polynomially convex hulls of the spectra of A and B, Lemma 2.6 implies that $\widetilde{\sigma(A_n)} \cap \widetilde{\sigma(B_n)} = \emptyset$, for n large enough. So we can apply Lemma 2.12 to obtain an inequality similar to (32) for A_n instead of A and B_n instead of B, which in turn implies that $\{\kappa(A_n \oplus B_n)\}_{n\geqslant 1}$ is bounded. Since $\kappa(A \oplus B) = \infty$, this last statement contradicts the fact that T is a point of continuity for κ. ∎

The last result from this section characterizes the set of points of continuity for κ in the case when the underlying space is infinite-dimensional.

THEOREM 2.14. *If H is a separable infinite-dimensional Hilbert space, then the points of continuity for $\kappa : \mathcal{B}(H) \to [1, \infty]$ form a dense G_δ set in $\mathcal{B}(H)$; i.e. a set of the second category.*

Proof. Let T be an arbitrary operator in $\mathcal{B}(H)$. Following the proof of Theorem 2.1, (i), we notice that, for λ in the boundary of the unbounded component of $\mathbf{C} \setminus \sigma_e(T)$, T can be approximated by operators similar to

$$S_{n,k} = \begin{bmatrix} \lambda_n & \frac{1}{k} \\ 0 & \lambda_n \end{bmatrix} \oplus \begin{bmatrix} \lambda^{(\infty)} & * \\ 0 & A \end{bmatrix}, \quad k \geqslant 1, n \geqslant 1, \tag{33}$$

with the polynomially convex hulls of the spectra of the two summands disjoint. By Theorem 1.6 the first summand of $S_{n,k}$ is a point of continuity for $\kappa : \mathcal{B}(\mathbf{C}^2) \to [1,\infty]$. Thus, using Theorem 2.13, we can conclude that $S_{n,k}$ itself is a point of continuity for the corresponding κ. Finally, making use of Theorem 2.7, we conclude that T can be approximated by operators that are points of continuity for κ. Thus the set of points of continuity for κ is dense in $\mathcal{B}(H)$.

To end the proof, use the fact that the set of points of continuity for a real valued function is a G_δ set in the domain. ∎

3. A RELATED PROBLEM

After undertaking the research concerning the problem studied in the previous two sections we learned about another paper about hyperreflexivity and stability properties. Even though, at the first sight, the two problems seemed to be similar, even if the functions whose continuity is studied are related by a simple condition, and even if we consider only the finite-dimensional case, the differences are meaningful and interesting. The following will provide a complete description of the relationship between the problems, showing, at the same time, the independence of our results from those in [23].

First, in [23], the authors consider the function $K : \mathcal{S}(\mathcal{B}(H)) \to [1,\infty]$, where $\mathcal{S}(\mathcal{B}(H))$ denotes the space of all (closed linear) subspaces of $\mathcal{B}(H)$ with the topology given by the distance

$$\begin{aligned} d(M,N) = \max\{&\sup\{\operatorname{dist}(x,N) : x \in M,\ \|x\| \leqslant 1\},\\ &\sup\{\operatorname{dist}(y,M) : y \in N,\ \|y\| \leqslant 1\}\}. \end{aligned} \tag{34}$$

and $K(\mathcal{S})$, for a subspace $\mathcal{S}$, is the constant of hyperreflexivity of $\mathcal{S}$.

The main results in [23] that relate to our results are Propositions 3.1 and 3.4 below.

PROPOSITION 3.1. ([23], Theorem 2.2) *If H is a separable Hilbert space, $\mathcal{S} \subseteq \mathcal{B}(H)$ is a ultra-weakly-closed subspace such that $\mathcal{S}_\perp$ is reflexive as a Banach space and $\{\mathcal{S}_n\}_{n\geqslant 1}$ is a sequence of hyperreflexive subspaces of $\mathcal{B}(H)$ such that $\lim\limits_{n\to\infty} d(\mathcal{S}_n,\mathcal{S}) = 0$, and $\{K(\mathcal{S}_n)\}_{n\geqslant 1}$ is bounded, then $\mathcal{S}$ is hyperreflexive and $K(\mathcal{S}) \leqslant \limsup\limits_{n\to\infty} K(\mathcal{S}_n)$.*

REMARK 3.2. From Proposition 3.1 it follows that the function K is continuous at every non-hyperreflexive subspace.

DEFINITION 3.3. If $\mathcal{S}$ is a subspace of $\mathcal{B}(H)$, we denote by $[\mathcal{S}]$ the set of pairs $(x, y) \in H^2$ such that $Sx \perp y$. A pair $(x, y) \in [\mathcal{S}]$ is called *regular* if the only $S \in \mathcal{S}$ which satisfies $Sx = S^*y = 0$ is the zero operator. The subspace $\mathcal{S}$ is called *regular* if every pair in $[\mathcal{S}]$ with non-zero components is a limit of regular pairs in $[\mathcal{S}]$.

PROPOSITION 3.4. ([23], Theorem 3.4) *If H is a finite-dimensional Hilbert space, then regular reflexive subspaces of $\mathcal{B}(H)$ are points of continuity for K.*

The following commutative diagram shows the set-theoretical relationship between K and κ; the connecting function f is given, obviously, by $f(T) = \mathcal{A}_w(T), \forall T \in \mathcal{B}(H)$.

$$\begin{array}{ccc} \mathcal{B}(H) & \xrightarrow{\kappa} & [1, \infty] \\ f \searrow & & \nearrow K \\ & \mathcal{S}(\mathcal{B}(H)) & \end{array}$$

$$\begin{array}{ccc} T & \xrightarrow{\kappa} & \kappa(T) \\ f \searrow & & \nearrow K \\ & \mathcal{A}_w(T) & \end{array}$$

In other words, $\kappa = K \circ f$. In consequence, it is natural to ask if, at least when H is finite-dimensional, we could "transfer" the continuity properties from K to κ by means of f, in which case part of Theorem 1.6 could be obtained as a corollary of the results in [23]. But, as it turns out, the situation is not so simple. The reason is that the points of continuity for f defined on a finite-dimensional Hilbert space are exactly the cyclic operators (Theorem 3.8), so the transfer is not so smooth: since not all non-reflexive operators on a finite-dimensional space are cyclic we cannot derive the continuity of κ at non-reflexive operators from Proposition 3.1. Also, we cannot derive the continuity of κ at operators in $\mathcal{E}$ from Proposition 3.4 without Corollary 3.11, which will be proven below. This corollary asserts that for finite-dimensional Hilbert spaces, $\mathcal{A}_w(T)$ is a regular subspace of $\mathcal{B}(H)$ for any $T \in \mathcal{E}$.

The next lemma will be used in the proof of Theorem 3.8, as well as in the proof of another auxiliary result used to prove Theorem 3.8 (Lemma 3.6). It is true in the context of arbitrary topological vector spaces and we shall state and prove it in that generality.

LEMMA 3.5. *Let X be a topological vector space and let n be a positive integer. Then the set $\mathcal{LI}_n$ defined by*

$$\mathcal{LI}_n = \{(x_1, x_2, \ldots, x_n) \in X^n : \{x_1, x_2, \ldots, x_n\} \text{ is linearly independent in } X\}$$

is an open set in X^n.

Proof. The set $\mathcal{LI}_n$ is open if and only if $X^n \setminus \mathcal{LI}_n$ is closed, and we prefer to show this last statement. Let

$$\{\vec{x}_\lambda\}_\lambda \subset X^n \setminus \mathcal{LI}_n \tag{35}$$

be a net such that

$$\lim_\lambda \vec{x}_\lambda = \vec{x} \tag{36}$$

for some $\vec{x} \in X^n$. Suppose that $\vec{x}_\lambda = (x_{1,\lambda}, x_{2,\lambda}, \ldots, x_{n,\lambda})$, $\forall\lambda$, and $\vec{x} = (x_1, x_2, \ldots, x_n)$. From (35) it follows that there exists a net $\{\vec{\alpha}_\lambda\}_\lambda \subset \mathbf{C}^n \setminus \{0\}$, $\vec{\alpha}_\lambda = (\alpha_{1,\lambda}, \alpha_{2,\lambda}, \ldots, \alpha_{n,\lambda})$, $\forall\lambda$, such that

$$\vec{\alpha}_\lambda \cdot \vec{x}_\lambda = \sum_{i=1}^{n} \alpha_{i,\lambda} x_{i,\lambda} = 0, \quad \forall\lambda. \tag{37}$$

Dividing through by $c_\lambda \overset{\text{def}}{=} \max\limits_{1 \leqslant i \leqslant n} |\alpha_{i,\lambda}| \neq 0$ in the relation corresponding to λ in (37), we can assume, without loss of generality, that $\|\vec{\alpha}_\lambda\|_\infty = 1, \forall\lambda$. Hence $\{\vec{\alpha}_\lambda\}_\lambda$ has a convergent subnet. So, again without loss of generality, we can assume that

$$\lim_\lambda \vec{\alpha}_\lambda = \vec{\alpha}, \tag{38}$$

for some $\vec{\alpha} \in \mathbf{C}^n$. Using now (36) and (38) in (37), it follows that

$$\vec{\alpha} \cdot \vec{x} = \sum_{i=1}^{n} \alpha_i x_i = 0. \tag{39}$$

But $\vec{\alpha} \neq \mathbf{0}$; as a matter of fact, it has norm one, as a limit of norm-one elements in $\mathbf{C}^n$. Thus, (39) shows that $\vec{x} \in X^n \setminus \mathcal{LI}_n$. ∎

If $\vec{x} = (x_1, x_2, \ldots, x_n)$ is an element of X^n, denote by $\operatorname{Sp}(\vec{x})$ the linear span of $\{x_1, x_2, \ldots, x_n\}$ in X.

LEMMA 3.6. *Let X be a normed space, n be a positive integer and $\mathcal{LI}_n$ be the set defined in Lemma 3.5. Define the function $\delta_n : X^n \to \mathbf{R} \cup \{\infty\}$, by*

$$\delta_n(\vec{x}) = \sup\{\|\vec{\alpha}\|_\infty : \vec{\alpha} \in \mathcal{P}_{\vec{x}}\}, \quad \forall \vec{x} \in X^n,$$

where

$$\mathcal{P}_{\vec{x}} \stackrel{\text{def}}{=} \{\vec{\alpha} \in \mathbf{C}^n : \|\vec{\alpha} \cdot \vec{x}\| \leqslant 1\}.$$

Then

(i) $\delta_n(\vec{x}) < \infty$ *if and only if* $\vec{x} \in \mathcal{LI}_n$,

(ii) δ_n *is continuous on all of* X^n.

Proof. (i) If $\vec{x} \in X^n \backslash \mathcal{LI}_n$, then there exists a $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbf{C}^n \backslash \{0\}$ such that

$$\vec{\beta} \cdot \vec{x} = \sum_{i=1}^{n} \beta_i x_i = 0. \tag{40}$$

For every $t \in \mathbf{R}, t \geqslant 1$, define $\vec{\beta}_t = t\vec{\beta}$. Since $\|\vec{\beta}\|_\infty \neq 0$, it follows that $\lim_{t\to\infty} \|\vec{\beta}_t\|_\infty = +\infty$. Multiplying through by t in (40), we deduce that $\vec{\beta}_t \in \mathcal{P}_{\vec{x}}$. And since $\sup\{\|\vec{\alpha}\|_\infty : \vec{\alpha} \in \mathcal{P}_{\vec{x}}\} \geqslant \|\vec{\beta}\|_\infty$, we deduce that $\delta_n(\vec{x}) = \infty$.

To show the other implication, let $\vec{x} \in \mathcal{LI}_n$ and assume that $\delta_n(\vec{x})$ is infinite. Then, for every $k \geqslant 1$, there exists a $\vec{\beta}_k = (\beta_{1,k}, \beta_{2,k}, \ldots, \beta_{n,k}) \in \mathcal{P}_{\vec{x}}$, such that

$$\|\vec{\beta}_k\|_\infty > k. \tag{41}$$

But $\{\vec{\beta}_k\}_{k\geqslant 1} \subset \mathcal{P}_{\vec{x}}$ is equivalent to saying that

$$\|\vec{\beta}_k \cdot \vec{x}\| = \left\| \sum_{i=1}^{n} \beta_{i,k} x_i \right\| \leqslant 1, \quad \forall k \geqslant 1. \tag{42}$$

Dividing through in (42) by the corresponding $\|\vec{\beta}_k\|$, which is non-zero by (41), it follows that:

$$\left\| \sum_{i=1}^{n} \frac{\beta_{i,k}}{\|\vec{\beta}_k\|_\infty} x_i \right\| \leqslant \frac{1}{\|\vec{\beta}_k\|_\infty}, \quad \forall k \geqslant 1. \tag{43}$$

Obviously, the sequence $\left\{ \frac{1}{\|\vec{\beta}_k\|_\infty} \vec{\beta}_k \right\}_{k\geqslant 1}$ is bounded in $\mathbf{C}^n$, having norm one, so it has a convergent subsequence. Hence, without loss of generality, assume that:

$$\lim_{k\to\infty} \frac{1}{\|\vec{\beta}_k\|_\infty} \vec{\beta}_k = \vec{\gamma}, \tag{44}$$

for some $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbf{C}^n$, with, of course, $\|\vec{\gamma}\|_\infty = 1$. Now taking the limit as $k \to \infty$ in (43) and using (44) and (41), it follows that $\sum_{i=1}^{n} \gamma_i x_i = 0$, which, in the context of $\vec{\gamma} \neq 0$, contradicts the fact that $\vec{x} \in \mathcal{LI}_n$. Thus $\delta_n(\vec{x}) < \infty$.

(ii) Let $\vec{x} \in X^n$, and $\{\vec{x}_k\}_{k \geqslant 1} \subset X^n$ such that:

$$\lim_{k \to \infty} \vec{x}_k = \vec{x}. \tag{45}$$

Claim 1. $\limsup_{k \to \infty} \delta_n(\vec{x}_k) \leqslant \delta_n(\vec{x})$. Assume that the claim is false, i.e., assume that $\limsup_{k \to \infty} \delta_n(\vec{x}_k) > \delta_n(\vec{x})$. It follows that there exists an $r_0 > 0$ such that

$$\limsup_{k \to \infty} \delta_n(\vec{x}_k) > r_0 > \delta_n(\vec{x}). \tag{46}$$

By the definition of the function δ_n, (46) implies that there exists a sequence $\{\vec{\gamma}_k\}_{k \geqslant 1}$ in $\mathcal{P}_{\vec{x}_k}$ such that

$$\|\vec{\gamma}_k\|_\infty \geqslant r_0, \quad \forall k \geqslant 1. \tag{47}$$

Because $\vec{\gamma}_k \in \mathcal{P}_{\vec{x}_k}$ for every $k \geqslant 1$, it follows that

$$\left\| \sum_{i=1}^{n} \gamma_{i,k} x_{i,k} \right\| \leqslant 1, \quad \forall k \geqslant 1. \tag{48}$$

From (47) and (48) we deduce that

$$\begin{aligned} \left\| \sum_{i=1}^{n} r_0 \frac{\gamma_{i,k}}{\|\vec{\gamma}_k\|_\infty} x_{ik} \right\| &= \left\| \frac{r_0}{\|\vec{\gamma}_k\|_\infty} \sum_{i=1}^{n} \gamma_{i,k} x_{i,k} \right\| \\ &= \frac{r_0}{\|\vec{\gamma}_k\|_\infty} \left\| \sum_{i=1}^{n} \gamma_{i,k} x_{i,k} \right\| \leqslant 1. \end{aligned} \tag{49}$$

Define $\vec{\beta}_k = \frac{r_0}{\|\vec{\gamma}_k\|_\infty} \vec{\gamma}_k, \forall k \geqslant 1$. From (49), it follows that $\{\vec{\beta}_k\}_{k \geqslant 1} \subset \mathcal{P}_{\vec{x}_k}$, and, by its definition,

$$\|\vec{\beta}_k\|_\infty = r_0. \tag{50}$$

Thus, without loss of generality, we can assume that there exists a $\vec{\beta} \in \mathbf{C}^n$, $\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, such that

$$\lim_{k \to \infty} \vec{\beta}_k = \vec{\beta}. \tag{51}$$

From (50) and (51) it follows immediately that

$$\|\vec{\beta}\|_\infty = r_0. \tag{52}$$

And using (51) and (45) we infer that

$$\lim_{k\to\infty} \sum_{i=1}^{n} \beta_{i,k} x_{i,k} = \sum_{i=1}^{n} \beta_i x_i.$$

Hence

$$\left\| \sum_{i=1}^{n} \beta_i x_i \right\| \leqslant 1,$$

which shows that $\vec{\beta} \in \mathcal{P}_{\vec{x}}$. Combining this with (52) we obtain a contradiction with the second inequality in (46). Thus *Claim* 1 is proved.

Claim 2. $\liminf_{k\to\infty} \delta_n(\vec{x}_k) \geqslant \delta_n(\vec{x})$. Let $r < \delta_n(\vec{x})$. By the definition of the function δ_n, it follows that there exists a $\vec{\beta} \in \mathcal{P}_{\vec{x}}$ such that

$$\left\| \sum_{i=1}^{n} \beta_i x_i \right\| < 1, \tag{53}$$

and

$$\|\vec{\beta}\|_\infty > r. \tag{54}$$

From (45) and (53) we deduce that

$$\lim_{k\to\infty} \left\| \sum_{i=1}^{n} \beta_i x_{ik} \right\| = \left\| \sum_{i=1}^{n} \beta_i x_i \right\| < 1.$$

Hence, for sufficiently large k's, $\vec{\beta} \in \mathcal{P}_{\vec{x}_k}$. So we can use (54) to obtain from this that $\delta_n(\vec{x}_k) > r, \forall k \geqslant k_0$, for some $k_0 \geqslant 1$. Thus $r \leqslant \liminf_{k\to\infty} \delta_n(\vec{x}_k)$. And, since r was an arbitrary number less than $\delta_n(\vec{x})$, *Claim* 2 is proved.

From *Claim* 1 and *Claim* 2 it follows that $\lim_{k\to\infty} \delta_n(\vec{x}_k) = \delta_n(\vec{x})$, and thus δ_n is continuous at $\vec{x}$. The proof ends once we remember that $\vec{x}$ was an arbitrary element of X^n. ∎

At this moment, we need to refer to the topology given on $\mathcal{S}(\mathcal{B}(H))$ by (34) and cite the following result (see [23], for example).

PROPOSITION 3.7. *If H is a finite-dimensional Hilbert space, then $(\mathcal{S}(\mathcal{B}(H)), d)$ is a compact space whose connected components are the subsets of $\mathcal{B}(H)$ made out of all subspaces of a prescribed dimension.*

The next theorem is the main result that helps to describe the relation between the problem we studied above and the problem studied in [23]. It characterizes the set of points of continuity for the connecting function f.

THEOREM 3.8. *If H is a finite-dimensional Hilbert space, and $f : \mathcal{B}(H) \to \mathcal{S}(\mathcal{B}(H))$ is given by $f(T) = \mathcal{A}_w(T)$, then f is continuous at T if and only if T is a cyclic operator.*

Proof. Let n be the dimension of H. If $T \in \mathcal{B}(H)$ and T is not cyclic, then the set $\{I, T, T^2, \ldots, T^{n-1}\}$ is linearly dependent in $\mathcal{B}(H)$. In other words,

$$\dim \operatorname{Sp}(I, T, T^2, \ldots, T^{n-1}) < n. \tag{55}$$

Since the set of cyclic $n \times n$ matrices is an (open) dense set in M_n ([13]), it follows that there exists a sequence $\{T_k\}_{k\geqslant 1}$ of cyclic operators on H such that $\lim_{k\to\infty} T_k = T$. Then

$$\lim_{k\to\infty} (I, T_k, T_k^2, \ldots, T_k^{n-1}) = (I, T, T^2, \ldots, T^{n-1}). \tag{56}$$

Also, the cyclicity of the terms in the sequence $\{T_k\}_{k\geqslant 1}$ can be translated into

$$\dim \operatorname{Sp}(I, T_k, T_k^2, \ldots, T_k^{n-1}) = n, \quad \forall k \geqslant 1. \tag{57}$$

From (55), (57) and Proposition 3.7 it follows that $\operatorname{Sp}(I, T, T^2, \ldots, T^{n-1})$ is in a different connected component of $\mathcal{S}(\mathcal{B}(H))$ than any of $\operatorname{Sp}(I, T_k, T_k^2, \ldots, T_k^{n-1})$, $k \geqslant 1$. But in view of (56), this shows that f is not continuous at T.

Conversely, let T be a cyclic operator in $\mathcal{B}(H)$ and let $\{T_k\}_{k\geqslant 1} \subset \mathcal{B}(H)$ such that $\lim_{k\to\infty} T_k = T$. Using the fact that the set of cyclic operators on a finite-dimensional space is an open set ([13]), we can assume, without loss of generality, that T_k is cyclic, $\forall k \geqslant 1$. Recall that the definition of the distance d, (34), is

$$d(\mathcal{A}_w(T_k), \mathcal{A}_w(T)) = \max\{\sup\{\operatorname{dist}(x, \mathcal{A}_w(T_k)) : x \in \mathcal{A}_w(T),\ \|x\| \leqslant 1\},$$

$$\sup\{\operatorname{dist}(y, \mathcal{A}_w(T)) : y \in \mathcal{A}_w(T_k),\ \|y\| \leqslant 1\}. \tag{58}$$

But it is obvious that

$$\mathcal{A}_w(T) = \operatorname{Sp}(I, T, T^2, \ldots, T^{n-1})$$

and

$$\mathcal{A}_w(T_k) = \mathrm{Sp}\,(I, T_k, T_k^2, \ldots, T_k^{n-1}).$$

With the notation in Lemma 3.5,

$$(59) \qquad (I, T, T^2, \ldots, T^{n-1}) \quad \text{and} \quad (I, T_k, T_k^2, \ldots, T_k^{n-1}), \quad \forall k \geqslant 1,$$

belong to $\mathcal{LI}_n$, where $\mathcal{LI}_n$ corresponds to the topological vector space $X = \mathcal{B}(H)$. Also, from the definition of the function δ_n is Lemma 3.6, it follows that there exists a constant $M > 0$ such that

$$(60) \qquad \begin{aligned} &\sup\{\mathrm{dist}\,(x, \mathcal{A}_w(T_k)) : x \in \mathcal{A}_w(T),\ \|x\| \leqslant 1\} \\ &\qquad \leqslant M\delta_n(I, T_k, T_k^2, \ldots, T_k^{n-1})\|T_k - T\| \end{aligned}$$

and

$$(61) \qquad \begin{aligned} &\sup\{\mathrm{dist}\,(y, \mathcal{A}_w(T)) : y \in \mathcal{A}_w(T_k),\ \|y\| \leqslant 1\} \\ &\qquad \leqslant M\delta_n(I, T, T^2, \ldots, T^{n-1})\|T_k - T\|. \end{aligned}$$

Using now Lemma 3.6 in the context of (58), (59), (60) and (61) we infer that there exists a constant $R > 0$ such that $d(\mathcal{A}_w(T_k), \mathcal{A}_w(T)) \leqslant R\|T_k - T\|$, which implies immediately that $\lim\limits_{k\to\infty} d(\mathcal{A}_w(T_k), \mathcal{A}_w(T)) = 0$. Thus f is continuous at T. ∎

REMARK 3.9. Theorem 3.8 shows that we cannot derive the continuity of κ at non-reflexive operators from Proposition 3.1, since there are non-reflexive operators which are not cyclic (for example $T = J_2 \oplus J_4$, where J_2 and J_4 are Jordan blocks with eigenvalue zero, of size two, respectively, four).

The following is a generalization of [23], Proposition 4.3 (i).

PROPOSITION 3.10. *Let H be an arbitrary separable Hilbert space. If $\mathcal{S}$ is a regular subspace of $\mathcal{B}(H)$ and A and B are invertible operators in $\mathcal{B}(H)$, then $A\mathcal{S}B$ is a regular subspace.*

Proof. Let $x, y \in H$. For any subspace $\mathcal{S} \subseteq \mathcal{B}(H)$ and for any operators $A, B \in \mathcal{B}(H)$,

$$(62) \qquad (x, y) \in [A\mathcal{S}B] \quad \text{if and only if} \quad (Bx, A^*y) \in [\mathcal{S}].$$

If, in addition, A and B are invertible, it follows from (62) that

(63) (x, y) is regular (in $[A\mathcal{S}B]$) if and only if (Bx, A^*y) is regular (in $[\mathcal{S}]$).

Assume $\mathcal{S}$ is regular and let $(x, y) \in [A\mathcal{S}B]$ such that $x \neq 0$, and $y \neq 0$. Using (62), we infer that $(Bx, A^*y) \in [\mathcal{S}]$. But $\mathcal{S}$ is regular and $Bx \neq 0$ and $A^*y \neq 0$, since A and B are invertible. Thus (Bx, A^*y) is a limit of regular pairs in $[\mathcal{S}]$. In other words, there exists a sequence $\{(u_k, v_k)\}_{k\geqslant 1} \subset [\mathcal{S}]$, such that (u_k, v_k) is regular $\forall k \geqslant 1$, and such that

$$\lim_{k\to\infty}(u_k, v_k) = (Bx, A^*y).$$

It follows immediately that

$$\lim_{k\to\infty}(B^{-1}u_k, (A^*)^{-1}v_k) = (x, y).$$

Since (63) implies that $(B^{-1}u_k, (A^*)^{-1}v_k)$ is a regular pair in $[A\mathcal{S}B]$, for every $k \geqslant 1$, the proof is complete. ∎

COROLLARY 3.11. *If H is a finite-dimensional Hilbert space and $T \in \mathcal{E}$, then $\mathcal{A}_w(T)$ is a regular subspace of $B(H)$.*

Proof. Let $T \in \mathcal{E}$. By direct calculation or by using [4], it follows that $\mathcal{A}_w(T)$ is similar to $\mathcal{D}_n$, the algebra of all $n \times n$-matrices. But, from [23], Proposition 3.2 ((i) and (ii)), it is easy to deduce that $\mathcal{D}_n$ is a regular subspace. Thus, a direct application of Proposition 3.10 leads to the conclusion that $\mathcal{A}_w(T)$ is regular. ∎

REMARK 3.12. From Theorem 3.8 and Corollary 3.11, it follows that we could use Proposition 3.4 above to prove that, in case H is a finite-dimensional Hilbert space, κ is continuous at any point $T \in \mathcal{E}$, i.e., to prove $\frac{1}{4}$ of Theorem 1.6. But one should compare the direct proof given in Theorem 1.6 with the (indirect) one using two non-trivial results, Proposition 3.4 and Theorem 3.8. Also, we should keep in mind that the remaining $\frac{3}{4}$ of Theorem 1.6 seems independent of the results in [23].

Acknowledgements. The results contained in this paper constitute part of the author's Ph.D. Dissertation ([16]). The author wishes to express her thanks to professor D. W. Hadwin for his very helpful discussions on these and related subjects (and for his ∞ enthusiasm "vis-à-vis de mathématiques").

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ILEANA IONAŞCU
Pure Mathematics Department
University of Waterloo
Waterloo, Ontario, N2L 3G1
CANADA

and

The Fields Institute
180 Columbia St. West
Waterloo, Ontario N2L 5Z5
CANADA

On leave from:

The Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764
RO-70700 Bucharest
ROMANIA

Received December 7, 1993; revised June 23, 1994.