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A DESCRIPTION OF SPATIALLY PROJECTIVE VON NEUMANN ALGEBRAS

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ABSTRACT. Let $\mathcal{R}$ be a von Neumann algebra on the Hilbert space H. Then H, as a Banach left module over $\mathcal{R}$ with the multiplication $a \cdot x = a(x)$, is projective if and only if the following conditions are satisfied: 1) $\mathcal{R}$ is of type I; 2) the center of $\mathcal{R}$ is the weak-operator-closed linear span of its minimal projections; and 3) in the standard decomposition $\mathcal{R} = \sum\limits_{m,n} \mathcal{R}_{m,n}$, where $\mathcal{R}_{m,n}$ is a von Neumann algebra of type I_m with the commutant of type I_n, there is no non-zero summand for which both m and n are finite. The most difficult part of the proof is to show that H is not projective in the case of an infinite type I factor in the standard form.

As an application, it is shown that the indicated conditions on $\mathcal{R}$ characterize the class of von Neumann algebras with the property of vanishing their cohomology groups with coefficents in certain "operator" $\mathcal{R}$-bimodules.

KEYWORDS: *Projective Banach module, von Neumann algebra, spatial projectivity.*

AMS SUBJECT CLASSIFICATION: Primary 46; Secondary 18.

INTRODUCTION: THE FORMULATION OF THE MAIN RESULTS AND SOME DISCUSSION

Let A be a Banach algebra. Recall that a left Banach A-module P is called *projective*, if every continuous morphism of left Banach A-modules with P as codomain has a right-inverse continuous A-module morphism, provided that it has a right-inverse continuous linear operator. For the equivalent definition in terms of cohomology groups, and for other alternative definitions, see e.g. [7] or [8].

Now suppose that E is a Banach space and that A is some Banach algebra of operators acting on E. Consider E as a left Banach A-module with the outer multiplication ax defined as $a(x)$ for all $a \in A, x \in E$.

DEFINITION. The algebra A is called *spatially projective* if the module E is projective.

Which operator algebras are spatially projective? Until recently, the available information was rather scanty. Apart from the trivial case of finite-dimensional semi-simple algebras, it was known that an operator algebra is spatially projective provided it contains a column of 1-dimensional operators (i.e. the set of all operators of the form $f(\cdot)x$, where $x \in E$ is fixed and f runs through E^*.) On the other hand, some counter-examples were known: these included certain nest algebras [4], and also $C[0,1]$ naturally represented on $L^2[0,1]$.

In the present paper, we concentrate on the 'classical' case of von Neumann algebras, that is on those operator algebras on Hilbert spaces which are self-adjoint and weak-operator closed. To formulate the result, let us recall the following known fact of their structure theory. Every von Neumann algebra $\mathcal{R}$ of type I acting on a Hilbert space H, has a unique decomposition into a direct sum of von Neumann algebras $\mathcal{R}_{m,n}$, where m and n are cardinalities not exceeding $\dim H$, such that $\mathcal{R}_{m,n}$ is of type I_m and its commutant is of type I_n (see e.g. [10], Section 9.3).

MAIN THEOREM. *A von Neumann algebra $\mathcal{R}$ is spatially projective if and only if it has the following properties:*

(i) *$\mathcal{R}$ is of type I;*

(ii) *the centre of $\mathcal{R}$ is the weak-operator-closed linear span of its minimal projections (in other words, the centre is isometrically $*$-isomorphic to the algebra $\ell^\infty(M)$ for some set M);*

(iii) *there is no non-zero summand $\mathcal{R}_{m,n}$ (see above) for which both m and n are infinite.*

Let us remark that the properties (i) and (ii) combined are equivalent to the property that $\mathcal{R}$ itself is the weak-operator-closed span of its minimal projections ([10], Exercise 6.9.37). (In what follows, such an algebra will be called *atomic*). Further, the property (ii) easily implies that $\mathcal{R}$ has a unique decomposition into a direct sum of a family of factors. Therefore the theorem could also be formulated in the following way:

A von Neumann algebra is spatially projective if and only if it is atomic, and its decomposition into a direct sum of factors contains no infinite factor having infinite commutant.

(In particular, if the algebra has, as a direct summand, an infinite factor in the standard form, it is not spatially projective).

The solved question, apart form its independent interest (when do such important algebras have such an important homological property?), is connected with the following known problem. As usual, we consider $\mathcal{B}(H)$, the algebra of all bounded operators on a Hilbert space H, as a Banach bimodule over an arbitrary Banach algebra of operators on H. Is it true that the 1-dimensional cohomology group $\mathcal{H}^1(\mathcal{R}, \mathcal{B}(H))$ vanishes for every von Neumann algebra $\mathcal{R}$ on H? In other words, is it true that every continuous derivation of such an $\mathcal{R}$ with values in $\mathcal{B}(H)$ is inner? This problem was raised by Christensen, who established its connections with various questions concerning operator algebras (in particular, with the similarity problem in representation theory). He also managed to obtain a positive answer for several important classes of algebras (see [1] and references therein).

In its full generality, the problem of Christensen still remains open. What will happen, however, if we consider a more burdensome cohomological condition on $\mathcal{R}$, namely, the vanishing of $\mathcal{H}^1(\mathcal{R}, \mathcal{B}(H, X))$ for all left Banach $\mathcal{R}$-modules X? (Recall that the space $\mathcal{B}(Y, X)$ of all bounded operators between left Banach modules Y and X has a structure of a Banach $\mathcal{R}$-bimodule with the outer multiplications defined by $(a \cdot \varphi)(x) = a \cdot \varphi(x)$ and $(\varphi \cdot a)(x) = \varphi(ax)$ for $a \in \mathcal{R}, \varphi \in \mathcal{B}(Y, X)$ and $x \in Y$ ([7], [8]). In the case $Y = X = H$, we have the already mentioned bimodule $\mathcal{B}(H)$).

As it turns out, the answer to such a 'more rigid' question is in the negative. The above formulated theorem, combined with standard facts of Banach homology ([7], Chapter III; [8] Chapter VII), immediately implies the following:

COROLLARY. *Let $\mathcal{R}$ be a von Neumann algebra on H. Then:*

(i) *if $\mathcal{R}$ satisfies the conditions of the main theorem, we have $\mathcal{H}^n(\mathcal{R}, \mathcal{B}(H, X)) = 0$ for all left Banach $\mathcal{R}$-modules X and all $n > 0$;*

(ii) *if $\mathcal{R}$ fails to satisfy these conditions, then there exists a left Banach $\mathcal{R}$-module X for which $\mathcal{H}^1(\mathcal{R}, \mathcal{B}(H, X)) \neq 0$ (or, equivalently, for which there exists an outer derivation on $\mathcal{R}$ with values in $\mathcal{B}(H, X)$.)*

Now we proceed to the proof of the main theorem. First, however, we need to discuss some notations. Our argument will use, sometimes simultaneously, different types of tensor products of Banach spaces. The *projective, injective (or weak)* and the *Hilbert* tensor products will be denoted respectively by the symbols $\widehat{\otimes}, \check{\otimes}$ and $\dot{\otimes}$. To avoid a possible ambiguity, we shall use the same symbols for elementary tensors in the respective spaces (e.g. $x \dot{\otimes} y \in H' \dot{\otimes} H''$ or $a \widehat{\otimes} x \in \mathcal{R} \widehat{\otimes} H$) and also for representative types of tensor products of operators (e.g. $a \dot{\otimes} b$ acting on $H'\dot{\otimes}H''$&c). The inner product on a Hilbert space is always denoted by $\langle\cdot,\cdot\rangle$. The identity operator on E is denoted by $\mathbf{1}_E$ (or simply $\mathbf{1}$ if it is clear which E is intended).

In what follows, we essentially use the following standard method to check the projectivity of a given left Banach module P over a unital Banach algebra A. Consider the so-called *canonical surjection* $\pi_P : A\widehat{\otimes}P \to P : a\widehat{\otimes}x \to a\cdot x (a \in A, x \in P)$. Then P is *projective* if and only if π_P has a right-inverse continuous A-module morphism.

1. THE BEGINNING OF THE PROOF: THE IMPORTANCE OF BEING ATOMIC

Let $\mathcal{R}$ be a von Neumann algebra on H. Denote the (lattice-theoretic) union of all its minimal projections by p_a, and put $p_c = \mathbf{1} - p_a$, $H_a = \operatorname{Im} p_a$ and $H_c = \operatorname{Im} p_c$. It is easy to see that the property $H_a = H$ (i.e. that $H_c = (0)$) means just that $\mathcal{R}$ is atomic.

Consider the free Banach left $\mathcal{R}$-module with basic space E: in other words, the module $\mathcal{R}\widehat{\otimes}E$ with the outer multiplication well-defined by $a\cdot(b\widehat{\otimes}y) = ab\widehat{\otimes}y$ $(a, b \in \mathcal{R}, y \in E)$ (see e.g. [8], Chapter VI).

THEOREM 1. *Suppose that at least one of the Banach spaces E and $\mathcal{R}$ has the approximation property. Then every continuous $\mathcal{R}$-module morphism $\varphi : H \to \mathcal{R}\widehat{\otimes}E$ is equal to zero on H_c.*

Proof. Assume, on the countrary, that we have $\varphi(x) \neq 0$ for some fixed $x \in H_c$. The following argument, which leads us to a contradiction, is divided into several stages. The first stage uses an observation actually due to Selivanov (cf. [7], Proposition 4.4).

(i) *There exists a continuous $\mathcal{R}$-module morphism $\psi : H \to \mathcal{R}$ such that $\psi(x) \neq 0$.*

Consider the 'canonical' operator $\mathcal{R}\widehat{\otimes}E \to \mathcal{R}\check{\otimes}E : a\widehat{\otimes}y \to a\check{\otimes}y (a \in \mathcal{R}, y \in E)$. The condition concerning the approximation property implies that the operator is injective (see e.g. [2], Theorem 3.4 a) $\Rightarrow$ b) and 3.5 a) $\Rightarrow$ b)). According to the definition of the norm in $\mathcal{R}\check{\otimes}E$, it now follows that there exist functionals $f \in \mathcal{R}^*$ and $g \in E^*$ such that $f\widehat{\otimes}g(\varphi(x)) \neq 0$. Therefore, since $f\widehat{\otimes}g = (f\widehat{\otimes}\mathbf{1})(\mathbf{1}\widehat{\otimes}g)$, there exists $g \in E^*$ such that the generator $\mathbf{1}\widehat{\otimes}g : \mathcal{R}\widehat{\otimes}E \to \mathcal{R} \cong \mathcal{R}\widehat{\otimes}\mathbb{C}$, which is well-defined by $(\mathbf{1}\widehat{\otimes}g)(a\widehat{\otimes}y) = g(y)a$, sends $\varphi(x)$ to a non-zero operator. It remains to notice that $\mathbf{1}\widehat{\otimes}g$ is a continuous morphism of $\mathcal{R}$-modules, and hence that $\psi := (\mathbf{1}\widehat{\otimes}g)\varphi$ has the desired properties.

Now put $a = \psi(x)$ and denote the projection onto $\operatorname{Im} a$ (which certainly belongs to $\mathcal{R}$) by p.

(ii) *There exists a sub-projection p' of p, belonging to $\mathcal{R}$, and $\theta > 0$, such that we have $\|qa\| > \theta$, for every sub-projection q of p' that belongs to $\mathcal{B}(H)$.*

Consider the polar decomposition $a = H\nu$, where $h = \sqrt{aa^*}$ is a positive operator and ν is a partial isometry. Then $\|a\| = \|h\|$ and $\operatorname{Im} a = \operatorname{Im} h$, so that $\operatorname{Im} h$ is the range of p; furthermore, ν and h, and hence all the resolution of the identity for h, belongs to $\mathcal{R}$. The spectral theorem implies that there exist $\theta > 0$ and a projection $p' \in \mathcal{R}$ such that $\theta p' \leqslant h$ and $p' \leqslant p$ (one can take, for example, every θ with $0 < \theta < \|h\|$ and $p' = 1 - p_\theta$, where $p_\lambda (\lambda \in \mathbf{R})$ is the resolution of the identity for h.)

Now suppose that $q \in \mathcal{B}(H)$ is a sub-projection of p'. Then

$$\begin{aligned}\|qa\| = \sqrt{\|(qa)(qa)^*\|} = \sqrt{\|qaa^*q\|} = \sqrt{\|qh^2q\|} = \sqrt{\|(qh)(qh)^*\|} \\ = \|qh\| = \|qh\|\|q\| \geqslant \|qhq\|.\end{aligned}$$

At the same time, it follows from $\theta p' \leqslant h$ that $qhq \geqslant q\theta p' q = \theta q$. Hence $\|qhq\| \geqslant \|\theta q\| = \theta$, and we have the desired inequality.

(iii) *The end of the proof.*

Since $x \in H_c$ we have $x = p_c(x) = p_c \cdot x$. Therefore, since ψ is a morphism of $\mathcal{R}$-modules, we have $a = \psi(x) = p_c \cdot \psi(x) = p_c a$. Hence $\operatorname{Im} a \subseteq H_c$, that is $p \leqslant p_c$ and thus $p' \leqslant p_c$. But it follows from the definition of p_c that it has no minimal sub-projections in $\mathcal{R}$. This obviously implies that for each natural number N, p' has the form $\sum_1^N q_i$ where q_i are non-zero, mutually orthogonal projections in $\mathcal{R}$. From this

$$\|x\|^2 \geqslant \|p'(x)\|^2 = \sum_{i=1}^{N} \|q_i(x)\|^2.$$

It follows that there exists $j \in \{1, \ldots, N\}$ such that $\|q_j(x)\|^2 \leqslant \|x\|^2/N$. Using, again, that ψ is a morphism, we have that $\psi(q_j(x)) = \psi(q_j \cdot x) = q_j \cdot \psi(x) = q_j a$. Hence, the previous inequality implies that

$$\|q_j a\| = \|\psi(q_j(x))\| \leqslant \|\psi\| \, \|q_j(x)\| \leqslant \left(\frac{\|\psi\|}{\sqrt{N}}\right) \|x\|.$$

At the same time, according to what was proved in (2), we have $\|q_i a\| \geqslant \theta$ for all $i = 1, \ldots, N$. Since N can be closed arbitrarily, we have come to a contradiction. ∎

2. CONTINUATION: THE SUFFICIENT CONDITION

THEOREM 2. *Let $\mathcal{R}$ be a von Neumann algebra, on a Hilbert space H, which has a decomposition into a direct sum of type-I factors. Suppose further that each of these summands has a least one of the following two properties: either the factor itself or its commutant is finite. Then $\mathcal{R}$ is spatially projective.*

Proof. Let us recall that the conditions of the theorem, in its detailed form, mean the following. There exists a family $p_\alpha (\alpha \in \Lambda)$ of the central projections of $\mathcal{R}$ such that H decomposes into a Hilbert direct sum of spaces $H_\alpha := p_\alpha H$. Furthermore, every H_α can be represented as a Hilbert tensor product $H'_a \dot{\otimes} H''_a$, such that the algebra $\mathcal{R}_\alpha := p_\alpha \mathcal{R}$, being considered as an operator algebra on H_α, has the form $\mathcal{B}(H'_a) \dot{\otimes} \mathbf{C}\mathbf{1}$; in other words, it consists of operators $a \dot{\otimes} \mathbf{1}$, where a runs through $\mathcal{B}(H'_a)$. Finally, the last condition of the theorem means that the cardinality $n(\alpha) := \min\{\dim H'_a, \dim H''_a\}$ is finite for every $\alpha \in \Lambda$.

According to what was said in the introduction, it is sufficient to show that the canonical surjection $\pi_H : \mathcal{R} \widehat{\otimes} H \to H$ has a right inverse that is a continuous morphism of $\mathcal{R}$-modules, say ρ. Moreover, it will turn out that we can construct ρ so that $\|\rho\| = 1$.

For every $\alpha \in \Lambda$, we choose an orthonormal set $e_1^\alpha, \dots, e_{n(\alpha)}^\alpha$ in H'_a and an orthonormal basis $e_m^\alpha (m \in I_\alpha)$ in H''_a (the cardinality of the index set I_α is, of course, $\dim H''_a$). In what follows, we write for brevity $e_{i,m}^\alpha = e_i^\alpha \dot{\otimes} e_m^\alpha \in H_\alpha$. Further, for every $\alpha \in \Lambda$ and vectors $x_1, x_2 \in H'_a$ we introduce the operator on H which acts as the zero operator on $H \ominus H_\alpha$ and is well-defined by $x' \dot{\otimes} y' \mapsto \langle x', x_1 \rangle x_2 \dot{\otimes} y' (x' \in H'_a, y' \in H''_a)$ on $H_\alpha = H'_a \dot{\otimes} H''_a$; (in other words, it acts on H_α as the tensor product of the one-dimensional operator $x' \mapsto \langle x', x_1 \rangle x_2$ on H'_a and the identity operator on H''_a). We denote this operator by $x_1 \diamond x_2$; it certainly belongs to $\mathcal{R}$ and is a partial isometry provided $\|x_1\| \|x_2\| = 1$. It is obvious that for any $a \in \mathcal{R}$ with the restriction $b \dot{\otimes} \mathbf{1}$ $(b \in \mathcal{B}(H'_a))$ on H_a we have $a(x_1 \diamond x_2) = x_1 \diamond b(x_2)$; in particular, $x_1 \diamond x_2$ depends linearly on x_2. Observe also that $(x_1 \diamond x_2)^* = x_2 \diamond x_1$.

Consider the set H_{00} of vectors $x \dot{\otimes} e_m^\alpha \in H^\alpha \subseteq H$, for all possible $\alpha \in \Lambda, x \in H'_a$ and $m \in I_\alpha$. Define the map $\rho_{00} : H_{00} \to \mathcal{R} \widehat{\otimes} H$ by

$$\rho_{00}(x \dot{\otimes} e_m^\alpha) = \frac{1}{n(\alpha)} \sum_{i=1}^{n(\alpha)} (e_i^\alpha \diamond x) \widehat{\otimes} e_{i,m}^\alpha . \tag{1}$$

Let H_0 be the dense subspace of H formed by arbitrary sums of vectors belonging to H_{00}. It is obvious that ρ_{00} has a unique extension to a linear operator $\rho_0 : H_0 \to \mathcal{R}\widehat{\otimes}H$. Observe that the definition of ρ_{00} easily implies the equations

$$\rho_0(\alpha \cdot \zeta) = \alpha \cdot \rho_0(z) \tag{3}$$

and

$$\pi_H \rho_0(z) = z \tag{4}$$

for every $a \in \mathcal{R}, z \in H_0$.

Let us fix, for a time, some $z \in H_0$. Obviously there are different indices $\alpha(j) \in \Lambda, j = 1, \ldots, M$ such that $z = \sum_{j=1}^{M} z_j$ with $z_j \in H_0 \cap H_{\alpha(j)}$. These summands, in their turn, have the form

$$z_j = \sum_{k=1}^{N(j)} x_{j,k} \dot{\otimes} \dot{e}^j_{m(k)} \tag{5}$$

where $x_{j,k} \in H'_{\alpha(j)}$ and $m(k)$ $(1 \leqslant k \leqslant N(j))$ are different indices in $I_{\alpha(j)}$. (Here, and in what follows, we write the upper index j instead of $\alpha(j)$ in extensions like $e^j_{m(k)}, e^j_l$, etc.).

We now want to rewrite $\rho_0(z) \in \mathcal{R}\widehat{\otimes}H$ in a form more convenient for the estimation of its norm. For every $j = 1, \ldots, M$ we choose an orthonormal basis, say $\hat{e}^j_1, \ldots, \hat{e}^j_{K(j)}$ in the linear span of the vectors $x_{j,1}, \ldots, x_{j,N(j)}$. Observe that $K(j)$, that is the dimension on this span, does not exceed both of $\dim H''_{\alpha(j)}$ (i.e. the cardinality of $I_{\alpha(j)}$) and $\dim H'_{\alpha(j)}$; hence $K(j) \leqslant n(j)$ (where, for brevity, we write $n(j)$ instead of $n(\alpha(j))$.)

Decompose every $x_{j,k}$ with respect to the chosen basis. Taking into account the linearity of the operation '$\diamond$' on the second variable, we have

$$\begin{aligned}
\rho_0(z_j) &= \sum_{k=1}^{N(j)} \frac{1}{n(j)} \sum_{i=1}^{n(j)} (e^j_i \diamond x_{j,k}) \widehat{\otimes} e^j_{i,m(k)} \\
&= \frac{1}{n(j)} \sum_{k=1}^{N(j)} \sum_{i=1}^{n(j)} \sum_{l=1}^{K(j)} \langle x_{j,k}, \hat{e}^j_l \rangle (e^j_i \diamond \hat{e}^j_l) \widehat{\otimes} e^j_{i,m(k)} \\
&= \frac{1}{n(j)} \sum_{i=1}^{n(j)} \sum_{l=1}^{K(j)} (e^j_i \diamond \hat{e}^j_l) \widehat{\otimes} \Big(\sum_{k=1}^{N(j)} \langle x_{j,k}, \hat{e}^j_l \rangle e^j_{i,m(k)} \Big) .
\end{aligned}$$

Observe that

$$(\hat{e}^j_l \diamond e^j_i) \cdot (x_{j,k} \dot{\otimes} e^j_{i,m(k)}) = \langle x_{j,k}, \hat{e}^j_l \rangle e^j_{i,m(k)}, \quad \text{and} \quad (\hat{e}^j_l \diamond e^j_i) \cdot z_r = 0 \quad \text{if} \quad j \neq r.$$

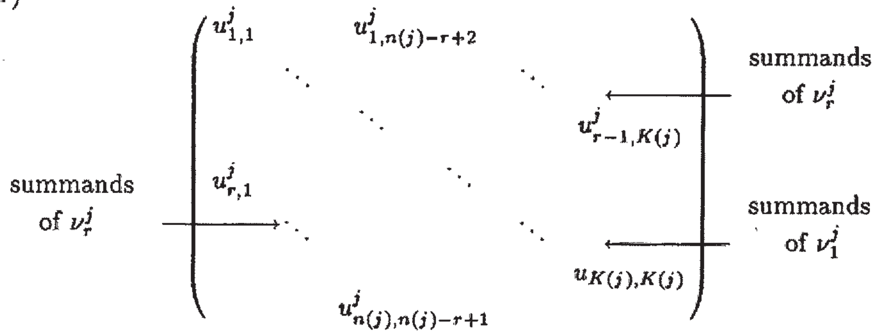
From this and (5), we have that

$$\rho_0(z) = \sum_{j=1}^{M} \rho_0(z_j) = \sum_{j=1}^{M} \frac{1}{n(j)} \sum_{i=1}^{n(j)} \sum_{l=1}^{K(j)} u^j_{i,l} \tag{6}$$

where $u^j_{i,l}$ is a brief notation for $(e^j_i \Diamond \hat{e}^j_l)\widehat{\otimes}(\hat{e}^j_l \Diamond e^j_i) \cdot z \in \mathcal{R}\widehat{\otimes}H$.

Regroup the summands in (6) as follows. At first, for every j and $r = 1, \ldots, n(j)$, we denote by ν^j_r the sum of all $u^j_{i,l}$ with $i - l = (r-1) \bmod n(j)$. The figure (7) shows the elements $u^j_{i,l}$ forming an $n(j) \times K(j)$ matrix:

(7)



Then we put $\nu(r_1, \ldots, r_M) = \nu^1_{r_1} + \cdots + \nu^M_{r_M}$ for every tuple $(r_1, \ldots, r_M)$ where $1 \leqslant r_j \leqslant n(j), j = 1, \ldots, M$. It is obvious that

$$\rho_0(z) = \frac{1}{n(1) \ldots n(M)} \sum_{(r_1, \cdots, r_M)} \nu(r_1, \ldots, r_M) \tag{8}$$

where the sum contains just $n(1) \cdots n(M)$ summands which run over all possible tuples of the indicated form.

Choose an arbitrary element, say ν, among these summands, Our problem now is to show that $\|\nu\| \leqslant \|z\|$. We see that ν is a sum of the form

$$\nu = \sum_{s=1}^{S} (e_s \Diamond \hat{e}_s)\widehat{\otimes}(\hat{e}_s \Diamond e_s) \cdot z,$$

where for every s, the vectors e_s, $\hat{e}_s$ belongs to the same H'_α (where $\alpha \in \Lambda$ and depends on s) and where $s' \neq s''$ implies that $e_{s'} \perp e_{s''}$ and $\hat{e}_{s'} \perp \hat{e}_{s''}$, provided that all four vectors belongs to the same H'_α. It follows that the partial isometries $e_s \Diamond \hat{e}_s (s = 1, \ldots, S)$ have mutually orthogonal initial, as well as final, subspaces. This easily implies that every linear combination

$$\sum_{s=1}^{S} \lambda_s (e_s \Diamond \hat{e}_s) \qquad (\lambda_s \in \mathbf{C}),$$

has $\max\{|\lambda_s| : s = 1, \ldots, S\}$ as its operator norm.

Now we use a certain trick from ([6], Lemma 4.1). Obvious calculations lead to the equality
$\nu = \frac{1}{S}\sum_{s=1}^{S} w_s \widehat{\otimes} w_s^* \cdot z$, where $w_s = \sum_{t=1}^{S} \zeta^{st}(e_t \Diamond \hat{e}_t)$ and ζ is a primitive S-th root of 1. It follows from what said in the previous passage that $\|w_s\| = \|w_s^*\| = 1$. But the definition of the projective norm in $\mathcal{R}\widehat{\otimes}H$ implies that

$$\|\nu\| \leqslant \frac{1}{S}\sum_{s=1}^{S} \|w_s\| \, \|w_s^* \cdot z\| \leqslant \frac{1}{S}\sum_{s=1}^{S} \|w_s\| \, \|w_s^*\| \, \|z\|,$$

and we have the desired estimate $\|\nu\| \leqslant \|z\|$. From this, in virtue of (8), we see that $\|\rho_0(z)\| \leqslant \|z\|$ for all $z \in H_0$. Combined with the obvious equation that $\|\rho_0(z)\| = \|z\|$ for all $z \in H_{00}$, we have that $\|\rho_0\| = 1$.

Since H_0 is dense in H, ρ_0 has a unique extension to a linear operator $\rho : H \to \mathcal{R}\widehat{\otimes}H$ of the same norm. Moreover, the equality (3), combined with the continuity of ρ, immediately implies that ρ is a morphism of $\mathcal{R}$-modules, and (4) similarly that it is a right inverse to the canonical surjection π_ρ. Thus, the map ρ with the desired properties is constructed and the theorem is proved. ∎

3. THE EXCEPTIONAL CASE: AN INFINITE TYPE-I FACTOR IN STANDARD FORM

We begin with several preparatory assertions; some of them, perhaps, have an independent interest. In what follows, H is a Hilbert space and $e \bigcirc x (e, x \in H)$ is the notation for the 1-dimensional operator $y \mapsto \langle y, e \rangle x (y \in H)$.

LEMMA 1. *Every continuous morphism of $\mathcal{B}(H)$-modules $\varphi : H \to \mathcal{B}(H)$ arises from an element $e \in H$ and acts as $x \mapsto e \bigcirc x$.*

Proof. Fix some non-zero $x \in H$ and consider the operator $\varphi(x)$. Take an arbitrary $z \in H$. Since φ is a morphism, we have

$$a \cdot [\varphi(x)(z)] = [a \cdot \varphi(x)](z) = [\varphi(a \cdot x)](z),$$

for every $a \in \mathcal{B}(H)$, and thus $a \cdot x = 0$ implies that $a \cdot [\varphi(x)(z)] = 0$. Hence, all vectors in the image of $\varphi(x)$ are proportional to x. As is well known, that means that $\varphi(x) = e \bigcirc x$ for some $e \in H$.

It remains to take an arbitrary $x' \in H$ and to choose $b \in \mathcal{B}(H)$ with $b \cdot x = x'$. Then

$$\varphi(x') = \varphi(b \cdot x) = b \cdot \varphi(x) = b \cdot (e \bigcirc x) = e \bigcirc (b \cdot x) = e \bigcirc x'. \quad ∎$$

Now let us consider, apart from H, an arbitrary Banach space E and the projective tensor product $E\widehat{\otimes}H$. Let $\{e_m : m \in I\}$, where I is an index set, be an orthonormal basis in H. For every m consider the functional $e_m^* : H \to \mathbb{C}$, given by $x \mapsto \langle x, e_m\rangle$, and the operator $\sigma_m : E\widehat{\otimes}H \to E$ which is the composition of $\mathbf{1}\widehat{\otimes}e_m^* : E\widehat{\otimes}H \to E\widehat{\otimes}\mathbb{C}$ and the canonical identification $E\widehat{\otimes}\mathbb{C} \cong E$. In other words, if $u \in E\widehat{\otimes}H$ is represented as $\sum_{k=1}^{\infty} y_k\widehat{\otimes}x_k$, $\sum_{k=1}^{\infty} \|y_k\|\,\|x_k\| < \infty$ (cf. [8], Section 0.3.3) we have

$$\sigma_m(u) = \sum_{k=1}^{\infty}\langle x_k, e_m\rangle y_k \tag{9}$$

LEMMA 2. *For every* $u \in E\widehat{\otimes}H$, *we have* $u = \sum_{m=I} \sigma_m(u)\widehat{\otimes}e_m$.

(In more detail, if $\lambda = (m_1, \ldots, m_k)$ runs through the family of all finite subsets of I, directed by the inclusion relation, then the net

$$u_\lambda = \sigma_{m_1}(u)\widehat{\otimes}e_{m_1} + \cdots + \sigma_{m_k}(u)\widehat{\otimes}e_{m_k}$$

converges to u.)

Proof. With λ as above, let q_λ be the projection of H onto the linear span of the vectors $e_{m_1}, \ldots, e_{m_k}$. It easily follows from (9), that $u_\lambda = (\mathbf{1} \otimes q_\lambda)u$ for all $u \in E\widehat{\otimes}H$. Furthermore, if u is an elementary tensor, say $u = y \otimes x$, then $(\mathbf{1}\otimes q_\lambda)(u) = y\otimes q_\lambda x$, and the Fourier expansion of x with respect to $\{e_m : m \in I\}$ implies that $\lim_\lambda u_\lambda = \lim_\lambda(\mathbf{1} \otimes q_\lambda)u = u$. Since $\|\mathbf{1} \otimes q_\lambda\| = \|q_\lambda\| = 1$, for all λ, and thus the family $\mathbf{1} \otimes q_\lambda$ is uniformly bounded, we have the same convergence for every u in the closure of the linear span of the elementary tensors, that is for all $u \in E\widehat{\otimes}H$. ∎

LEMMA 3. *For every* $u \in E\widehat{\otimes}H$, *we have* $\sum_{m=I} \|\sigma_m(u)\|^2 \leqslant \|u\|^2$.

Proof. Consider the representation of u as a convergent series, which was used in (9); we can always assume that $\|y_k\| = \|x_k\| (k = 1, 2, \ldots)$. Then (9) implies that

$$\begin{aligned}\sum_m \|\sigma_m(u)\|^2 &\leqslant \sum_m \Big(\sum_{k=1}^{\infty} |\langle x_k, e_m\rangle|\,\|y_k\|\Big)^2 \\ &\leqslant \sum_m \Big[\Big(\sum_{k=1}^{\infty} |\langle x_k, e_m\rangle|^2\Big)\Big(\sum_{k=1}^{\infty} \|y_k\|^2\Big)\Big] \\ &\leqslant \Big(\sum_{k=1}^{\infty} \|y_k\|^2\Big)\Big(\sum_{k=1}^{\infty} \|x_k\|^2\Big) = \Big(\sum_{k=1}^{\infty} \|y_k\|\,\|x_k\|\Big)^2.\end{aligned}$$

It remains to recall that

$$\|u\| = \inf \sum_{k=1}^{\infty} \|y_k\| \, \|x_k\|,$$

where the lower bound is taken over all possible representations of u in the indicated form. ∎

THEOREM 3. *Let $\mathcal{R}$ be an infinite type-I factor on H with an infinite commutant. Then the left Banach $\mathcal{R}$-module H is not projective.*

Proof. We assume the contrary. Then in accordance with what was said in the introduction, there exists a continuous morphism of $\mathcal{R}$-modules, $\rho : H \to \mathcal{R}\widehat{\otimes}H$ such that $\pi_H \rho$ is the identity morphism of H.

By the hypothesis, H can be represented in the form $H = H'\dot{\otimes}H''$, such that $\mathcal{R} = \{a \otimes \mathbf{1} : a \in \mathcal{B}(H')\}$. Let $\{e'_i : i \in \Lambda'\}$ and $\{e''_i \in \Lambda''\}$ be orthonormal bases in H' and H'' respectively. Then $\{e_{i,j} := e'_i\dot{\otimes}e''_j : (i,j) \in \Lambda' \times \Lambda''\}$ is an orthonormal basis of H.

Consider, for every triple $(i \in \Lambda', j, k \in \Lambda'')$ the chain of operators

$$H' \xrightarrow{\alpha_j} H \xrightarrow{\rho} \mathcal{R}\widehat{\otimes}H \xrightarrow{\sigma_{i,k}} \mathcal{R} \xrightarrow{\sim} \mathcal{B}(H'),$$

where $\alpha_j : x \mapsto x\dot{\otimes}e''_j$, ρ is our hypothetical morphism, $\sigma_{i,k}$ is a special case of $\sigma_m : E\widehat{\otimes}H \to E$ (see above) with $\mathcal{R}$ as E and $e_{i,k}$ as e_m, and finally, the right arrow denotes the isomorphism $a \otimes \mathbf{1} \to a$. It is easy to check that the composition of these operators is a continuous morphism of $\mathcal{B}(H')$-modules. Thus, it follows from Lemma 1 that it acts as $x \to y^j_{i,k} \bigcirc x$, for some $y^j_{i,k} \in H'$. This immediately implies that $\sigma_{i,k}(\rho(x\dot{\otimes}e''_j)) = y^j_{i,k} \Diamond x$, where the latter is the notation for the operator $(y^j_{i,k} \bigcirc x)\dot{\otimes}\mathbf{1}$ (cf. the proof of Theorem 2).

Thus, starting with ρ, we have arrived at a certain family of elements $y^j_{i,k} \in H'$. It follows from Lemma 2 that

$$\rho(x\dot{\otimes}e''_j) = \sum_{i,k} (y^j_{i,k} \Diamond x)\widehat{\otimes}e_{i,k} \tag{10}$$

for every $x \in H'$ and $j \in \Lambda''$.

The further argument will be divided into several stages. To speak informally, in the first three stages we fix $i \in \Lambda'$ and consider the 'matrix' $y^j_{i,k} \in H'(j, k \in \Lambda'')$. We show that the entries of this matrix converge weakly to zero in the following three directions: 'along the diagonal', 'down' and 'to the right'. As usual, we write $\lim\limits_k \lambda_k = 0$ for a family $\{\lambda_k \in \mathbb{C} : k \in \Lambda''\}$, if for every $\varepsilon > 0$, the set $\{k \in \Lambda'' : |\lambda_k| \geqslant \varepsilon\}$ is finite.

(i) *For any fixed* $i_0, m_0 \in \Lambda'$ *we have* $\sum_{j\in\Lambda''} |\langle e'_{m_0}, y^j_{i_0,j}\rangle|^2 \leqslant \|\rho\|^2$, *and hence* $\lim_j \langle e'_{m_0}, y^j_{i_0,j}\rangle = 0$.

To prove this it is sufficient to show that

$$\sum_{n=1}^{\infty} |\langle e'_{m_0}, y^{j_n}_{i_0,j_n}\rangle|^2 \leqslant \|\rho\|^2,$$

for any given sequence $j_n \in \Lambda''$, $n = 1, 2, \ldots$.

Choose an arbitrary sequence $i_n \in \Lambda'$, $n = 1, 2, \ldots$, and put

$$\mathcal{F} : \mathcal{R} \otimes H \to \mathbf{C} : (a\dot{\otimes}\mathbf{1}, z) \mapsto \sum_{n=1}^{\infty} \langle a \cdot e'_{m_0}, e'_{i_n}\rangle \langle z, e_{i_0,j_n}\rangle.$$

It follows, from the Cauchy and Bessel inequalities that

$$\begin{aligned} |\mathcal{F}(a\dot{\otimes}\mathbf{1}, z)|^2 &\leqslant \sum_{n=1}^{\infty} |\langle a \cdot e'_{m_0}, e'_{i_n}\rangle|^2 \sum_{n=1}^{\infty} |\langle z, e_{i_0,j_n}\rangle|^2 \\ &\leqslant \|a \cdot e'_{m_0}\|^2 \|z\|^2 \leqslant \|a\dot{\otimes}\mathbf{1}\|^2 \|z\|^2. \end{aligned}$$

Hence, $\mathcal{F}$ is a bilinear functional with norm not exceeding 1, and hence it generates a functional F on $\mathcal{R}\widehat{\otimes}H$ with the same norm; put $f := F \circ \rho$. Since it is a continuous functional on H and since e_{i_l,j_l} $(l = 1, 2)$ is an orthonormal set in H, we have

$$\sum_{n=1}^{\infty} |f(e_{i_l,j_l})|^2 \leqslant \|f\|^2 \leqslant \|F\|^2 \|\rho\|^2 \leqslant \|\rho\|^2.$$

At the same time, it follows from (10) that, for any l, we have

$$\begin{aligned} f(e_{i_l,j_l}) &= F(\rho(e'_{i_l}\dot{\otimes}e''_{j_l}) = \sum_{i,k} F[(y^{j_l}_{i,k} \Diamond e'_{i_l}) \widehat{\otimes} e'_{i,k}] \\ &= \sum_{i,k} \Big(\sum_{n=1}^{\infty} \langle (y^{j_l}_{i,k} \circ e'_{i_l}) \cdot e'_{m_0}, e'_{i_n}\rangle \langle e_{i,k}, e_{i_0,j_n}\rangle \Big) \\ &= \sum_{i,k} \Big(\sum_{n=1}^{\infty} \langle e'_{m_0}, y^{j_l}_{i,k}\rangle \langle e'_{i_l}, e'_{i_n}\rangle \langle e_{i,k}, e_{i_0,j_n}\rangle \Big) \\ &= \langle e'_{m_0}, y^{j_l}_{i_0,j_l}\rangle. \end{aligned}$$

The rest is clear.

(ii) *For any fixed* $i_0, m_0 \in \Lambda'$ *and* $l_0 \in \Lambda''$, *we have* $\sum_{j\in\Lambda''} |\langle e'_{m_0}, y^j_{i_0,l_0}\rangle|^2 \leqslant$

$\|\rho\|^2$, *and hence* $\lim_j \langle e'_{m_0}, y^j_{i_0,l_0}\rangle = 0$.
Choose an arbitrary $x \in H'$, $\|u\| = 1$ and put

$$\mathcal{F} : \mathcal{R} \times H \to \mathbf{C} : (a\dot{\otimes}1, z) \mapsto \langle a \cdot e_{m_0}, x\rangle\langle z, e_{i_0,l_0}\rangle.$$

Obviously $\mathcal{F}$ is a bilinear functional of norm not exceeding 1; as such it generates a functional F on $\mathcal{R}\widehat{\otimes}H$, with the same norm; put $f := F\bigcirc\rho$. Since $\{x\otimes e''_j : j \in \Lambda''\}$ is an orthonormal set, we have

$$\sum_{j\in\Lambda''} |f(x\dot{\otimes}e''_j)|^2 \leqslant \|f\|^2 \leqslant \|\rho\|^2.$$

At the same time, we have from (10) that

$$f(x\dot{\otimes}e''_j) = F\Big(\sum_{i,k}(y^j_{i,k}\diamond x)\widehat{\otimes}e_{i,k}\Big) = \sum_{i,k}\langle e_{m_0}, y^j_{i,k}\rangle\langle x, x\rangle\langle e_{i,k}, e_{i_0,l_0}\rangle = \langle e_{m_0}, y^j_{i_0,j_0}\rangle,$$

and again, the rest is clear.

(iii) *For any fixed* $m_0, i_0 \in \Lambda'$ *and* $j_0 \in \Lambda''$ *we have* $\sum_{l\in\Lambda''} |\langle e'_{m_0}, y^{j_0}_{i_0,l}\rangle|^2 \leqslant \|\rho\|^2$, *and hence* $\lim_j \langle e'_{m_0}, y^{j_0}_{i_0,l}\rangle = 0$.
Choose an arbitrary $x \in H$, $\|x\| = 1$ and put $\mathcal{R}$ as E and $\rho(x\dot{\otimes}e''_{j_0})$ as u in Lemma 3. Then we have

$$\sum_{i,k} \|y^{j_0}_{i,k}\diamond x\|^2 = \sum_{i,k} \|\sigma_{i,k}(\rho(x\dot{\otimes}e''_{j_0}))\|^2 \leqslant \|\rho(x\dot{\otimes}e''_{j_0}\|^2 \leqslant \|\rho\|^2,$$

and it remains to observe that

$$|\langle e_{m_0}, y^{j_0}_{i_0,l}\rangle| \leqslant \|y^{j_0}_{i_0,l}\|,$$

for all $l \in \Lambda''$.

Notice that in the assertions of (i) - (iii), we could take an arbitrary continuous morphism from H into $\mathcal{R}\widehat{\otimes}H$ as ρ. Now the condition $\pi_H\bigcirc\rho = 1$ comes into action.

(iv) *We have*

$$\sum_{i\in\Lambda'} \langle e'_i, y^j_{ij}\rangle = 1 \tag{11}$$

for all $j \in \Lambda''$, *and*

$$\sum_{i\in\Lambda'} \langle e'_j, y^j_{il}\rangle = 0 \tag{12}$$

for all $j, l \in \Lambda''$, $j \neq l$.

It follows from (10) that

$$\begin{aligned} x\dot{\otimes}e''_j = \pi_H(\rho(x\dot{\otimes}e''_j)) &= \sum_{i,k} \pi_H[(y^j_{i,k}\Diamond x)\widehat{\otimes}e_{i,k}] \\ &= \sum_{i,k}(y^j_{i,k}\Diamond x)\cdot e_{i,k} = \sum_{i,k}\langle e'_i, y^j_{i,k}\rangle x\dot{\otimes}e''_k, \end{aligned}$$

for every $x \in H'$. It remains to use the mutual orthogonality of the vectors $x\dot{\otimes}e''_k$ $(k \in \Lambda'')$.

We now come to the crucial stage of the proof.

(v) ('Lacunization') *There exist sequences* $i_l \in \Lambda'$ *and* $j_l \in \Lambda''$ $(l = 1, 2, \ldots)$ *and a sequence of integers* $0 < r_1 < r_2 < \cdots$ *such that we have:*

$$\Big|\Big(\sum_{n=r_{l-1}+1}^{r_l} \langle e'_{i_n}, y^{j_l}_{i_n,j_l}\rangle\Big) - 1\Big| < \frac{1}{4^l} \tag{13}$$

for all $l = 1, 2, \ldots$, *and*

$$\Big|\Big(\sum_{n=r_{l-1}+1}^{r_l} \langle e'_{i_n}, y^{j_s}_{i_n,j_l}\rangle\Big)\Big| < \frac{1}{4^{\max\{s,l\}}} \tag{14}$$

for all $l, s = 1, 2, \ldots; l \neq s$.

We shall construct the desired indices and numbers by induction. As the first step we take an arbitrary j_1. It follows from (11) that we can choose r_1 and a set $i_1, i_2, \ldots, i_{r_1}$ such that

$$\Big|\Big(\sum_{n=1}^{r_1} \langle e'_{i_n}, y^{j_1}_{i_n,j_1}\rangle\Big) - 1\Big| < \frac{1}{4}\,.$$

Now suppose that for some positive integer l we already have $j_1, \ldots, j_l \in \Lambda''$, integers $r_1 < \cdots < r_l$ and $i_1, \ldots, i_{r_l} \in \Lambda'$, which satisfy the corresponding inequalities among (13) and (14). (We mean those $l \times l$ 'first' inequalities, where only these indices and numbers are involved.)

The assertions (i) - (iii) imply that we have

$$\lim_j \langle e_m, y^j_{m,j}\rangle = \lim_j \langle e_m, y^j_{m,j_s}\rangle = \lim_k \langle e_m, y^{j_s}_{m,k}\rangle = 0,$$

for all $s = 1, \ldots, l$ and $m = i_1, \ldots, i_{r_l}$. Since the number of these relations is finite, there exists $j_{l+1} \in \Lambda''$ such that all numbers $|\langle e'_m, y^{j_{l+1}}_{m,j_{l+1}}\rangle|$, $|\langle e'_m, y^{j_{l+1}}_{m,j_s}\rangle|$ and $|\langle e'_m, y^{j_s}_{m,j_{l+1}}\rangle|$ for all indicated s and m are less than $1/r_l 4^{l+1}$; this, in particular implies that

$$\Big|\Big(\sum_{n=r_{s-1}+1}^{r_s} \langle e'_{i_n}, y^{j_{l+1}}_{i_n,j_s}\rangle\Big)\Big| < \frac{1}{4^{l+1}} \quad \text{for} \quad s = 1, \ldots, l.$$

Furthermore, if we consider the index set $\Lambda' \setminus \{i_1, \ldots, i_{r_l}\}$, then it follows from (11), and (12) that

$$\Big|\Big(\sum_{i\in\Lambda'_l} \langle e'_i, y^{j_{l+1}}_{i,j_{l+1}}\rangle\Big) - 1\Big| \leqslant \sum_{n=1}^{r_l} |\langle e'_i, y^{j_{l+1}}_{i,j_{l+1}}\rangle| < \frac{1}{4^{l+1}},$$

and

$$\Big|\sum_{i\in\Lambda'_l} \langle e'_i, y^{j_s}_{i,j_{l+1}}\rangle\Big| \leqslant \sum_{n=1}^{r_l} |\langle e'_i, y^{j_s}_{i,j_{l+1}}\rangle| < \frac{1}{4^{l+1}},$$

for all $s = 1, \ldots, l$. Hence there exists a finite index set

$$i_{r_l+1}, i_{r_l+2}, \ldots, i_{r_{l+1}}$$

such that

$$\Big|\Big(\sum_{n=r_l+1}^{r_{l+1}} \langle e'_{i_n}, y^{j_{l+1}}_{i_n,j_{l+1}}\rangle\Big) - 1\Big| < \frac{1}{4^{l+1}},$$

and

$$\Big|\sum_{n=r_l+1}^{r_{l+1}} \langle e'_{i_n}, y^{j_s}_{i_n,j_{l+1}}\rangle\Big| < \frac{1}{4^{l+1}},$$

for the same s. Thus, we have $j_1, \ldots, j_{l+1} \in \Lambda''$, integers $r_1 < \cdots < r_{l+1}$ and $i_1, \ldots, i_{r_{l+1}}$ which satisfy the corresponding (now $(l+1) \times (l+1)$) relations from (13) and (14). The rest is clear.

(vi) *The end of the proof.* Let j_l etc. be as in (v). Put $k_l = j_l (l = 1, 2)$, provided $l \in \{r_{l-1}, r_{l-1}+1, \ldots, r_l\}$ ($l = 1, 2, \ldots$; here $r_0 := 0$). Furthermore, choose an arbitrary $x \in H'$, $\|x\| = 1$ and put

$$\mathcal{F} : \mathcal{R} \times H \to \mathbf{C} : (a \otimes 1, z) \mapsto \sum_{n=1}^{\infty} \langle a \cdot e'_{i_n}, x\rangle \langle z, e_{i_n,k_n}\rangle.$$

Since

$$|\mathcal{F}(a\otimes \mathbf{1}, z)|^2 \leqslant \sum_{n=1}^{\infty} |\langle e'_{i_n}, a^*\cdot x\rangle|^2 \sum_{n=1}^{\infty} \langle z, e_{i_n,k_n}\rangle|^2 \leqslant \|a^* x\|^2\|z\|^2 \leqslant \|a\otimes \mathbf{1}\|^2\|z\|^2,$$

$\mathcal{F}$ is a bilinear functional of norm not exceeding 1, and thus it generates a functional F on $\mathcal{R}\otimes H$ of the same norm. Hence, $f = F\bigcirc\rho$ is a functional on H with $\|f\| \leqslant \|\rho\|$, and therefore, taking into account that $\{x\dot{\otimes}e'_l\}$ is an orthonormal set in H, we have

$$\sum_{l=1}^{\infty} |f(x\dot{\otimes}e_{j_l})|^2 \leqslant \|\rho\|^2. \tag{15}$$

However, it follows from (10) that for every l we have

$$f(x\otimes e_{j_l}) = F\Big(\sum_{i,k}(y^{j_l}_{i,k}\Diamond x)\otimes e_{i,k}\Big) = \sum_{i,k}\Big(\sum_{n=1}^{\infty}\langle (y^{j_l}_{i,k}\bigcirc x)\cdot e'_{i_n}, x\rangle\langle e_{i,k}, e_{i_n,k_n}\rangle\Big)$$
$$= \sum_{i,k}\Big(\sum_{n=1}^{\infty}\langle e'_{i_n}, y^{j_l}_{i,k}\rangle\langle x,x\rangle\langle e_{i,k}, e_{i_n,k_n}\rangle\Big) = \sum_{n=1}^{\infty}\langle e'_{i_n}, y^{j_l}_{i_n,k_n}\rangle.$$

Since the special choice of k_n, we have from this that

$$f(x\dot{\otimes}e_{j_l}) = \sum_{s=1}^{\infty}\Big(\sum_s\Big), \quad \text{where} \quad \sum_s = \sum_{n=r_{s-1}+1}^{r_s} \langle e'_{i_n}, y^{j_l}_{i_n,j_s}\rangle.$$

Hence, the inequalities (13) and (14) imply that

$$|f(x\dot{\otimes}e_{j_l}) - 1| \leqslant \sum_{s=1}^{l-1}\Big|\sum_s\Big| + \Big|\sum_l -1\Big| + \sum_{s=l+1}^{\infty}\Big|\sum_s\Big|$$
$$\leqslant (l-1)\frac{1}{4^l} + \frac{1}{4^l} + \sum_{s=l+1}^{\infty}\frac{1}{4^s} \leqslant \sum_{s=1}^{\infty}\frac{1}{4^l} = \frac{1}{3}.$$

Thus $f(x\dot{\otimes}e_{j_l}| \leqslant \frac{2}{3}$ for all $l = 1, 2, \ldots$, and we have come to a contradiction of (15). The proof is complete. ∎

4. THE CONCLUDING ARGUMENT AND REMARKS

It follows from what was said in the introduction, that the conditions of Theorem 2 are just the conditions of the main theorem, in their detailed form. So we already have a proof of the sufficiency part of that theorem; we turn now to the necessity.

Assume that H is a projective $\mathcal{R}$-module. Then there certainly exists a continuous $\mathcal{R}$-module morphism, $\rho : H \to \mathcal{R}\widehat{\otimes}H$, with $\ker \rho = 0$: we can take, for example, a right inverse to π_H. Since H has the approximation property, it follows from Theorem 1 that $H_c = 0$; in order words, that $\mathcal{R}$ is atomic. As was mentioned in the introduction, this implies the conditions (i) and (ii) of the main theorem or, otherwise, that $\mathcal{R}$ decomposes into a direct sum of type-I factors. It remains to establish condition (iii) or, equivalently, to prove that there is no infinite factor with infinite commutant among these summands.

We shall use a standard argument of homological algebra in its 'Banach' packing. In what follows, A-mod is the category of left Banach modules over a Banach algebra A and their continuous morphisms.

So, let $\mathcal{R}_\alpha = p_\alpha \mathcal{R}$ be a factor-summand of $\mathcal{R}$ (cf. the introduction). First, we notice that every object in $\mathcal{R}$-mod is automatically an object in $\mathcal{R}_\alpha$-mod (with the same outer multiplication), and every object in $\mathcal{R}$-mod is an object in $\mathcal{R}$- mod (with $a \cdot x$ defined as $p_\alpha a \cdot x$). Moreover, every morphism in $\mathcal{R}$-mod turns out to be a morphism in $\mathcal{R}_\alpha$-mod, and vice versa.

Now observe that the space $H_\alpha = p_\alpha H$ is a direct summand of the assumedly projective object H in $\mathcal{R}$-mod, and it is itself a projective object in that category. Therefore, if we take in $\mathcal{R}_\alpha$ -mod the canonical surjection $\pi_\alpha : \mathcal{R}_\alpha \widehat{\otimes} H_\alpha \to H_\alpha$, and then consider it in $\mathcal{R}$-mod, we have an admissible surjection onto a projective module. It follows that π_α has a right-inverse morphism ρ in $\mathcal{R}$-mod (see, for example ([8], Proposition VII.1.5). Then ρ is obviously a right-inverse morphism for π_α in $\mathcal{R}_\alpha$-mod as well, and H_α is thus a projective object in $\mathcal{R}_\alpha$-mod.

Therefore, in virtue of Theorem 3 (with $\mathcal{R}_\alpha$ as $\mathcal{R}$), either $\mathcal{R}_\alpha$ itself or its commutant is finite. Since $\mathcal{R}_\alpha$ is an arbitrary factor-direct summand of $\mathcal{R}$, the main theorem is proved.

REMARK 1. Recently, Yu. O. Golovin [5] had succeeded in giving a characterization of spatially projective algebras within a substantial class of non-self-adjoint operator algebras on Hilbert spaces. These are so-called *indecomposable* CSL-*algebras*; such a class includes all nest algebras and many others (see, e.g. [3]). The relevant criterion is formulated in terms of the invariant subspace lattice of the given algebra.

REMARK 2. As should be expected, the class of spatially flat von Neumann algebra (that is $\mathcal{R}$ with a flat H) is much larger than that of spatially projective algebras. In fact every Connes ('injective') algebra is spacially flat. The proof uses techniques which are different from those presented here, and is the subject of the separate paper [9].

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REFERENCES

1. E. CHRISTENSEN, Derivations and their relation to perturbations of operator algebras, in *Operator Algebras and Applications*, R. V. Kadison Ed., *Proc. Symp. Pure Math.*, Amer. Math. Soc. Part **2**(1982), 261–274.
2. J. CIGLER, V. LOSERT, P.MICHOR, *Banach modules and functors on categories of Banach spaces*, Marcel Dekker, New York 1979.
3. K.R. DAVIDSON, Commutative subspace lattices, *Indiana Univ. Math. J.* **27**(1978), 479–490.
4. YU.O. GOLOVIN, Homological properties of Hilbert modules over nest operator algebras and over their generalizations [Russian], *Math. Zametki* **41**(1987), 769–775.
5. YU.O. GOLOVIN, A criterion of the spacial projectivity of indecomposable SCL-algebra. *Russian Math. Surveys* **49**:4(1994), 161–162.
6. A.YA. HELEMSKII, On the homological dimension of normed modules over Banach algebras, *Math. USSR Sb.* **10**(1970), 399–411.
7. A.YA. HELEMSKII, *The homology of Banach and topological algebras*, Kluwer Academic Press, Dordrecht 1989.
8. A.YA. HELEMSKII, *Banach and locally convex algebras*, OUP, Oxford 1993.
9. A.YA. HELEMSKII, The spatial flatness and injectivity of Connes operator algebras, *Extracta Math.* **9**:1(1994), 75–81.
10. R.V. KADISON, J.R. RINGROSE, *Fundamentals of the theory of operator algebras*, Vol.II: Advanced Theory, Academic Press, London 1986.

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