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# ON THE EXTENSION OF THE PROPERTIES $(\mathbb{A}_{m,n})$

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ABSTRACT. Here we show that we can extend the properties $(\mathbb{A}_{m,n})$ from a given weak*-closed subspace to a larger one in some cases. Our technique yields examples of weak*-closed subspaces $\mathcal{A}$ having the property $(\mathbb{A}_{\aleph_0})$ without having any of the properties $X_{0,\gamma}$, in contrast to the case where $\mathcal{A}$ is the dual algebra generated by a contraction in the class $\mathbb{A}$ (for which it is well known that the two properties are equivalent).

KEYWORDS: *Dual algebra, weak*-closed subspace, compact operator, property* $(\mathbb{A}_{m,n})$.

AMS SUBJECT CLASSIFICATION: 47D15, 47D25, 47A55.

## 1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, then $\mathcal{A}_T$ denotes the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that contains $T$ and $I_{\mathcal{H}}$ and is closed in the weak*-topology. Let $Q_T$ denote the quotient space $\mathcal{C}^1(\mathcal{H})/{}^{\perp}\mathcal{A}_T$, where $\mathcal{C}^1(\mathcal{H})$ is the trace-class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}^{\perp}\mathcal{A}_T$ denotes the preannihilator of $\mathcal{A}_T$ in $\mathcal{C}^1(\mathcal{H})$. One knows that $\mathcal{A}_T$ is the dual space of $Q_T$ and the duality is given by

$$\langle A, [L]\rangle = \operatorname{tr}(AL), \quad A \in \mathcal{A}_T, \quad [L] \in Q_T,$$

where $[L]$ is the image in $Q_T$ of the operator $L$ in $\mathcal{C}^1(\mathcal{H})$. If $x$ and $y$ are vectors in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank-one operator in $\mathcal{C}^1(\mathcal{H})$ defined by

$$(x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H}.$$

Then, of course, $[x \otimes y] \in Q_T$, and it is easy to see that

$$\langle A, [x \otimes y] \rangle = \operatorname{tr}(A(x \otimes y)) = (Ax, y).$$

Suppose now that $m$ and $n$ are any cardinal numbers less than or equal to $\aleph_0$, and let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$. We say that $\mathcal{A}$ has the property $(\mathbb{A}_{m,n})$ if for every doubly indexed family $([L_{i,j}])_{0 \leqslant i < m, 0 \leqslant j < n}$ in $Q_{\mathcal{A}}$, there exist sequences $(x_i)_{0 \leqslant i < m}$ and $(y_j)_{0 \leqslant j < n}$ in $\mathcal{H}$ such that:

$$[L_{ij}] = [x_i \otimes y_j] \quad \text{for} \quad 0 \leqslant i < m \quad \text{and} \quad 0 \leqslant j < n. \tag{1.1}$$

Furthermore, if for every $s > \rho$ ($\rho$ fixed) we can solve (1.1) and also the inequalities:

$$\|x_i\| < \left( s \sum_{0 \leqslant j < n} \|[L_{ij}]\| \right)^{\frac{1}{2}}, \quad 0 \leqslant i < m,$$

$$\|y_j\| < \left( s \sum_{0 \leqslant i < m} \|[L_{ij}]\| \right)^{\frac{1}{2}}, \quad 0 \leqslant j < n,$$

then $\mathcal{A}$ is said to have property $(\mathbb{A}_{m,n}(\rho))$. The class $\mathbb{A} = \mathbb{A}(\mathcal{H})$ is defined to be the set of all absolutely continuous contractions for which the Sz.-Nagy–Foias functional calculus is an isometry (cf [6], Chapter 3).

We also define the classes $\mathbb{A}_{m,n} = \{T \in \mathbb{A} \mid \mathcal{A}_T \text{ has the property } (\mathbb{A}_{m,n})\}$, $\mathbb{A}_{m,n}(\rho) = \{T \in \mathbb{A} \mid \mathcal{A}_T \text{ has the property } (\mathbb{A}_{m,n}(\rho))\}$. The class $\mathbb{A}_{n,n}$ is also denoted by $\mathbb{A}_n$.

Let $0 \leqslant \theta < 1$. Then the following subsets of the predual of $\mathcal{A}$ were defined in [1] and [4] : $\mathcal{X}_\theta(\mathcal{A})$ denotes the set of all $[L] \in Q_{\mathcal{A}}$ such that there exist $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $(\mathcal{H})_1$ (the closed unit ball of $\mathcal{H}$) which converge weakly to 0 and satisfy (1.2), (1.3) and (1.4):

$$\limsup_{n \to \infty} \|[L] - [x_n \otimes y_n]\| \leqslant \theta; \tag{1.2}$$

$$\forall w \in \mathcal{H} \quad \lim_{n \to \infty} \|[x_n \otimes w]\| = 0; \tag{1.3}$$

$$\forall w \in \mathcal{H} \quad \lim_{n \to \infty} \|[w \otimes y_n]\| = 0. \tag{1.4}$$

$\mathcal{E}_\theta^r(\mathcal{A})$ is the set of all $[L] \in Q_{\mathcal{A}}$ such that there exist $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $(\mathcal{H})_1$ which converge weakly to 0 and satisfy (1.2) and (1.3). We define also $\mathcal{E}_\theta^l(\mathcal{A})$ by symmetry on the second condition.

Note that $\mathcal{X}_\theta(\mathcal{A})$ is closed and absolutely convex; on the other hand the convexity of the sets $\mathcal{E}_\theta^r(\mathcal{A})$ and $\mathcal{E}_\theta^l(\mathcal{A})$ is an open question (cf. [7] for partial results).

DEFINITION 1.1. ([1]) Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$; $\mathcal{A}$ is said to have the property $X_{\theta,\gamma}$, $0 \leqslant \theta < \gamma \leqslant 1$ if $\mathcal{X}_\theta(\mathcal{A}) \supset (Q_\mathcal{A})_\gamma$ (the closed ball in $Q_\mathcal{A}$ centered at 0 with radius $\gamma$).

DEFINITION 1.2. ([4]) Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$; $\mathcal{A}$ is said to have the property $E^r_{\theta,\gamma}$, $(0 \leqslant \theta < \gamma \leqslant 1)$ (respectively $E^l_{\theta,\gamma}$) if $\overline{\text{aco}}(\mathcal{E}^r_\theta(\mathcal{A}))$, the closed absolutely convex hull of the set $\mathcal{E}^r_\theta(\mathcal{A})$, (respectively $\overline{\text{aco}}(\mathcal{E}^l_\theta(\mathcal{A}))$ contains $(Q_\mathcal{A})_\gamma$.

The following theorem is established in [3], Chapter 3.

THEOREM 1.3. *Let $\mathcal{A}$ be a dual subalgebra of $\mathcal{L}(\mathcal{H})$. If $\mathcal{A}$ has the property $X_{\theta,\gamma}$, $(0 \leqslant \theta < \gamma \leqslant 1)$, then $\mathcal{A}$ has the property $(\mathbb{A}_{\aleph_0})$.*

This theorem is still true when $\mathcal{A}$ is a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$.

In the case of a dual subalgebra of $\mathcal{L}(\mathcal{H})$ generated by an operator in the class $\mathbb{A}$, we have the following result ([5], Theorem 6.2):

THEOREM 1.4. *Assume $T \in \mathbb{A} = \mathbb{A}(\mathcal{H})$. Then $T \in \mathbb{A}_{1,\aleph_0}$ (respectively $T \in \mathbb{A}_{\aleph_0,1}$) if and only if there exists $\gamma\,(0 < \gamma \leqslant 1)$ such that $\mathcal{A}_T$ has the property $E^r_{0,\gamma}$ (respectively $E^l_{0,\gamma}$).*

This result is similar to one of the characterizations of the class $\mathbb{A}_{\aleph_0}$ given in [1], Chapter 4.

THEOREM 1.5. *Assume $T \in \mathbb{A} = \mathbb{A}(\mathcal{H})$. Then $T \in \mathbb{A}_{\aleph_0}$ if and only if there exists $\gamma$, $(0 < \gamma \leqslant 1)$ such that $\mathcal{A}_T$ has the property $X_{0,\gamma}$.*

We are interested here in the extension of the properties $(\mathbb{A}_{m,n})$. In [2] we obtained a definite result when we "add" a finite rank operator R to $\mathcal{A}$, where $\mathcal{A}$ is a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$. If $\mathcal{R}(\mathcal{H})$ denotes the set of all finite rank operators on $\mathcal{H}$ the main result is:

THEOREM 1.6. ([2]) *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$, and $R \in \mathcal{R}(\mathcal{H}) \setminus \{0\}$ such that* $\operatorname{rank}(R) = n$. *Then $\mathcal{A} + \mathbb{C}R$ has the properties $\mathbb{A}_1(1 + 2/\gamma)$ and $(\mathbb{A}_{n,\aleph_0}) \cap (\mathbb{A}_{\aleph_0,n}) \setminus (\mathbb{A}_{n+1})$ without having any property $E^r_{0,\rho}$ or $E^l_{0,\rho}$.*

This result implies that Theorem 1.4 fails in the general case. The purpose of this paper is to give a similar result with any compact operator $K$.

## 2. PRELIMINARIES

The following result is proved in [1], Proposition 3.1.

PROPOSITION 2.1. *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$; then $M_n(\mathcal{A})$ has the property $X_{0,\gamma/n^2}$, for every $n \geqslant 1$, where $M_n(\mathcal{A}) = \{(A_{ij})_{1\leqslant i,j\leqslant n} \mid A_{ij} \in \mathcal{A}\}$ which is naturaly identified with a subspace of $\mathcal{H}^{(n)}$ and $Q_{M_n(Q_\mathcal{A})}$ is identified with $M_n(Q_\mathcal{A})$.*

We have the following result, given in [3], Chapter 1.

PROPOSITION 2.2. *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$. Suppose that we are given $[L] \in Q_\mathcal{A}$, vectors $a \in \mathcal{H}$, $b \in \mathcal{H}$, $(w_k)_{1\leqslant k\leqslant n}$ in $\mathcal{H}$, a finite codimensional subspace $\mathcal{L}$ of $\mathcal{H}$ and $\delta, \varepsilon > 0$ such that $\|[L] - [a \otimes b]\| < \delta$ ; then there exist $x$ and $y$ in $\mathcal{H}$ such that :*

$$[L] = [x \otimes y], \ (x - a) \in \mathcal{L}, \ (y - b) \in \mathcal{L},$$

$$\sup(\|x - a\|, \|y - b\|) < \sqrt{\frac{\delta}{\gamma}},$$

$$\|[w_k \otimes (y - b)]\| < \varepsilon \quad \textit{and} \quad \|[(x - a) \otimes w_k]\| < \varepsilon, \ 1 \leqslant k \leqslant n.$$

## 3. EXTENSION OF THE PROPERTIES $(\mathbb{A}_{m,n})$

Let $\mathcal{K}(\mathcal{H})$ denote the set of all compact operators on $\mathcal{H}$. We have the following result.

THEOREM 3.1. *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$, and $K \in \mathcal{K}(\mathcal{H}) \setminus \{0\}$. Then $\tilde{\mathcal{A}} = \mathcal{A} + \mathbb{C}K$ has the property $(\mathbb{A}_1(1 + 2/\gamma))$.*

*Proof.* We may suppose that $\|K\| = 1$.

Let $K = \sum_{i\geqslant 1} \lambda_i \varepsilon_i \otimes e_i$ be the canonical writing of K, where $(e_i)_i$, $(\varepsilon_i)_i$ are orthonormal systems and $(\lambda_i)_i$ a nonincreasing sequence of positive numbers. Then $\lambda_1 = 1$ and $\lambda_i \searrow 0$, if $K$ is not of finite rank. Let $\eta > 0$ and take

$$R_k = \sum_{i=1}^{p_k - 1} \lambda_i \varepsilon_i \otimes e_i, \quad k \geqslant 1,$$

where $p_k$ satisfies

$$\sum_{k\geqslant 1}(\lambda_{p_k})^{\frac{1}{2}} < \frac{\sqrt{1+\eta}-1}{1+\eta}\left(1+\frac{2}{\gamma}\right)^{-\frac{3}{2}}.$$

We have $R_k \to K$ in $\mathcal{L}(\mathcal{H})$ and, if $r_k = \|R_k - R_{k-1}\| = \lambda_{p_{k-1}}$, $k \geqslant 2$,

$$\left(1+\frac{2}{\gamma}\right)^{\frac{3}{2}} \sum_{k\geqslant 2} r_k{}^{\frac{1}{2}} < \frac{\sqrt{1+\eta}-1}{1+\eta}. \tag{3.1}$$

Let $\psi \in Q_{\tilde{\mathcal{A}}}$. Thus $\psi$ is well defined by its action on $\mathcal{A}$ and on $\mathbb{C}K$; then we write $\psi = ([L], d)$ where $[L] = \psi|\mathcal{A}$ and $\psi(K) = d$. We may suppose that $\max(\|[L]\|, |d|) < 1$.

Let us denote $a = e_1$ and $b = \overline{d}\varepsilon_1$. We have $(R_1 a, b) = d$ and

$$\begin{aligned}\|[L] - [a\otimes b]\| &\leqslant \|[L]\| + \|a\|\,\|b\| \\ &\leqslant \|[L]\| + |d| \\ &< 2.\end{aligned}$$

Proposition 2.2 provides vectors $x_1,\ y_1 \in \mathcal{H}$ such that

$$[L] = [x_1 \otimes y_1], \quad \sup(\|x_1 - a\|^2, \|y_1 - b\|^2) < \frac{2}{\gamma}$$

and

$$(x_1 - a),\ (y_1 - b) \in (R_1\mathcal{H} \cup R_1^*\mathcal{H})^{\perp} \cap \{a, b\}^{\perp}.$$

It follows from this that $(R_1 x_1, y_1) = d$ and $(x_1, e_1) = 1$.

Suppose now that we can find $(x_k)_{2\leqslant k\leqslant n}$ and $(y_k)_{2\leqslant k\leqslant n} \in \mathcal{H}$ such that

$$\begin{aligned}&[L] = [x_k \otimes y_k],\ (R_k x_k, y_k) = d, \quad (x_k, e_1) = 1, \\ &\|x_k - x_{k-1}\|^2 < \left(\frac{(1+\eta)^{\frac{3}{2}}}{\gamma}\right)\left(1+\frac{2}{\gamma}\right)^{\frac{3}{2}} r_k, \\ &\|x_k\|^2 < \left(1+\frac{2}{\gamma}\right) + \frac{(1+\eta)^{\frac{3}{2}}}{\gamma}\left(1+\frac{2}{\gamma}\right)^{\frac{3}{2}} \sum_{j=2}^{n} r_j, \\ &\|y_k\| < \left(1+\frac{2}{\gamma}\right)^{\frac{1}{2}} + (1+\eta)\left(1+\frac{2}{\gamma}\right)^{2} \sum_{j=2}^{n} r_j^{\frac{1}{2}}.\end{aligned}$$

We will have occasion to use $\max(\|x_k\|^2, \|y_k\|^2) < (1+\eta)\left(1+\frac{2}{\gamma}\right)$ which may be deduced from the induction hypothesis.

Note that

$$(R_{n+1}x_n, y_n) = d + ((R_{n+1} - R_n)x_n, y_n).$$

Let $u_n = -\overline{((R_{n+1} - R_n)x_n, y_n)}e_1$. Since $(x_n, e_1) = 1$ we have $(R_{n+1}x_n, y_n + u_n) = d$.

By using (3.1) and the fact that $\| u_n \| \leqslant r_{n+1} \| x_n \| \| y_n \|$ we have

$$\begin{aligned}\|[L] - [x_n \otimes (y_n + u_n)]\| &\leqslant \|x_n\| \|u_n\| \\ &\leqslant \|x_n\|^2 \|y_n\| r_{n+1} \\ &< (1+\eta)^{\frac{3}{2}} \left(1 + \frac{2}{\gamma}\right)^{\frac{3}{2}} r_{n+1}.\end{aligned}$$

Thus there exist, by Proposition 2.2, $x_{n+1}$ and $y_{n+1} \in \mathcal{H}$ such that

$$[L] = [x_{n+1} \otimes y_{n+1}],$$

$$\max\left(\|x_{n+1} - x_n\|^2, \|y_{n+1} - (y_n + u_n)\|^2\right) < \frac{(1+\eta)^{\frac{3}{2}}}{\gamma}\left(1 + \frac{2}{\gamma}\right)^{\frac{3}{2}} r_{n+1},$$

and

$$((x_{n+1} - x_n), (y_{n+1} - (Y_n + u_n)) \in (R_{n+1}\mathcal{H} \cup R_{n+1}^*\mathcal{H})^{\perp} \cup \{x_n\}^{\perp}.$$

From this we deduce that $(R_{n+1}x_{n+1}, y_{n+1}) = d$, $(x_{n+1}, e_1) = 1$,

$$\begin{aligned}\|x_{n+1}\|^2 &= \|x_{n+1} - x_n\|^2 + \|x_n\|^2 \\ &< \frac{(1+\eta)^{\frac{3}{2}}}{\gamma}\left(1 + \frac{2}{\gamma}\right)^{\frac{3}{2}} r_{n+1} + \frac{(1+\eta)^{\frac{3}{2}}}{\gamma}\left(1 + \frac{2}{\gamma}\right)^{\frac{3}{2}} \sum_{j=2}^{n} r_j + \left(1 + \frac{2}{\gamma}\right) \\ &< \left(1 + \frac{2}{\gamma}\right) + \frac{(1+\eta)^{\frac{3}{2}}}{\gamma}\left(1 + \frac{2}{\gamma}\right)^{\frac{3}{2}} \sum_{j=2}^{n+1} r_j,\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1}\| &< \|y_{n+1} - (y_n + u_n)\| + \|y_n\| + \|u_n\| \\ &< \frac{(1+\eta)^{\frac{3}{4}}}{\sqrt{\gamma}}\left(1 + \frac{2}{\gamma}\right)^{\frac{3}{4}} r_{n+1}^{\frac{1}{2}} + \left(1 + \frac{2}{\gamma}\right)^{\frac{1}{2}} + \\ &\quad + (1+\eta)\left(1 + \frac{2}{\gamma}\right)^{2} \sum_{j=2}^{n} r_j^{\frac{1}{2}} + (1+\eta)\left(1 + \frac{2}{\gamma}\right) r_{n+1} \\ &< \left(1 + \frac{2}{\gamma}\right)^{\frac{1}{2}} + (1+\eta)\left(1 + \frac{2}{\gamma}\right)^{2} r_{n+1}^{\frac{1}{2}} + (1+\eta)\left(1 + \frac{2}{\gamma}\right)^{\frac{1}{2}} \sum_{j=2}^{n} r_j^{\frac{1}{2}} \\ &< \left(1 + \frac{2}{\gamma}\right)^{\frac{1}{2}} + (1+\eta)\left(1 + \frac{2}{\gamma}\right)^{2} \sum_{j=2}^{n+1} r_j^{\frac{1}{2}}.\end{aligned}$$

It is easy to see that $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences and thus converge. If $x$ and $y$ are their respective limits we have

$$\begin{aligned} \|x\|\,\|y\| &< (1+2\eta)\left(1+\frac{2}{\gamma}\right), \\ [L] &= \lim_{n\to\infty}[x_n \otimes y_n] = [x \otimes y], \\ d &= \lim_{n\to\infty}(R_n x_n, y_n) = (Kx, y). \end{aligned}$$

It follows that

$$\psi = [x \otimes y]_{\mathcal{A}+\mathbb{C}K} \quad \text{and} \quad \|x\|\,\|y\| < (1+2\eta)\left(1+\frac{2}{\gamma}\right)$$

and the proof is complete. ∎

Here, we establish a result similar to the finite rank case.

THEOREM 3.2. *Let $\mathcal{A}$ be a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ with the property $X_{0,\gamma}$ $(0 < \gamma \leqslant 1)$ and $K \in \mathcal{K}(\mathcal{H}) \setminus \mathcal{R}(\mathcal{H})$. Then $\mathcal{A} + \mathbb{C}K$ has the property $(\mathbb{A}_{\aleph_0})$.*

*Proof.* As before let $K = \sum_{i\geqslant 1} \lambda_i \varepsilon_i \otimes e_i$ be the canonical writing of $K$ and $\|K\| = 1$. Then $\lambda_1 = 1$ and $\lambda_i \searrow 0$. Take:

$$R_n = \sum_{i=1}^{p_n-1} \lambda_i \varepsilon_i \otimes e_i, \; \overline{R}_n = R_n - R_{n-1} \quad \text{and} \quad r_n = \left\|\overline{R}_n\right\| = \lambda_{p_{n-1}} \quad \text{for} \quad n \geqslant 2.$$

The conditions on the choice of the sequence $(p_n)_n$ will be given later, now we only assume that $p_n > 2n+1$. Since the proof is fairly technical we first give a general outline: given a doubly infinite matrix $(\psi_{ij})_{i\geqslant 1, j\geqslant 1}$ of elements in $Q_{\tilde{\mathcal{A}}}$ we want to find sequences of vectors $(x_i)_{i\geqslant 1}$ and $(y_j)_{j\geqslant 1}$ in $\mathcal{H}$ such that

$$\psi_{ij} = [x_i \otimes y_j] \quad i, j \geqslant 1.$$

It is well known (and easy to prove by standard scaling argument) that we may assume $\|\psi_{ij}\| < \delta_i \delta_j$ where $(\delta_n)_{n\geqslant 1}$ is a given sequence of strictly positive numbers (to be chosen later). As in the previous theorem we "split" $\psi_{ij}$ into its actions $[L_{ij}](= \psi_{ij}|\mathcal{A})$ on $\mathcal{A}$ and $d_{ij}(= \psi_{ij}(K))$ on $\mathbb{C}K$. For any $k \geqslant 1$, $[\overline{L}]_k$ will denote the $k \times k$ matrix with entries $[L_{ij}]$ in $Q_{\mathcal{A}}$ (which as usual, we identify with an element in the predual $Q_{\mathcal{M}_k(\mathcal{A})}$ of the weak*-closed subspace $\mathcal{M}_k(\mathcal{A})$ in $\mathcal{L}(\mathcal{H}^{(k)})$.

The idea of the proof is to build by induction on $k$ vectors $X^k = (X_1^k, \ldots, X_k^k)$, $Y^k = (Y_1^k, \ldots, Y_k^k)$ in $\mathcal{H}^k$ such that

(3.2) $$[\overline{L}]_k = \left[X^k \otimes Y^k\right],$$

and

$$(3.3) \qquad \left(R_k X_i^k, Y_j^k\right) = d_{ij} \quad 1 \leqslant i,j \leqslant k,$$

$$(3.4) \qquad \sup\left(\left\|X^k - (X^{k-1},0)\right\|, \left\|Y^k - (Y^{k-1},0)\right\|\right) < \frac{1}{2^{k-1}} \quad (k \geqslant 2).$$

Suppose this has been done; then clearly (3.2) becomes

$$(3.5) \qquad [L_{ij}] = \left[X_i^k \otimes Y_j^k\right], \quad k \geqslant \max(i,j),$$

and the sequences

$$\left(X_i^k\right)_{k\geqslant i}, \ \left(Y_i^k\right)_{k\geqslant i}$$

are Cauchy sequences for each $i$. Denoting by $X_i$ , $Y_i$ their respective limits we have, by going to the limits in (3.2) and (3.3),

$$\begin{cases} [L_{ij}] = [X_i \otimes Y_j] \\ (KX_i, Y_j) = d_{ij}, \end{cases} \qquad i,j \geqslant 1,$$

that is the desired conclusion.

The main difficulty in implementing the induction procedure (that is, assuming $X^k, Y^k$ are defined up to $k = n$, define $X^{n+1}$ and $Y^{n+1}$) is to obtain condition (3.3). We proceed in two steps:

(a) First, find vectors $U^n \in \mathcal{H}$, $V^n \in \mathcal{H}^{(n+1)}$ such that the vectors

$$\overline{X}^n = (X^n, U^n), \quad \overline{Y}^n = (Y^n, 0) + V^n$$

satisfy

$$\left(R_{n+1}\overline{X}_i^n, \overline{Y}_j^n\right) = d_{ij}, \quad 1 \leqslant i,j \leqslant n+1,$$

with "reasonable" bounds

$$\|U^n\| (= \|\overline{X}^n - (X^n,0)\|) \quad \text{and} \quad \|V^n\|.$$

(b) Then, Proposition 2.1 allows us to apply a matricial version of Proposition 2.2 to conclude the induction step. (Note that, to facilitate step (a), some additional technicalities have been included in the induction hypothesis.)

Let $(\alpha_n)_{n\geqslant 1}$ be a decreasing non negative sequence such that $\alpha_1 = 1$:

$$(3.6) \qquad \alpha_{2n+1} < \lambda_{2n+1} \frac{1}{\left(1 + \dfrac{3}{\alpha_{2n}}\right)^{n+1}} \frac{\gamma}{3^4(n+1)^2} \frac{1}{2^{2(n+1)+1}}, \quad n \geqslant 1,$$

$$\alpha_{2n+2} < c_n \stackrel{\text{def}}{=} \frac{1}{\left(1+\dfrac{3}{\alpha_{2n+1}}\right)^{n+1}} \frac{\gamma}{3^4(n+1)^2} \frac{1}{2^{2(n+1)+1}}, \quad n \geqslant 0. \tag{3.7}$$

Let $(\psi_{ij})_{i\geqslant 1, j\geqslant 1} \subset Q_{\mathcal{A}+\mathbb{C}K}$. As before $\psi_{ij} = ([L_{ij}], d_{ij})$ when $[L_{ij}] = \psi_{ij}|\mathcal{A}$ and $d_{ij} = \psi_{ij}(K)$; we may suppose that

$$\|\psi_{ij}\| < \delta_i\delta_j$$

where $(\delta_n)_{n\geqslant 1}$ is a sequence satisfying

$$\delta_{n+1} < c_n, \quad n \geqslant 0. \tag{3.8}$$

Put $a = e_1$ and $b = \overline{d}_{11}\varepsilon_1 + \alpha_2\varepsilon_2$. We have $(R_1 a, b) = d_{11}$ and

$$\begin{aligned}\|[L_{11}] - [a \otimes b]\| &< \delta_1^2 + \|a\|\,\|b\| \\ &< \delta_1^2 + \sqrt{\delta_1^2 + \alpha_2^2} \\ &< \gamma.\end{aligned}$$

Proposition 2.2 provides a vector $X^1$ and $Y^1 \in \mathcal{H}$ such that

$$\begin{aligned}&[L_{11}] = [X^1 \otimes Y^1], \\ &\max(\|X^1 - a\|, \|Y^1 - b\|) < 1, \\ &(X^1 - a),\ (Y^1 - b) \in (R_1\mathcal{H} \cup R_1^*\mathcal{H})^\perp \cap \operatorname{span}\{a, b\}^\perp.\end{aligned}$$

This implies $(R_1 X^1, Y^1) = d_{11}$ and $\max(\|X^1\|, \|Y^1\|) < 2$.

Suppose now that we can find $(X^k)_{k\geqslant 1}$ and $(Y^k)_{k\geqslant 1}$ when $X^k, Y^k$ are in $\mathcal{H}^{(k)}$ such that (3.2), (3.3) and (3.4) hold, and

$$(\lambda_{2j-1}X_i^k, e_{2j-1}) = \begin{cases} \alpha_{2j-1} & \text{if } j = i, \\ 0 & \text{if } j < i; \end{cases} \tag{3.9}$$

$$(\varepsilon_{2i}, Y_j^k) = \begin{cases} \alpha_{2j} & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases} \tag{3.10}$$

Suppose that the vectors $(X^k)_{1\leqslant k\leqslant n}$ and $(Y^k)_{1\leqslant k\leqslant n}$ have been found for $n \geqslant 1$.

Let

$$[\overline{L}]_{n+1} = ([L_{ij}])_{1\leqslant i,j\leqslant n+1},$$

$$\overline{X}^n = (X^n, U^n), \quad \overline{Y}^n = (Y_j^n + V_j^n, V_{n+1}^n),$$

$$U^n = \mu_{n+1}\frac{e_{2n+1}}{\lambda_{2n+1}} + \sum_{k=1}^n \mu_k \frac{e_{2k}}{\lambda_{2k}},\ V_j^n = \sum_{k=0}^n \beta_k^{n,j}\varepsilon_{2k+1},\ V_{n+1}^n = W_n + \sum_{k=1}^n s_k\varepsilon_{2k-1},$$

where

$$\mu_{n+1} = \alpha_{2n+1},\ \mu_k = -\frac{1}{\alpha_{2k}}\left(\sum_{l=k+1}^{n} \mu_l(\varepsilon_{2l}, Y_k^n) + \mu_{n+1}(\varepsilon_{2n+1}, Y_k^n)\right) \text{ for } 1 \leqslant k \leqslant n,$$

$$\beta_n^{n,j} = \frac{\overline{d}_{n+1,j}}{\alpha_{2n+1}},$$

and, for $0 \leqslant k \leqslant n$ and $1 \leqslant j \leqslant n$,

$$\overline{\beta}_k^{n,j} = -\frac{1}{\alpha_{2k+1}}\left((\overline{R}_{n+1}X_{k+1}^n, Y_j^n) + \sum_{l=k+1}^{n}\left(R_{n+1}X_{k+1}^n, \beta_l^{n,j}\varepsilon_{2l+1}\right)\right),$$

$$W_n = \frac{\overline{d}_{n+1,n+1}}{\alpha_{2n+1}}\varepsilon_{2n+1} + \alpha_{2n+2}\varepsilon_{2n+2}\ ,\ \overline{s}_n = \frac{1}{\alpha_{2n-1}}(d_{n,n+1} - (R_{n+1}X_n^n, W_n)),$$

$$\overline{s}_k = \frac{1}{\alpha_{2k-1}}\left(d_{k,n+1} - (R_{n+1}X_k^n, W_n) - \sum_{l=k+1}^{n}(R_{n+1}X_k^n, s_l\varepsilon_{2l-1})\right) \text{ for } 1 \leqslant k \leqslant n.$$

We shall verify that

$$\left(R_{n+1}\overline{X_i^n}, \overline{Y_j^n}\right) = d_{ij} \quad \text{for all} \quad 1 \leqslant i, j \leqslant n.$$

We start to establish that

$$\left(R_{n+1}X_i^n, Y_j^n + V_j^n\right) = \left(R_{n+1}X_i^n, Y_j^n\right) + \left(R_{n+1}X_i^n, V_j^n\right) = d_{ij}.$$

For this let us calculate

$$\begin{aligned}
\left(R_{n+1}X_i^n, V_j^n\right) &= \sum_{k=0}^{n}\left(R_{n+1}X_i^n, \beta_k^{n,j}\varepsilon_{2k+1}\right) \\
&= \sum_{k=i-1}^{n}\left(R_{n+1}X_i^n, \beta_k^{n,j}\varepsilon_{2k+1}\right) \\
&= \alpha_{2i-1}\overline{\beta_{i-1}^{n,j}} + \sum_{k=i}^{n}\left(R_{n+1}X_i^n, \beta_k^{n,j}\varepsilon_{2k+1}\right) \\
&= -\left(\overline{R}_{n+1}X_i^n, Y_j^n\right)
\end{aligned}$$

which run down by the definition of $\beta_{i-1}^{n,j}$. Thus

$$\begin{aligned}
\left(R_{n+1}X_i^n, Y_j^n + V_j^n\right) &= ((R_{n+1} - \overline{R}_{n+1})X_i^n, Y_j^n) \\
&= \left(R_n X_i^n, Y_j^n\right) = d_{ij}; \\
\left(R_{n+1}X_i^n, V_{n+1}^n\right) &= (R_{n+1}X_i^n, W_n) + \sum_{k=1}^{n} (R_{n+1}X_i^n, s_k\varepsilon_{2k-1}) \\
&= (R_{n+1}X_i^n, W_n) + \sum_{k=i+1}^{n} (R_{n+1}X_i^n, s_k\varepsilon_{2k-1}) + \alpha_{2i-1}\overline{s}_i \\
&= (R_{n+1}X_i^n, W_n) - \alpha_{2i-1}\overline{s}_i + d_{i,n+1} \\
&\qquad - (R_{n+1}X_i^n, W_n) + \alpha_{2i-1}\overline{s}_i \\
&= d_{i,n+1}.
\end{aligned}$$

We have also

$$\begin{aligned}
\left(R_{n+1}U^n, Y_j^n + V_j^n\right) &= \left(R_{n+1}U^n, Y_j^n\right) + \left(R_{n+1}U^n, V_j^n\right) \\
&= \mu_{n+1}\left(\varepsilon_{2n+1}, Y_j^n\right) + \sum_{k=1}^{n} \mu_k\left(\varepsilon_{2k}, Y_j^n\right) + d_{n+1,j} \\
&= \mu_{n+1}\left(\varepsilon_{2n+1}, Y_j^n\right) + \sum_{k=j+1}^{n} \mu_k\left(\varepsilon_{2k}, Y_j^n\right) + \alpha_{2j}\mu_j + d_{n+1,j} \\
&= \alpha_{2j}\mu_j - \alpha_{2j}\mu_j + d_{n+1,j} = d_{n+1,j}
\end{aligned}$$

and

$$\left(R_{n+1}U^n, V_{n+1}^n\right) = (R_{n+1}U^n, W_n) = d_{n+1,n+1}.$$

We remark from the induction hypothesis that $\max(\|X^k\|, \|Y^k\|) < 3$. Now, we seek upper bounds for $\|U^n\|$, $\|V_j^n\|$ and $\|V_{n+1}^n\|$. It easy to check that

$$\begin{cases} |\mu_{n+1}| \leqslant \alpha_{2n+1}, \\ |\mu_k| \leqslant 3\dfrac{\alpha_{2n+1}}{\alpha_{2n}}\left(1 + \dfrac{3}{\alpha_{2n}}\right)^{n-k}, \quad 1 \leqslant k \leqslant n, \end{cases}$$

$$\begin{cases} |\beta_n^{n,j}| \leqslant \dfrac{\delta_{n+1}\delta_j}{\alpha_{2n+1}}, \\ |\beta_k^{n,j}| \leqslant \dfrac{3}{\alpha_{2n+1}} t\left(1 + \dfrac{3}{\alpha_{2n+1}}\right)^{n-(k+1)}, 1 \leqslant k \leqslant n-1, \text{ where } t = 3r_{n+1} + \dfrac{\delta_{n+1}\delta_j}{\alpha_{2n+1}}, \end{cases}$$

$$|s_k| \leqslant \frac{h}{\alpha_{2n+1}}\left(1 + \frac{3}{\alpha_{2n+1}}\right)^{n-k}, 1 \leqslant k \leqslant n, \text{ where } h = \delta_{n+1} + 3\|W_n\|.$$

Then

$$\|U^n\| \leqslant \frac{1}{\lambda_{2n+1}}\left(1 + \frac{3}{\alpha_{2n}}\right)^n \alpha_{2n+1}$$

and

$$\|V_j^n\| \leqslant |\beta_n^{n,j}| + \sum_{k=0}^{n-1} \left|\beta_k^{n,j}\right| \leqslant t\left(1 + \frac{3}{\alpha_{2n+1}}\right)^n,$$

$$\|V_{n+1}^n\| \leqslant \|W_n\| + \sum_{k=1}^{n} |s_k| \leqslant \frac{1}{3}h\left(1 + \frac{3}{\alpha_{2n+1}}\right)^n.$$

Put $V^n = \left(V_j^n, V_{n+1}^n\right)$. Thus we have

$$\begin{aligned}\|V^n\| &\leqslant \sum_{j=1}^{n} \|V_j^n\| + \|V_{n+1}^n\| \\ &\leqslant \left(1 + \frac{3}{\alpha_{2n+1}}\right)^n \left(3nr_{n+1} + \frac{\delta_{n+1}}{\alpha_{2n+1}}\right) + \frac{1}{3}h\left(1 + \frac{3}{\alpha_{2n+1}}\right)^n \\ &\leqslant \left(1 + \frac{3}{\alpha_{2n+1}}\right)^{n+1} (3nr_{n+1} + \delta_{n+1} + \alpha_{2n+2}).\end{aligned}$$

Hence

$$\begin{aligned}\|U^n\| &\leqslant \frac{1}{\lambda_{2n+1}}\left(1 + \frac{3}{\alpha_{2n}}\right)^n \alpha_{2n+1}, \\ \|V^n\| &\leqslant \left(1 + \frac{3}{\alpha_{2n+1}}\right)^{n+1} (3nr_{n+1} + \delta_{n+1} + \alpha_{2n+2}).\end{aligned} \tag{3.11}$$

$$\begin{aligned}\left\|[\overline{L}]_{n+1} - \left[\overline{X^n} \otimes \overline{Y^n}\right]\right\| &\leqslant \sum_{i=1}^{n+1} \|[L_{i,n+1}]\| + \sum_{j=1}^{n} \|[L_{n+1,j}]\| \\ &\quad + \|X^n\|\,\|V^n\| + \|U^n\|\,\|Y^n\| + \|U^n\|\,\|V^n\| \\ &\leqslant \delta_{n+1} + 3(\|U^n\| + \|V^n\|) + \|U^n\|\,\|V^n\|.\end{aligned}$$

The above considerations ((3.6), (3.7), (3.8) and (3.11)) and the condition $r_{n+1} < \frac{c_n}{n}$ (see the definition in (3.7)) give us

$$\begin{aligned}\delta_{n+1} &< \frac{\gamma}{3(n+1)^2}\,\frac{1}{2^{2(n+1)}}, \\ \|U^n\| &< \frac{\gamma}{9(n+1)^2}\,\frac{1}{2^{2(n+1)+1}}, \\ \|V^n\| &< \frac{\gamma}{9(n+1)^2}\,\frac{1}{2^{2(n+1)+1}}.\end{aligned} \tag{3.12}$$

From (3.12) we have

$$\left\|[\overline{L}]_{n+1} - \left[\overline{X^n} \otimes \overline{Y^n}\right]\right\| < \frac{\gamma}{(n+1)^2}\,\frac{1}{2^{2(n+1)}}.$$

Proposition 2.2 provides a vector $X^{n+1}$ and $Y^{n+1} \in \mathcal{H}^{(n+1)}$ such that

$$[\overline{L}]_{n+1} = [X^{n+1} \otimes Y^{n+1}],$$
$$\max(\|X^{n+1} - \overline{X}^n\|, \|Y^{n+1} - \overline{Y}^n\|) < \frac{1}{2^{n+1}},$$
$$(X^{n+1} - \overline{X}^n) \quad \text{and} \quad (Y^{n+1} - \overline{Y}^n) \in \left((R_{n+1}\mathcal{H} \cup R^*_{n+1}\mathcal{H})^{\perp}\right)^{n+1}.$$

This implies

$$\left(R_{n+1}X_i^{n+1}, Y_j^{n+1}\right) = d_{ij} \quad \text{for } 1 \leqslant i, j \leqslant n+1,$$
$$\left(X_i^{n+1}, \frac{e_{2j-1}}{\lambda_{2j-1}}\right) = \begin{cases} \alpha_{2i-1} & \text{if } j = i, \\ 0 & \text{if } j < i, \end{cases}$$
$$(\varepsilon_{2i}, Y_j^{n+1}) = \begin{cases} \alpha_{2j} & \text{if } i = j, \\ 0 & \text{if } i < j. \end{cases}$$

Furthermore

$$\left\|X^{n+1} - (X^n, 0)\right\| \leqslant \left\|X^{n+1} - \overline{X}^n\right\| + \|U^n\| < \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} < \frac{1}{2^n}$$

and

$$\left\|Y^{n+1} - (Y^n, 0)\right\| < \frac{1}{2^n}.$$

Thus the sequences $(X_i^n)_{n \geqslant i}$ and $\left(Y_j^n\right)_{n \geqslant j}$ converge in norm. Let respectively $X_i$ and $Y_j$ be their limits. Then we have by going to the limits

$$\begin{cases} [L_{ij}] = [X_i \otimes Y_j] \\ (RX_i, Y_j) = d_{ij} \end{cases} \quad i, j \geqslant 1.$$

Thus $\psi_{ij} = [X_i \otimes Y_j]_{\mathcal{A}+\mathbb{C}K}$ for all $i, j \geqslant 1$, and the proof is complete. ∎

We have shown in [2], Proposition 3.3 one consequence of the properties $E^r_{0,\gamma}$ and $E^l_{0,\gamma}$.

PROPOSITION 3.3. *Assume $\mathcal{A}$ a is weak*-closed subspace of $\mathcal{L}(\mathcal{H})$. If $\mathcal{A}$ has one of the properties $E^r_{0,\gamma}$ or $E^l_{0,\gamma}(0 < \gamma \leqslant 1)$, then $\mathcal{A} \cap \mathcal{K}(\mathcal{H}) = \{0\}$.*

*Proof.* Without loss of generality, we may suppose that $\mathcal{E}^r_0(\mathcal{A})$ *or* $\mathcal{E}^l_0(\mathcal{A})$ contains $(Q_{\mathcal{A}})_\gamma = \{[L] \in Q_{\mathcal{A}} / \|[L]\| < \gamma\}$. Then for every $[L] \in (Q_{\mathcal{A}})_\gamma$, there exist $(x_n)_{n \geqslant 1}$ and $(y_n)_{n \geqslant 1}$ *in* $(\mathcal{H})_1$ which converge weakly to 0, and $\lim_{n\to\infty} \|[L] - [x_n \otimes y_n]\| = 0$.

Let $K \in \mathcal{A} \cap \mathcal{K}(\mathcal{H})$, then we have

$$\lim_{n\to\infty} \langle K, [L] - [x_n \otimes y_n] \rangle = 0,$$
$$\lim_{n\to\infty} (Kx_n, y_n) = \operatorname{tr}(KL).$$

Since $K$ is compact and $(x_n)_{n \geqslant 1}$ converge weakly to 0, then $\|Kx_n\| \to 0$. As $\|y_n\| \leqslant 1$ for $n \geqslant 1$, $|(Kx_n, y_n)| \leqslant \|Kx_n\| \to 0$, we have $\operatorname{tr}(KL) = 0$ for every $[L] \in (Q_{\mathcal{A}})_\gamma$. Then $K = 0$. ∎

It is obvious that the property $X_{0,\gamma}$ implies the properties $E^r_{0,\gamma}$ and $E^l_{0,\gamma}$ $(0 < \gamma \leqslant 1)$. Thus we obtain the following corollary.

COROLLARY 3.4. *Suppose $\mathcal{A}$ is a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$, and $K \in \mathcal{K}(\mathcal{H}) \setminus \mathcal{R}(\mathcal{H})$. Then $\mathcal{A} + \mathbb{C}K$ has not the property $X_{0,\gamma}$.*

We conclude that for every weak*-closed subspace $\mathcal{A}$ with the property $X_{0,\gamma}$, $(0 < \gamma \leqslant 1)$ and $K \in \mathcal{K}(\mathcal{H}) \setminus \mathcal{R}(\mathcal{H})$, $\mathcal{A} + \mathbb{C}K$ has the property $(\mathbb{A}_{\aleph_0})$ without having any property $X_{0,\tau}$; this proves that Theorem 1.5 fails in a general case.

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