J. OPERATOR THEORY
34(1995), 57–89

A PATH MODEL FOR CIRCLE ALGEBRAS

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Communicated by Norberto Salinas

ABSTRACT. Using groupoid theory, we construct a path model for *finite type* embeddings of circle algebras that generalizes the path model of Ocneanu and Sunder for Bratteli diagrams. The Jones-Watatani index is computed using the maps induced on K_0-theory by the embedding and its dual. The analysis is based on imprimitivity groupoids associated to the embeddings. Taking inductive limits, we obtain generalizations of the Bunce-Deddens algebras.

KEYWORDS: *Path model, circle algebras, imprimitivity groupoid, Jones-Watatani index, K-theory.*

AMS SUBJECT CLASSIFICATION: 46L05, 46L55.

0. INTRODUCTION

The notion of Bratteli diagram was introduced in [4], and since then, it has played an important role in the study of AF-algebras. Subsequently, Ocneanu and Sunder introduced a path model for inclusions of finite dimensional algebras and used it to analyze the index theory for subfactors. In this paper, we investigate algebras that arise if one replaces the points in a Bratteli diagram with more general spaces. More precisely, we use groupoid theory to construct path models for certain inclusions of circle algebras. We compute the Watatani index for such a pair of algebras, using the imprimitivity groupoid construction and the transfer map in K-theory. It turns out that these inclusions are built using two basic homomorphisms, the composition with a covering map, and the so called k-times around embedding, maps which are dual to each other in a certain sense made explicit using imprimitivity groupoids. We consider inductive limits determined by such inclusions and we obtain a class of C^*-algebras that generalize the Bunce-Deddens algebras. However, in general, they are neither simple nor have real rank zero. These inductive limits are obtained either from a given infinite diagram, or from a diagram with two levels. In the second case, iterating the imprimitivity groupoid construction, we get an infinite diagram, in which the consecutive floors are symmetric.

Inductive limits of various homogeneous algebras have been studied intensively in recent years by many people, and generalized Bratteli diagrams have appeared also in [1] and [27]. Our use of groupoids provides a new perspective on these and highlights the relations among the basic construction, groupoid actions and imprimitivity groupoids.

1. ACTIONS OF GROUPOIDS ON SPACES

In this section, we will give some basic definitions and some examples of groupoid actions that are necessary to develop our theory. The fundamental philosophy to keep in mind is that groups act on spaces, groupoids act on *fibered* spaces. More precisely, let Γ be a locally compact Hausdorff groupoid with unit space Γ^0, and let r and s denote the range and source maps. We say that Γ acts (to the left) on a locally compact space X if there are a continuous, open surjection

$$\rho : X \to \Gamma^0$$

and a continuous map

$$\Gamma * X \to X, \ (\gamma, x) \to \gamma \cdot x,$$

where

$$\Gamma * X = \{(\gamma, x) \in \Gamma \times X \mid s(\gamma) = \rho(x)\},$$

that satisfy

(i) $\rho(\gamma \cdot x) = \rho(\gamma), \ \forall (\gamma, x) \in \Gamma * X$;

(ii) $(\gamma_1, x) \in \Gamma * X, \ (\gamma_2, \gamma_1) \in \Gamma^{(2)}$ implies $(\gamma_2\gamma_1, x), (\gamma_2, \gamma_1 \cdot x) \in \Gamma * X$ and

$$\gamma_2 \cdot (\gamma_1 \cdot x) = (\gamma_2\gamma_1) \cdot x;$$

(iii) $\rho(x) \cdot x = x, \ \forall x \in X$.

Right actions are defined in a similar way.

We say that the action is *free* if $\gamma \cdot x = x$ only when γ is a unit. The action is called *proper* if the map

$$\Gamma * X \to X \times X, \ (\gamma, x) \to (\gamma \cdot x, x)$$

is proper. The fibered space X is called a *principal* Γ-space if the action is both free and proper. This case is of particular interest, since one can then define what is called *the imprimitivity groupoid.* Start with

$$X * X = \{(x, y) \in X \times X \mid \rho(x) = \rho(y)\},$$

and let

$$X *_\Gamma X = \Gamma \setminus X * X$$

be the orbit space under the diagonal action of Γ :

$$\gamma \cdot (x, y) = (\gamma \cdot x, \gamma \cdot y).$$

The elements in $X *_\Gamma X$ will be denoted by $[x, y]$, and the elements in $\Gamma \setminus X$ by $[x]$. Then $X *_\Gamma X$ has a groupoid structure with multiplication

$$[x, y] \cdot [y', z] = [x, \gamma^{-1} \cdot z],$$

where we require $[y] = [y']$, and γ is the unique element with $y' = \gamma \cdot y$. The unit space is $\Gamma \setminus X$, and the range and source maps are given by

$$r([x, y]) = [x], \ s([x, y]) = [y].$$

The groupoid $X *_\Gamma X$ acts on X to the right via

$$\sigma : X \to \Gamma \setminus X,$$

the quotient map. If

$$X * (X *_\Gamma X) = \{(z, [x, y]) \in X \times (X *_\Gamma X) \mid [z] = [x]\},$$

then the action is given by

$$z \cdot [x, y] = \gamma \cdot y,$$

where γ is the unique element of Γ such that $z = \gamma \cdot x$. The action is well defined: if $[x', y'] = [x, y]$, then there is a unique $\beta \in \Gamma$ such that $x' = \beta \cdot x$ and $y' = \beta \cdot y$. It follows that $[x'] = [z]$ when $[x] = [z]$, and $\gamma\beta^{-1} \cdot x' = z$. Therefore,

$$z \cdot [x', y'] = \gamma\beta^{-1} \cdot y' = \gamma \cdot y = z \cdot [x, y].$$

Note that the two actions commute and that X realizes what is called a $(\Gamma, X *_\Gamma X)$-equivalence. Moreover, if both Γ and $X *_\Gamma X$ have Haar systems, then the C^*-algebras $\mathbf{C}^*(\Gamma)$ and $\mathbf{C}^*(X *_\Gamma X)$ are strongly Morita equivalent (see [16]). It should be noted that given Γ acting freely and properly on X, there is no reason *a priori* to expect that there be a Haar system on $X *_\Gamma X$. One needs additional hypotheses; for example, the assumption that there is a Γ-invariant ρ-system on X in the sense of Renault ([22]) guaranties that $X *_\Gamma X$ has a Haar system. Fortunately for the purposes of the present paper, the existence of a Haar system on $X *_\Gamma X$ will be evident from the contexts considered. Let us mention also that, although all the groupoids we consider will be Hausdorff, the notions of groupoid action and equivalence of groupoids make sense also in the non-Hausdorff case (see [22]).

Some of the following examples can be found in [16].

EXAMPLE 1.1. Let $\Gamma = G$ be a group acting freely and properly on X. Then $X \to G \setminus X$ is a principal G-bundle and $X *_G X$ is a transitive groupoid (it has only one orbit). Indeed, since G is a group, $G^0 = \{e\}$ and $X * X = X \times X$. Given $[x], [y] \in G \setminus X, [x, y] \in G \setminus (X \times X)$ is well defined and maps $[x]$ onto $[y]$.

To specialize this example, let X be a locally compact space that is connected, locally arcwise connected and semilocally simply connected. If the fundamental group $G = \pi_1(X, x_0)$ acts on $\tilde{X}$, the universal covering space, in the usual way, then the imprimitivity groupoid is the fundamental groupoid of X, obtained from the homotopy classes of paths in X, with the usual structure.

EXAMPLE 1.2. Recall that for any space X we have the trivial and the cotrivial groupoids $X \times X$ and $\Delta_X \subset X \times X$, respectively, the second being identified with X. Let $X \xrightarrow{\sigma} B$ be a covering map. Then B, viewed as the cotrivial groupoid, acts on X via the formula

$$b \cdot x = x \quad \text{if} \quad \sigma(x) = b.$$

The resulting imprimitivity groupoid

$$X *_B X = \{(x, y) \in X \times X \mid \sigma(x) = \sigma(y)\}$$

is an equivalence relation. It carries a Haar system, namely, the counting measures on the σ-fibers, and the associated C^*-algebra was first studied by Kumjian in [14].

EXAMPLE 1.3. Let G be a group with a subgroup H, and let H act on G by left multiplication. Then $G *_H G \simeq (H \setminus G) \times G$, where G acts on $H \setminus G$ by right multiplication and $(H \backslash G) \times G$ is the corresponding transformation group groupoid. Indeed, in this case $G * G = G \times G$ and the map

$$[g, g'] \mapsto ([g], g^{-1}g')$$

is the desired isomorphism.

EXAMPLE 1.4. (*Bi-transformation groups*, see [25]) Let G and H be groups acting freely and properly on a locally compact space X, and assume that the actions commute. It follows that the orbit spaces X/G and X/H are locally compact spaces and the commutativity assumption implies that X/G carries an H action, while X/H carries a G action. The groupoid $\Gamma = (X/G) \times H$ acts on X, and the imprimitivity groupoid is isomorphic to $(X/H) \times G$. Indeed, let $\rho : X \to X/G$ be the canonical map. The action is

$$(\rho(x), h) \cdot y = h \cdot y \quad \text{if} \quad \rho(y) = \rho(x),$$

and the orbit space $\Gamma \setminus X$ is homeomorphic to X/H. Then $X *_\Gamma X$ is isomorphic to $(X/H) \times G$ via the map

$$[x, y] \mapsto (\sigma(x), g),$$

where $\sigma : X \to X/H$ is the canonical map, and $g \in G$ is the unique element such that $y = g \cdot x$. Therefore $C(X/G) \times H$ and $C(X/H) \times G$ are strongly Morita equivalent.

EXAMPLE 1.5. Let $X = \Gamma$ be a groupoid and let Γ act on itself by left multiplication. Then $\Gamma *_\Gamma \Gamma \simeq \Gamma$ by the map $[\gamma, \eta] \to \gamma^{-1}\eta$, with inverse $\gamma \to [r(\gamma), \gamma]$.

2. GENERALIZED BRATTELI DIAGRAMS AND THEIR C^*-ALGEBRAS

The notion of Bratteli diagram introduced in [4] has a groupoid approach in the path model introduced independently by Ocneanu (see [18]) and Sunder (see [26]). In an attempt to construct a path model for inclusions of C^*-algebras other than the finite dimensional ones, the first step is to replace the discrete spaces of vertices at each level by more general spaces. Edges, then, become relations between consecutive pairs of vertex spaces. Problems about Haar systems and continuity arise and special conditions need to be placed on the spaces of edges. We have found the situation when the edge spaces are the graphs of covering maps to be the most tractable. The following construction is inspired by the works of Ocneanu and Sunder, and will be applied here mainly for inclusions of circle algebras.

Consider a sequence of compact Hausdorff spaces, $L_0, L_1, \ldots, L_n, \ldots$. These will be the spaces of *vertices* on each level. Suppose that L_0 is connected, and that $L_1, L_2, \ldots, L_n, \ldots$ each have finitely many components. The assumption on L_0 is not essential, but is convenient in some cases. Let E_i^{i+1} be a compact subset of $L_i \times \mathbf{N} \times L_{i+1}$; we call the elements of E_i^{i+1} *edges* between level L_i and level L_{i+1}. We need to assume several things about E_i^{i+1}. For $\gamma = (x_i, m_i, x_{i+1}) \in E_i^{i+1}$, we write $s(\gamma) = x_i$ for the *source* of γ and $r(\gamma) = x_{i+1}$ for the *range* of γ. We shall assume that for each i, r is a surjective local homeomorphism, but we require only that each s be continuous and onto. However, in our most important applications, s will also be a local homeomorphism. Since E_i^{i+1} is compact, the cardinality of $r^{-1}(\cdot)$ is finite and constant on each connected component of L_i. The number m_i in (x_i, m_i, x_{i+1}) should be viewed as indexing a particular edge from x_i to x_{i+1}. The number of distinct m_i that occur for a given pair (x_i, x_{i+1}) is the *multiplicity* of the edges with the prescribed vertices. It follows that each multiplicity is finite. If the multiplicity is one for each pair of vertices, we shall drop reference to m_i and $\mathbb{N}$ altogether, and simply view E_i^{i+1} as a subset of $L_i \times L_{i+1}$.

By a *path* from L_0 to L_n we mean a concatenation of edges

$$\gamma = \gamma_1 \cdots \gamma_n,$$

where $\gamma_i \in E_{i-1}^i$ and the equation $r(\gamma_i) = s(\gamma_{i+1})$ is satisfied for all i, $i = 1, \ldots, n-1$. Let $X_0 = L_0$ and for $n \geqslant 1$, let X_n be the space of paths from L_0 to L_n, with the relative topology from $E_0^1 \times \ldots \times E_{n-1}^n$. It follows that the maps

$$\begin{aligned} &\rho_n : X_{n+1} \to X_n,\ \rho_n(\gamma_1 \cdots \gamma_{n+1}) = \gamma_1 \cdots \gamma_n, \quad n \geqslant 1, \\ &\rho_0 : X_1 \to X_0,\ \rho_0(\gamma_1) = s(\gamma_1) \end{aligned} \tag{2.1}$$

are all continuous.

DEFINITION 2.1. By a *generalized Bratteli diagram* we mean a sequence of vertex spaces $\{L_n\}$ and edge spaces $\{E_n^{n+1}\}$ together with the spaces of paths $\{X_n\}$, satisfying the above properties.

In order to associate a sequence of C^*-algebras to such a diagram, consider the equivalence relations

$$R_n \subset X_n \times X_n,\ R_n = \{(\gamma, \gamma') \mid r(\gamma) = r(\gamma')\}.$$

Here for $\gamma = \gamma_1 \ldots \gamma_n$, we take $r(\gamma) := r(\gamma_n)$. Each R_n is endowed with the relative topology. Since r is a local homeomorphism, counting measures give a Haar system (see [21]). The range and source maps for the principal r-discrete groupoids R_n are denoted also by r, s:

$$r(\gamma, \gamma') = \gamma,\ s(\gamma, \gamma') = \gamma'.$$

Note that while there is a multiplicity of uses for r and s, it will be clear from the context which maps we are talking about.

THEOREM 2.2. *With the above notation,* $\mathbf{C}^*(R_n)$ *is a unital continuous trace* C^**-algebra, and is associated to a continuous field of matrix algebras over* L_n.

Proof. Since the spaces X_n and R_n are compact, the equivalence relation R_n is proper, and the structure of its C^*-algebra is determined in [17]. In particular, it follows that the spectrum of $\mathbf{C}^*(R_n)$ is the orbit space $L_n \simeq R_n \backslash X_n$. ∎

REMARK 2.3. Because $\mathbf{C}^*(R_n)$ is unital and continuous trace, it can be decomposed into a finite direct sum of homogeneous C^*-algebras, where the summands are indexed by the connected components of L_n, denoted $L_{n,j}$. From the above theorem, it follows that the internal structure of the algebras $\mathbf{C}^*(R_n)$ may be quite complicated, depending on some cohomology properties of the spaces L_n. If the Brauer group (the torsion part of $H^3(L_n, \mathbb{Z})$) is trivial, then each direct summand is the C^*-algebra of sections of a complex vector bundle over the corresponding component $L_{n,j}$ (see [8], [10]). In particular, if every such vector bundle is trivial, then

$$\mathbf{C}^*(R_n) \simeq \bigoplus_j C(L_{n,j}) \otimes M_{[n,j]},$$

where $[n,j]$ is an integer depending on n and j. It is in this particular case that we are doing concrete computations.

PROPOSITION 2.4. *Consider the map*

$$\Phi_n : \mathbf{C}^*(R_n) \to \mathbf{C}^*(R_{n+1})$$

given by

$$(\Phi_n f)(\gamma, \gamma') = \begin{cases} f(\rho_n(\gamma), \rho_n(\gamma')) & \text{if } \gamma_{n+1} = \gamma'_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

where $f \in C(R_n)$*, and* ρ_n *is defined in* (2.1). *Then* Φ_n *is a unital,* 1-1, $*$*-homomorphism which takes* $C(X_n)$ *into* $C(X_{n+1})$.

Proof. Note that if $\gamma_{n+1} = \gamma'_{n+1}$, then $(\rho_n(\gamma), \rho_n(\gamma')) \in R_n$, and $f(\rho_n(\gamma), \rho_n(\gamma'))$ makes sense. The map is obviously linear. If $(\Phi_n f)(\gamma, \gamma') = 0$ for $f \in \mathbf{C}^*(R_n)$, then $f(\rho_n(\gamma), \rho_n(\gamma')) = 0$ and $f = 0$ since ρ_n is onto. Note that as spaces of functions, $C(R_n) = \mathbf{C}^*(R_n)$. This is because R_n is compact and amenable (see [21], Proposition II.4.2, page 99). To check that it is unital, note that the unit of $\mathbf{C}^*(R_n)$ is the constant function 1 supported on the diagonal of $X_n \times X_n$, and $(\Phi_n(1))(\gamma, \gamma') = 1$ if $\rho_n(\gamma) = \rho_n(\gamma')$ and $\gamma_{n+1} = \gamma'_{n+1}$ (i.e. $\gamma = \gamma'$), and is 0 otherwise. To check that Φ_n is multiplicative, let $f, g \in C(R_n)$. Then

$$\Phi_n(f * g)(\gamma, \gamma') = \sum_\alpha f(\rho_n(\gamma), \alpha) g(\alpha, \rho_n(\gamma')), \quad \text{if} \quad \gamma_{n+1} = \gamma'_{n+1},$$

and is 0 otherwise. On the other hand,

$$(\Phi_n f) * (\Phi_n g)(\gamma, \gamma') = \sum_\eta (\Phi_n f)(\gamma, \eta)(\Phi_n g)(\eta, \gamma')$$
$$= \sum_\eta f(\rho_n(\gamma), \rho_n(\eta))g(\rho_n(\eta), \rho_n(\gamma'))$$

if $\gamma_{n+1} = \eta_{n+1} = \gamma'_{n+1}$, and is 0 otherwise. It follows that

$$\Phi_n(f * g) = (\Phi_n f) * (\Phi_n g),$$

since $\alpha = \rho_n(\eta)$ does not depend on γ_{n+1}. It follows that Φ_n is an embedding of $C^*(R_n)$ into $C^*(R_{n+1})$. Finally, if $f \in C^*(R_n)$ is supported on the diagonal, then $\Phi_n f$ is also supported on the diagonal. Therefore

$$\Phi_n(C(X_n)) \subset C(X_{n+1}). \quad \blacksquare$$

We now assume s is a local homeomorphism. This fact will allow us to consider the imprimitivity groupoid for a groupoid action and moreover, to built an infinite diagram and a tower of C^*-algebras from a diagram with two levels. The groupoid $\Gamma = R_n$ acts to the left on the space $X = R_{n+1}$ in the following way: observe that

$$\rho_n \circ r : R_{n+1} \to R_n^0,$$

is open. Thus we may set

$$R_n * R_{n+1} = \{((\gamma, \gamma'), (\eta, \eta')) \mid \rho_n(\eta) = \gamma' \},$$

and define the action via the formula

$$(\gamma, \gamma') \cdot (\gamma' \eta_{n+1}, \eta') = (\gamma \eta_{n+1}, \eta'),$$

see Figure 1. Here L_0 is on the top of the diagram, the paths γ and γ' are from level 0 to level n, the path η' is from level 0 to level $n+1$, and η_{n+1} is an edge from level n to level $n+1$. The edge $\tilde{\eta}_{n+1}$ from level $n+1$ to level $n+2$ is the mirror image of η_{n+1}.

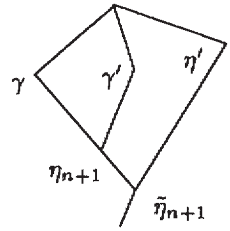

Figure 1.

The action is free: if

$$(\gamma,\gamma')\cdot(\gamma'\eta_{n+1},\eta')=(\gamma'\eta_{n+1},\eta'),$$

then $\gamma=\gamma'$, and (γ,γ') is a unit. The action is proper since all the spaces are compact. In order to identify $R_{n+1}*_{R_n}R_{n+1}$, note that the orbit space $R_n\backslash R_{n+1}$ may be identified with $X_{n+1}\tilde{E}_n^{n+1}$, where $\tilde{E}_n^{n+1}$ is the mirror image of E_n^{n+1}. That means, each $(x_n,m,x_{n+1})\in E_n^{n+1}$ becomes the element $(x_{n+1},m,x_n)\in\tilde{E}_n^{n+1}$. Therefore a path in X_{n+1} may be followed by an edge in $\tilde{E}_n^{n+1}$. The identification map is

$$[\gamma\tau,\eta]\mapsto\eta\tilde{\tau},$$

where $(\gamma\tau,\eta)\in R_{n+1},\tau\in E_n^{n+1}$, and $[\gamma\tau,\eta]$ denotes its class in $R_n\backslash R_{n+1}$. The map is well defined: if $[\gamma\tau,\eta]=[\gamma'\tau',\eta']$, then there is $(\alpha,\beta)\in R_n$ with

$$(\gamma'\tau',\eta')=(\alpha,\beta)\cdot(\gamma\tau,\eta),$$

so that

$$\gamma'=\alpha,\ \eta'=\eta,\ \tau'=\tau.$$

It follows that $R_{n+1}*_{R_n}R_{n+1}$ may be identified with an equivalence relation on $X_{n+1}\tilde{E}_n^{n+1}$,

$$R_{n+1}*_{R_n}R_{n+1}=\{(\eta\tilde{\tau},\eta'\tilde{\tau}')\mid s(\tau)=s(\tau')\}.$$

Note that the counting measures form a Haar system for $R_{n+1}*_{R_n}R_{n+1}$ when

$$s:E_n^{n+1}\to L_n$$

is a local homeomorphism, since in that case $s^{-1}(x)$ has the same (finite) number of elements for each connected component of L_n. Moreover, R_{n+1} acts on $R_{n+1}*_{R_n}R_{n+1}$ to the left by

$$(\xi,\eta)\cdot(\eta\tilde{\tau},\eta'\tilde{\tau}')=(\xi\tilde{\tau},\eta'\tilde{\tau}'),$$

where $X_{n+1}\tilde{E}_n^{n+1}$ is fibered over X_{n+1} via $\eta\tilde{\tau}\mapsto\eta$. One may continue, consider

$$(R_{n+1}*_{R_n}R_{n+1})*_{R_{n+1}}(R_{n+1}*_{R_n}R_{n+1}),$$

and iterate the previous analysis. Note the analogy with the Jones basic construction for a pair of II_1 factors.

THEOREM 2.5. *The C^*-algebra of the imprimitivity groupoid $R_{n+1} *_{R_n} R_{n+1}$ has the same spectrum as $\mathbf{C}^*(R_n)$, and the set of edges from L_{n+1} to $L_{n+2} = L_n$ is obtained by reversing the edges in E_n^{n+1}. Moreover, the map*

$$\widehat{\Phi}_n : \mathbf{C}^*(R_{n+1}) \to \mathbf{C}^*(R_{n+1} *_{R_n} R_{n+1})$$

defined by the formula

$$(\widehat{\Phi}_n f)(\gamma\tilde{\tau}, \gamma'\tilde{\tau}') = \begin{cases} f(\gamma,\gamma') & \text{if } \tau = \tau' \\ 0 & \text{otherwise} \end{cases}$$

is an embedding, called the dual of Φ_n.

Proof. This follows from the above discussion and from a result in [16] asserting that the C^*-algebras of two equivalent groupoids are strongly Morita equivalent. In particular, $\mathbf{C}^*(R_n)$ and $\mathbf{C}^*(R_{n+1} *_{R_n} R_{n+1})$ have the same spectrum. Note that if we reverse once again the edges, we get back E_n^{n+1}. ∎

REMARK 2.6. In the case $\mathbf{C}^*(R_n)$ and $\mathbf{C}^*(R_{n+1})$ are direct sums of homogeneous C^*-algebras of the form $C(Y) \otimes M_k$, the structure of the homomorphisms Φ_n was studied by Dădârlat, Thomsen (see [6], [27]) and others. Using the same technique, we will consider an irreducible representation ω of $\mathbf{C}^*(R_{n+1})$ and we will decompose $\omega \circ \Phi_n$ into irreducible representations of $\mathbf{C}^*(R_n)$. Recall from [17] that $\mathbf{C}^*(R_n)$ may be represented on $L^2(R_n, \lambda_u)$ for $u = u_1 \cdots u_n \in X_n$ by the formula

$$((\pi^u f)\varphi)(\gamma,\gamma') = \begin{cases} \sum\limits_{\alpha} f(\gamma,\alpha)\varphi(\alpha,u) & \text{if } \gamma' = u \\ 0 & \text{otherwise,} \end{cases}$$

where $f \in \mathbf{C}^*(R_n), \varphi \in L^2(R_n,\lambda_u), \lambda_u(P) := \lambda^u(P^{-1})$, and $\{\lambda^u\}$ is the Haar system on R_n given by the counting measures. Note that $\operatorname{supp}(\lambda_u) = s^{-1}(u)$. By Lemma 2.4 in [17], π^u is irreducible, and $\pi^u, \pi^{u'}$ are unitarily equivalent iff $[u] = [u'] \in L_n$, i.e. iff u, u' have the same orbit.

PROPOSITION 2.7. *Fix a unit $v \in X_{n+1}$, and let E^v be the set of edges in E_n^{n+1} with the same range as v. Then the irreducible representation π^v, when restricted to $\mathbf{C}^*(R_n)$, decomposes into a direct sum of irreducible representations*

$$\bigoplus_{u \in E^v} \pi^u,$$

where for each $e \in E^v$, we choose only one $u \in X_n$ with $r(u) = s(e)$. Therefore, to each point $[v] \in L_{n+1}$ there corresponds a finite set $\{[u^1],[u^2],\ldots,[u^k]\} \subset L_n$, where we count also multiplicities (see Figure 2).

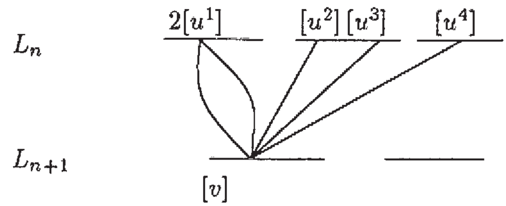

Figure 2.

Proof. The finite dimensional Hilbert space $L^2(R_{n+1}, \lambda_v)$ has basis

$$\{\varepsilon_\beta \mid \beta \in X_{n+1}, (\beta, v) \in R_{n+1}\}, \;\; \varepsilon_\beta(\eta, v) = \delta_{\beta\eta},$$

therefore $\dim L^2(R_{n+1}, \lambda_v)$ is the number of paths ending in $r(v)$. Hence $L^2(R_{n+1}, \lambda_v)$ decomposes into a direct sum

$$\bigoplus_{e \in E^v} L^2(R_n, \lambda_u),$$

where we choose only one u for each $s(e)$. For fixed $e \in E^v$ and $u \in X_n$ with $r(u) = s(e)$, $L^2(R_n, \lambda_u)$ is the Hilbert space generated by $\{\varepsilon_\alpha \mid \alpha \in [u]\}$, and $L^2(R_n, \lambda_u)$ is embedded as a subspace of $L^2(R_{n+1}, \lambda_v)$ by $\varepsilon_\alpha \mapsto \varepsilon_{\alpha e}$. Now

$$\begin{aligned}((\pi^v \circ \Phi_n)f)\varphi(\beta, v) &= \sum_\eta (\Phi_n f)(\beta, \eta)\varphi(\eta, v) \\ &= \sum_e \sum_\alpha (\Phi_n f)(\beta, \alpha e)\varphi(\alpha e, v) \\ &= \sum_e \sum_\alpha f(\rho_n(\beta), \alpha)\varphi(\alpha e, v) \\ &= \Big(\sum_e \pi^u f\Big)\varphi(\beta, v). \quad \blacksquare\end{aligned}$$

DEFINITION 2.8. For a fixed point $[v] \in L_{n+1}$, the set $\{[u^1], [u^2], \ldots, [u^k]\} \subset L_n$ which appears in Proposition 2.7 is called the *spectrum* of Φ_n at $[v]$, denoted by $SP(\Phi_n, [v])$.

Let's see now what happens if we remove some levels of the diagram. If we delete the level L_i, we obtain another diagram, where the edges from L_{i-1} to L_{i+1} are just concatenations of edges in E^i_{i-1} and E^{i+1}_i. Let's check that $\Phi_i \circ \Phi_{i-1} = \Phi^{i+1}_{i-1}$, where $\Phi^{i+1}_{i-1} : \mathrm{C}^*(R_{i-1}) \to \mathrm{C}^*(R_{i+1})$,

$$(\Phi^{i+1}_{i-1} f)(\xi, \xi') = \begin{cases} f(\rho_{i-1}\rho_i\xi, \rho_{i-1}\rho_i\xi') & \text{if } \xi_i\xi_{i+1} = \xi'_i\xi'_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed,

$$(\Phi_i \circ \Phi_{i-1} f)(\xi,\xi') = \begin{cases} \Phi_{i-1} f(\rho_i \xi, \rho_i \xi') & \text{if } \xi_{i+1} = \xi'_{i+1} \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} f(\rho_{i-1}\rho_i \xi, \rho_{i-1}\rho_i \xi') & \text{if } \xi_{i+1} = \xi'_{i+1} \text{ and } \xi_i = \xi'_i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, it follows that the inductive limit

$$\varinjlim \mathbf{C}^*(R_n)$$

does not change if we compress (or expand) a diagram. In the sequel, for $k < n$, let

$$\rho_k^n := \rho_k \circ \cdots \circ \rho_{n-1}, \quad \Phi_k^n := \Phi_{n-1} \circ \cdots \circ \Phi_k.$$

REMARK 2.9. In the case we have an infinite diagram, we may consider the space of infinite paths

$$X := \varprojlim (X_n, \rho_n)$$

with the projective limit topology, the equivalence relations

$$R^n := \{(\gamma,\gamma') \in X \times X \mid \gamma_p = \gamma'_p \ \ \forall p \geqslant n+1\},$$

with the induced topologies, and

$$R^\infty := \bigcup_n R^n,$$

with the inductive limit topology. Note that each R^n has a Haar system given by the counting measures when the maps $r : E_n^{n+1} \to L_n$ are local homeomorphisms. Also, note that R^n is open in R^{n+1}, and their Haar systems are compatible. It follows that R^∞ has a Haar system (see [21], page 122). We have maps

$$r_n : X \to X_n,\ r_n(\gamma_1\gamma_2\cdots) = \gamma_1\cdots\gamma_n,\ n \geqslant 1,$$

$$r_0 : X \to X_0,\ r_0(\gamma) = s(\gamma),$$

and $R_n = r_n^{(2)}(R^n)$. Therefore $\mathbf{C}^*(R^\infty)$ is isomorphic to

$$A := \varinjlim \mathbf{C}^*(R_n).$$

Consider the conditional expectation

$$E : \mathbf{C}^*(R^\infty) \to C(X),\ (Ef)(\gamma) = f(\gamma,\gamma) \ \text{ for } f \in C_c(R).$$

PROPOSITION 2.10. (i) *Let φ be a state on A. Then there is a unique probability measure μ defined on the Borel sets of X, identified with the diagonal in R^∞, such that*

$$\varphi(f) = \int f(\gamma, \gamma)\, \mathrm{d}\mu(\gamma) \quad \forall f \in C(X). \tag{2.2}$$

(ii) *If μ is a probability measure defined on the Borel sets of X, then there is a unique state φ on A which satisfies* (2.2) *and $\varphi = \varphi \circ E$.*

Thus there is a bijection between the probability measures on X and the states φ that satisfy $\varphi = \varphi \circ E$.

Proof. (i) Considering $\varphi \mid C(X_n)$, it follows that for each n there is a unique probability measure μ_n defined on the Borel sets of X_n such that

$$\varphi(f) = \int_{X_n} f(\gamma, \gamma)\, \mathrm{d}\mu_n(\gamma), \quad f \in C(X_n).$$

Since $(\varphi \mid C(X_{n+1})) \mid C(X_n) = \varphi \mid C(X_n)$, the sequence of measures $\{\mu_n\}$ is consistent, and there is a unique probability measure μ on X such that

$$\mu(r_n^{-1}(F)) = \mu_n(F) \quad \forall n\, \forall F \subset X_n.$$

(ii) Given μ, let

$$\varphi_0(f) := \int f(\gamma, \gamma)\, \mathrm{d}\mu(\gamma), \quad f \in C(X).$$

Simply define φ to be $\varphi_0 \circ E$. ∎

3. EXAMPLES

EXAMPLE 3.1. The usual Bratteli diagrams are obtained when each L_i consists of a finite number of points. In this case, the $\mathrm{C}^*(R_n)$ are finite dimensional and the Φ_n are precisely the inclusions of multimatrix algebras given by the diagram. The dual homomorphism $\widehat{\Phi}_n$, obtained by the basic construction, induces the matrix at the level of K_0-theory that is equal to the transpose of the matrix induced by Φ_n (see [11], Lemma 3.3.1).

EXAMPLE 3.2. In this example we associate a diagram to a free action of a finite group. Let G be such a group, let $k = |G|$, and suppose G is acting freely on a connected compact space L. Let $L_0 = L/G$, let $L_1 = L$, and let $\sigma : L \to L/G$

be the corresponding covering map. Consider the space of paths $X_1 = E_0^1 = \{(\sigma(x), x) \mid x \in L\}$. Then X_1 is homeomorphic to L, and the equivalence relation R_1 may be identified with the diagonal in $L \times L$. The embedding

$$\mathbf{C}^*(R_0) \xrightarrow{\Phi_0} \mathbf{C}^*(R_1)$$

is just the homomorphism induced by σ,

$$C(L/G) \ni f \longmapsto f \circ \sigma \in C(L).$$

Let's consider the dual embedding. The imprimitivity groupoid $R_1 *_{R_0} R_1$ may be identified with

$$R_2 := \{(x, x') \in L \times L \mid \sigma(x) = \sigma(x')\},$$

which is an r-discrete groupoid, and $\mathbf{C}^*(R_2)$ is associated to a continuous field of $k \times k$ matrices over L/G. Here

$$\tilde{E}_0^1 = \{(x, \sigma(x)) \mid x \in L\}$$

is the graph of σ. In the case $\mathbf{C}^*(R_2) \simeq C(L/G) \otimes M_k$, $\tilde{\Phi}_0$ is the homomorphism compatible with the covering $\sigma : L \to L/G$,

$$(\hat{\Phi}_0 \circ \Phi_0)(f)(x) = \begin{pmatrix} f(x) & & \\ & \ddots & \\ & & f(x) \end{pmatrix}, \quad f \in C(L/G).$$

Note that in general $\mathbf{C}^*(R_2)$ is not a tensor product as above (see the example before Situation 3 in [25]). Note also that one can obtain an infinite diagram, by repeating infinitely many times the above diagram. We have been unable to identify explicitly the corresponding inductive limit C^*-algebra, except in some particular cases.

EXAMPLE 3.3. This is a particular case of the previous example. It is very important for understanding the building blocks we are considering for the circle diagrams in the next section. Consider the diagram which, at each level $0, 1, 2$, has a circle, and in which we join each point $z \in L_1$ with $z^k \in L_0$ ($k \neq 0$ is an integer), and also with $z^k \in L_2$, using a single edge. That is to say, a point $w \in L_0$ is joined with all its k-roots in L_1.

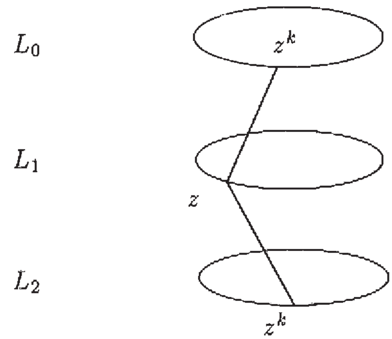

Figure 3.

We have $X_0 = \mathbb{S}^1$,

$$X_1 = \{(z^k, z) \mid z \in \mathbb{S}^1\}$$

is homeomorphic to $\mathbb{S}^1$ by the map $(z^k, z) \mapsto z$, and

$$X_2 = \{(z^k, z, z^k) \mid z \in \mathbb{S}^1\}$$

is also homeomorphic to $\mathbb{S}^1$ by $(z^k, z, z^k) \mapsto z$. The equivalence relation $R_0 \subset X_0 \times X_0$ is the diagonal, $R_1 \subset X_1 \times X_1$ is also the diagonal, since just one path ends at each point on L_1, and $R_2 \subset X_2 \times X_2$ may be identified with

$$\{(z, w) \in \mathbb{S}^1 \times \mathbb{S}^1 \mid w^k = z^k\},$$

with the relative topology. Note that exactly k paths end at each point on L_2. The action of R_0 on R_1 is given, after identifying them with $\mathbb{S}^1$, by

$$z \cdot y = y \quad \text{if} \quad y^k = z.$$

The map

$$\rho_0 : X_1 \to X_0, \ (z^k, z) \mapsto z^k$$

is a covering map and the inclusion

$$\mathbf{C}^*(R_0) \xrightarrow{\Phi_0} \mathbf{C}^*(R_1)$$

is the homomorphism denoted by σ_k:

$$\sigma_k : C(\mathbb{S}^1) \to C(\mathbb{S}^1), \quad (\sigma_k f)(z) = f(z^k).$$

The action of R_1 on R_2 is given by

$$z \cdot (z, w) = (z, w),$$

where

$$\rho_1 : X_2 \to X_1, \quad \rho_1(z^k, z, z^k) = (z^k, z)$$

is the identity map, after identifying X_1 and X_2 with $\mathbb{S}^1$. The homomorphism

$$\mathbf{C}^*(R_1) \xrightarrow{\Phi_1} \mathbf{C}^*(R_2)$$

is given by

$$(\Phi_1 f)(z, w) = \begin{cases} f(z) & \text{if } z = w \\ 0 & \text{otherwise,} \end{cases}$$

for $f \in C(R_1)$. Now $\mathbf{C}^*(R_2)$ is isomorphic to $C(\mathbb{S}^1) \otimes M_{|k|}$ (see [14], Example (iii)), and this isomorphism may be realized in the following way. Endow $\mathbf{C}^*(R_1)$ with a $\mathbf{C}^*(R_0)$-Hilbert module structure via the inner product

$$\langle f_1 \mid f_2 \rangle(z) = \frac{1}{|k|} \sum_{w^k = z} \bar{f}_1(w) f_2(w).$$

An element $h \in C(R_2)$ is viewed as a compact operator T_h on $C(R_1) = C(\mathbb{S}^1)$ via the equation

$$(T_h g)(z) = \sum_v h(z, v) g(v), \ \ g \in C(\mathbb{S}^1).$$

Associate to $T \in \mathcal{K}(C(\mathbb{S}^1))$ the matrix $(T_{jl}) \in C(\mathbb{S}^1) \otimes M_{|k|}$, given by the formula

$$(T_{jl})(z) = \langle g_j \mid T g_l \rangle(z) = \frac{1}{|k|} \sum_{w^k = z} w^{1-j} (T g_l)(w),$$

where $g_j(z) = z^{j-1}$, $j = 1, \ldots, |k|$. In particular, a function f supported on the diagonal of R_2 has the matrix

$$(T_f)_{jl}(z) = \frac{1}{|k|} \sum_{w^k = z} w^{l-j} f(w).$$

Therefore, denoting by u the generator of $\mathbf{C}^*(R_1) = C(\mathbb{S}^1), u(z) = z$, $\Phi_1 u$ has the matrix

$$(T_{jl})(z) = \frac{1}{|k|} \sum_{w^k = z} w^{l-j+1} = \begin{cases} z & \text{if } j = 1, \ l = k, \ k > 0 \\ \bar{z} & \text{if } j = 1, \ l = -k, \ k < 0 \\ 1 & \text{if } j = l + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the embedding

$$\mathbf{C}^*(R_1) \xrightarrow{\Phi_1} \mathbf{C}^*(R_2)$$

is the k-times around embedding, denoted by $\widehat{\sigma}_k$,

$$\widehat{\sigma}_k : C(\mathbb{S}^1) \to C(\mathbb{S}^1) \otimes M_{|k|}.$$

Note that in this case $R_2 = R_1 *_{R_0} R_1$, and the diagram is symmetric with respect to level 1. In the triple

$$C(\mathbb{S}^1) \xrightarrow{\sigma_k} C(\mathbb{S}^1) \xrightarrow{\widehat{\sigma}_k} C(\mathbb{S}^1) \otimes M_{|k|},$$

the second map is the dual of the first and at the level of K-theory those maps induce

$$\text{(3.1)} \qquad \text{at } K_0,\ \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{|k|} \mathbb{Z};\quad \text{and at } K_1,\ \mathbb{Z} \xrightarrow{k} \mathbb{Z} \xrightarrow{\operatorname{sign}(k)} \mathbb{Z}.$$

Indeed, $K_0(C(\mathbb{S}^1)) \simeq \mathbb{Z}$, and is generated by the constant function 1, while $K_1(C(\mathbb{S}^1)) \simeq \mathbb{Z}$ and is generated by $u(z) = z$. Also, $K_0(C(\mathbb{S}^1) \otimes M_{|k|}) \simeq \mathbb{Z}$, and is generated by

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

while $K_1(C(\mathbb{S}^1) \otimes M_{|k|}) \simeq \mathbb{Z}$ is generated by

$$\begin{pmatrix} z & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Now

$$\sigma_k(1) = 1;\ \widehat{\sigma}_k(1) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix};$$

$$\sigma_k(z) = z^k;\ \widehat{\sigma}_k(z) = \begin{pmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad \text{if } k \geqslant 1,$$

which is equivalent to

$$\begin{pmatrix} z & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix};$$

and

$$\widehat{\sigma}_k(z) = \begin{pmatrix} 0 & 0 & \dots & 0 & \bar{z} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad \text{if } k \leqslant 1,$$

which is equivalent to

$$\begin{pmatrix} \bar{z} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Note that if we add a new level to this diagram, by reversing the last floor, we get the bidual embedding

$$C(\mathbb{S}^1) \otimes M_{|k|} \xrightarrow{\widehat{\widehat{\sigma}}_k} C(\mathbb{S}^1) \otimes M_{|k|}, \widehat{\widehat{\sigma}}_k = \sigma_k \otimes \text{id}.$$

Also, by repeating infinitely many times the above diagram, we obtain a tower of circle algebras, where the embeddings $\sigma_k \otimes \text{id}$ and $\widehat{\sigma}_k \otimes \text{id}$ alternate. Taking into account the fact that

$$(\widehat{\sigma}_k \circ \sigma_k)(f) = f \otimes 1,$$

we deduce that the corresponding inductive limit is isomorphic to $C(\mathbb{S}^1) \otimes \text{UHF}(k^\infty)$.

EXAMPLE 3.4. The *Bunce-Deddens algebras* fit in our setting. Let $L_n = \mathbb{S}^1$ for $n = 0, 1, \dots$ and let $\{p_n\}$ be a strictly increasing sequence of positive integers with $p_0 = 1$, and $p_n \mid p_{n+1}$ $\forall n$. Let

$$E_n^{n+1} = \{(z, z^{k_{n+1}}) \mid z \in L_n\},$$

where $k_n := p_n/p_{n-1}$, $n \geqslant 1$. We get an infinite diagram in which R_n may be identified with

$$\{(z, w) \in \mathbb{S}^1 \times \mathbb{S}^1 \mid z^{p_n} = w^{p_n}\},$$

$\mathbf{C}^*(R_n) \simeq C(\mathbb{S}^1) \otimes M_{p_n}$, and the maps Φ_n are the k_n-times around embeddings $\widehat{\sigma}_{k_n}$. Therefore,

$$A := \varinjlim(\mathbf{C}^*(R_n), \Phi_n)$$

is the Bunce-Deddens algebra $\mathrm{BD}(\{p_n\})$.

EXAMPLE 3.5. This example illustrates a more complex circle diagram. Let $L_0 = \mathbb{S}^1$, let $L_1 = L_{1,1} \cup L_{1,2}$, and let $L_2 = L_{2,1} \cup L_{2,2}$ where each $L_{i,j}$ is a copy of $\mathbb{S}^1$. Let

$$E_0^1 = \{(z, m, w) \in L_0 \times \mathbf{N} \times L_{1,1} \mid w = z, m = 1, 2\} \cup \{(z, w) \in L_0 \times L_{1,2} \mid w = z\},$$

and let

$$\begin{aligned} E_1^2 = \{(z, w) \in L_{1,1} \times L_{2,1} \mid w = z^{-2} \text{ or } w = z\} &\cup \{(z, w) \in L_{1,1} \times L_{2,2} \mid w = z^2\} \\ &\cup \{(z, w) \in L_{1,2} \times L_{2,1} \mid w = z^{-3}\} \\ &\cup \{(z, m, w) \in L_{1,2} \times \mathbf{N} \times L_{2,2} \mid z = w^4, m = 1, 2\}. \end{aligned}$$

Here X_0 may be identified with a copy of $\mathbb{S}^1$, X_1 with three copies of $\mathbb{S}^1$, and X_2 with nine copies of $\mathbb{S}^1$, with the usual topology.

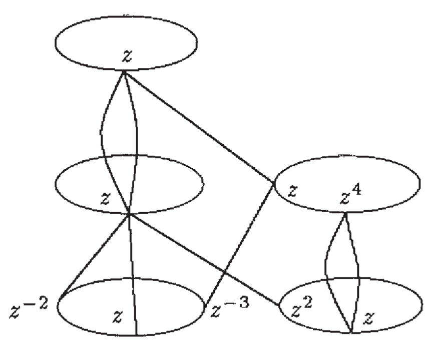

Figure 4.

We can see that

$$\mathbf{C}^*(R_0) = C(\mathbb{S}^1), \ \mathbf{C}^*(R_1) = M_2(C(\mathbb{S}^1)) \oplus C(\mathbb{S}^1),$$

and the inclusion

$$\mathbf{C}^*(R_0) \to \mathbf{C}^*(R_1)$$

is obtained by tensoring the inclusion

$$\mathbf{C} \to M_2 \oplus \mathbf{C},\ \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \oplus \lambda,$$

with $C(\mathbb{S}^1)$. Therefore

$$\Phi_0 f = \begin{pmatrix} \sigma_1 f & 0 \\ 0 & \sigma_1 f \end{pmatrix} \oplus \sigma_1 f.$$

Now $\mathbf{C}^*(R_2) \simeq M_9(C(\mathbb{S}^1)) \oplus M_6(C(\mathbb{S}^1))$ and for $z \in L_{2,1}$,

$$\mathrm{SP}(\Phi_1, z) = \{w \in L_{1,1} \mid w = z \text{ or } w^{-2} = z\} \cup \{w \in L_{1,2} \mid w^{-3} = z\},$$

and for $z \in L_{2,2}$,

$$\mathrm{SP}(\Phi_1, z) = \{w \in L_{1,1} \mid w^2 = z\} \cup \{w \in L_{1,2} \text{ with multiplicity } 2 \mid w = z^4\}.$$

Recall that two $*$-homomorphisms Φ, $\Psi : A \to B$ with B unital are called *approximately inner equivalent* when there is a sequence of unitaries $\{u_n\} \subset B$ such that

$$\lim_{n\to\infty} u_n \Phi(a) u_n^* = \Psi(a),\ \forall a \in A.$$

By a result of Thomsen (see [27], Theorem 2.1), two $*$-homomorphisms between circle algebras are approximately inner equivalent iff they have the same spectrum (see the Definition 2.8). Therefore Φ_1 is approximately inner equivalent to

$$\Psi_1 : M_2(C(\mathbb{S}^1)) \oplus C(\mathbb{S}^1) \to M_9(C(\mathbb{S}^1)) \oplus M_6(C(\mathbb{S}^1)),$$

given by the formula

$$\Psi_1(f \oplus g) = \begin{pmatrix} \sigma_1(f) & 0 & 0 \\ 0 & \widehat{\sigma}_{-2}(f) & 0 \\ 0 & 0 & \widehat{\sigma}_{-3}(g) \end{pmatrix} \oplus \begin{pmatrix} \widehat{\sigma}_2(f) & 0 & 0 \\ 0 & \sigma_4(g) & 0 \\ 0 & 0 & \sigma_4(g) \end{pmatrix}.$$

Since the maps s and r are local homeomorphisms, we can consider $R_2 *_{R_1} R_2$, and the corresponding embedding $\widehat{\Phi}_1$ will be approximately inner equivalent to

$$\widehat{\Psi}_1 : M_9(C(\mathbb{S}^1)) \oplus M_6(C(\mathbb{S}^1)) \to M_{33}(C(\mathbb{S}^1)) \oplus M_{57}(C(\mathbb{S}^1)),$$

where

$$\widehat{\Psi}_1(f \oplus g) = \begin{pmatrix} \widehat{\sigma}_1(f) & 0 & 0 \\ 0 & \sigma_{-2}(f) & 0 \\ 0 & 0 & \sigma_2(g) \end{pmatrix} \oplus \begin{pmatrix} \sigma_{-3}(f) & 0 & 0 \\ 0 & \widehat{\sigma}_4(g) & 0 \\ 0 & 0 & \widehat{\sigma}_4(g) \end{pmatrix}.$$

Computing the maps induced at K-theory by Ψ_1 and $\widehat{\Psi}_1$, we find that at K_0:

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 3 & 1 \\ 1 & 8 \end{pmatrix}} \mathbb{Z}^2;$$

while at K_1:

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & -1 \\ 1 & 8 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & 2 \\ -3 & 2 \end{pmatrix}} \mathbb{Z}^2.$$

EXAMPLE 3.6. In this example we consider a stationary diagram related to a branched covering. The map s will no longer be a local homeomorphism, so that the construction of the imprimitivity groupoid will not be possible. Let $\sigma : I \longrightarrow I$, obtained by stretching and folding the interval $I = [0, 1]$:

$$\sigma(t) = \begin{cases} 2t & 0 \leqslant t \leqslant \frac{1}{2} \\ 2(1-t) & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

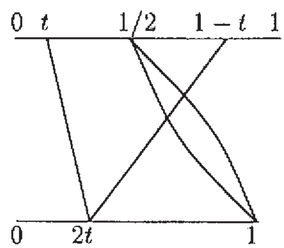

Figure 5.

Let $L_n = I$ and let

$$E_n^{n+1} = \Big\{(t, 1, 2t) \mid 0 \leqslant t \leqslant \frac{1}{2}\Big\} \cup \Big\{(t, 2, 2(1-t)) \mid \frac{1}{2} \leqslant t \leqslant 1\Big\} \quad \forall n \geqslant 0.$$

Note that we have two edges joining $1/2 \in L_n$ with $1 \in L_{n+1}$. It is easily seen that X_n is homeomorphic to a disjoint union of 2^n segments and that $r : X_n \longrightarrow L_n$ are local homeomorphism. It follows that

$$\mathbf{C}^*(R_n) \simeq C(I) \otimes M_{2^n}.$$

Dădârlat (see [6], Remark 2.5) proves that, in the case Y is simply connected and locally pointwise connected, the homomorphisms $C(X) \longrightarrow C(Y) \otimes M_k$ are diagonal up to a unitary equivalence. Taking into account the fact that $\mathrm{SP}(\Phi_n, t) = \{t/2, 1 - t/2\}$, we deduce that

$$\Phi_n : \mathbf{C}^*(R_n) \to \mathbf{C}^*(R_{n+1})$$

is given by

$$(\Phi_n f)(t) = \begin{pmatrix} f\left(\frac{t}{2}\right) & 0 \\ 0 & f\left(1 - \frac{t}{2}\right) \end{pmatrix},$$

up to a unitary equivalence. This example appears in [2], where it is proven that the inductive limit $\varinjlim \mathbf{C}^*(R_n)$ is isomorphic to the UHF-algebra of type 2^∞.

Note that the map $s : X_1 \to L_0$ is not open, therefore one can not form the imprimitivity groupoid $R_1 *_{R_0} R_1$. Another way to see this is that each $t \neq 1/2$ is the starting point of a single edge, but for $t = 1/2$ there are two edges with this initial point.

4. CIRCLE DIAGRAMS AND INDUCTIVE LIMITS OF CIRCLE ALGEBRAS

In this section we consider a diagram in which L_n is a finite disjoint union of unit circles

$$L_n = \bigcup_{j=1}^{k_n} L_{n,j} \quad \forall n \geqslant 0,$$

with $L_{n,j} = \mathbb{S}^1$ for every n, j. We set $k_0 = 1$ for convenience. If

$$E_n^{n+1} = \bigcup_{j,l} E_{n,j}^{n+1,l},$$

where $E_{n,j}^{n+1,l}$ is the space of edges from $L_{n,j}$ to $L_{n+1,l}$, then we require that each $E_{n,j}^{n+1,l}$ be either empty, or consists of edges of the form

$$\{(z, h, w) \in L_{n,j} \times \mathbf{N} \times L_{n+1,l} \mid w^p = z,\ h = 1, \ldots, m_p\}$$

or

$$\{(z, h', w) \in L_{n,j} \times \mathbf{N} \times L_{n+1,l} \mid w = z^q,\ h' = 1, \ldots, m'_q\},$$

for some nonzero integers p, q and some multiplicities m_p, m'_q. We may have different p's and q's for the same $z \in L_{n,j}$, but once p or q are fixed, the multiplicities are independent of z. We suppose that each point in L_n is the source and each point in L_{n+1} is the range of at least one edge. Hence, the maps r, s are onto, and note that they are local homeomorphisms. It follows that the $X_n, n \geqslant 1$ are disjoint unions of circles, and the equivalence relations R_n all have Haar systems given by the counting measures. The C^*-algebra $\mathbf{C}^*(R_n)$ is isomorphic to

$$\bigoplus_{j=1}^{k_n} C(\mathbb{S}^1) \otimes M_{[n,j]},$$

where $[n,j]$ is the number of paths ending at each point of $L_{n,j}$. Indeed, any homogeneous C^*-algebra with spectrum $\mathbb{S}^1$ is isomorphic to $C(\mathbb{S}^1) \otimes M_n$ (recall Remark 2.3). This is true because $H^3(\mathbb{S}^1) = 0$, and every complex vector bundle over $\mathbb{S}^1$ is trivial; or see Lemma 2.8 in [13]. The partial embeddings of

$$\mathbf{C}^*(R_n) \xrightarrow{\Phi_n} \mathbf{C}^*(R_{n+1}),$$

are approximately inner equivalent to maps of the form

$$\sum_p m_p \sigma_p + \sum_q m'_q \widehat{\sigma}_q,$$

where the sums are finite, and $m \cdot \sigma$ means $\sigma \otimes \mathrm{id}_m$ (see [27], Theorem 2.1). Note that for any $z \in L_{n,j}$, $\mathrm{SP}(\Phi_n, z)$ contains powers of z or roots of z, with certain multiplicities.

Taking compositions of such maps (i.e., compressing the diagram), we get embeddings which will have in their spectra (nonzero) rational powers of z or elements of the form εz, where ε is a root of 1. On the other hand, taking into account the fact that

$$\widehat{\sigma}_k \sigma_k = k \cdot \sigma_1,$$

we may expand the diagram, introducing new levels, such that the generic partial embedding is a sum of words in $\sigma_p, \widehat{\sigma}_q$ with multiplicity 1.

DEFINITION 4.1. By a *circle diagram* we mean a diagram in which each level is a disjoint union of circles, and the edges between two consecutive levels are disjoint unions of sets of the form

$$\{(z, m, w) \in \mathbb{S}^1 \times F \times \mathbb{S}^1 \mid w^p = z^q\},$$

where p and q are nonzero integers and F is a finite subset of $\mathbf{N}$. That is, in the notation at the beginning of this section, they are obtained from the compact sets $E_{n,j}^{n+1,l}$, by compressing or expanding in a finite number of steps.

From our discussion to this point, the proof of the following theorem is immediate.

THEOREM 4.2. *To each circle diagram there corresponds a tower of circle algebras*

$$A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots,$$

where

$$A_n = \bigoplus_{j=1}^{k_n} C(\mathbb{S}^1) \otimes M_{[n,j]},$$

and the embeddings may be expressed as compositions of σ_p's and $\widehat{\sigma}_q$'s. Conversely, to each tower of circle algebras with these particular embeddings we may associate a path model.

REMARK 4.3. Note that

$$\sigma_p \sigma_{p'} = \sigma_{pp'} \quad \text{and} \quad \widehat{\sigma}_q \widehat{\sigma}_{q'} = \widehat{\sigma}_{qq'},$$

up to unitary equivalence. Note, too, that σ_p commutes with $\widehat{\sigma}_q$ iff $(p,q) = 1$. Recall that $\widehat{\sigma}_q \sigma_q = q \cdot \sigma_1$, and $\sigma_1 = \widehat{\sigma}_1 = \text{id}$. Taking inductive limits of towers as above, we get a class $\mathcal{C}$ of C^*-algebras which includes the Bunce-Deddens algebras and the C^*-algebras considered in Section 2 of [1], where only $\widehat{\sigma}_q$'s are allowed. Note finally that replacing the embeddings with other maps in the same approximately inner equivalence class preserves the inductive limit.

The ideals in the algebras in $\mathcal{C}$ are characterized pretty much in a fashion analoguous to the characterization of ideals in AF-algebras:

PROPOSITION 4.4. *Let J be a closed two-sided ideal in $A = \varinjlim A_n \in \mathcal{C}$. Then there are open subsets $\Omega_n \subset L_n$ satisfying the following conditions: if $z \in \Omega_n$, then all points in L_{n+1} joined with z are in Ω_{n+1}, and if $w \in L_{n-1}$ is joined with some points in L_n, with all of them in Ω_n, then $w \in \Omega_{n-1}$.*

Proof. It is known that

$$J = \overline{\bigcup J_n},$$

where $J_n := J \cap A_n$ (see [5]). Now we can use the structure of ideals in a circle algebra and the diagram of the inclusions. ∎

Note that in general $J \notin \mathcal{C}$ in this setting, since all algebras in the class $\mathcal{C}$ are unital.

Recall that a C^*-algebra A has *real rank zero* if any selfadjoint element can be approximated arbitrarily close by an invertible selfadjoint element or, equivalently, by one with finite spectrum. Recall also that, for

$$\Phi : C(\mathbb{S}^1) \otimes M_n \to C(\mathbb{S}^1) \otimes M_{nk}$$

with

$$\mathrm{SP}(\Phi, z) = \{w_1(z), \ldots, w_k(z)\},$$

where we count multiplicities, the *spectrum variation* is defined by

$$\mathrm{SPV}(\Phi) := \max_{z,z' \in \mathbb{S}^1} \min_{\pi \in S_k} \max_{1 \leqslant j \leqslant k} |w_j(z) - w_{\pi(j)}(z')|,$$

where S_k is the symmetric group. Note that the multiplicity does not affect the variation of the spectrum. Following Corollary 2.25 in [7], a C^*-algebra $A \in C$ has real rank zero iff for any $n \in \mathbf{N}$, for any $\varepsilon > 0$, there is an $m \in \mathbf{N}$ such that

$$\mathrm{SPV}(\Phi_{n,j}^{m,l}) < \varepsilon,$$

for any partial embedding of Φ_n^m.

DEFINITION 4.5. To each σ_Λ, a word in σ_p's and $\widehat{\sigma}_q$'s, we associate the rational number

$$|\sigma_\Lambda| := \frac{\prod_{\sigma_p} p}{\prod_{\widehat{\sigma}_q} q},$$

where the product over the empty set is 1.

PROPOSITION 4.6. *Consider*

$$A = \varinjlim(\mathrm{C}^*(R_n), \Phi_n)$$

as in the Theorem 4.2. If for any n and for any $\varepsilon > 0$ there is an $m > n$ such that $|\sigma_\Lambda| < \varepsilon$ for any word σ_Λ appearing in the partial embeddings of Φ_n^m, then A has real rank zero.

Proof. Note that $\mathrm{SPV}(\sigma_p) = 2$ and $\mathrm{SPV}(\widehat{\sigma}_q) < 2\pi/q$. Hence, if for any n there is an m such that the partial embeddings of Φ_n^m have many $\widehat{\sigma}_q$'s with q big, then the spectrum variation can be made less than ε, and we may apply the result in [7]. ∎

5. TRACES, CONDITIONAL EXPECTATIONS AND INDEX

We are now in the position to study the index associated to a pair of circle algebras, and to see how it is related to the K-theory of the embedding. The notion of index for subfactors of a type II_1 factor was introduced by Jones (see [11]) and generalized to arbitrary factors by Kosaki, in terms of conditional expectations (see [12]). Watatani defined an index in the category of C^*-algebras, using the works of Jones, Pimsner and Popa ([20]), and the theory of Morita equivalence of Rieffel ([24]). The C^*-notion of index coincides with the number of sheets of a covering map in the case the embedding of C^*-algebras is induced by this map. From our point of view, a pair $A \subset B$ of circle algebras may be viewed as a generalized covering. We consider here only pairs of unital C^*-algebras, with the same unit.

In this section, we follow the standard definitions and notation in [28] and [9]. We recall the salient terms as we need them.

DEFINITION 5.1. By a *trace* on a unital C^*-algebra A we mean a faithful positive linear functional τ satisfying

$$\tau(xy) = \tau(yx).$$

A trace is called normalized if $\tau(1) = 1$.

DEFINITION 5.2. Consider a pair of unital C^*-algebras $A \subset B$ with the same unit. A *conditional expectation* $E : B \to A$ is a linear, positive, surjective map satisfying the equations

$$E(ba) = E(b)a, \quad E(ab) = aE(b), \quad E(a) = a, \quad \forall\, a \in A,\ \forall\, b \in B.$$

We will suppose that E is faithful, that is

$$E(b^*b) = 0 \ \text{ implies } \ b = 0, \quad \forall\, b \in B.$$

A conditional expectation is said to be *compatible* with a trace τ on B if $\tau \circ E = \tau$.

DEFINITION 5.3. A finite family $\{u_1, \ldots, u_n\} \in B$ is called a *quasi-basis* for E if

$$\sum_{i=1}^{n} u_i E(u_i^* x) = x = \sum_{i=1}^{n} E(x u_i) u_i^* \quad \forall\, x \in B.$$

A conditional expectation $E : B \to A$ is of *index-finite type* if there exists a quasi-basis for E. The *index* of E is then defined by the equation

$$\mathrm{Ind}(E) := \sum_{i=1}^{n} u_i u_i^* \in B.$$

REMARK 5.4. The element $\mathrm{Ind}(E)$ is in $Z(B)$, the center of B. In many interesting cases, we will see that it is a scalar. In the case $A = N \subset M = B$ are type II_1 factors, and $E : M \to N$ is the canonical conditional expectation determined by the trace, the Jones index $[M : N]$ coincides with $\mathrm{Ind}(E)$ (see [20]). If M is an arbitrary factor and $E : M \to N$ is a faithful normal conditional expectation, then E is of index-finite type if and only if $\mathrm{Ind}(E)$ is finite in the sense of Kosaki (see [12]).

EXAMPLE 5.5. Let X, Y be compact Hausdorff spaces, with Y connected, and let $\sigma : X \to Y$ be a local homeomorphism. Then σ is a covering map and the conditional expectation

$$E : C(X) \to C(Y), \ \ (Ef)(y) = \frac{1}{n} \sum_{\sigma x = y} f(x),$$

where n is the number of sheets of σ, is of index-finite type; in fact, $\mathrm{Ind}(E) = n$. In particular, this is the case when $X = Y = \mathbb{S}^1$ and $\sigma(z) = z^n$. If we consider on both copies of $C(\mathbb{S}^1)$ the trace τ induced by Lebesgue measure on $\mathbb{S}^1$,

$$\tau(f) = \int_0^1 f(\mathrm{e}^{2\pi i t})\mathrm{d}t,$$

then $\tau \circ E = \tau$. Indeed,

$$\begin{aligned} \tau(Ef) &= \int_0^1 Ef(\mathrm{e}^{2\pi i t})\mathrm{d}t = \frac{1}{n}\sum_{k=0}^{n-1}\int_0^1 f(\mathrm{e}^{\frac{2\pi i(t+k)}{n}})\mathrm{d}t \\ &= \frac{1}{n}\sum_{k=0}^{n-1}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(\mathrm{e}^{2\pi i s})n\,\mathrm{d}s = \int_0^1 f(\mathrm{e}^{2\pi i s})\mathrm{d}s = \tau(f). \end{aligned}$$

EXAMPLE 5.6. Let $A \subset B$ be a pair of finite dimensional C^*-algebras,

$$A = \bigoplus_{j=1}^{p} M_{k_j}, \quad B = \bigoplus_{i=1}^{q} M_{l_i},$$

and let $\Phi = (\varphi_{ij})$ be the inclusion matrix. Let τ be a faithful trace on B, and define $\bar{s} \in \mathbf{C}^q$, by $\bar{s} = (s_1, \ldots, s_q)$, where $s_i := \tau(e_i)$, and e_i is a minimal projection in M_{l_i}. Let $\bar{t} \in \mathbf{C}^p$ be the vector determining the trace $\tau|A$. Then $\bar{t} = \bar{s}\Phi$. Let $E : B \to A$ be the conditional expectation determined by τ, with $\tau(E(x)y) = \tau(xy)$ for $x \in B$ and $y \in A$. Then E is of index-finite type and

$$\mathrm{Ind}(E) = \sum_{i=1}^{q} \left(\frac{1}{s_i} \sum_{j=1}^{p} \varphi_{ij} t_j \right) f_i,$$

where f_i are the minimal central projections of B (see the Proposition 2.4.2 in [28]). Moreover, $\mathrm{Ind}(E)$ is a scalar iff there is $\beta > 0$ such that

$$\bar{s}\Phi\Phi^t = \beta\bar{s}.$$

In this case $\mathrm{Ind}(E) = \beta$ ([28], Corollary 2.4.3).

EXAMPLE 5.7. Let $\Phi : C(I) \to C(I), \Phi(f) = f \circ \sigma$, where $\sigma : I \to I$ is the branched covering in Example 3.6, and let

$$E : C(I) \to C(I), \ (Ef)(t) = \frac{1}{2}\left(f\left(\frac{t}{2}\right) + f\left(1 - \frac{t}{2}\right)\right).$$

Then E is not of index-finite type, since the branched covering is associated to a nonfree action of $\mathbf{Z}_2$ on I (see Proposition 2.8.2 in [28]).

DEFINITION 5.8. (*The C^*-basic construction*) Let $E : B \to A$ be a conditional expectation. Consider $\mathcal{E}_0 := B$ as a pre-Hilbert module over A with the A-valued inner product

$$\langle b_1 \mid b_2 \rangle = E(b_1^* b_2), \ b_1, b_2 \in B.$$

Let $\mathcal{E}$ be the completion of $\mathcal{E}_0$ with the norm

$$||b|| = ||E(b^* b)||^{\frac{1}{2}}.$$

Then $\mathcal{E}$ is a Hilbert C^*-module over A. Let $\mathcal{K}(\mathcal{E})$ be the algebra of compact operators. Then $\mathcal{K}(\mathcal{E})$ is called the *C^*-basic construction* associated to (A, B, E) (see the Chapter II in [28]).

REMARK 5.9. Observe that $\mathcal{K}(\mathcal{E})$ is the closure of the linear span of

$$\{\lambda(b_1)e_A\lambda(b_2^*) \in \mathcal{L}_A(\mathcal{E}) \mid b_1, b_2 \in B\},$$

where $\lambda(b) \in \mathcal{L}_A(\mathcal{E})$ is the operator of left multiplication by b, and e_A is the projection induced by E. Indeed, $\lambda(x)e_A\lambda(y^*)$ is the rank one operator $\theta_{x,y}$, where

$$\theta_{x,y}(b) = x\langle y \mid b\rangle.$$

For this reason, the C^*-basic construction will be denoted also by $\mathbf{C}^*(B, e_A)$. Moreover, if $E : B \to A$ is of index-finite type, then

$$\mathcal{K}(\mathcal{E}) \simeq \mathcal{L}_A(\mathcal{E}) \simeq B \otimes_A B,$$

where the norm on the tensor product is $\|\cdot\|_{\max}$ (see Proposition 2.1.5 and Lemma 2.2.9 in [28]). It is known that $\mathcal{K}(\mathcal{E})$ is strongly Morita equivalent to A, therefore they have the same K-theory.

DEFINITION 5.10. In the above notation, the transfer map

$$T_E : K_0(B) \to K_0(A)$$

is defined to be the composition

$$K_0(B) \xrightarrow{j_*} K_0(\mathbf{C}^*(B, e_A)) \xrightarrow{\simeq} K_0(A),$$

where the first map is induced by the inclusion $B \subset \mathbf{C}^*(B, e_A)$. When one views K_0 as formal differences of isomorphism classes of finitely generated projective modules, the transfer map coincides with the restriction $M_B \mapsto M_A$, where M is a finitely generated projective B-module (see the Proposition 3.3.7 in [28]).

PROPOSITION 5.11. *Let $A = \mathbf{C}^*(R_n)$, $B = \mathbf{C}^*(R_{n+1})$, where the R_n are the equivalence relations associated to a circle diagram as in Section 4. We view A to be contained in B via the embedding Φ_n. Let*

$$E : \mathbf{C}^*(R_{n+1}) \to \mathbf{C}^*(R_n)$$

be defined by the formula

$$(Ef)(\alpha, \alpha') = \sum_{\omega} \frac{1}{k(\omega)} f(\alpha\omega, \alpha'\omega), \quad f \in C(R_{n+1}),$$

where $k(\omega)$ is the number of edges starting at $r(\alpha) = r(\alpha')$. Then E is a conditional expectation of index-finite type, the C^*-basic construction $\mathbf{C}^*(B, e_A)$ is the C^*-algebra of the imprimitivity groupoid $R_{n+1} *_{R_n} R_{n+1}$, and the inclusion $B \subset \mathbf{C}^*(B, e_A)$ is given by the dual map $\widehat{\Phi}_n$.

Proof. Note that the number $k(\omega)$ is finite, and is the same for each connected component of L_n, since the map $s : E_n^{n+1} \to L_n$ is a local homeomorphism. Now

$$\begin{aligned}
E(f * \Phi_n g)(\alpha, \alpha') &= \sum_{\omega} \frac{1}{k(\omega)} (f * \Phi_n g)(\alpha\omega, \alpha'\omega) \\
&= \sum_{\omega} \frac{1}{k(\omega)} \sum_{\gamma, \gamma_n = \omega} f(\alpha\omega, \gamma) g(\rho_n(\gamma), \alpha') \\
&= \sum_{\rho_n(\gamma)} \left(\sum_{\omega} \frac{1}{k(\omega)} f(\alpha\omega, \rho_n(\gamma)\omega) g(\rho_n(\gamma), \alpha' \right) \\
&= (E(f) * g)(\alpha, \alpha'), \quad f \in C(R_{n+1}),\ g \in C(R_n),
\end{aligned}$$

and in a similar way $E(\Phi_n g * f) = g * E(f)$. Also,

$$E(\Phi_n g)(\alpha, \alpha') = \sum_{\omega} \frac{1}{k(\omega)} \Phi_n g(\alpha\omega, \alpha'\omega) = \frac{1}{k(\omega)} k(\omega) g(\alpha, \alpha') = g(\alpha, \alpha').$$

Since E is associated to a local homeomorphism, it is of index-finite type, and by a result of Watatani (see Proposition 2.1.5 in [28]), the A-module $\mathcal{E}_0 = B$ is already complete. Therefore $\mathcal{K}(B)$ coincides with $\mathbf{C}^*(R_{n+1} *_{R_n} R_{n+1})$. We have seen before (Theorem 2.5) that the embedding $\mathbf{C}^*(R_{n+1}) \to \mathbf{C}^*(R_{n+1} *_{R_n} R_{n+1})$ is given by $\widehat{\Phi}_n$. ∎

It is known that the traces on $\mathbf{C}^*(R_n)$ and $\mathbf{C}^*(R_{n+1} *_{R_n} R_{n+1})$ are related in the following way (see [23]): any trace τ on the first algebra corresponds to a trace $\widehat{\tau}$ on the second such that

$$\widehat{\tau}(\theta_{f,g}) = \tau(\langle f \mid g \rangle),$$

where $\langle \cdot \mid \cdot \rangle$ is a fixed inner product on $\mathbf{C}^*(R_{n+1})$ with values in $\mathbf{C}^*(R_n)$, and $\theta_{f,g}$ is the rank one operator determined by f and g. Now a trace on $\mathbf{C}^*(R_{n+1} *_{R_n} R_{n+1})$ determines a trace on $\mathbf{C}^*(R_n)$ via the map $[\widehat{\Phi}_n]_0[\Phi_n]_0$, where $[\Phi]_0$ is the map induced by Φ on K_0-theory.

Let us specialize at this point to circle diagrams and consider the trace tr on $C(\mathbf{S}^1) \otimes M_k$ given by the formula

$$\mathrm{tr}(f_{jl}) = \sum_j \int_0^1 f_{jj}(\mathrm{e}^{2\pi i t}) \mathrm{d}t.$$

On $\mathbf{C}^*(R_{n+1})$, let τ be a faithful trace which is a linear combination of these traces. Let $E : \mathbf{C}^*(R_{n+1}) \to \mathbf{C}^*(R_n)$ be a conditional expectation of index-finite type with $\tau \circ E = \tau$, and assume $\mathrm{Ind}(E)$ is a scalar. Since the K_0 groups of both algebras are finite direct sums of copies of $\mathbb{Z}$, $\mathrm{Ind}(E)$ is the Perron-Frobenius eigenvalue of $[\widehat{\Phi}_n]_0[\Phi_n]_0$ (see [28]).

Recall that the center of $\bigoplus_k C(\mathbb{S}^1) \otimes M_{p_k}$ is $\bigoplus_k C(\mathbb{S}^1)$. An inclusion of circle algebras $A \xrightarrow{\Phi} B$ will be called *connected*, if $\Phi(Z(A)) \cap Z(B) = C(\mathbb{S}^1)$ (see the definition on page 32 in [9] for inclusions of finite dimensional algebras).

DEFINITION 5.12. Consider a *connected* inclusion $A \xrightarrow{\Phi} B$ of circle algebras. Then a *Markov trace* on B is the trace with weights given by $\bar{s}$, where

$$\bar{s}[\Phi]_0[\widehat{\Phi}]_0 = \beta\bar{s},$$

and where β is defined to be the spectral radius of $[\Phi]_0[\widehat{\Phi}]_0$. The index $[B : A]$ is also defined to be β.

REMARKS 5.13. The maps $[\widehat{\Phi}]_0$ and $[\Phi]_1^t$ are related in the following way. Fix an entry

$$\sum_{\sigma_p} p + \sum_{\widehat{\sigma}_q} \mathrm{sign}(q)$$

of $[\Phi]_1^t$, where the sums are over the σ_p's and the $\widehat{\sigma}_q$'s involved in the corresponding partial embedding (see (3.1) in Example 3.3). Then the corresponding entry in $[\widehat{\Phi}]_0$ is

$$\sum_{\sigma_p} |p| + \sum_{\widehat{\sigma}_q} 1.$$

The maps $[\widehat{\Phi}]_1$ and $[\Phi]_0^t$ are related in a similar way.

Note that in the computation of the index, both maps induced by Φ on K_0 and K_1 are involved, in contrast to the finite dimensional case. Also, different homomorphisms between the same pair of circle algebras may induce the same maps at the level of K-theory. For example,

$$\Phi : C(\mathbb{S}^1) \to C(\mathbb{S}^1) \otimes M_3,\ \Phi f = \begin{pmatrix} \widehat{\sigma}_2(f) & 0 \\ 0 & \sigma_2(f) \end{pmatrix}$$

and

$$\Psi : C(\mathbb{S}^1) \to C(\mathbb{S}^1) \otimes M_3,\ \Psi f = \begin{pmatrix} \sigma_1(f) & 0 & 0 \\ 0 & \sigma_1(f) & 0 \\ 0 & 0 & \sigma_1(f) \end{pmatrix}$$

both induce multiplication by 3 on K_0 and K_1.

The maps Φ, Ψ have the diagrams

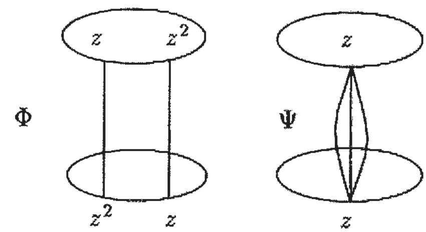

Figure 6.

The corresponding equivalence relations have different unit spaces: one is a union of two circles, the other is a union of three circles, therefore Φ and Ψ are not unitarily equivalent. Note also that they have different spectra.

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Received November 29, 1994; revised February 16, 1995.