J. OPERATOR THEORY
33(1995), 91–124

TWISTED GROUP C^*-ALGEBRAS FOR TWO-STEP NILPOTENT AND GENERALIZED DISCRETE HEISENBERG GROUPS

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Communicated by William B. Arveson

ABSTRACT. Twisted group C^*-algebras associated to two-step nilpotent groups are studied and their primitive ideal spaces are described as fibre bundles over an abelian group with fibre spaces being quasi-orbit spaces for affine group actions. Under appropriate conditions a $*$-isomorphism is constructed between such a C^*-algebra and the C^*-algebra of continuous sections of a C^*-bundle over an abelian group with fibres stably isomorphic to twisted abelian group C^*-algebras, thus simplifying the description of the corresponding primitive ideal spaces. These results are then applied to the study of twisted group C^*-algebras associated to generalized discrete Heisenberg groups. The multiplier groups are computed, and a setwise parametrization of the primitive ideal spaces is given. For discrete Heisenberg groups of rank greater than or equal to five it is shown that the associated twisted group C^*-algebras can always be decomposed as C^*-algebras of sections of C^*-bundles over a torus with fibres being matrix algebras over non-commutative tori.

KEYWORDS: *Locally compact group, C^*-algebra, primitive ideal space.*

AMS SUBJECT CLASSIFICATION: 22D25, 46L05.

0. INTRODUCTION

Twisted group C^*-algebras corresponding to locally compact abelian groups arise in a wide variety of situations in mathematics, including the representation theory of nilpotent groups and connected Lie groups ([2], [14], [28], [7], [6]), non-commutative differential geometry ([8]) and the study of continuous trace C^*-algebras ([9], [29], [30]). These C^*-algebras have been intensively studied over the past few decades, and, as a result, much is already known about their structure, including the topology of their primitive ideal spaces ([4], [17], [12]) and, in certain cases, a description of their K-groups ([10], [27]). Recall that if A is a locally compact abelian group with multiplier $\sigma \in Z^2(A, \mathbf{T})$, then $\mathrm{Prim}(C^*(A,\sigma))$ is homeomorphic to $\widehat{S}_\sigma$, where S_σ is the symmetrizer subgroup of A corresponding to σ, i.e. $S_\sigma = \{s \in A : \widetilde{\sigma}(a,s) = \sigma(a,s)\overline{\sigma(s,a)} = 1, \ \forall a \in A\}$. Very recently, S. Echterhoff and J. Rosenberg have completed an extensive study of twisted abelian group C^*-algebras, particularly as related to the theory of crossed products of continuous trace C^*-algebras by abelian groups. They have shown that if σ is a type I multiplier on A, then $C^*(A,\sigma)$ is strongly Morita equivalent to $C_o(\widehat{S}_\sigma)$ and if in addition A is second countable and S_σ has infinite index in A, then $C^*(A,\sigma) \cong C_o(\widehat{S}_\sigma) \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space ([9]).

For nonabelian groups, much less is known about the structure of the corresponding twisted group C^*-algebras. A general program to study this problem was initiated in [27], where certain techniques for studying the structure of the C^*-algebra in terms of normal abelian subgroups were provided. Extensive studies of certain particular cases, including the rank 3 discrete Heisenberg group and non-twisted two-step nilpotent group C^*-algebras, were carried out in [24], [6], [22], [31], among other places.

In this paper, which can be regarded as a sequel to both [24] and [27], we intend to apply the techniques of [27] and [13] to study the twisted group C^*-algebras associated to two-step nilpotent groups, the untwisted algebras having already been studied in [28], [5] and [6]. We first turn our attention to general locally compact second countable (hereafter, abbreviated by l.c.s.c.) two-step nilpotent groups, where by generalizing some results from [27] we are able to describe their primitive ideal spaces via bundle-theoretic means. We recall that the quasi-orbit space $Q^G(X)$ for a topological transformation group (X, G) is defined to be the quotient space $X/\sim$, where for $x_1, x_2 \in X$, $x_1 \sim x_2$ if $\overline{Gx_1} = \overline{Gx_2}$.

THEOREM 1.1. *Let N be a* l.c.s.c. *two-step nilpotent group with center Z, and let $\sigma \in Z^2(N, \mathbf{T})$ be a multiplier on N. Define subgroups $Z_o \subseteq Z$ and $K \subseteq N$ by*

$$Z_o = \{z \in Z : \widetilde{\sigma}(z,n) = 1, \ \forall n \in N\}$$

and

$$K = \{k \in N : \widetilde{\sigma}(c,k) = 1, \ \forall c \in [N,N]\}.$$

Then there is a one parameter family $\{(X_\gamma, N/K) : \gamma \in \widehat{Z}_o\}$ of free affine actions of N/K on l.c.s.c. *abelian groups X_γ, giving rise to a continuous open surjection* *from* $\mathrm{Prim}(C^*(N,\sigma))$ *onto* $\widehat{Z}_o$ *with fibre over* $\gamma \in \widehat{Z}_o$ *given by the quasi-orbit space* $Q^{N/K}(X_\gamma)$. *Here* $X_\gamma = \widehat{S}_\gamma$ *for an abelian subgroup* S_γ *of* N/Z_o *depending only on* $[\sigma] \in H^2(N,\mathbb{T})$ *and* $\gamma \in \widehat{Z}_o$.

For $\sigma \equiv 1$, $Z_o = Z$ and $K = N$, so that for nontwisted group C^*-algebras, this result reduces to Corollary 2.3 of [5], which described $\mathrm{Prim}\,(C^*(N))$ as a fibre space over $\widehat{Z}$.

After proving this general result, we discuss situations under which the above theorem can be combined with techniques of P. Green ([13]) to describe certain twisted group C^*-algebras even more precisely as the section algebras of C^*-bundles whose fibre C^*-algebras are stably isomorphic to twisted abelian group C^*-algebras (and thus, by using results on twisted abelian group algebras, whose primitive ideal spaces can be parametrized in an even more straightforward fashion).

We then specialize our study to consider twisted group C^*-algebras for generalized discrete Heisenberg groups. These groups are finitely generated non-torsion two-step nilpotent groups of rank $2n+1$ with rank 1 center; explicitly, given an n-tuple $(d_1, d_2, \ldots, d_n)$ of positive integers with $d_1|d_2|\cdots|d_n$ we write $H(d_1, d_2, \ldots, d_n) = \{(r,s,t) : r \in \mathbb{Z},\ s,t \in \mathbb{Z}^n\}$ where the group operation is defined by

$$(r, s_1, \ldots, s_n, t_1, \ldots, t_n)(r', s_1', \ldots, s_n', t_1', \ldots, t_n') = \Big(r + r' + \sum_{j=1}^{n} d_j s_j' t_j, s_1 + s_1', \ldots, t_n + t_n'\Big).$$

By using results from [6], it can be seen that these are exactly the finitely generated discrete non-torsion two-step nilpotent groups with rank one center. Therefore, by applying the classical theory of A. Maltsev ([21]), for fixed $n \in \mathbb{Z}^+$ these groups can be viewed as a parametrization of the isomorphism classes of cocompact discrete subgroups of the $(2n+1)$-dimensional Heisenberg Lie group. For the case $n = 1$ and $d_1 = 1$, these C^*-algebras have already been exhaustively studied in [24], and for the case $n \geqslant 2$, $d_1 = d_2 = \cdots = d_n = 1$ and $\sigma \equiv 1$, the primitive ideal space of the corresponding group C^*-algebra was described in [5].

In order to understand the twisted group C^*-algebras for these groups, one must first understand what sort of multipliers for these groups can arise; recall for a discrete group G the multiplier group for G is defined to be the second cohomology group for G with coefficients in the trivial G-module $\mathbb{T}$, written $H^2(G,\mathbb{T}) = Z^2(G,\mathbb{T})/B^2(G,\mathbb{T})$. We have explicitly calculated the multiplier group for generalized discrete Heisenberg groups in the following theorem:

THEOREM 2.11. *Let $N = H(d_1, \ldots, d_n)$ be a generalized discrete Heisenberg group. Then the multiplier group of N, $H^2(N, \mathbf{T})$ is given by*

$$H^2(N,\mathbf{T}) = \begin{cases} \mathbf{T}^2 & n = 1 \\ \mathbf{Z}^2_{d_2} \times \mathbf{Z}^{2(n-1)}_{d_1} \times \mathbf{T}^{(2n+1)(n-1)} & n \geqslant 2. \end{cases}$$

The proof of the theorem uses Mackey's analysis of the multiplier group of semidirect product groups ([20]), and in the course of the proof, we provide an explicit parametrization of the multiplier groups involved in terms of cocycles in $Z^2(N, \mathbf{T})$ representing each cohomology class. Thus we are able to show:

COROLLARY 2.14. *Let $N = H(d_1, \ldots, d_n)$. For $n \geqslant 2$, every multiplier on H is cohomologous to a multiplier inflated from the quotient group N/L where*

$$L = d_2 Z = \{(d_2 r, 0, 0) : r \in \mathbf{Z}\} \subseteq Z.$$

Hence, for $n \geqslant 2$ *no* twisted group C^*-algebra of the form $C^*(N, \sigma)$ can be simple; in fact by applying some of our general structure results we are able to prove:

THEOREM 3.4. *Let $n \geqslant 2$ and let $N = H(d_1, \ldots, d_n)$. Then for any $\sigma \in Z^2(N, \mathbf{T})$, there exists $\ell \in \mathbf{Z}^+$, a subgroup R of $\mathbf{Z}^{2n}$ of index ℓ, and a one parameter family of multipliers of R, $\{\tau_\gamma : \gamma \in \widehat{\ell Z} \cong \mathbf{T}\}$ such that $C^*(N, \sigma)$ is $*$-isomorphic to the C^*-algebra of continuous sections of a C^*-bundle over $\widehat{Z}_o = \widehat{\ell Z}$ with fibre over $\gamma \in \widehat{Z}_o$ $*$-isomorphic to $M_\ell(C^*(R, \tau_\gamma))$ the C^*-algebra of $\ell \times \ell$ matrices with entries in the higher-dimensional rotation algebra $C^*(R, \tau_\gamma)$.*

Thus when $n \geqslant 2$, the above theorem, together with the classical results on twisted abelian group C^*-algebras mentioned previously, can be used to give a description of $\mathrm{Prim}(C^*(N, \sigma))$ as a fibre space over $\widehat{Z}_o = \widehat{\ell Z} \cong \mathbf{T}$ with fibre over $\gamma \in \mathbf{T}$ given by $\widehat{S}_\gamma$, where S_γ is the symmetrizer subgroup of R associated to the multiplier τ_γ. Therefore, in a natural way one can view the $C^*(N, \sigma)$ as associated to one parameter families of rotation algebras, as in the well-known result of Anderson and Paschke ([1]), and one can view the above theorem as one more example of the ubiquity of rotation algebras, or "non-commutative tori", as they are popularly known.

For the case $n = 1$, taking $N = H(d)$ and $\sigma \in Z^2(N, \mathbf{T})$, the C^*-algebras $C^*(N, \sigma)$ can be analyzed completely using the method of [25] and [27]; in this case not all multipliers are inflated from quotient spaces, and indeed for a generic subset of $H^2(N, \mathbf{T})$, these C^*-algebras are simple, as already seen in the case $d = 1$ in [24], [25], and [27].

At first glance, this essential distinction between the cases $n = 1$ and $n \geqslant 2$ may appear surprising. We note however, that a natural explanation for this difference can be obtained by examining the Lyndon-Hochschild-Serre spectral sequence ([15]) for $H^*(N, \mathbb{T})$ corresponding to the exact sequence of groups

$$1 \longrightarrow L \longrightarrow N \longrightarrow N/L \longrightarrow 1,$$

where L is a central subgroup of rank 1, so that N/L is a central extension of $\mathbb{Z}^{2n}$ by a finite cyclic group.

The paper is organized into three sections. The first section gives our general results on the primitive ideal spaces for twisted two-step nilpotent group C^*-algebras, the second section contains the main result on the structure of the multiplier groups for generalized discrete Heisenberg groups, and the third section contains the description of the primitive ideal spaces of the twisted generalized discrete Heisenberg group C^*-algebras.

Some of these results first appeared in the first author's M.Sc. thesis written at the National University of Singapore ([19]), and we would like to thank Professor I. Raeburn for suggesting to us that it might be possible to generalize some of the results there.

1. THE PRIMITIVE IDEAL SPACE FOR GENERAL TWISTED TWO-STEP NILPOTENT GROUP C^*-ALGEBRAS

Let N be a l.c.s.c. two-step nilpotent group and $\sigma \in Z^2(N, \mathbb{T})$ a Borel multiplier on N. In [27], Corollary 1.6, necessary and sufficient conditions were given for simplicity of the corresponding twisted group C^*-algebra $C^*(N, \sigma)$, and in [28] and [5], Corollary 2.3, descriptions of the primitive ideal space of the untwisted group C^*-algebra $C^*(N)$ were given. In this section, we will obtain a setwise description for $\mathrm{Prim}\,(C^*(N, \sigma))$ for general σ.

We first recall that for $\sigma \in Z^2(N, \mathbb{T})$, the twisted group C^*-algebra $C^*(N, \sigma)$ is defined as the C^*-enveloping algebra for the Banach $*$-algebra $L^1(N, \sigma)$ with its usual L^1-norm, and product and involution defined by

$$f_1 \cdot f_2(n) = \int_N f_1(m) f_2(m^{-1}n)\sigma(m, m^{-1}n)\, dm \tag{1.1}$$

and

$$f^*(n) = \overline{\sigma(n, n^{-1}) f(n^{-1})} \tag{1.2}$$

for $f, f_1, f_2 \in L^1(N)$. (Since N is nilpotent, it is unimodular, and the modular function drops out of the standard formulae.) Recall also that the $*$-isomorphism class of $C^*(N,\sigma)$ depends only on the cohomology class of σ in the Moore cohomology group $H^2(N,\mathbf{T}) = Z^2(N,\mathbf{T})/B^2(N,\mathbf{T}) \cong \underline{Z}^2(N,\mathbf{T})/\underline{B}^2(N,\mathbf{T})$ (see [22] for technical details on cohomology groups).

To state the main theorem of this section, we first need to recall some facts about the symmetrizer $\widetilde{\sigma}$ associated to $\sigma \in Z^2(N,\mathbf{T})$, first introduced by K. Hannabuss in [14].

Given $\sigma \in Z^2(N,\mathbf{T})$, let $\widetilde{\sigma} : N \times N \to \mathbf{T}$ be the symmetrizer, or symmetrized form of σ, defined by

$$\widetilde{\sigma}(m,n) = \sigma(m,n)\overline{\sigma(n,n^{-1}mn)}, \qquad m,n \in N. \tag{1.3}$$

Recall from [27] the symmetrizer identities

$$\widetilde{\sigma}(m,n)\widetilde{\sigma}(n^{-1}mn,p) = \widetilde{\sigma}(m,np) \tag{1.4}$$

$$\widetilde{\sigma}(mn,p) = \overline{\sigma(m,n)}\sigma(p^{-1}mp,p^{-1}np)\widetilde{\sigma}(m,p)\widetilde{\sigma}(n,p), \qquad m,n,p \in N. \tag{1.5}$$

These identities imply that for D any central closed subgroup of N, the map $\varphi_D(\sigma) : N \to \widehat{D}$, defined by

$$\varphi_D(\sigma)(n)(d) = \widetilde{\sigma}(d,n) = \sigma(d,n)\overline{\sigma(n,d)}, \qquad n \in N, d \in D \tag{1.6}$$

is a group homomorphism.

For future reference, we note that the map $\varphi_D : Z^2(N,\mathbf{T}) \to \mathrm{Hom}\,(N,\widehat{D})$ factors through the Moore group $H^2(N,\mathbf{T}) = Z^2(N,\mathbf{T})/B^2(N,\mathbf{T})$ and this defines a group homomorphism (also denoted by φ_D) $\varphi_D : H^2(N,\mathbf{T}) \to \mathrm{Hom}\,(N,\widehat{D}) = H^1(N,H^1(D,\mathbf{T}))$. Now let Z denote the center of N, and define the closed subgroup $Z_o \subseteq Z$ by

$$Z_o = \{z \in Z : \varphi_Z(\sigma)(n)(z) = 1, \ \forall n \in N\}. \tag{1.7}$$

By [27], Proposition A2 there is a multiplier ω on N/Z_o such that σ is cohomologous to the inflated multiplier Inf ω defined on N, where if $\pi : N \to N/Z_o$ is the canonical projection, Inf ω is defined to be $\omega \circ (\pi,\pi)$. For nontrivial Z_o we can then deduce using [27], Theorem 1.2, that $C^*(N,\sigma)$ is $*$-isomorphic to the section algebra $\Gamma_0(E)$ of a C^*-bundle E over $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by $C^*(N/Z_o,\omega_\gamma = \omega \cdot d_2(\gamma))$ where

$$d_2(\gamma)(s,t) = \gamma(\eta(s)\eta(t)\eta(st)^{-1}), \qquad s,t \in N/Z_o \tag{1.8}$$

for a Borel cross-section $\eta : N/Z_o \to N$ satisfying $\eta(\mathbf{1}_{N/Z_o}) = \mathbf{1}_N$.

By [18], Theorem 4, there will be a continuous open surjection from $\mathrm{Prim}(C^*(N,\sigma))$ onto $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by $\mathrm{Prim}(C^*(N/Z_o, \omega_\gamma))$. Thus to understand $\mathrm{Prim}(C^*(N,\sigma))$, we first need to understand $\mathrm{Prim}(C^*(N/Z_o, \omega_\gamma))$.

By following through on the above strategy, we will prove:

THEOREM 1.1. *Let N be a* l.c.s.c. *two-step nilpotent group with center Z and multiplier $\sigma \in Z^2(N, \mathbf{T})$. Define $Z_o \subseteq Z$ as in* (1.7). *Then there is a normal subgroup $K \subseteq N$ with N/K abelian, and a one parameter family $\{(X_\gamma, N/K) : \gamma \in \widehat{Z}_o\}$ of free affine actions of N/K on* l.c.s.c. *abelian groups X_γ, such that there is a continuous open surjection from* $\mathrm{Prim}(C^*(N,\sigma))$ *onto $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by the quasi-orbit space $Q^{N/K}(X_\gamma)$. Here $X_\gamma = \widehat{S}_\gamma$ for an abelian subgroup S_γ of N/Z_o depending only on $[\sigma] \in H^2(N, \mathbf{T})$ and $\gamma \in \widehat{Z}_o$.*

The proof of Theorem 1.1 will be accomplished by a succession of lemmas and propositions.

LEMMA 1.2. *Let $N, Z, \sigma, Z_o, \{\omega_\gamma : \gamma \in \widehat{Z}_o\}$ be as above. Let $N_1 = N/Z_o$, and let $Z_1 = Z/Z_o \subseteq N_1$. Then the homomorphism $\varphi_{Z_1}(\omega_\gamma) : N_1 \to \widehat{Z}_1$ defined as usual by $\varphi_{Z_1}(\omega_\gamma)(n)(z) = \widetilde{\omega}_\gamma(z,n)$, is independent of $\gamma \in \widehat{Z}_o$ and has dense range in $\widehat{Z}_1$.*

Proof. Let $\mathrm{Res} : H^2(N_1, \mathbf{T}) \to H^2(Z_1, \mathbf{T})$ be the restriction map. Easy calculations show that $d_2(\gamma) \in \ker \mathrm{Res}$, and hence $\varphi_{Z_1}(d_2(\gamma)) = 1 \in H^2(N_1, H^1(Z, \mathbf{T}))$, $\forall \gamma \in \widehat{Z}_o$. Therefore,

$$\begin{aligned}\varphi_{Z_1}([\omega_\gamma]) = \varphi_{Z_1}([\omega \cdot d_2(\gamma)]) &= \varphi_{Z_1}([\omega]) \cdot \varphi_{Z_1}([d_2(\gamma)]) \\ &= \varphi_{Z_1}([\omega]) \cdot \mathbf{1} = \varphi_{Z_1}([\omega]), \qquad \forall \gamma \in \widehat{Z}_o,\end{aligned}$$

and is thus independent of $\gamma \in \widehat{Z}_o$. We now show that $\varphi_{Z_1} : N_1 \to \widehat{Z}_1$ has dense range.

Suppose that $\varphi_{Z_1}(\omega)$ maps N_1 into a proper closed subgroup $X \subsetneq \widehat{Z}_1$. Then Pontryagin duality shows that there exists $z_1 \in X^\perp \subseteq Z_1$, $z_1 \neq \mathbf{1}_{Z_1}$, with $\varphi_{Z_1}(\omega)(m)(z_1) = \mathbf{1}$, $\forall m \in N_1$, which contradicts the maximality of the subgroup Z_o. Thus the range of φ_{Z_1} is dense in $\widehat{Z}_1$, as we desired to show. ∎

Now let N, Z, σ, Z_o, N_1 and ω_γ be as in Lemma 1.2, and let $C = [N, N]$ and $C_1 = [N_1, N_1]$ be the commutator subgroups of N and N_1 respectively. It is clear that $\pi(C) = C_1$, and since N is two-step nilpotent, $C \subseteq Z$, so that $C_1 = \pi(C) \subseteq \pi(Z) = Z_1$. Let Res : $\widehat{Z}_1 \to \widehat{C}_1$ denote the restriction map on characters. Then it is clear from (1.6) that $\varphi_{C_1}(\omega_\gamma) : N_1 \to \widehat{C}_1$ satisfies $\varphi_{C_1}(\omega_\gamma) = \mathrm{Res} \circ \varphi_{Z_1}(\omega_\gamma)$, so that as a consequence of Lemma 1.2, $\varphi_{C_1}(\omega_\gamma)$ has a dense range in $\widehat{C}_1$, which is independent of $\gamma \in \widehat{Z}_o$.

We now consider multipliers ω on N such that the range $\varphi_C(\omega) : N \to \widehat{C}$ is dense:

PROPOSITION 1.3. *Let N be a* l.c.s.c. *two-step nilpotent group with commutator subgroup C, and suppose that $\omega \in Z^2(N, \mathbf{T})$ is such that $\varphi_C(\omega) : N \to \widehat{C}$ has dense range. Let $K = \ker \varphi_C(\omega) \subseteq N$, and define $S \subseteq K$ by $S = \{s \in K : \widetilde{\omega}(s, k) = \mathbf{1}, \forall k \in K\}$, i.e. S is the symmetrizer subgroup of K corresponding to $\omega/K \times K$. Then $C \subseteq K$, and* Prim $(C^*(N, \omega))$ *is homeomorphic to the quasi-orbit space corresponding to the affine action of the abelian group N/K on $\widehat{S}$ defined by*

$$n.\chi(s) = \widetilde{\omega}(s, n)\chi(n^{-1}sn), \qquad n \in N, s \in S, \chi \in X = \widehat{S}.$$

Proof. Corollary 1.6 of [27] shows that K is a normal abelian subgroup of N with $C \subseteq K$, so that N/K is abelian, and applications of [27], Proposition 1.1 and Theorem 1.5, and [26], Proposition 5.1, imply that $C^*(N, \omega)$ is $*$-isomorphic to a twisted crossed product $C^*(K, \mathrm{Res}\ \omega) \times_{\alpha,\tau} N/K$, where the action α of N/K on Prim $(C^*(K, \mathrm{Res}\ \omega)) = \widehat{S}$ is topologically conjugate to the affine action of N/K given in the statement of our proposition. We will prove that this action is *free*, thus implying the results by an application of Gootman and Rosenberg's proof of the Effros-Hahn conjecture ([11], Theorem 4.2).

Let S be the symmetrizer subgroup of ω in K. A calculation using the definition of K shows that $C \subseteq S$. Now let χ be an arbitrary element of $X = \widehat{S}$, and suppose that $n \in N$ stabilizes χ. To show that the action of N/K on $\widehat{S}$ is free, it suffices to show that $n \in K$. For $s = c \in C \subseteq S$ we obtain $\widetilde{\omega}(c, n) = \chi(c)\overline{\chi(n^{-1}cn)} = \chi(c)\overline{\chi(c)} = 1$, since $c \in C \subseteq Z$. So $\widetilde{\omega}(c, n) = \mathbf{1}$, $\forall c \in C$. By the definition of K, this implies that $n \in K$. Hence the action of N/K on $X = \widehat{S}$ is topologically free. Since N/K is l.c.s.c. abelian, we see that $(C^*(K, \omega), N/K, \alpha, \tau)$ is an Effros-Hahn regular free dynamical system ([11], Theorem 4.2) and it follows by [12], Theorem 17 that Prim $(C^*(N, \omega))$ is homeomorphic to the quasi-orbit space $Q^{N/K}(\widehat{S})$. ∎

Proposition 1.3 can be immediately applied to give a slight variant of [26], Corollary 1.6:

COROLLARY 1.4. *Let N be a* l.c.s.c. *two-step nilpotent group with multiplier $\omega \in Z^2(N, \mathbf{T})$ and commutator subgroup C. Then the twisted group C^*-algebra $C^*(N, \omega)$ is simple if and only if*

(i) *$\varphi_C(\omega) : N \to \widehat{C}$ has dense range, and*

(ii) *the action of N/K on $\widehat{S}$, where K, S and the action are defined in Proposition* 1.3, *is minimal.*

Proof. The direction ($\Leftarrow$) follows immediately from Proposition 1.3. As for ($\Rightarrow$), if (i) does not hold, by [27], Theorem 1.2, $C^*(N, \omega)$ is not simple. If (i) holds but (ii) does not hold, then Proposition 1.3 shows that $C^*(N, \omega)$ is not simple. ∎

We can now combine Lemma 1.2 and Proposition 1.3 to prove Theorem 1.1:

Proof of Theorem 1.1. We assume that $\sigma = \operatorname{Inf}\omega$, where ω is a multiplier defined on $N_1 = N/Z_o$, so that $C^*(N, \sigma)$ is $*$-isomorphic to the section algebra of a C^*-bundle over $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by $C^*(N_1, \omega_\gamma)$ for $\omega_\gamma = \omega \cdot d_2(\gamma)$. Thus by [18], Theorem 4, there is a continuous open surjection from $\operatorname{Prim}(C^*(N, \sigma))$ onto $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by $\operatorname{Prim}(C^*(N_1, \omega_\gamma))$. Now let $C_1 = [N_1, N_1]$. By Proposition 1.3, letting $K_1 = \ker \varphi_{C_1}(\omega_\gamma) \subseteq N_1$, K_1 is independent of γ and N_1/K_1 is abelian, and putting $S_\gamma = \{s \in K_1 : \widetilde{\omega}_\gamma(s, k) = 1, \forall k \in K_1\}$, $\operatorname{Prim}(C^*(N_1, \omega_\gamma))$ is homeomorphic to the quasi-orbit space $Q^{N_1/K_1}(X_\gamma)$ for the affine action of N_1/K_1 on $X_\gamma = \widehat{S}_\gamma$ defined by

$$n.\chi(s) = \widetilde{\omega}_\gamma(s, n)\chi(n^{-1}sn), \qquad n \in N_1, s \in S_\gamma, \chi \in X_\gamma. \tag{1.9}$$

Now define $K \subseteq N$ by $K = \{k \in N : \varphi_C(\sigma)(k)(c) = \mathbf{1}, \forall c \in C\}$. Then K is a normal subgroup of N and $N/K \cong (N/Z_o)/(K/Z_o) \cong N_1/K_1$ is abelian. Combining our results, we have shown that there is a continuous open surjection from $\operatorname{Prim}(C^*(N, \sigma))$ onto $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by a quasi-orbit space for an affine group action, $Q^{N/K}(X_\gamma)$, as we desired to show. ∎

EXAMPLE 1.5. To illustrate the use of Theorem 1.1, we compute the primitive ideal space of $C^*(N, \sigma)$ for a straightforward example; more examples will be given in Section 3. Let $N = \{(r, s, t, n) : r, s, t, n \in \mathbb{Z}\}$, where the group operation is given by

$$(r_1, s_1, t_1, n_1) \cdot (r_2, s_2, t_2, n_2) = (r_1 + r_2 + t_1 s_2, s_1 + s_2, t_2 + t_2, n_1 + n_2).$$

Note that N is just the direct product $H \times \mathbf{Z}$, where H represents the standard integer Heisenberg group. Fix an irrational $\alpha \in [0,1)$ and $q \in \mathbf{Z}^+$, $q > 1$, and define a multiplier σ on N by

$$\sigma((r_1, s_1, t_1, n_1), (r_2, s_2, t_2, n_2)) = e^{2\pi i\alpha\left[r_2 t_1 + s_2 \frac{t_1(t_1-1)}{2}\right]}\left[e^{\frac{2\pi i}{q}}\right]^{s_2 n_1}.$$

A calculation shows that $Z_o = \{(0,0,0,qn) : n \in \mathbf{Z}\} \subseteq Z = \{(r,0,0,n) : r, n \in \mathbf{Z}\}$, and $\sigma = \operatorname{Inf}\ \omega$ for ω defined on $N_1 = N/Z_o = H \times \mathbf{Z}_q$ in the obvious way. For any $\gamma \in \widehat{Z_o} = \mathbf{T}$, another easy calculation shows that $d_2(\gamma)$ as defined in (1.8) is a multiplier on N_1 lifted from a multiplier on $N_1/H \times \{0\} = \mathbf{Z}_q$, hence is trivial. Hence $[\omega_\gamma] = [\omega] \in H^2(N_1, \mathbf{T})$, $\forall \gamma \in \mathbf{T}$. We now calculate $\operatorname{Prim}(C^*(N_1, \omega))$. Using Proposition 1.3, one checks that $C_1 = \{(r,0,0,\dot{0}) : r \in \mathbf{Z}\}$ and $\varphi_{C_1}(\omega)(r_1, s_1, t_1, \dot{n}_1)(r,0,0,\dot{0}) = \left[e^{-2\pi i\alpha t_1}\right]^r$, so that $K_1 = \ker \varphi_{C_1}(\omega) = \{(r,s,0,\dot{n}) : r, s \in \mathbf{Z},\ \dot{n} \in \mathbf{Z}_q\}$, and $\omega | K_1 \times K_1$ is defined by

$$\omega\left((r_1, s_1, 0, \dot{n}_1), (r_2, s_2, 0, \dot{n}_2)\right) = \left[e^{\frac{2\pi i}{q}}\right]^{s_2 \dot{n}_1}.$$

Thus the symmetrizer subgroup S of K_1 corresponding to ω is $\{(j, kq, 0, 0) : j, k \in \mathbf{Z}\}$, and the map $c : N_1/K_1 \cong \mathbf{Z} \to N_1$ given by $c(t) = (0,0,t,\dot{0})$ is a splitting for the projection $N_1 \longrightarrow N_1/K_1$. The action of $\mathbf{Z} \cong N_1/K_1$ on $\widehat{S} \cong \widehat{\mathbf{Z}^2} = \mathbf{T}^2$ is calculated to be given by

$$t \cdot (\lambda, \mu) = \left(e^{-2\pi i\alpha t}\lambda, e^{-2\pi i\alpha q\frac{t(t-1)}{2}}\lambda^{-qt}\mu\right), \qquad (t \in \mathbf{Z},\ \lambda, \mu \in \mathbf{T}).$$

Since α is irrational, this is a minimal action of $\mathbf{Z}$ on $\mathbf{T}^2$, so that, for each $\gamma \in \mathbf{T}$, the quasi-orbit space for the action of N_1/K_1 on X_γ reduces to a point. Thus Theorem 1.1 tells us that Prim $(C^*(N, \sigma))$ is homeomorphic to the circle group $\mathbf{T}$. ∎

In certain situations, we can combine Theorem 1.1 with a result of P. Green ([13]) to simplify considerably our description of $\operatorname{Prim}(C^*(N, \sigma))$, via the following reduction method:

PROPOSITION 1.6. *Suppose that N is a* l.c.s.c. *two-step nilpotent group with center Z, commutator subgroup C, and multiplier $\omega \in Z^2(N, \mathbf{T})$. Suppose there is a closed central subgroup D, such that*

(i) *$\omega/D \times D$ is trivial.*

(ii) *$\varphi_D(\omega) : N \to \widehat{D}$ is surjective.*

Let $M = \ker \varphi_D(\omega)$. Then M is a normal subgroup of N containing D, and there is a multiplier τ on the group M/D such that

$$C^*(N, \omega) \cong C^*(M/D, \tau) \otimes \mathcal{K}(L^2(N/M)).$$

Here $\mathcal{K}(L^2(N/M))$ represents the C^*-algebra of compact operators on the Hilbert space $L^2(N/M)$. If in addition $C \subseteq D$, then both M and M/D are abelian groups.

Proof. M is normal since it is the kernel of the group homomorphism $\varphi_D(\omega) : N \to \widehat{D}$. Since $\varphi_D(\omega)$ is surjective, N/M is isomorphic to $\widehat{D}$. Let N^ω be the central group extension of N by $\mathbf{T}$ defined as in [4], p. 301 (setwise by $N^\omega = \mathbf{T} \times N$, with product $(\lambda_1, n_1)(\lambda_2, n_2) = (\lambda_1\lambda_2\omega(n_1, n_2), n_1 n_2)$, $\lambda_i \in \mathbf{T}$, $n_i \in N, i = 1, 2$); define central extensions M^ω and D^ω of M and D similarly. By [26], Proposition 1.1, we can write $C^*(N, \omega)$ as a twisted covariance algebra, in the sense of P. Green ([12]), $C^*(N^\omega, C^*(D, \text{Res }\omega), \mathcal{T}_{D^\omega})$, where the action α of N^ω on $C^*(D, \text{Res }\omega)$ is given by

$$(1.10) \quad \alpha((\lambda, n))f(d) = \widetilde{\omega}(n^{-1}dn, n^{-1})f(d) = \varphi_D(\omega)(n^{-1})(d)f(d), \;\; f \in L^1(D)$$

and the twist $\mathcal{T} : D^\omega \to \mathcal{UM}(C^*(D, \text{Res }\omega))$ is given by

$$(1.11) \qquad \mathcal{T}((\lambda, d_1))f(d) = \lambda f(d_1^{-1}d), \qquad f \in L^1(D).$$

Since ω is trivial on $D \times D$, this twisted covariance algebra can be viewed as $C^*(N^\omega, C_o(\widehat{D}), \mathcal{T}_{D^\omega})$ where the action $\widetilde{\alpha}$ of N^ω on $C_o(\widehat{D})$ is given by

$$(1.12) \qquad \widetilde{\alpha}((\lambda, n)) \cdot f(\chi) = f(\varphi_D(\omega)(n^{-1}).\chi), \quad f \in C_o(\widehat{D}).$$

Given $(\lambda, n) \in N^\omega$, and $n \in N$, let $\widetilde{(\lambda, n)}$ and $\dot{n}$ denote their images in N^ω/M^ω and N/M respectively. Note that under the canonical isomorphism $N^\omega/M^\omega \cong N/M$, $\widetilde{(\lambda, n)}$ is identified with $\dot{n}$, $\forall\,(\lambda, n) \in N^\omega$. Let Φ be the isomorphism from $N^\omega/M^\omega \cong N/M$ to $\widehat{D}$ defined by $\Phi(\dot{n}) = \varphi_D(\omega)(n)$. It is easy to check that Φ gives an N^ω-equivariant isomorphism of the N^ω-spaces N^ω/M^ω and $\widehat{D}$. By [13], Theorem 2.13 (i), it follows that $C^*(N^\omega, C_o(\widehat{D}), \mathcal{T}_{D^\omega})$ is $*$-isomorphic to

$$C^*(M^\omega, C_o(\widehat{D})/I, \mathcal{T}_{D^\omega}^{C_o(\widehat{D})/I}) \otimes \mathcal{K}(L^2(N^\omega/M^\omega)),$$

where $I = \{f \in C_o(\widehat{D}) : f(\mathbf{1}_{\widehat{D}}) = 0\}$. Now $C_o(\widehat{D})/I \cong \mathbf{C}$, and easy calculations using [26], Proposition 5.1 show that $C^*(M^\omega, \mathbf{C}, \mathcal{T}_{D^\omega}^{\mathbf{C}})$ is $*$-isomorphic to the twisted group C^*-algebra $C^*(M/D, \tau)$ where $\tau : M/D \times M/D \to \mathbf{T}$ is defined by

$$(1.13) \qquad \tau(x, y) = \omega(c(x), c(y))\,\overline{\omega(c(x)c(y)c(xy)^{-1}, c(xy))}, \quad x, y \in M/D$$

for some Borel cross-section $c : M/D \to M$ satisfying $c(\mathbf{1}_{M/D}) = \mathbf{1}_M$. Upon using the isomorphism $N^\omega/M^\omega \cong N/M$, we obtain $C^*(N, \omega) \cong C^*(M/D, \tau) \otimes \mathcal{K}(L^2(N/M))$, as desired.

If D contains C, one easily calculates that $D \subseteq M \subseteq K$ where K is the normal abelian subgroup of Proposition 1.3. Thus M and M/D are also abelian. ∎

REMARK 1.7. We note here that if we take $D = C$, then condition (i) will automatically be satisfied. Hence to apply the theorem with $D = C$ it is enough to verify condition (ii). The simple example of the rank three Heisenberg group given in the first paragraph of [27], Corollary 1.7, shows that the surjectivity of condition (ii) is a necessary condition which cannot be replaced by a dense range condition, for example.

Combining Theorem 1.1 with Proposition 1.6 gives:

THEOREM 1.8. *Let N be a* l.c.s.c. *two-step nilpotent group with center Z, commutator subgroup C and multiplier $\sigma \in Z^2(N, \mathbf{T})$. Let $Z_o, N_1 = N/Z_o$, and $\omega \in Z^2(N, \mathbf{T})$ be as in the paragraph preceeding the statement of Theorem* 1.1. *Let $C_1 = [N_1, N_1]$ and $Z_1 = Z/Z_o$. Suppose that there exists a closed subgroup D_1 of N_1 with $C_1 \subseteq D_1 \subseteq Z_1 \subseteq N_1$ such that*

(i) $\omega/D_1 \times D_1$ *is trivial;*

(ii) $\varphi_{D_1}(\omega) : N_1 \to \widehat{D}_1$ *is surjective.*

Then there is a one parameter family of abelian groups $\{X_\gamma : \gamma \in \widehat{Z}_o\}$ and a continuous open surjection from $\mathrm{Prim}(C^*(N, \sigma))$ *onto $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by X_γ.*

Proof. By the proof of Theorem 1.1, $C^*(N, \sigma)$ can be viewed as the C^*-algebra of continuous sections of a C^*-bundle over $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o$ given by $C^*(N_1, \omega_\gamma)$, where $\omega_\gamma = \omega \cdot d_2(\gamma) \in Z^2(N_1, \mathbf{T})$. The argument of Lemma 1.2 shows that $\varphi_{D_1}(\omega_\gamma) = \varphi_{D_1}(\omega)$, $\forall\, \gamma \in \widehat{Z}_o$, and therefore, by our hypothesis on $\varphi_{D_1}(\omega)$, is surjective, $\forall\, \gamma \in \widehat{Z}_o$. By Proposition 1.6, there exists a normal abelian subgroup M_1 of N_1 containing D_1, depending only on ω and a one parameter family of multipliers on the quotient group M_1/D_1, $\{\tau_\gamma \in Z^2(M_1/D_1, \mathbf{T}),\ \gamma \in \widehat{Z}_o\}$ (these multipliers τ_γ will each depend on $\gamma \in \widehat{Z}_o$), such that $C^*(N_1, \omega_\gamma)$ is $*$-isomorphic to $C^*(M_1/D_1, \tau_\gamma) \otimes \mathcal{K}(L^2(N_1/M_1))$, for each $\gamma \in \widehat{Z}_o$. Thus $C^*(N_1, \omega_\gamma)$ is stably isomorphic to the twisted abelian group C^*-algebra $C^*(M_1/D_1, \tau_\gamma)$, and consequently $\mathrm{Prim}\,(C^*(N_1, \omega_\gamma))$ is homeomorphic to X_γ, where X_γ is the dual group of the symmetrizer subgroup $S_{\tau_\gamma} \subseteq M_1/D_1$ associated to the multiplier $\tau_\gamma \in Z^2(M_1/D_1, \mathbf{T})$. Applying [18], Theorem 4 once again, we obtain the desired result. ∎

We now note that if C_1 is compact, the decomposition of Theorem 1.8 can always be obtained:

COROLLARY 1.9. *Let N, Z, σ, Z_o be as in Theorem 1.8 and suppose that $C_1 = [N/Z_o,\ N/Z_o]$ is compact. Then the conclusion of Theorem 1.8 holds.*

Proof. Since $C_1 = [N_1, N_1]$ is compact, $\widehat{C}_1$ is discrete, so that any map having dense range in $\widehat{C}_1$ will be automatically surjective. Since ω is automatically trivial on C_1 (recall $C_1 \subseteq \ker \varphi_{C_1}(\omega)$ where $\sigma = \operatorname{Inf}\ \omega$), we can therefore apply Theorem 1.8 with $D_1 = C_1$. ∎

Corollary 1.9 will be used in Section 3.

2. THE MULTIPLIER GROUPS FOR GENERALIZED DISCRETE HEISENBERG GROUPS

In this section we will compute the multiplier group $H^2(N, \mathbf{T})$ where N belongs to the class of generalized discrete Heisenberg groups described in our introduction, i.e. $N \in \{H(d_1, \ldots, d_n) : n \in \mathbb{Z}^+,\ d_1, \ldots, d_n \in \mathbb{Z}^+ \text{ with } d_1|d_2|\cdots|d_n\}$, where recall setwise $H(d_1, \ldots, d_n) = \mathbb{Z} \times \mathbb{Z}^n \times \mathbb{Z}^n$ with multiplication defined by

$$(2.1)\quad (r,s,t)\cdot(r',s',t') = \Big(r+r'+\sum_{i=1}^{n} d_i t_i s_i', s+s', t+t'\Big),\quad r, r' \in \mathbb{Z},\ s, s', t, t' \in \mathbb{Z}^n.$$

Though these groups appear somewhat specialized, they occur in a wide variety of situations, as indicated in our introduction. For example, if $B \in M_k(\mathbb{Z})$ has rank $n \leqslant k$, and if the group $H(B)$ is defined setwise by $H(B) = \mathbb{Z} \times \mathbb{Z}^k \times \mathbb{Z}^k$ with multiplication given by

$$(2.2)\quad (r,s,t)\cdot(r',s',t') = (r + t^{\mathrm{T}} B s', s+s', t+t'),\quad r, r' \in \mathbb{Z},\ s, s', t, t' \in \mathbb{Z}^k,$$

then one can show using [16], Theorem 3.8 that there exists $d_1, d_2, \ldots, d_n \in \mathbb{Z}^+$ with $d_1|d_2|\cdots|d_n$ such that

$$H(B) \cong H(d_1, d_2, \ldots, d_n) \times \mathbb{Z}^{2(k-n)}.$$

More generally, the analysis of [6], Corollary 3.4 shows that the collection of groups $\{H(d_1, \ldots, d_n) : n \in \mathbb{Z}^+,\ d_1, \ldots, d_n \in \mathbb{Z}^+,\ d_1|d_2|\cdots|d_n\}$ is a parametrization of the isomorphism classes of finitely generated discrete non-torsion two-step nilpotent groups with rank one center. By the celebrated work of A. Maltsev ([21]), all such groups embed as cocompact subgroups of simply connected two-step nilpotent Lie groups with rank one center, and arguments similar to those used in [6], Section 3, can be used to show that such Lie groups are all isomorphic to the classical Heisenberg Lie groups of dimension $2n+1$, $n \in \mathbf{Z}^+$, i.e. the groups $H_n = \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$ with multiplication given by

$$(2.3)\quad (r,s,t).(r',s',t') = (r+r'+\langle t,s'\rangle, s+s', t+t'), \quad r,r' \in \mathbf{R},\ s,s',t,t' \in \mathbf{R}^n.$$

(Here $\langle t,s\rangle$ represents the standard inner product on $\mathbf{R}^n$).

Thus we view our groups $\{H(d_1,\ldots,d_n) : n \in \mathbf{Z}^+,\ d_1,\ldots,d_n \in \mathbf{Z}^+,\ d_1|d_2|\cdots|d_n\}$ as natural candidates to put forth as generalizations of the 3-dimensional integer Heisenberg group.

We now want to compute the second cohomology groups $H^2(H(d_1,\ldots,d_n), \mathbf{T})$. In addition to knowing the group structure of this cohomology group, it will be useful for our results in the next section to construct explicitly a multiplier in each cohomology class.

From this point on we fix $n \in \mathbf{Z}^+$ and $d_1,\ldots,d_n \in \mathbf{Z}^+$ with $d_i|d_j$ for $i \leqslant j$. Throughout the remainder of this section, N will denote the generalized discrete Heisenberg group, $N = H(d_1,\ldots,d_n)$. Consider the subgroups M and K of N defined by $M = \{(r,s,0) : r \in \mathbf{Z}, s \in \mathbf{Z}^n\} \cong \mathbf{Z}^{n+1}$, $K = \{(0,0,t) : t \in \mathbf{Z}^n\} \cong \mathbf{Z}^n$. In what follows we shall frequently abuse notation and write $(r,s,0) \in M$ and $(0,0,t) \in K$ as (r,s) and t respectively. For later use, we let $\{m_1,\ldots,m_{n+1}\}$ and $\{k_1,\ldots,k_n\}$ be the standard generators for M and K respectively, i.e. $m_1 = (1,0,\ldots,0)$, $m_2 = (0,1,0,\ldots,0)$, etc. and similarly for the $\{k_i\}$. Clearly, M is a normal subgroup of N, $M \cap K = \{1_N\}$, and $N = MK$. Consequently N is a semidirect product of M and K and we can compute $H^2(N,\mathbf{T})$ by using results from [3] and [20].

Since N is a semidirect product, there is an action $\pi : K \to \operatorname{Aut} M$ obtained by conjugation which one can calculate to be:

$$\pi(t)(r,s) = \left(r + \sum_{i=1}^{n} d_i t_i s_i, s\right), \qquad (r,s) \in M, t \in K.$$

Note that if we write (r,s) as the column vector $(r,s_1,\ldots,s_n)^{\mathrm{T}}$, then

$$(2.4)\qquad \pi(t)(r,s_1,\ldots,s_n)^{\mathrm{T}} = \begin{pmatrix} 1 & d_1t_1 & d_2t_2 & \cdot & \cdot & \cdot & d_nt_n \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} r \\ s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_n \end{pmatrix}.$$

For convenience, we shall also denote the $(n+1)\times(n+1)$ matrix on the right hand side of (2.4) by $\pi(t)$. The action π of K on M also induces an action of K on the group of multipliers for M, $Z^2(M,\mathbb{T})$ (which one can check, factors through the cohomology group $H^2(M,\mathbb{T})$), given by

$$(\pi(t)\gamma)((r,s),(r',s')) = \gamma(\pi(t)(r,s),\pi(t)(r',s')), \quad t\in K,\ (r,s),(r',s')\in M,\ \gamma\in Z^2(M,\mathbb{T}). \tag{2.5}$$

Let $[H^2(M,\mathbb{T})]^K$ denote the subgroup of those elements of $H^2(M,\mathbb{T})$ fixed by the above action of K.

Since M and K are finitely generated free abelian groups, their second cohomology groups are easily described. In fact for any $k\in\mathbb{Z}^+$, it is well known that $H^2(\mathbb{Z}^k,\mathbb{T})\cong\mathbb{T}^{\frac{k(k-1)}{2}}$. Here we give a modification of the description of $H^2(\mathbb{Z}^k,\mathbb{T})$ due to Backhouse ([3]). Let $M(k)$ denote the additive group of $k\times k$ real matrices. Let $\mathcal{S}(k)$ and $\mathcal{T}(k)$ denote the subgroups of $M(k)$ consisting of all symmetric matrices and all upper triangular matrices with 0's along the diagonal. (We use here upper triangular matrices rather than the more standard choice of lower triangular matrices since it makes some of our upcoming calculations simpler.)

For each $A\in M(k)$ the function

$$\gamma_A(u,v) = e^{2\pi i u^{\mathrm{T}}Av}, \qquad u,v\in\mathbb{Z}^k$$

defines a multiplier of $\mathbb{Z}^k$. Each multiplier on $\mathbb{Z}^k$ is similar to some γ_A where $A\in\mathcal{T}(k)$. Moreover if $A=(a_{ij})$ and $B=(b_{ij})\in M(k)$, then γ_A is cohomologous to γ_B if and only if there exists $C=(c_{ij})\in\mathcal{S}(k)$ such that $a_{ij}-b_{ij}\equiv c_{ij} \pmod 1$, $\forall\,(i,j)$, i.e. if and only if $A-B\equiv C \pmod 1$. Consequently if $\mathcal{E}(k)=\{A\in M(k): A\equiv (0)\pmod 1\}$, then the map from $\mathcal{T}(k)$ into $Z^2(\mathbb{Z}^k,\mathbb{T})$ described above induces an isomorphism of $\mathcal{T}(k)/\mathcal{E}(k)\cap\mathcal{T}(k)$ onto $H^2(\mathbb{Z}^k,\mathbb{T})$. It follows that $\{\gamma_A : A=(a_{ij})\in\mathcal{T}(k),\ a_{ij}\in[0,1),\ \forall\, i,j\}$ gives a complete set of inequivalent multipliers for $\mathbb{Z}^k$.

Since $N=M\rtimes K$, by results in [20], each multiplier of N is cohomologous to a multiplier of the form

$$\sigma_{\gamma,\beta,\alpha}((r,s,t),(r',s',t')) = \gamma((r,s),\pi(t)(r',s'))\beta(t,(r',s'))\alpha(t,t') \tag{2.6}$$

where $\gamma\in\mathbb{Z}^2(M,\mathbb{T}),\alpha\in Z^2(K,\mathbb{T})$ and $\beta: K\times M\to\mathbb{T}$ satisfy the following compatibility conditions:

$$\pi(t)(\gamma)\cdot\gamma^{-1}((r,s),(r',s')) = \beta(t,(r+r',s+s'))\overline{\beta(t,(r,s))\beta(t,(r',s'))}, \tag{2.7}$$

(2.8) $\beta(t+t',(r,s)) = \beta(t,\pi(t')(r,s))\beta(t',(r,s)), \quad (r,s),(r',s') \in M,\ t,t' \in K.$

Conversely, if $\gamma \in Z^2(M,\mathbf{T})$ and $\beta : K \times M \to \mathbf{T}$ satisfy (2.7) and (2.8), then for any $\alpha \in Z^2(K,\mathbf{T})$, equation (2.6) defines a multiplier $\sigma_{\gamma,\alpha,\beta} \in Z^2(N,\mathbf{T})$. We now let $\widehat{M}$ be the group of characters of M. The action π of K induces an action π of K on $\widehat{M}$ given by

$$(\pi(t)(a))(v) = a(\pi(t)v), \qquad a \in \widehat{M},\ t \in K,\ v \in M.$$

Let the group of 1-cocycles $Z^1(K,\widehat{M})$, of 1-coboundaries $B^1(K,\widehat{M})$, and the first cohomology group of K with values in the K-module $\widehat{M}$, $H^1(K,\widehat{M}) = Z^1(K,\widehat{M})/B^1(K,\widehat{M})$ be defined relative to this action. Then Mackey's result (as modified in Appendix 2 of [27]) implies that there is an exact sequence

$$(2.9) \qquad 0 \longrightarrow H^2(K,\mathbf{T}) \times H^1(K,\widehat{M}) \xrightarrow{\zeta} H^2(N,\mathbf{T}) \xrightarrow{\text{Res}} [H^2(M,\mathbf{T})]^K,$$

where the map ζ is given by

$$(2.10) \qquad \zeta([\alpha],[\chi])((r,s,t),(r',s',t')) = [\alpha(t,t')\chi(t,(r',s'))],$$

for $(r,s,t),(r',s',t') \in N, \alpha \in Z^2(K,\mathbf{T}), \chi \in Z^1(K,\widehat{M})$.

We shall show later in this section that (2.9) can be extended to a split exact sequence

$$(2.11) \qquad 0 \longrightarrow H^2(K,\mathbf{T}) \times H^1(K,\widehat{M}) \xrightarrow{\zeta} H^2(N,\mathbf{T}) \xrightarrow{\text{Res}} [H^2(M,\mathbf{T})]^K \longrightarrow 0.$$

LEMMA 2.1. *Let $A = (a_{ij}) \in \mathcal{T}(n+1)$. If $n = 1$, $[\gamma_A] \in [H^2(M,\mathbf{T})]^K$, $\forall A \in \mathcal{T}(2)$. For $n \geqslant 2$, $[\gamma_A] \in [H^2(M,\mathbf{T})]^K$ if and only if $d_1 a_{1j} \equiv 0 \pmod 1$ for all $3 \geqslant j \geqslant n+1$ and $d_2 a_{12} \equiv 0 \pmod 1$.*

Proof. Recall that $[\gamma] \in [H^2(M,\mathbf{T})]^K$ if and only if $\pi(t)(\gamma)\cdot\gamma^{-1}$ is an exact multiplier for every $t \in K$. Thus we must show that $\pi(t)(\gamma_A)\gamma_A^{-1}$ is exact if and only if the stated conditions concerning the entries of A are satisfied. It is easy to check that $(\pi(t)\gamma_A)\gamma_A^{-1} = \gamma_{M_t^A}$ where $M_t^A = \pi(t)^t A \pi(t) - A$.

For the $n = 1$ case one sees that M_t^A is a symmetric matrix for all t and thus $\gamma_{M_t^A}$ is exact for all t, so that $H^2(M,\mathbf{T}) = [H^2(M,\mathbf{T})]^K$. Thus we concentrate on the case $n \geqslant 2$. We write

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1,n+1} \\ 0 & & & & & & \\ 0 & & & & & & \\ \cdot & & & A' & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 0 & & & & & & \end{pmatrix}$$

where A' is an $n \times n$ upper triangular matrix. One then calculates

$$\pi(t)^{\mathrm{T}} A \pi(t) = A + \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & & & & \\ \cdot & & & & & \\ \cdot & & & B & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & & & & & \end{pmatrix} \quad \text{where } B = \begin{pmatrix} d_1 t_1 \\ \cdot \\ \cdot \\ \cdot \\ d_n t_n \end{pmatrix} (a_{12}, \ldots, a_{1,n+1}),$$

so that $\gamma_{M_t^A}$ will be exact if and only if the $n \times n$ matrix B is equivalent modulo 1 to a symmetric matrix. Now $B = (b_{ij})_{1 \leqslant i,j \leqslant n}$ where $b_{ij} = d_i t_i a_{1,j+1}$. Thus a necessary and sufficient condition for $\pi(t)^{\mathrm{T}} A \pi(t) - A$ to be exact is that $b_{ij} - b_{ji} \equiv 0 \pmod 1$, $\forall\, i,j$, for every choice of $t \in K$, i.e. we want

$$d_i t_i a_{1,j+1} - d_j t_j a_{1,i+1} \equiv 0 \pmod 1, \qquad \forall\, 1 \leqslant i,j \leqslant n,\ t \in K. \tag{2.12}$$

Suppose $\gamma_{M_t^A}$ is exact for all $t \in K$. Then putting $t = k_1 = (1, 0, \ldots, 0)$, and taking $i = 1$ and $2 \leqslant j \leqslant n$ in (2.12), we obtain $d_1 a_{1,j+1} \equiv 0 \pmod 1$, $\forall\, 2 \leqslant j \leqslant n$, i.e. $d_1 a_{1j} \equiv 0 \pmod 1$, $\forall\, 3 \leqslant j \leqslant n+1$. Putting $t = k_2 = (0, 1, 0, \ldots, 0)$ and taking $i = 2$ and $j = 1$ in (2.12), we get $d_2 a_{12} = 0 \pmod 1$, proving the necessity of the stated condition.

Conversely, if $d_2 a_{12} \equiv 0 \pmod 1$ and $d_1 a_{1j} = 0$, $3 \leqslant j \leqslant n+1$, then since $d_1 | d_i$, $\forall\, 1 \leqslant i \leqslant n$, we also have $d_i a_{1j} = 0 \pmod 1$, $\forall\, 1 \leqslant i \leqslant n$, $3 \leqslant j \leqslant n+1$ and $d_i a_{12} \equiv 0 \pmod 1$, $\forall\, 2 \leqslant i \leqslant n$. Therefore $(d_i t_i a_{1,j+1})_{1 \leqslant i,j \leqslant n}$ is equivalent modulo 1 to the symmetric matrix whose $(1,1)$-entry is $d_1 t_1 a_{12}$ and all of whose other entries are zero. Thus in this case $\gamma_{M_A^t}$ can be calculated exactly as $\gamma_{M_A^t}((r,s),(r',s')) = e^{2\pi i (d_1 t_1 a_{12}) s_1 s_1'}$, which is easily seen to be exact for all $t \in K$. ∎

COROLLARY 2.2. *For $n = 1$, let $T = \mathcal{T}(2)$, and for $n \geqslant 2$, let*

$$T = \{A = (a_{ij}) \in \mathcal{T}(n+1) : d_1 a_{1j} \equiv 0 \pmod 1,\ 3 \leqslant j \leqslant n+1,\ d_2 a_{12} \equiv 0 \pmod 1\}.$$

Then

$$[H^2(M, \mathbf{T})]^K \cong T/\mathcal{E}(n+1) \cap \mathcal{T}(n+1) \cong \begin{cases} \mathbf{T} & n = 1, \\ \mathbf{Z}_{d_2} \times \mathbf{Z}_{d_1}^{n-1} \times \mathbf{T}^{\frac{n(n-1)}{2}} & n \geqslant 2, \end{cases}$$

where $\mathbf{Z}_k = \mathbf{Z}/k\mathbf{Z}$ is the cyclic group of order k.

Proof. Lemma 2.1 shows that for $n = 1$, $[H^2(M, \mathbf{T})]^K = H^2(M, \mathbf{T}) = H^2(\mathbf{Z}^2, \mathbf{T}) \cong \mathbf{T}$, and for $n \geqslant 2$, it shows that the map $A \longrightarrow \gamma_A$ of T into $Z^2(M, \mathbf{T})$ induces an isomorphism of $T/\mathcal{E}(n+1) \cap \mathcal{T}(n+1)$ onto $[H^2(M, \mathbf{T})]^K$. The conclusion follows upon observing that $T/\mathcal{E}(n+1) \cap \mathcal{T}(n+1)$ is isomorphic to $\mathbf{Z}_{d_2} \times \mathbf{Z}_{d_1}^{n-1} \times \mathbf{T}^{\frac{n(n-1)}{2}}$. ∎

We now define for each $A \in T$, the function $\beta_A : K \times M \longrightarrow \mathbf{T}$ by

$$\beta_A(t,(r,s)) = e^{2\pi i d_1 t_1 \frac{s_1(s_1-1)}{2} a_{12}}. \tag{2.13}$$

LEMMA 2.3. *For each $A \in T$ the pair (γ_A, β_A) satisfies the Mackey compatibility conditions (2.7) and (2.8). Hence by Mackey's results, the function $\sigma_A : N \times N \to \mathbf{T}$ defined by*

$$\begin{aligned}\sigma_A((r,s,t),(r',s',t')) &= \gamma_A((r,s),\pi(t)(r',s'))\beta_A(t,(r',s'))\\ &= \gamma_A((r,s),(r',s'))\beta_A(t,(r',s'))\end{aligned}$$

is a multiplier of N.

Proof. It is straightforward to check that (γ_A, β_A) satisfies (2.7) and (2.8) so that σ_A as defined in the statement of the lemma is a multiplier. We note for future reference that because A is upper-triangular with 0's along the diagonal, $A \cdot \pi(t) = A$, $\forall t \in K$, so that $\gamma_A((r,s),(r',s')) = \gamma_A((r,s),\pi(t)(r',s'))$, $\forall t \in K$, which completes the proof. ∎

LEMMA 2.4. *Let $\eta : T \to H^2(N,\mathbf{T})$ be defined by $\eta(A) = [\sigma_A]$, where σ_A is as defined in Lemma 2.3. Then η induces a monomorphism*

$$\widetilde{\eta} : T/\mathcal{E}(n+1) \cap \mathcal{T}(n+1) \to H^2(N,\mathbf{T})$$

such that the composition

$$[H^2(M,\mathbf{T})]^K \overset{\Omega}{\cong} T/\mathcal{E}(n+1) \cap \mathcal{T}(n+1) \xrightarrow{\widetilde{\eta}} H^2(N,\mathbf{T}) \xrightarrow{\text{Res}} [H^2(M,\mathbf{T})]^K$$

is the identity map on $[H^2(M,\mathbf{T})]^K$.

Proof. It is straightforward to check that η is a homomorphism. Now let $C = (c_{ij}) \in \mathcal{E}(n+1) \cap \mathcal{T}(n+1)$, that is, $c_{ij} = 0$, $i \geqslant j$ and $c_{ij} \equiv 0 \pmod 1$, $i < j$. We shall show that not only is $\sigma_C \in B^2(N,\mathbf{T})$, but in fact, $\sigma_C \equiv 1$. One easily calculates that

$$\sigma_C((r,s,t),(r',s',t')) = e^{2\pi i d_1 t_1 c_{12} \frac{[(s_1')^2 - s_1']}{2}}$$

which is automatically equal to 1, since d_1, t_1, s_1', c_{12}, $\frac{(s_1')^2 - s_1'}{2} \in \mathbf{Z}$. Thus the map η factors through $T/\mathcal{E}(n+1) \cap \mathcal{T}(n+1)$ giving thus a homomorphism $\widetilde{\eta} : T/\mathcal{E}(n+1) \cap \mathcal{T}(n+1) \to H^2(N,\mathbf{T})$. By Mackey's results, $\text{Res}(\widetilde{\eta}([A])) = [\gamma_A]$. Hence the composition $\text{Res} \circ \widetilde{\eta} \circ \Omega$ is the identity map on $[H^2(M,\mathbf{T})]^K$ as desired. ∎

The next corollary follows immediately.

COROLLARY 2.5. *We have the following short exact sequence, which is split:*

$$0 \longrightarrow H^2(K,\mathbb{T}) \times H^1(K,\widehat{M}) \xrightarrow{\zeta} H^2(N,\mathbb{T}) \xrightarrow{\text{Res}} [H^2(M,\mathbb{T})]^K \longrightarrow 0.$$

Consequently $H^2(N,\mathbb{T}) \cong H^2(K,\mathbb{T}) \times H^1(K,\widehat{M}) \times [H^2(M,\mathbb{T})]^K$.

Since $H^2(K,\mathbb{T})$ and $[H^2(M,\mathbb{T})]^K$ are known, to complete our calculation of $H^2(N,\mathbb{T})$ it remains only to calculate the cohomology group $H^1(K,\widehat{M})$, which we do now.

LEMMA 2.6. *Let* $n \geqslant 2$. *For any* $\chi \in Z^1(K,\widehat{M})$ *we have* $\chi(k_j,m_1)^{d_1} = 1$, $\forall\, 2 \leqslant j \leqslant n$ *and* $\chi(k_1,m_1)^{d_2} = 1$.

Proof. Let $\chi \in Z^1(K,\widehat{M})$. For $1 \leqslant i,j \leqslant n$, we have

$$\chi(k_i + k_j,(r,s)) = \chi(k_i,m_1)^{d_j s_j}\chi(k_i,(r,s))\chi(k_j,(r,s)).$$

By interchanging the roles of i and j, we obtain

$$\chi(k_i,m_1)^{d_j s_j} = \chi(k_j,m_1)^{d_i s_i}, \qquad 1 \leqslant i,\, j \leqslant n.$$

Putting $i = 1$ and $s = (1,0,\ldots,0)$, we obtain

$$\chi(k_j,m_1)^{d_1} = 1, \qquad 2 \leqslant j \leqslant n.$$

Putting $i = 2$, $j = 1$ and $s = (0,1,0,\ldots,0)$ we obtain

$$\chi(k_1,m_1)^{d_2} = 1. \quad \blacksquare$$

We now define for $n \geqslant 1$ a map

$$\Psi : Z^1(K,\widehat{M}) \longrightarrow \begin{cases} \mathbb{T}^2 & n = 1, \\ \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_1}^{n-1} \times \mathbb{T}^{n^2} & n \geqslant 2, \end{cases}$$

given by

$$\Psi(\chi) = \begin{cases} (\chi_{11},\chi_{12}) & n = 1, \\ (\chi_{11},\chi_{21},\ldots,\chi_{n,1},(\chi_{ij} : 1 \leqslant i \leqslant n, 2 \leqslant j \leqslant n+1)) & n \geqslant 2, \end{cases}$$

where $\chi_{ij} = \chi(k_i,m_j)$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$.

The following lemma shows that Ψ is surjective.

LEMMA 2.7. *For $n = 1$, choose arbitrary $\chi_{1,j} \in \mathbf{T}$, $j = 1, 2$ and for $n \geqslant 2$, choose $\chi_{11}, \chi_{21}, \ldots, \chi_{n1}$ such that $\chi_{11}^{d_2} = 1$ and $\chi_{i1}^{d_1} = 1$, $2 \leqslant i \leqslant n$, and choose arbitrary $\chi_{ij} \in \mathbf{T}$, $1 \leqslant i \leqslant n$, $2 \leqslant j \leqslant n+1$. Define $\chi : K \times M \to \mathbf{T}$ by*

$$\chi(t,(r,s)) = \begin{cases} \chi_{11}^{[d_1 s \frac{t(t-1)}{2} + rt]} \chi_{12}^{st} & n = 1, \quad (2.14) \\ \chi_{11}^{d_1 s_1 \frac{t_1(t_1-1)}{2}} \prod_{i=1}^{n} \chi_{i1}^{rt_i} \prod_{i=1}^{n} \prod_{j=2}^{n+1} \chi_{ij}^{s_{j-1} t_i} & n \geqslant 2. \quad (2.15) \end{cases}$$

Then $\chi \in Z^1(K, \widehat{M})$ and

$$\Psi(\chi) = \begin{cases} (\chi_{11}, \chi_{12}) & n = 1, \\ (\chi_{11}, \chi_{21}, \ldots, \chi_{n1}, \ (\chi_{ij} : 1 \leqslant i \leqslant n, 2 \leqslant j \leqslant n+1)) & n \geqslant 2. \end{cases}$$

Proof. It is clear that for fixed $n \in \mathbf{Z}^+$ and $t \in K$, $\chi(t, \cdot)$ is a character of M. We check that the 1-cocycle identity $\chi(t, \pi(t')(r,s))\chi(t', (r,s)) = \chi(t+t', (r,s))$ holds for the separate cases $n = 1$ and $n \geqslant 2$. For $n = 1$, $t, t' \in K$, $(r,s) \in M$,

$$\begin{aligned} \chi(t+t', (r,s)) &= \chi_{11}^{\frac{d_1}{2} s(t+t')(t+t'-1)} \chi_{11}^{r(t+t')} \chi_{12}^{(t+t')s} \\ &= \chi_{11}^{\frac{d_1}{2} st(t-1)} \chi_{11}^{rt} \chi_{11}^{\frac{d_1}{2} stt'} \chi_{11}^{\frac{d_1}{2} st't} \chi_{11}^{\frac{d_1}{2} st'(t'-1)} \chi_{11}^{rt'} \chi_{12}^{ts} \chi_{12}^{t's} \\ &= \chi(t,(r,s))\chi(t',(r,s))\chi_{11}^{d_1 stt'} \\ &= \chi(t, (r + st'd_1, s))\chi(t', (r,s)) \\ &= \chi(t, \pi(t')(r,s))\chi(t', (r,s)). \end{aligned}$$

For $n \geqslant 2$, $t, t' \in K$, $(r,s) \in M$, we have

$$\begin{aligned} \chi(t, \pi(t')(r,s)) &= \chi\left(t, \left(r + \sum_{i=1}^{n} d_i t'_i s_i, s\right)\right) \\ &= \chi(t,(r,s))\chi(t, m_1)^{\sum_{i=1}^{n} d_i t'_i s_i} \\ &= \chi(t,(r,s)) \left(\prod_{i=1}^{n} \chi_{i1}^{t_i}\right)^{\sum_{i=1}^{n} d_i t'_i s_i}, \end{aligned}$$

which, since $\chi_{i1}^{d_1} = 1$ for $i \geqslant 2$ and since $d_1 | d_2 | \cdots | d_n$, is equal to $\chi(t,(r,s))\chi_{11}^{t_1 d_1 t'_1 s_1}$. Therefore,

$$\begin{aligned} \chi(t+t', (r,s)) &= \chi_{11}^{\frac{d_1}{2} s_1 (t_1 + t'_1)(t'_1 + t'_1 - 1)} \prod_{i=1}^{n} \chi_{i1}^{r(t_i + t'_i)} \prod_{i=1}^{n} \prod_{j=2}^{n+1} \chi_{ij}^{s_{j-1}(t_i + t'_i)} \\ &= \chi_{11}^{t_1 t'_1 s_1 d_1} \chi(t,(r,s))\chi(t',(r,s)) \\ &= \chi(t, \pi(t')(r,s))\chi(t',(r,s)). \quad ∎ \end{aligned}$$

COROLLARY 2.8.

$$Z^1(K,\widehat{M}) \cong \begin{cases} \mathbb{T}^2 & n=1, \\ \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_1}^{n-1} \times \mathbb{T}^{n^2} & n \geqslant 2. \end{cases}$$

LEMMA 2.9.

$$\Psi(B^1(K,\widehat{M})) = \begin{cases} \{(1,\lambda) : \lambda \in \mathbb{T}\} & n=1, \\ \{(\underbrace{1,1,\ldots,1}_{n},(\lambda,\lambda^{d_2/d_1},\ldots,\lambda^{d_n/d_1},\underbrace{1,\ldots,1}_{n^2-n}) : \lambda \in \mathbb{T}\} & n \geqslant 2. \end{cases}$$

[*For the case* $n \geqslant 2$, *we have placed* $(\chi_{ij})_{1\leqslant i \leqslant n+1,\ 2\leqslant j\leqslant n+1}$ *in the following order:* $(\chi_{12}, \chi_{23}, \ldots, \chi_{n,n+1}$, *followed by lexicographically ordering the remaining coordinates).*]

Proof. For $a \in \widehat{M}$, define $\chi^a : K \times M \to \mathbb{T}$ by

$$\chi^a(t,v) = \frac{a(v)}{a(\pi(t)v)}, \qquad t \in K, v \in M.$$

By definition, $B^1(K,\widehat{M}) = \{\chi^a : a \in \widehat{M}\}$.

For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$, we have $\pi(k_i)(m_j) = m_j + \delta_{i+1,j} d_i m_1$, so that for all $a \in \widehat{M}$,

$$\chi^a(k_i,m_j) = \begin{cases} \frac{a(m_j)}{a(m_j)} = 1 & j \neq i+1, \\ \frac{a(m_{i+1})}{a(m_{i+1})[a(m_1)]^{d_i}} = \left[\frac{1}{a(m_1)}\right]^{d_1} & j = i+1. \end{cases}$$

Let $\lambda = \left[\frac{1}{a(m_1)}\right]^{d_1}$. Then $\chi^a(k_i,m_{i+1}) = \left[\frac{1}{a(m_1)}\right]^{d_i} = \lambda^{d_i/d_1}$, so that

$$\Psi(\chi^a) = \begin{cases} (1,\lambda) & n=1, \\ ((\underbrace{1,\ldots,1}_{n}),(\underbrace{\lambda,\lambda^{d_2/d_1},\ldots,\lambda^{d_n/d_1}}_{n},\underbrace{1,\ldots,1}_{n^2-n})) & n \geqslant 2, \end{cases}$$

as desired. ∎

LEMMA 2.10.

$$H^1(K,\widehat{M}) = \begin{cases} \mathbb{T} & n=1, \\ \mathbb{Z}_{d_2} \times \mathbb{Z}_{d_1}^{n-1} \times \mathbb{T}^{(n+1)(n-1)} & n \geqslant 2. \end{cases}$$

Proof. Let

$$U = \begin{cases} \mathbb{T} & n=1, \\ \{(\lambda,\lambda^{d_2/d_1},\ldots,\lambda^{d_n/d_1}) : \lambda \in \mathbb{T}\} \subseteq \mathbb{T}^n & n \geqslant 2. \end{cases}$$

Then by Corollary 2.8 and Lemma 2.9,

$$\begin{aligned} H^1(K,\widehat{M}) &= Z^1(K,\widehat{M})/B^1(K,\widehat{M}) \cong \Psi(Z^1(K,\widehat{M}))/\Psi(B^1(K,\widehat{M})) \\ &\cong \begin{cases} \mathbf{T}^2/\{1\} \times U & n=1, \\ \mathbf{Z}_{d_2} \times \mathbf{Z}_{d_1}^{n-1} \times \mathbf{T}^n \times \mathbf{T}^{n^2-n}/\{1\} \times \{1\} \times U \times \{1\} & n \geqslant 2, \end{cases} \\ &\cong \begin{cases} \mathbf{T} & n=1, \\ \mathbf{Z}_{d_2} \times \mathbf{Z}_{d_1}^{n-1} \times \mathbf{T}^n/U \times \mathbf{T}^{n^2-n} & n \geqslant 2. \end{cases} \end{aligned}$$

Hence it remains to determine $\mathbf{T}^n/U$ for $n \geqslant 2$. We now define $\nu : \mathbf{T}^n \to \mathbf{T}^{n-1}$ for $n \geqslant 2$ by

$$\nu(u_1, u_2, \ldots, u_n) = \left(u_1^{-d_2/d_1} u_2, u_1^{-d_3/d_1} u_3, \ldots, u_1^{-d_n/d_1} u_n\right).$$

Then ν is clearly a surjective homomorphism with kernel U. Consequently $\mathbf{T}^n/U \cong \mathbf{T}^{n-1}$ and the lemma follows. ∎

We have completed the proof of:

THEOREM 2.11. *Let $N = H(d_1, \ldots, d_n)$ be a generalized discrete Heisenberg group. Then the multiplier group $H^2(N, \mathbf{T})$ is given by*

$$H^2(N,\mathbf{T}) = \begin{cases} \mathbf{T}^2 & n=1, \\ \mathbf{Z}_{d_2}^2 \times \mathbf{Z}_{d_1}^{2(n-1)} \times \mathbf{T}^{\frac{n(n-1)}{2}} \times \mathbf{T}^{\frac{n(n-1)}{2}} \times \mathbf{T}^{n^2-1} & \\ \qquad \cong \mathbf{Z}_{d_2}^2 \times \mathbf{Z}_{d_1}^{2(n-1)} \times \mathbf{T}^{(2n+1)(n-1)} & n \geqslant 2. \end{cases}$$

Proof. This follows immediately from Corollary 2.2, Corollary 2.5, and Lemma 2.10. ∎

REMARK 2.12. We note that for $n = 1$, $H^2(N, \mathbf{T})$ is always path connected, whereas for $n \geqslant 2$, this will no longer be the case, unless $d_1 = d_2 = 1$. An explanation for this can be found in the proof of [27], Corollary 2.12 and for the reader's convenience we summarize that explanation here. The group N will have as a classifying space a $2n+1$ dimensional nilmanifold X, and the path component of $H^2(N, \mathbf{T})$ will correspond exactly to the torsion subgroup of $\check{H}^3(X, \mathbf{Z})$. When $n = 1$, $\dim X = 3$, and $\check{H}^3(X, \mathbf{Z}) \cong \mathbf{Z}$ by Poincaré duality, showing that $H^2(N, \mathbf{T})$ must be path connected.

For our results in the next section, we will want to use explicit cocycles in $Z^2(N, \mathbf{T})$, so we now construct a monomorphism from $H^2(N, \mathbf{T})$ into $Z^2(N, \mathbf{T})$.

PROPOSITION 2.13. *Write* $H^2(N,\mathbb{T})$ *as* $\mathbb{T}^2$ *for* $n=1$ *and as* $\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_1}^{n-1}\times\mathbb{Z}_{d_2}\times\mathbb{Z}_{d_1}^{n-1}\times\mathbb{T}^{\frac{n(n-1)}{2}}\times\mathbb{T}^{\frac{n(n-1)}{2}}\times\mathbb{T}^{n^2-1}$ *for* $n\geqslant 2$ *and define a map* c *from* $H^2(N,\mathbb{T})$ *into* $Z^2(N,\mathbb{T})$ *by*

$$c((\lambda,\mu))((r,s,t),(r',s',t')) = \lambda^{s'r+d_1ts'\frac{(s'-1)}{2}}\mu^{r't+d_1s'\frac{t(t-1)}{2}}, \qquad n=1 \tag{2.16}$$

and

$$c((\lambda_1,\dots,\lambda_n),(\mu_1,\dots,\mu_n),(\gamma_{jk}:1\leqslant j<k\leqslant n),(\beta_{jk}:1\leqslant j<k\leqslant n),$$
$$(\alpha_{ij}:1\leqslant i\leqslant n, 1\leqslant j\leqslant n,(i,j)\neq(1,0))) = \sigma$$

where

$$\sigma((r,s,t),(r',s',t')) = \prod_{i=1}^n \lambda_i^{s_i'r}\prod_{i=1}^n \mu_i^{r't_i}\lambda_1^{d_1t_1s_1'\frac{(s_1'-1)}{2}}\mu_1^{d_1s_1'\frac{t_1(t_1-1)}{2}} \cdot \prod_{1\leqslant j<k\leqslant n}\gamma_{jk}^{s_js_k'}\prod_{1\leqslant j<k\leqslant n}\beta^{t_jt_k'}\prod_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n\\ (i,j)\neq(1,1)}}\alpha_{ij}^{s_j't_i}, \quad n\geqslant 2. \tag{2.17}$$

Then c *is a monomorphism which is a cross-section for the canonical projection from* $Z^2(N,\mathbb{T})$ *onto* $H^2(N,\mathbb{T})$.

Proof. For $n=1$ the result is easily seen to be true and hence we concentrate on $n\geqslant 2$. For this case the proof just amounts to a change in the notation previously used. In particular, $\lambda_i = e^{2\pi i a_{1,i+1}}$, $1\leqslant i\leqslant n$, and $\gamma_{ij}=e^{2\pi i a_{i+1,j+1}}$, $1\leqslant i<j\leqslant n$, where the (a_{ij}) are as in Lemma 2.1. The $(\beta_{ij},\ 1\leqslant i<j\leqslant n)$ are the standard upper-triangular parametrization for $H^2(K,\mathbb{T})$ where $K\cong\mathbb{Z}^n$. The $\mu_i=\chi_{i1}$, $1\leqslant i\leqslant n$, and the $\alpha_{ij}=\chi_{i,j+1}$, $1\leqslant i\leqslant j\leqslant n$, $(i,j)\neq(1,1)$, where the (χ_{ij}) are as in Lemma 2.7. (Lemma 2.9 shows that without loss of generality we can choose $\alpha_{11}=\chi_{12}=1$.) It is clear that the map c is a monomorphism, and Lemmas 2.1, 2.2, 2.3 and 2.9 combine to complete the proof of the proposition. ∎

From the proposition we immediately obtain the following:

COROLLARY 2.14. *Let* $N=H(d_1,\dots,d_n)$ *for* $n\geqslant 2$ *and let* $L=\{(d_2m,0,0): m\in\mathbb{Z}\}\subseteq Z=\{(r,0,0):r\in\mathbb{Z}\}\subseteq N$. *Then every multiplier on* N *is cohomomlogous to a multiplier of the form* Inf ω *where* ω *is a multiplier on the quotient group* N/L.

Proof. We note that L is a central subgroup of N, and, that if σ is as in equation (2.17), then $\sigma/L\times L=1$ in $Z^2(L,\mathbb{T})$ and $\widetilde{\sigma}((d_2m,0,0),(r,s,t))=1$,

$\forall (r,s,t) \in N$ and $\forall m \in \mathbf{Z}$, so that by [27], Proposition A2, $[\sigma] = [\text{Inf } \omega]$ for some $\omega \in H^2(N/L, \mathbf{T})$. (One can check that L is the largest subgroup of Z that will have this property for all σ.) Indeed, one can easily verify that σ itself is a lift of a multiplier on N/L, by considering N/L as the semidirect product $(\mathbf{Z}_{d_2} \times \mathbf{Z}^n) \rtimes \mathbf{Z}^n$ and noting that σ is made up from components as in (2.6), (2.7) and (2.8) which satisfy the Mackey's compatibility conditions for the quotient group. We will use this fact in the next section. ∎

REMARK 2.15. It is easily seen that Corollary 2.14 does not extend to the case $n = 1$. We now attempt to indicate why there should be this significant difference between the cases $n = 1$ and $n \geqslant 2$, as already seen in Remark 2.12. Let N be a discrete group with normal subgroup L, and let A be a N-module. The Lyndon-Hochschild-Serre spectral sequence [15] gives a first quadrant spectral sequence converging to $\bigoplus_{n \geqslant 0} H^n(N, A)$, with E_2-term given by

$$E_2^{p,q} \cong H^p(N/L, H^q(L, A)).$$

Therefore $H^2(N, A)$ can be obtained via the filtration $H^2 \cong F_0 > F_1 > F_2 > \{1\}$, where $F_0/F_1 \cong E_\infty^{0,2}$, $F_1/F_2 \cong E_\infty^{1,1}$ and $F_2 \cong E_\infty^{2,0}$. Now let $N = H(d_1, \ldots, d_n)$ and let L be some nontrivial subgroup of the center $Z = \{(r, 0, 0) : r \in \mathbf{Z}\}$ and let $A = \mathbf{T}$ viewed as a trivial N-module. Since L has rank 1, $H^2(L, \mathbf{T}) = \{1\}$ and consequently $E_\infty^{0,2}$ (a subgroup of $E_2^{0,2} \cong H^2(L, \mathbf{T}) = \{1\}$) is also $\{1\}$. The map $\pi_0 : F_0 \to F_0/F_1$ corresponds to the restriction map Res $: H^2(N, \mathbf{T}) \to H^2(L, \mathbf{T})$ so that $H^2(N, \mathbf{T}) = F_0 = F_1 = \ker \text{Res}$. The map $\varphi_L : \ker \text{Res} = H^2(N, \mathbf{T}) \to H^1(N/L, H^1(L, \mathbf{T}))$ discussed in Section 1 corresponds to the projection $\pi_1 : F_1 \to F_1/F_2$ in the filtration, upon identifying $E_\infty^{1,1}$ with a subgroup of $E_2^{1,1} = H^1(N/L, H^1(L, \mathbf{T}))$. The spectral sequence shows that $\ker \varphi_L$ is exactly the image of the inflation map Inf $: H^2(N/L, \mathbf{T}) \to H^2(N, \mathbf{T})$ (see [27], Proposition A2 for a detailed proof). If we can show that $F_1/F_2 \cong E_\infty^{1,1} = \{1\}$, it will follow that $F_1 = F_2$ and $\text{Im}\,\text{Inf} = \ker \varphi_L$ will equal all of $H^2(N, \mathbf{T})$. Now $E_\infty^{1,1} = E_3^{1,1} = \ker \partial_2 : E_2^{1,1} \to E_2^{3,0}$. So in order that Inf $: H^2(N/L, \widehat{L}) \to H^3(N/L, \mathbf{T})$ be onto, it is necessary and sufficient that $\partial_2 : H^1(N/L, \widehat{L}) \to H^3(N/L, \mathbf{T})$ be one-to-one. One computes that N/L is a finite extension of $\mathbf{Z}^{2n}$, so that $H^1(N/L, \widehat{L})$ will be isomorphic to $\mathbf{T}^{2n} \times F$ for some finite abelian group F. On the other hand, $H^3(N/L, \mathbf{T})$ will be a finite group for $n = 1$, and will have a $\frac{2n(2n-1)(2n-2)}{3!}$ dimensional torus as a subgroup for $n \geqslant 2$. It follows that for $n = 1$, ∂_2 must have a large kernel, thus it is impossible for Inf to be onto. On the other hand, for $n \geqslant 2$, there is no such obvious contradiction to prevent $\partial_2 : E_2^{1,1} \to E_2^{3,0}$ from being one-to-one. In fact [15], Theorem 4 gives an explicit formula for ∂_2 in terms of a cup product construction, so that it should be possible to verify directly that in the case $n \geqslant 2$, $N = H(d_1, \ldots, d_n)$, $L = d_2 Z$, the map $\partial_2 : H^1(N/L, \widehat{L}) \to H^3(N/L, \mathbf{T})$ is one-to-one. Since this follows from our Corollary 2.11, we will not elaborate on this point further.

3. THE PRIMITIVE IDEAL SPACES OF TWISTED GENERALIZED HEISENBERG GROUP C^*-ALGEBRAS

In this section we use the results obtained so far to give a setwise parametrization for the primitive ideal spaces of twisted group C^*-algebras constructed from multipliers on generalized discrete Heisenberg groups.

Let $N = H(d_1, \ldots, d_n)$ and let σ be a multiplier on N as given in (2.16) for $n = 1$ and (2.17) for $n \geqslant 2$. In order to obtain a setwise parametrization of $\mathrm{Prim}(C^*(N, \sigma))$, our first step will be to calculate $Z_o = \{z \in Z : \varphi_Z(\sigma)(n)(z) = 1, \forall n \in N\}$.

PROPOSITION 3.1. *Let σ be as in* (2.16) *for $n = 1$ and* (2.17) *for $n \geqslant 2$. Then*

$$Z_o = \begin{cases} \{(m, 0, 0) : [\lambda^s \mu^{-t}]^m = 1,\ \forall (s, t) \in \mathbb{Z}^2\} & n = 1, \\ \left\{(m, 0, 0) : \left[\prod_{i=1}^{n} \lambda_i^{s_i} \mu_i^{-t_i}\right]^m = 1,\ \forall (s, t) \in \mathbb{Z}^{2n}\right\} & n \geqslant 2. \end{cases}$$

Proof. Here $Z = \{(r, 0, 0) : r \in \mathbb{Z}\}$, and one calculates using (2.16) and (2.17) that

$$\varphi_Z(\sigma)((r, s, t))((m, 0, 0)) = \widetilde{\sigma}((m, 0, 0), (r, s, t))$$

$$= \begin{cases} \lambda^{ms} \mu^{-mt} \left[\lambda^s \mu^{-t}\right]^m & n = 1, \quad (3.1) \\ \left(\prod_{i=1}^{n} \lambda_i^{s_i} \mu_i^{-t_i}\right)^m & n \geqslant 2, \quad (3.2) \end{cases}$$

giving the desired result. ∎

Easy calculations using Proposition 2.13, Proposition 3.1 and Pontryagin duality gives the following proposition.

PROPOSITION 3.2. *For $n = 1$, Z_o is non-trivial if and only if both λ and μ are torsion elements of $\mathbf{T}$ and in this case, $[Z, Z_o] = \#\Gamma$, where Γ is the (finite) subgroup of $\mathbf{T}$ generated by λ and μ. For $n \geqslant 2$, Z_o is a non-trivial subgroup of Z containing $\{(d_2 r, 0, 0) : r \in \mathbf{Z}\}$, and $[Z, Z_o] = \#\Gamma$, where Γ is the subgroup of $\mathbf{T}$ generated by the elements $\{\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n\}$.*

Proof. For the case $n \geqslant 2$, by Proposition 2.13, $\lambda_1, \mu_1 \in \mathbf{Z}_{d_2}$ and $\{\lambda_i, \mu_i : 2 \leqslant i \leqslant n\} \subseteq \mathbf{Z}_{d_1}$. Since $d_1 | d_2$, $\mathbf{Z}_{d_1} \subseteq \mathbf{Z}_{d_2} \subseteq \mathbf{T}$, and $\{\lambda_i, \mu_i : 1 \leqslant i \leqslant n\} \subseteq \mathbf{Z}_{d_2}$. Therefore $\left[\prod_{i=1}^{n} \lambda_i^{s_i} \mu_i^{-t_i}\right]^{d_2} = 1$ regardless of which $(s, t) \in \mathbf{Z}^{2n}$ we choose, and it follows that $\{d_2 r, 0, 0)\} \subseteq Z_o$ so that Z_o must be non-trivial. Since $\{\lambda_i, \mu_i : 1 \leqslant i \leqslant n\}$ are all torsion elements of $\mathbf{T}$, the subgroup of $\mathbf{T}$ generated by them, Γ, is finite, hence closed, and therefore can be described by $\Gamma = \left\{\prod_{i=1}^{n} [\lambda_i^{s_i} \mu_i^{-t_i}] : (s, t) \in \mathbf{Z}^{2n}\right\}$. Now viewing $Z \cong \mathbf{Z}$ and $\widehat{Z} \cong \widehat{\mathbf{Z}} \cong \mathbf{T}$, it follows that Z_o as defined in Proposition 3.1 can be viewed as $Z_o = \Gamma^{\perp} \subseteq \widehat{\mathbf{T}} = \mathbf{Z} \cong Z$. It follows by Pontryagin duality that $\#\Gamma = \#\widehat{\Gamma} = \#(Z/\Gamma^{\perp}) = \#(Z/Z_o) = [Z : Z_o]$, proving the desired result for $n \geqslant 2$. For $n = 1$, we see from the calculations of Proposition 3.1 that if either λ and μ is a non-torsion element of $\mathbf{T}$, then Z_o must be trivial. Conversely, if Z_o is non-trivial so that $Z_o = \{(\ell r, 0, 0) : r \in \mathbf{Z}\}$ for some $\ell \in \mathbf{Z}^+$, then taking first $s = 1$ and $t = 0$ and then $s = 0$ and $t = 1$ in (3.1) gives $\lambda^{\ell} = 1$ and $\mu^{-\ell} = 1$ respectively so that λ and μ are both torsion elements of $\mathbf{T}$. The proof that $[Z : Z_o] = \#(\Gamma)$ is the same as in the case $n \geqslant 2$. ∎

We now concentrate on the case $n \geqslant 2$. For future reference we introduce some notation concerning the character group of $\ell\mathbf{Z} = \{\ell j : j \in \mathbf{Z}\}$. Since $\ell\mathbf{Z} \cong \mathbf{Z}$, by duality we have a corresponding isomorphism $\mathbf{T} = \widehat{\mathbf{Z}} \cong \widehat{\ell\mathbf{Z}}$, and this isomorphism is given by $\gamma \longrightarrow \chi_\gamma \in \widehat{\ell\mathbf{Z}}$ where $\chi_\gamma(\ell j) = \gamma^j$. To simplify notation, hereafter we will identify $\gamma \in \mathbf{T}$ with $\chi_\gamma \in \widehat{\ell\mathbf{Z}}$.

LEMMA 3.3. *Let $N = H(d_1, \ldots, d_2)$ be a generalized Heisenberg group for $n \geqslant 2$ and let σ be the multiplier on N defined as in (2.17), with parameters $\{\lambda_1, \mu_i : 1 \leqslant i \leqslant n\}$, $\{\gamma_{jk} : 1 \leqslant j < k \leqslant n\}$, $\{\beta_{jk} : 1 \leqslant j < k \leqslant n\}$ and $\{\alpha_{ij} : 1 \leqslant i, j \leqslant n,\ (i, j) \neq (1, 1)\}$ (recall that $\lambda_1, \mu_1 \in \mathbf{Z}_{d_2}$ and $\{\lambda_i, \mu_i : 2 \leqslant i \leqslant n\} \subseteq \mathbf{Z}_{d_1}$). Define the subgroups $Z_o \subseteq Z \subseteq N$ and $\Gamma \subseteq \mathbf{T}$ as in Proposition 3.2, and let $\ell = \#(\Gamma)$. Then σ is the lift of a multiplier ω defined on $N/Z_o \cong (\mathbf{Z}_\ell \times \mathbf{Z}^n) \rtimes \mathbf{Z}^n$, and $C^*(N, \sigma)$ is $*$-isomorphic to the C^*-algebra of continuous sections of a C^*-bundle*

E over $Z_o = \widehat{\ell\mathbb{Z}} \cong \mathbb{T}$ with fibre over $\gamma \in \mathbb{T} \cong \mathbb{Z}_o$ given by $C^*((\mathbb{Z}_\ell \times \mathbb{Z}^n)\rtimes\mathbb{Z}^n, \omega\cdot\delta_\gamma)$, where $\delta_\gamma \in Z^2(N/Z_o, \mathbb{T})$ is defined by

$$\delta_\gamma((\dot r, s, t), (\dot r', s', t')) = \rho^{\sum_{i=1}^{n} d_i t_i s_i'} \tag{3.3}$$

where $\rho \in \mathbb{T}$ satisfies $\rho^\ell = \gamma$, and where $\dot r \in \mathbb{Z}_\ell$ is the image of $r \in \mathbb{Z}$ in $\mathbb{Z}/\ell\mathbb{Z} \cong \mathbb{Z}_\ell$.

Proof. Arguments using Pontryagin duality show that we can view $Z_o \subseteq \mathbb{Z}$ as $\Gamma^\perp \subseteq \mathbb{T} \cong \widehat{\mathbb{T}} \cong \widehat{\widehat{\mathbb{Z}}} \cong \widehat{\widehat{Z}} \cong Z$, and therefore, that we can view Z_o as $(\ell\mathbb{Z} \times \{0\})\rtimes\{0\} \cong \ell\mathbb{Z}$, so that σ is cohomologous to a multiplier lifted from N/Z_o. Since every element of $\{\lambda_i, \mu_i : 1 \leqslant i \leqslant n\}$ has order dividing ℓ, (2.17) shows that σ is a lift of a multiplier ω on $N/Z_o \cong (\mathbb{Z}_\ell \times \mathbb{Z}^n)\rtimes\mathbb{Z}^n$, obtaining by viewing the latter group as a semi-direct product of $\mathbb{Z}_\ell \times \mathbb{Z}^n$ and $\mathbb{Z}$ and repeating the analysis of Corollary 2.14 with the group L there replaced by Z_o; the parameters defining ω remain unchanged. The desired result will follow from the proof of Theorem 1.1, upon showing that the multiplier $d_2(\gamma)$ on N/Z_o corresponding to $\gamma \in \widehat{Z}_o$ defined in (1.8) is cohomologous to the multiplier δ_γ defined in the statement of the present lemma. Define a cross-section $\tau : \mathbb{Z}_\ell \to \mathbb{Z}$ by $\tau : \mathbb{Z}_\ell \to \{0, 1, \ldots, \ell-1\} \subseteq \mathbb{Z}$, and use τ to define a cross-section $\eta : (\mathbb{Z}_\ell \times \mathbb{Z}^n)\rtimes\mathbb{Z}^n \to N$ given by $\eta((\dot r, s, t)) = (\tau(\dot r), s, t)$, $(\dot r, s, t) \in (\mathbb{Z}_\ell \times \mathbb{Z}^n)\rtimes\mathbb{Z}^n$. Then computing $d_2(\gamma)$ with respect to the cross-section η we obtain from (1.8) that

$$d_2(\gamma)((\dot r, s, t), (\dot r', s', t')) = \rho^{\left[\sum_{i=1}^{n} d_i t_i s_i' + \tau(\dot r) + \tau(\dot r') - \tau\left(\dot r + \dot r' + (\overbrace{\sum_{i=1}^{n} d_i t_i s_i'}^{\bullet})\right)\right]} \tag{3.4}$$

where $(\dot r, s, t), (\dot r', s', t') \in (\mathbb{Z}_\ell \times \mathbb{Z}^n)\rtimes\mathbb{Z}^n$ and $\rho \in \mathbb{T}$ satisfies $\rho^\ell = \gamma$ (the cohomology class of $d_2(\gamma)$ in $H^2((\mathbb{Z}_\ell \times \mathbb{Z})\rtimes\mathbb{Z}^n)$ is independent of the choice of ρ). Now define $b : (\mathbb{Z}_\ell \times \mathbb{Z})\rtimes\mathbb{Z} \to \mathbb{T}$ by $b((\dot r, s, t)) = \rho^{\tau(\dot r)}$. It is easy to calculate that $\partial b \cdot \delta_\gamma = d_2(\gamma)$, where δ_γ is as in (3.3) of the Lemma 3.3. Hence δ_γ is cohomologous to $d_2(\gamma)$, and the result of Lemma 3.3 follows. ∎

We are now ready to prove the main theorem of this section:

THEOREM 3.4. *Let $n \geqslant 2$ and let $N = H(d_1, \ldots, d_n)$. Then for any $\sigma \in Z^2(N, \mathbb{T})$ there exists $\ell \in \mathbb{Z}^+$, a subgroup R of $\mathbb{Z}^{2n}$ of index ℓ, and a one parameter family $\{\tau_\gamma : \gamma \in \widehat{\ell\mathbb{Z}} \cong \mathbb{T}\}$ of multipliers of R such that $C^*(N, \sigma)$ is $*$-isomorphic to the C^*-algebra of continuous sections of a C^*-bundle over $\widehat{Z}_o$ with fibre over $\gamma \in \widehat{Z}_o = \widehat{\ell\mathbb{Z}}$ $*$-isomorphic to $M_\ell(C^*(R, \tau_\gamma))$, the C^*-algebra of $\ell \times \ell$ matrices with entries in the rotation algebra $C^*(R, \tau_\gamma)$.*

Proof. By Proposition 2.13, we can assume that σ is a multiplier on N parametrized in the standard fashion, as in (2.17). Using Lemma 3.3, and letting $\ell \in \mathbf{Z}^+$ be as in the statement of that lemma, to prove our theorem it suffices to construct a subgroup R of $\mathbf{Z}^{2n}$ of index ℓ and for every $\gamma \in \widehat{Z}_o \cong \widehat{\ell\mathbf{Z}} \cong \mathbf{T}$ a multiplier τ_γ on R, such that $C^*((\mathbf{Z}_\ell \times \mathbf{Z}^n) \rtimes \mathbf{Z}^n, \omega \cdot \delta_\gamma)$ is $*$-isomorphic to $M_\ell(C^*(R, \tau_\gamma))$. Let $N_1 = N/Z_o = (\mathbf{Z}_\ell \times \mathbf{Z}^n) \rtimes \mathbf{Z}^n$ and let D let the subgroup of the center of N_1 given by $(\mathbf{Z}_\ell \times \{0\}) \rtimes \{0\} = Z/Z_o$. D clearly contains the commutator subgroup C_1 of N_1, and direct computations with (2.17) and (3.6) show that $\omega \cdot \delta_\gamma$ is trivial on $D \times D$ and $\varphi_D(\omega \cdot \delta_\gamma) : N_1 \to \widehat{D}$ is surjective, $\forall \gamma \in \widehat{Z}_o$. Letting $M = \ker \varphi_D$ and $R = M/D$, we have R a subgroup of $N_1/D \cong \mathbf{Z}^{2n}$ and $[\mathbf{Z}^{2n} : R] = [N_1 : M] = \#\Gamma = \ell$, as desired. Applying Corollary 1.9, we immediately obtain that $C^*((\mathbf{Z}_\ell \times \mathbf{Z}^n) \rtimes \mathbf{Z}^n, \omega \cdot \delta_\gamma)$ is $*$-isomorphic to $C^*(R, \tau_\gamma) \otimes \mathcal{K}(L^2(N_1/M))$ which in turn is $*$-isomorphic to $C^*(R, \tau_\gamma) \otimes M_\ell(\mathbf{C}) \cong M_\ell(C^*(R, \tau_\gamma))$, since $[N_1 : M] = \ell$. Here τ_γ is the multiplier defined on $R \times R$ by using (1.13). Let $c : \mathbf{Z}^{2n} \to \mathbf{Z}_\ell \times \mathbf{Z}^{2n}$ be the cross-section given by $c((s,t)) = (\dot{0}, s, t)$, and define the multiplier $\Omega : \mathbf{Z}^{2n} \times \mathbf{Z}^{2n} \to \mathbf{T}$ by

$$(3.5) \qquad \Omega((s,t),(s',t')) = \prod_{1 \leqslant j < k \leqslant n} \gamma_{jk}^{s_j s'_k} \prod_{1 \leqslant j < k \leqslant n} \beta_{jk}^{t_j t'_k} \prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n \\ (i,j) \neq (1,1)}} \alpha_{ij}^{s'_j t_i}$$

where the $\{\gamma_{jk}, \beta_{jk}, \alpha_{ij}\}$ are those elements of $\mathbf{T}$ associated to ω and σ as in Proposition 2.13. Then upon calculation, we obtain for $\gamma \in \widehat{Z}_o \cong \widehat{\ell\mathbf{Z}} \cong \mathbf{T}$, and $\rho \in \mathbf{T}$ with $\rho^\ell = \gamma$,

$$(3.6) \qquad \tau_\gamma((s,t),(s',t')) = \lambda_1^{-d_1 t_1 \frac{s'_1(s'_1+1)}{2}} \mu_1^{-d_1 s'_1 \frac{t_1(t_1+1)}{2}} \Omega((s,t),(s',t')) \rho^{\sum_{i=1}^{n} d_i t_i s'_i},$$

and $C^*(N, \sigma)$ is $*$-isomorphic to the C^*-algebra of continuous cross-sections of a C^*-bundle over $\mathbf{T}$ with fibres given by C^*-algebras which are $*$-isomorphic to matrix algebras of constant dimension over (varying) twisted $\mathbf{Z}^{2n}$ algebras, as we desired to show. ∎

COROLLARY 3.5. *Let $n \geqslant 2$ and let $H = H(d_1, \ldots, d_n)$. Then for any $\sigma \in \mathbf{Z}^2(H, \mathbf{T})$, there is a continuous open surjection from* $\mathrm{Prim}(C^*(H,\sigma))$ *onto $\mathbf{T}$ with fibres over $\gamma \in \mathbf{T}$ homeomorphic to $\mathbf{T}^{n_\gamma}$, $n_\gamma \in \mathbf{Z}^+$, $n_\gamma \leqslant 2n$. Thus in particular for $n \geqslant 2$ the twisted group C^*-algebras $C^*(H,\sigma)$ are not simple.*

Proof. The proof follows from Theorem 3.4 upon using [18], Theorem 4, and the fact that $\mathrm{Prim}(M_\ell(C^*(\mathbf{Z}^{2n}, \tau_\gamma))) = \mathrm{Prim}(C^*(\mathbf{Z}^{2n}, \tau_\gamma)) = \widehat{S}_{\tau_\gamma}$, where S_{τ_γ} is the symmetrizer subgroup of $\mathbf{Z}^{2n}$ associated to the multiplier τ_γ. In general, the positive integers $\{n_\gamma \in \mathbf{Z}^+ : \gamma \in \mathbf{T}\}$ will vary with $\gamma \in \mathbf{T}$, as we will see in upcoming examples. ∎

We illustrate some sample calculations in the following fairly straightforward example (the basic strategy would be followed in more complicated examples as well).

EXAMPLE 3.6. Let $\lambda_1 = e^{2\pi i \frac{p}{d_2}}$, $(p, d_2) = 1, \mu_1 = 1, \lambda_i = \mu_i = 1,\ 2 \leqslant i \leqslant n$, let $\gamma_{jk} = \beta_{jk} = 1,\ 1 \leqslant j, k \leqslant n$, let $\alpha_{ij} = 1,\ 1 \leqslant i \neq j \leqslant n$ and $(i, j) = (1, 1)$, and let $\{\alpha_{kk} = e^{2\pi i \theta_k} :\ 2 \leqslant k \leqslant n\}$ be arbitrary elements of $\mathbb{T}$. Let σ be the multiplier on $N = H(d_1, \ldots, d_n)$ given in (2.17) corresponding to the above choice of parameters. Then, following the notation of Proposition 3.2, $\Gamma = \{e^{2\pi i \frac{m}{d_2}} : m \in \mathbb{Z}\}$, $\ell = d_2$, σ is the lift of a multiplier ω defined on $N_1 = (\mathbb{Z}_{d_2} \times \mathbb{Z}^n) \rtimes \mathbb{Z}^n$, $R = (\mathbb{Z}_{d_2} \times \{0\}) \rtimes \{0\}$, and a calculation shows that $R \subseteq N_1/D \cong \mathbb{Z}^{2n}$ is given by $d_2\mathbb{Z} \times \mathbb{Z}^{2n-1}$. For each $\rho \in \mathbb{T} = \widehat{\mathbb{Z}} \cong \widehat{d_2\mathbb{Z}}$, choose $\rho \in \mathbb{T}$ with $\rho^{d_2} = \gamma$. Find $\ell_j \in \mathbb{Z}^+$, $2 \leqslant j \leqslant n$ with $d_j = \ell_j d_2$, $2 \leqslant j \leqslant n$ (note $\ell_2 = 1$). Then we compute that τ_γ on $R \times R$ is given by

$$\begin{aligned}
&\tau_\gamma(((s_1 = d_2 j_1, s_2, \ldots, s_n), t), ((s_1' = d_2 j_1', s_2', \ldots, s_n'), t')) \\
&\quad = e^{-\frac{2\pi i}{d_2} p d_1 t_1 d_2 j_1' \frac{(d_2 j_1' + 1)}{2}} \Omega((s,t),(s',t')) \rho^{d_1 t_1 d_2 j_1'} \rho^{\sum_{i=2}^{n} d_i t_i s_i'} \\
&\quad = e^{-2\pi i d_1 p t_1 \frac{(d_2 (j_1')^2 + (j_1'))}{2}} \Omega((s,t),(s',t')) \gamma^{d_1 t_1 s_1'} \gamma^{\sum_{i=2}^{n} \ell_i t_i s_i'} \\
&\quad = e^{2\pi i \frac{(d_2+1)}{2} t_1 j_1'} \Omega((s,t),(s',t')) \gamma^{d_1 t_1 s_1'} \gamma^{\sum_{i=2}^{n} \ell_i t_i s_i'}
\end{aligned}$$

(recalling that $(j_1')^2 = j_1' \bmod 2$, $j_1' \in \mathbb{Z}$ and that p can only be even if d_2 is odd).

Now write $j = s_i$, $j_i' = s_i$, $2 \leqslant i \leqslant n$, and we see that τ_γ on $R \times R$ is given by $\tau_\gamma((s,t),(s',t')) = \Omega_\gamma'((j,t),(j',t'))$ where $(j,t),(j',t') \in \mathbb{Z}^{2n}$ and $\Omega_\gamma'((j,t),(j',t'))$ $= \prod_{k=1}^{n} \alpha_k^{j_k' t_k}$ where

$$\alpha_k = \begin{cases} e^{2\pi i \frac{(d_2+1)}{2}} \gamma^{d_1} & k = 1, \\ \gamma^{\ell_k} e^{2\pi i \theta_k} & 2 \leqslant k \leqslant n. \end{cases}$$

For $\gamma \in \mathbb{T}$, write $\gamma = e^{2\pi i \xi}$, $\xi \in [0,1]$. Define $m_\gamma \in \{0, 1, 2, \ldots, n\}$ by

$$m_\gamma = \begin{cases} \#\{k : \ell_k \xi + \theta_k \text{ irrational}\} & \xi \text{ irrational}, \\ \#\{k : \theta_k \text{ irrational}\} + 1 & \xi \text{ rational}, \end{cases}$$

and let $n_\gamma = 2m_\gamma$. One easily calculates that $S_{\Omega_\gamma'} \subseteq \mathbb{Z}^{2n}$ is given by $S_{\Omega_\gamma'} \cong \mathbb{Z}^{n_\gamma}$, so that $\widehat{S}_{\Omega_\gamma'} \cong \mathbb{T}^{n_\gamma}$, giving us the desired setwise parametrization of $\operatorname{Prim}(C^*(N, \sigma))$. ∎

To complete our discussion of generalized discrete Heisenberg C^*-algebras, we focus on the case $n = 1$. Recall in this situation that the group $H(d_1) = H(d)$ has its second cohomology group given by $\mathbf{T}^2$, and a complete parametrization of inequivalent multipliers is given by $\{\sigma_{\lambda,\mu} : \lambda, \mu \in \mathbf{T}\}$ where

$$\sigma_{\lambda,\mu}((r,s,t),(r',s',t')) = \lambda^{s'r+dt\frac{s'(s'-1)}{2}}\mu^{r't+ds'\frac{t(t-1)}{2}}$$

(see (2.16)). By Proposition 3.2, there are two cases to consider: (i) at least one of the pair (λ, μ) is a non-torsion element of $\mathbf{T}$ and (ii) both λ and μ are torsion elements of $\mathbf{T}$. We concentrate on Case (i) first, Case (ii) being very similar to our results for $n \geqslant 2$. The next result was first proved for the special case $d = 1$ in [24].

THEOREM 3.7. (*See* [24], *Theorem* 1.6). *Let $n = 1$, $N = H(d)$ and $\sigma = \sigma_{\lambda,\mu}$ as in* (2.16) *where either λ or μ is non-torsion. Then $C^*(N, \sigma_{\lambda,\mu})$ is simple and has a unique trace.*

Proof. Here, one alternative would be to use Corollary 1.4 to prove simplicity; however, it is easier to use [25], Theorem 1.7, to get the desired result. The finite conjugacy classes of N are exactly the elements in the center $Z = \{(m,0,0) : m \in \mathbf{Z}\}$. For $(m,0,0) \in \mathbf{Z}$ and $(r,s,t) \in N$, using the notation of [25], Definition 1.1, we obtain

$$\chi^{\sigma,(m,0,0)}((r,s,t)) = \tilde{\sigma}((m,0,0),(r,s,t)) = [\lambda^s\mu^{-t}]^m.$$

Since either λ or μ is nontorsion, it follows that if $m \neq 0$, the conjugacy class $\{(m,0,0)\} \subseteq N$ is not σ-regular, and therefore by [25], Theorem 1.7, $C^*(N,\sigma)$ is simple and has a unique trace. ∎

We now consider the case $n = 1$, λ and μ both torsion. Up to this point we have stressed the differences between $n = 1$ and $n \geqslant 2$, but as mentioned earlier surprisingly enough this case could serve as a prototype for the work on $n \geqslant 2$ already done. Let $\lambda = e^{2\pi i\frac{p_1}{q_1}}$ and $\mu = e^{2\pi i\frac{p_2}{q_2}}$ where $(p_i, q_i) = 1$, $i = 1, 2$. Then $Z_o = \{(\ell j, 0, 0) : j \in \mathbf{Z}\} \subseteq \{(m,0,0) : m \in \mathbf{Z}\} = \mathbf{Z}$, where $\ell = \text{l.c.m.}(q_1, q_2)$, so that $N/Z_o = (\mathbf{Z}_\ell \times \mathbf{Z})\rtimes\mathbf{Z}$ with multiplier $\omega_{\lambda,\mu}$ defined on this quotient group in the obvious way; i.e.,

$$\omega_{\lambda,\mu}((\dot{r},s,t),(\dot{r}',s',t')) = \lambda^{s'\dot{r}+dt\frac{s'(s'-1)}{2}}\mu^{\dot{r}'t+ds'\frac{t(t-1)}{2}}, \quad (\dot{r},s,t),(\dot{r}',s',t') \in (\mathbf{Z}_\ell \times \mathbf{Z})\rtimes\mathbf{Z}. \tag{3.7}$$

Then we obtain the following analogue of Lemma 3.3.

LEMMA 3.8. *Let λ and μ be torsion elements of $\mathbb{T}$ and let Γ be the finite subgroup of $\mathbf{T}$ generated by $\langle\lambda,\mu\rangle$ and $\ell = \#(\Gamma)$. Then $C^*(H(d),\sigma_{\lambda,\mu})$ is $*$-isomorphic to the C^*-algebra of continuous sections of a C^*-bundle over $\widehat{Z}_o \cong \widehat{\ell\mathbb{Z}} \cong \mathbf{T}$, with fibre over $\gamma \in \widehat{Z}_o \cong \mathbf{T}$ given by $C^*((\mathbb{Z}_\ell \times \mathbb{Z})\rtimes\mathbb{Z}, \omega_{\lambda,\mu}\cdot\delta_\gamma)$, where $\delta_\gamma \in Z^2(N/Z_o,\mathbf{T})$ is defined by*

$$\delta_\gamma((\dot r,s,t),(\dot r',s',t')) = \rho^{dts'} \tag{3.8}$$

for $\rho \in \mathbf{T}$ with $\rho^\ell = \gamma$.

Proof. The proof of Lemma 3.3 carries through almost without change, and we omit details. ∎

We then use Lemma 3.8 to prove:

THEOREM 3.9. *Let λ, μ, Γ, ℓ, $\sigma_{\lambda,\mu}$, $\omega_{\lambda,\mu}$ be as in Lemma 3.8. Then $C^*(H(d),\sigma_{\lambda,\mu})$ is $*$-isomorphic to the C^*-algebra of continuous sections of a C^*-bundle over $\widehat{Z}_o \cong \widehat{\ell\mathbb{Z}} \cong \widehat{\mathbb{Z}} = \mathbf{T}$ with fibre over $\gamma \in \mathbf{T}$ $*$-isomorphic to $M_\ell(C^*(R,\tau_\gamma))$, the C^*-algebra of $\ell\times\ell$ matrices over the twisted abelian group C^*-algebra $C^*(R,\tau_\gamma)$. Here R is a subgroup of $\mathbb{Z}^2$ of index ℓ and τ_γ is defined on R by*

$$\tau_\gamma((s,t),(s',t')) = \lambda^{-dt\frac{s'(s'+1)}{2}}\mu^{-ds'\frac{t(t+1)}{2}}\rho^{dts'} \tag{3.9}$$

for some choice of $\rho \in \mathbf{T}$ with $\rho^\ell = \gamma$.

Proof. As in the proof of Theorem 3.4, we can view Γ as $\widehat{D}$ where $D \subseteq (\mathbb{Z}_\ell\times\mathbb{Z})\rtimes\mathbb{Z} = N_1$ is given by $(\mathbb{Z}_\ell\times\{0\})\rtimes\{0\}$. Then for every $\gamma \in \mathbf{T}$, $\varphi_D(\omega_{\lambda,\mu}\partial_\gamma) = \varphi_D(\omega_{\lambda,\mu}d_2(\gamma)) = \varphi_D(\omega_{\lambda,\mu})$ is a surjective map of N_1 onto $\widehat{D} = \Gamma$; a computation shows that

$$\varphi_D(\omega_{\lambda,\mu})((\dot r,s,t))(\dot m,0,0)) = \left[\lambda^s\mu^{-t}\right]^{m}, \quad (\dot r,s,t)\in N_1,\ (\dot m,0,0)\in D. \tag{3.10}$$

Let $M = \ker\varphi_D(\omega_{\lambda,\mu}) \subseteq N_1$, and set $R = M/D \subseteq N_1/D \cong \mathbb{Z}^2$; one easily sees that $R = \{(s,t)\in\mathbb{Z}^2 : \lambda^s\mu^{-t} = 1\}$. Then one shows as in Theorem 3.4 that $C^*((\mathbb{Z}_\ell\times\mathbb{Z})\rtimes,\omega_{\lambda,\mu}\partial_\gamma) \cong M_\ell(C^*(R,\tau_\gamma))$, where τ_γ is defined as in (3.9); we omit details. This $*$-isomorphism together with Lemma 3.9 competes the proof. ∎

EXAMPLE 3.10. As in Example 3.6 we now consider a special case where $\lambda = e^{2\pi i \frac{p}{q}}$, $(p,q) = 1$, and $\mu = 1$, for arbitrary $d \in \mathbf{Z}^+$. Then Γ is the set of all q-th root of unity, and $\ell = q$, so that $D = (\mathbf{Z}_q \times \{0\}) \rtimes \{0\}$, $M = (\mathbf{Z}_q \times q\mathbf{Z}) \rtimes \mathbf{Z}$, and $R = M/D \cong q\mathbf{Z} \times \mathbf{Z}$. For $\gamma \in \mathbf{T} \cong \widehat{\mathbf{Z}} \cong \widehat{q\mathbf{Z}} = \widehat{Z}_o$, choosing $\rho \in \mathbf{T}$ with $\rho^q = \gamma$, the multiplier τ_γ is defined on $R \times R$ by

$$\begin{aligned}\tau_\gamma((qj,k),(qj',k')) &= e^{-2\pi i \frac{p}{q} dk \frac{qj'(qj+1)}{2}} \rho^{dkq} \\ &= e^{-2\pi i dpk \frac{(q(j')^2 + j')}{2}} \gamma^{d_1 kj'} \\ &= [e^{2\pi i (\frac{q+1}{2})} \gamma]^{dkj'},\end{aligned}$$

(recalling that $(j')^2 = j' (\mathrm{mod}\, 2)$, $\forall j' \in \mathbf{Z}$, and that p can only be even if q is odd).

Using the isomorphism $R \cong \mathbf{Z}^2$ given by $(qj,k) \longrightarrow (j,k)$, we obtain from Theorem 3.9 that $C^*\left(H(d), \sigma_{(e^{2\pi i \frac{p}{q}},1)}\right)$ is $*$-isomorphic to the C^*-algebra of continuous sections of a C^*-bundle over $\mathbf{T}$ whose fibre over $e^{2\pi i\theta} = \gamma \in \mathbf{T}$ is given by $M_q\left(A_{[\theta + \frac{q+1}{2}]d}\right)$ (here A_α represents the standard notation for the rotation algebra with phase factor $\alpha \in [0,1)$). Therefore by [18], Theorem 4 there is a continuous open surjection from $C^*\left(H(d), \sigma_{(e^{2\pi i \frac{p}{q}},1)}\right)$ onto $\mathbf{T}$ with fibre over $\gamma \in \mathbf{T}$ given by a 2-torus if γ is torsion and a point for γ non-torsion. We remark that in the course of proving this result we have shown that $C^*\left((\mathbf{Z}_q \times \mathbf{Z}) \rtimes \mathbf{Z}, \omega_{(e^{2\pi i \frac{p}{q}}, e^{2\pi i\beta})}\right)$ (where $\omega_{(e^{2\pi i \frac{p}{q}}, e^{2\pi i\beta})}$ is as defined as in (3.7)) is $*$-isomorphic to $M_q\left(A_{[q\beta + \frac{q+1}{2}]d}\right)$, as first shown for the case $d = 1$ by P. Milnes and S. Walters ([22], Theorem 3).

By using similar methods to those outlined in the previous example, one can prove the following result, which we state without proof.

COROLLARY 3.11. *Fix $d \in \mathbf{Z}^+$ and let λ, μ be torsion elements of $\mathbf{T}$ generating a subgroup Γ of order ℓ. Then there is a continuous open surjection from* Prim $(C^*(H(d), \sigma_{\lambda,\mu}))$ *onto $\mathbf{T}$ whose fibre over $\gamma \in \mathbf{T}$ is given by $\mathbf{T}^2$ if γ is torsion and a point if γ is non-torsion.*

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Received November 10, 1994.