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SPACES CONTRACTIVELY INVARIANT FOR THE BACKWARD SHIFT

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ABSTRACT. We classify those Hilbert spaces, contractively contained in a vector-valued H^2-space, that are carried into themselves contractively by the backward shift. We then show when a completely non-unitary operator is unitarily equivalent to the action of the backward shift on one of its contractively invariant spaces.

KEYWORDS: *Shifts, Hardy spaces, contractively contained spaces, invariant spaces.*

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1. INTRODUCTION

We are interested in classifying those Hilbert spaces, contractively contained in vector-valued H^2-spaces, that are contractively invariant for the backward shift. A Hilbert space $\mathcal{H}$ is contractively contained in a Hilbert space $\mathcal{K}$ if $\mathcal{H} \subset \mathcal{K}$ and $\|h\|_{\mathcal{H}} \geqslant \|h\|_{\mathcal{K}}$ for every $h \in \mathcal{H}$. We write $\mathcal{H} \prec \mathcal{K}$. We assume that all Hilbert spaces are over $\mathbb{C}$ and that they are separable.

For a Hilbert space $\mathcal{E}$, we let $L^2(\mathcal{E})$ and $H^2(\mathcal{E})$ denote the standard Lebesgue and Hardy spaces of functions Lebesgue measurable on $\partial\mathbb{D}$ with values in $\mathcal{E}$. The inner product on both spaces is given by

$$\langle f, g \rangle = \int_{\partial\mathbb{D}} \langle f(\lambda), g(\lambda) \rangle_{\mathcal{E}} \, \mathrm{d}\sigma(\lambda).$$

Here σ is normalized Lebesgue measure on $\partial\mathbf{D}$. We will denote the orthogonal projection of $L^2(\mathcal{E})$ onto $H^2(\mathcal{E})$ by P_+. If $\mathfrak{X}$ is a Banach space, then $L^\infty(\mathfrak{X})$ and $H^\infty(\mathfrak{X})$ denote the Lebesgue and Hardy spaces of σ-essentially bounded functions on $\partial\mathbf{D}$ with values in $\mathfrak{X}$. Both spaces have the norm

$$\|A\|_\infty = \operatorname*{ess\,sup}_{\lambda\in\partial\mathbf{D}} \|A(\lambda)\|_{\mathfrak{X}}.$$

For $f \in L^2(\mathcal{E})$ and $A \in L^\infty(\mathbf{L}(\mathcal{E},\mathcal{E}_*))$, Af denotes the function in $L^2(\mathcal{E})$ defined by $(Af)(\lambda) = A(\lambda)f(\lambda)$. This determines a multiplication operator $M_A : L^2(\mathcal{E}) \to L^2(\mathcal{E}_*)$ with norm $\|M_A\| \leqslant \|A\|_\infty$.

We can think of the above H^p-spaces as either the subspaces of the corresponding L^p-spaces consisting of those functions with vanishing negative Fourier coefficients, or, as H^p-spaces of functions defined in the open unit disk, $\mathbf{D}$, the former being the non-tangential boundary values of the latter. In the H^2 case we can take norm limits, but in the H^∞ case, where $\mathfrak{X}$ will usually be the space of bounded linear operators between two Hilbert spaces, we need to take limits in the strong operator topology. For details about these functions, consult [12] or [16].

For a Hilbert space $\mathcal{E}$, we define the bilateral shift, $U_\mathcal{E}$, on $L^2(\mathcal{E})$ by

$$(U_\mathcal{E} f)(\lambda) = \lambda f(\lambda).$$

The space $H^2(\mathcal{E})$ is obviously invariant for $U_\mathcal{E}$; let $S_\mathcal{E} = U_\mathcal{E}|H^2(\mathcal{E})$. This is the unilateral, or *forward*, shift. The *backward shift* is $S_\mathcal{E}^*$, the adjoint of $S_\mathcal{E}$. It can be represented as

$$(S_\mathcal{E}^* f)(z) = \frac{1}{z}(f(z) - f(0))$$

for all $f \in H^2(\mathcal{E})$.

A vector space lying in a Hilbert space will be called a *linear manifold*. A *subspace* is a closed linear manifold.

By an *operator*, we mean a bounded linear operator. The range of an operator T can be made into a Hilbert space, $\mathcal{M}(T)$, by equipping it with the norm that makes T a co-isometry:

$$\|Th\|_{\mathcal{M}(T)} = \|g\|_\mathcal{H}$$

where $g \in (\ker T)^\perp$ and $Th = Tg$. Note that if $\|T\| \leqslant 1$, then $\mathcal{M}(T) \prec \mathcal{K}$. Conversely, if $\mathcal{H} \prec \mathcal{K}$ and T is the operator that embeds $\mathcal{H}$ into $\mathcal{K}$, then $\mathcal{H} = \mathcal{M}(T)$. If the operators T_1 and T_2 have ranges lying in the same Hilbert space, then $\mathcal{M}(T_1)$ is the same Hilbert space as $\mathcal{M}(T_2)$ if and only if $T_1T_1^* = T_2T_2^*$. This is shown in [13]. A consequence is that for any operator, $\mathcal{M}(T) = \mathcal{M}(T|(\ker T)^\perp)$.

In the sequel, we take $\mathcal{H}$ to be a space contained contractively in $H^2(\mathcal{E})$ and $T : \mathcal{H} \to H^2(\mathcal{E})$ to be the embedding map. If $\mathcal{H}$ is invariant for $S_{\mathcal{E}}^*$, let $C_{\mathcal{H}} \in \mathbf{L}(\mathcal{H})$ be the adjoint of $S_{\mathcal{E}}^*$ as an operator on $\mathcal{H}$. We then have

$$S_{\mathcal{E}}^* T = T C_{\mathcal{H}}^*. \tag{1.1}$$

This notation will be used throughout the rest of the paper. We will say that $\mathcal{H}$ is *contractively* (unitarily) *invariant* for $S_{\mathcal{E}}^*$ if $S_{\mathcal{E}}^*\mathcal{H} \subset \mathcal{H}$ and $C_{\mathcal{H}}^*$ is contractive (unitary).

The first result classifying shift invariant spaces was the following theorem of A. Beurling ([2]).

BEURLING'S THEOREM. *If $\mathcal{F}$ is a subspace of H^2 then $\mathcal{F}$ is invariant for the forward shift S on H^2 if and only if $\mathcal{F} = \varphi H^2$ for an inner function $\varphi \in H^\infty$.*

This theorem was subsequently extended to the spaces $H^2(\mathcal{E})$ by P. Lax ([9]) for finite dimensional $\mathcal{E}$, and by P. Halmos ([6]) and H. Helson and D. Lowdenslager ([7]) for infinite dimensional $\mathcal{E}$.

THEOREM. *If $\mathcal{F}$ is a subspace of $H^2(\mathcal{E})$ then it is invariant for $S_{\mathcal{E}}$ if and only if $\mathcal{F} = \Omega H^2(\mathcal{L})$ for an inner function $\Omega \in H^\infty(\mathbf{L}(\mathcal{L},\mathcal{E}))$ and a Hilbert space $\mathcal{L}$.*

Here *inner* means that $\Omega(\lambda)$ is an isometry for almost every $\lambda \in \partial\mathbf{D}$.

L. de Branges ([3]) extended this result by considering Hilbert spaces contractively contained in $H^2(\mathcal{E})$. A proof is found in [10].

THEOREM. *Suppose $\mathcal{H} \prec H^2(\mathcal{E})$. Then $\mathcal{H}$ is carried into itself contractively by $S_{\mathcal{E}}$ if and only if $\mathcal{H} = \mathcal{M}(M_B|H^2(\mathcal{L}))$ for some B in the closed unit ball of $H^\infty(\mathbf{L}(\mathcal{L},\mathcal{E}))$.*

In this paper we obtain an analogous result for $S_{\mathcal{E}}^*$. That is, we classify those $\mathcal{H}$ which are contractively invariant for $S_{\mathcal{E}}^*$. The first step is to classify those $\mathcal{H}$ for which $C_{\mathcal{H}}^*$ is unitary. We then treat the case where $C_{\mathcal{H}}^*$ is a completely non-unitary (cnu) operator. That is, where $C_{\mathcal{H}}^*$ fails to be unitary on any of its invariant subspaces. Such an $\mathcal{H}$ is said to be *cnu-invariant* for $S_{\mathcal{E}}^*$. A theorem of Sz.-Nagy and Foiaş, which states that any contraction can be decomposed into the direct sum of a unitary and a cnu-operator, will be used to reduce the general case to the previous cases. Next we investigate when there is a space $\mathcal{H} \prec H^2(\mathcal{E})$ contractively invariant for $S_{\mathcal{E}}^*$ such that $C_{\mathcal{H}}^*$ is unitarily equivalent to a given cnu-operator. We then discuss contractively invariant spaces in H^2. We conclude with a characterization of spaces contractively contained in $H^2(\mathcal{E})$ and invariant for $S_{\mathcal{E}}^*$, but not necessarily contractively.

The above theorems can be applied to special cases of our problem. If $\mathcal{H}$ is closed and contractively invariant for $S_{\mathcal{E}}^*$, then $H^2(\mathcal{E}) \ominus \mathcal{H}$ is an invariant subspace for $S_{\mathcal{E}}$. Thus $\mathcal{H}$ can be characterized as $H^2(\mathcal{E}) \ominus \Omega H^2(\mathcal{E}_*)$ for some $\Omega \in H^\infty(\mathbf{L}(\mathcal{E}_*, \mathcal{E}))$ that is inner.

In [10], the spaces $\mathcal{H}$ such that $S_{\mathcal{E}}^* \mathcal{H} \subset \mathcal{H}$ and

$$\|f(0)\|_{\mathcal{E}}^2 \leqslant \|f\|_{\mathcal{H}}^2 - \|S_{\mathcal{E}}^* f\|_{\mathcal{H}}^2, \tag{1.2}$$

for all $f \in \mathcal{H}$, are classified as the spaces $\mathcal{M}((1 - T_B T_B^*)^{1/2})$ where B is a function in $H^\infty(\mathbf{L}(\mathcal{K}, \mathcal{E}))$, $\|B\|_\infty \leqslant 1$ and $T_B = M_B | H^2(\mathcal{K})$. This is a *de Branges-Rovnyak space*, usually denoted by $\mathcal{H}(B)$. It is also shown in [10] that $S_{\mathcal{E}}^* \mathcal{H} \subset \mathcal{H}$, together with the condition that (1.2) is satisfied on $\mathcal{H}$, is equivalent to the complementary space of $\mathcal{H}$ being contractively invariant for $S_{\mathcal{E}}$. (For a contraction T, the complementary space of $\mathcal{M}(T)$ is $\mathcal{H}(T) = \mathcal{M}((1 - TT^*)^{1/2})$. For details, consult [13].) A space $\mathcal{H}$ that is unitarily invariant for $S_{\mathcal{E}}^*$ cannot satisfy (1.2). This is because $\|S_{\mathcal{E}}^* f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$ for all $f \in \mathcal{H}$ and $S_{\mathcal{E}}^* \mathcal{H} \subset \mathcal{H}$, combined with the above inequality, would give $\mathcal{H} = \{0\}$. We will see in Section 6, however, that there are nontrivial spaces $\mathcal{H}$ on which $S_{\mathcal{E}}^*$ acts unitarily. Such a space, then, cannot have its complementary space contractively invariant for $S_{\mathcal{E}}$. We will see such a space that is not even invariant for $S_{\mathcal{E}}$.

F. Suarez ([15]) has made a detailed study of a special case of our problem. He has classified the invariant subspaces of S^* as an operator on the de Branges-Rovnyak spaces $\mathcal{H}(b)$. These spaces will be discussed in Section 6.1, although we do not investigate them as deeply as Suarez has.

2. UNITARILY INVARIANT SPACES

Our first step is to classify those spaces which are unitarily invariant for $S_{\mathcal{E}}^*$. To express our results in a convenient form, we define a class of operators as follows. Let $A \in L^\infty(\mathbf{L}(\mathcal{E}))$ be positive valued. That is, $A(\lambda) \geqslant 0$ for almost every λ in the unit circle. Let $\mathcal{L}_A = \overline{M_A H^2(\mathcal{E})}$. Define $\Gamma_A : \mathcal{L}_A \to H^2(\mathcal{E})$ by

$$\Gamma_A = P_+ M_A | \mathcal{L}_A.$$

Then

$$\Gamma_A^* = M_A | H^2(\mathcal{E}).$$

The operator Γ_A is injective as Γ_A^* clearly has dense range. It is obvious that $U_{\mathcal{E}}\mathcal{L}_A \subset \mathcal{L}_A$ and that $U_{\mathcal{E}}\Gamma_A^* = \Gamma_A^* S_{\mathcal{E}}$. Let $U_A = U_{\mathcal{E}}|\mathcal{L}_A$, so $U_A \in \mathbf{L}(\mathcal{L}_A)$. We now have that $S_{\mathcal{E}}^*\Gamma_A = \Gamma_A U_A^*$. Thus $\mathcal{M}(\Gamma_A)$ is contractively invariant for $S_{\mathcal{E}}^*$ (because Γ_A implements a unitary equivalence between U_A^* and $C_{\mathcal{M}(\Gamma_A)}^*$).

THEOREM 2.1. *Suppose $\mathcal{H} \prec H^2(\mathcal{E})$. Then $\mathcal{H}$ is unitarily invariant for $S_{\mathcal{E}}^*$ if and only if $\mathcal{H} = \mathcal{M}(\Gamma_Q)$ for some positive-valued $Q \in L^\infty(\mathbf{L}(\mathcal{E}))$ such that $\mathcal{L}_Q$ reduces $U_{\mathcal{E}}$ and $\|Q\|_\infty \leqslant 1$.*

Proof. Suppose first that $\mathcal{H}$ is unitarily invariant for $S_{\mathcal{E}}^*$. Since $S_{\mathcal{E}}^*T = TC_{\mathcal{H}}^*$ and $C_{\mathcal{H}}$ is unitary, it follows that

$$S_{\mathcal{E}}^*TT^*S_{\mathcal{E}} = TC_{\mathcal{H}}^*C_{\mathcal{H}}T^* = TT^*.$$

Hence TT^* is a positive Toeplitz operator on $H^2(\mathcal{E})$; so Theorem 6.2.A of [12] provides a positive valued Q in the closed unit ball of $L^\infty(\mathbf{L}(\mathcal{E}))$ such that $TT^* = T_{Q^2}(= P_+M_{Q^2}|H^2(\mathcal{E}))$. This gives

$$\Gamma_Q\Gamma_Q^* = T_{Q^2} = TT^*,$$

so that $\mathcal{H} = \mathcal{M}(T) = \mathcal{M}(\Gamma_Q)$.

It remains to show that $\mathcal{L}_Q$ reduces $U_{\mathcal{E}}$, which is equivalent to showing that U_Q is unitary. This is immediate since U_Q^* is unitarily equivalent to $C_{\mathcal{H}}^*$, which is assumed to be unitary.

For the converse, assume $\mathcal{H} = \mathcal{M}(\Gamma_Q)$ for some Γ_Q where $\mathcal{L}_Q$ reduces $U_{\mathcal{E}}$. We still have that U_Q^* is unitarily equivalent to $C_{\mathcal{M}(\Gamma_Q)}^*$. Now U_Q is assumed to be unitary, so the theorem follows. ∎

A discussion of when $\mathcal{L}_A$ reduces $U_{\mathcal{E}}$, for a positive operator $A \in L^\infty(\mathbf{L}(\mathcal{E}))$, can be found in [10].

The referee has pointed out that a related characterization is contained in the work L. de Branges and J. Rovnyak. See Theorem 8 of [4].

3. COMPLETELY NON-UNITARILY INVARIANT SPACES

To treat the case where $S_{\mathcal{E}}^*$ acts completely non-unitarily on $\mathcal{H}$, we will employ the Sz.-Nagy–Foiaş functional model for the operator $C_{\mathcal{H}}$. Our technique is an adaptation of those used in [10]. Note that $C_{\mathcal{H}}$ is cnu if we assume $C_{\mathcal{H}}^*$ is. To construct the model, we first form the characteristic function of $C_{\mathcal{H}}$. Let $D_{C_{\mathcal{H}}} = (1 - C_{\mathcal{H}}^* C_{\mathcal{H}})^{1/2}$ and $\mathcal{D}_C = \overline{D_{C_{\mathcal{H}}}\mathcal{H}}$. Define $D_{C_{\mathcal{H}}^*}$ and $\mathcal{D}_{C^*}$ analogously for $C_{\mathcal{H}}^*$ in place of $C_{\mathcal{H}}$. The characteristic function of $C_{\mathcal{H}}$ is the function $\Theta \in H^\infty(\mathbf{L}(\mathcal{D}_C, \mathcal{D}_{C^*}))$ given by

$$\Theta(z) = (-C_{\mathcal{H}} + zD_{C_{\mathcal{H}}^*}(1 - zC_{\mathcal{H}}^*)^{-1}D_{C_{\mathcal{H}}})|\mathcal{D}_C. \tag{3.1}$$

The characteristic function determines the cnu contraction to within unitary equivalence. Let

$$\Delta \in L^\infty(\mathbf{L}(\mathcal{D}_C))$$

be given by $\Delta(\lambda) = (1 - \Theta(\lambda)^*\Theta(\lambda))^{1/2}$ for $\lambda \in \partial\mathbf{D}$. We define

$$\begin{aligned} \mathcal{H}_\Theta &= H^2(\mathcal{D}_{C_{\mathcal{H}}^*}) \oplus \overline{M_\Delta L^2(\mathcal{D}_{C_{\mathcal{H}}})}; \quad \text{and} \\ \mathcal{K}_\Theta &= \mathcal{H}_\Theta \ominus \begin{pmatrix} M_\Theta \\ M_\Delta \end{pmatrix} H^2(\mathcal{D}_{C_{\mathcal{H}}}). \end{aligned} \tag{3.2}$$

We will denote the orthogonal projection of $\mathcal{H}_\Theta$ onto $\mathcal{K}_\Theta$ by P_Θ. We will use U_Θ to denote the isometry on $\mathcal{H}_\Theta$ given by

$$(U_\Theta(u \oplus v))(\lambda) = \lambda u(\lambda) \oplus \lambda v(\lambda)$$

and we will let $S_\Theta = P_\Theta U_\Theta | \mathcal{K}_\Theta$. The operator S_Θ is the Sz.-Nagy–Foiaş model for $C_{\mathcal{H}}$ and U_Θ is the minimal isometric dilation of S_Θ. Thus there is a unitary $W : \mathcal{K}_\Theta \to \mathcal{H}$ such that $C_{\mathcal{H}}W = WS_\Theta$. The U_Θ^*-invariance of $\mathcal{K}_\Theta$ gives the intertwining

$$S_\Theta P_\Theta = P_\Theta U_\Theta.$$

Suppose now we begin with a function $\Theta \in H^\infty(\mathbf{L}(\mathcal{G}, \mathcal{G}_*))$ such that $\|\Theta\|_\infty \leqslant 1$ and $\|\Theta(0)f\| < \|f\|$ for all $f \in \mathcal{G}$ (a function satisfying these conditions is said to be *purely contractive*). For such a Θ we can construct Δ, $\mathcal{H}_\Theta$, $\mathcal{K}_\Theta$, U_Θ and S_Θ as before. The operator S_Θ is then a cnu contraction with characteristic function Θ (up to a constant unitary factor). Notice that Θ as constructed in (3.1) is purely contractive. This theory is the subject of [16].

THEOREM 3.1. *Suppose $\mathcal{H} \prec H^2(\mathcal{E})$. Then $\mathcal{H}$ is completely non-unitarily invariant for $S_{\mathcal{E}}^*$ if and only if $\mathcal{H} = \mathcal{M}(Y|\mathcal{K}_\Theta)$ where*

(i) *$\Theta \in H^\infty(\mathbf{L}(\mathcal{G}, \mathcal{G}_*))$ is purely contractive with $\|\Theta\|_\infty \leqslant 1$;*

(ii) *$Y : \mathcal{H}_\Theta \to H^2(\mathcal{E})$ is given by*

$$Y = P_+(M_F^* \quad M_G^*)|\mathcal{H}_\Theta$$

for some $F \in H^\infty(\mathbf{L}(\mathcal{E}, \mathcal{G}_*))$ and $G \in L^\infty(\mathbf{L}(\mathcal{E}, \mathcal{G}))$ satisfying $GH^2(\mathcal{E}) \subset \overline{\Delta L^2(\mathcal{G})}$; and

(iii) *$Y|\mathcal{K}_\Theta$ is an injective contraction.*

Proof. Suppose first that $\mathcal{H}$ is cnu-invariant for $S_{\mathcal{E}}^*$. Then $C_{\mathcal{H}}^*$ is a cnu contraction satisfying

$$S_{\mathcal{E}}^* T = T C_{\mathcal{H}}^*. \tag{3.3}$$

Let Θ be the characteristic function of $C_{\mathcal{H}}$, let $\mathcal{G} = \mathcal{D}_C$, $\mathcal{G}_* = \mathcal{D}_{C^*}$ and let $W : \mathcal{K}_\Theta \to \mathcal{H}$ be the unitary operator satisfying $C_{\mathcal{H}} W = W S_\Theta$. If we let $X = TW$, then (3.3) implies that

$$X^* S_{\mathcal{E}} = W^* T^* S_{\mathcal{E}} = W^* C_{\mathcal{H}} T^* = S_\Theta W^* T^* = S_\Theta X^*.$$

We may now employ the Commutant Lifting Theorem (Theorem II.2.3 of [16]) to obtain an operator $Y : \mathcal{H}_\Theta \to H^2(\mathcal{E})$ such that

$$\|X\| = \|Y\|;$$

$$Y^* S_{\mathcal{E}} = U_\Theta Y^*; \tag{3.4}$$

and

$$X = Y|\mathcal{K}_\Theta. \tag{3.5}$$

Note that the last equality gives that $Y|\mathcal{K}_\Theta$ is an injective contraction.

We can write

$$Y = (A \quad B) : \mathcal{H}_\Theta \to H^2(\mathcal{E}).$$

Note that this implies $B^* H^2(\mathcal{E}) \subset \overline{\Delta L^2(\mathcal{D}_{C_{\mathcal{H}}})}$. Because of (3.4), we have the intertwining relations

$$A^* S_{\mathcal{E}} = S_{\mathcal{D}_{C^*}} A^*;$$

and

$$B^* S_{\mathcal{E}} = (U_\Theta | \overline{\Delta L^2(\mathcal{D}_{C_{\mathcal{H}}})}) B^*.$$

The first of these gives that $A^* = M_F|H^2(\mathcal{E})$ for some $F \in H^\infty(\mathbf{L}(\mathcal{E}, \mathcal{D}_{C^*}))$. The second gives that $B^* = M_G|H^2(\mathcal{E})$ for some $G \in L^\infty(\mathbf{L}(\mathcal{E}, \mathcal{D}_C))$. Finally, the equality $X = TW$ gives $\mathcal{H} = \mathcal{M}(Y|\mathcal{K}_\Theta)$.

For the converse, suppose Θ, F, G, X and Y are as in the statement of the theorem. Form $\mathcal{K}_\Theta$, $\mathcal{H}_\Theta$, U_Θ and S_Θ. It is clear that $Y^* S_\mathcal{E} = U_\Theta Y^*$, thus $S_\mathcal{E}^*(Y|\mathcal{K}_\Theta) = (Y|\mathcal{K}_\Theta)S_\Theta^*$. This gives a unitary equivalence between $S_\mathcal{E}^*$ acting on $\mathcal{M}(Y|\mathcal{K}_\Theta)$ and S_Θ, since $Y|\mathcal{K}_\Theta$ is injective. This completes the proof as S_Θ is a cnu-contraction. ∎

4. CONTRACTIVELY INVARIANT SPACES

We treat the general case by employing a theorem of Sz.-Nagy and Foiaş which reduces the general case to our two previous cases.

THEOREM 4.1. *Suppose $\mathcal{H} \prec H^2(\mathcal{E})$. Then $\mathcal{H}$ is contractively invariant for $S_\mathcal{E}^*$ if and only if $\mathcal{H} = \mathcal{M}((\,Y|\mathcal{K}_\Theta \quad \Gamma_Q\,))$ where:*

(i) *$Q \in L^2(\mathcal{E})$ is positive-valued and $\|Q\|_\infty \leqslant 1$;*

(ii) *$\mathcal{L}_Q$ reduces $U_\mathcal{E}$;*

(iii) *$\Theta \in H^\infty(\mathbf{L}(\mathcal{G}, \mathcal{G}_*))$ is purely contractive with $\|\Theta\|_\infty \leqslant 1$;*

(iv) *$Y : \mathcal{H}_\Theta \to H^2(\mathcal{E})$ is given by*

$$Y = P_+(M_F^* \quad M_G^*)|\mathcal{H}_\Theta$$

for some $F \in H^\infty(\mathbf{L}(\mathcal{E}, \mathcal{G}_*))$ and $G \in L^\infty(\mathbf{L}(\mathcal{E}, \mathcal{G}))$ satisfying $GH^2(\mathcal{E}) \subset \overline{\Delta L^2(\mathcal{G})}$;

(v) *$Y|\mathcal{K}_\Theta$ is an injective contraction;*

(vi) *$(\,Y|\mathcal{K}_\Theta \quad \Gamma_Q\,) : \mathcal{H}_\Theta \oplus \mathcal{L}_Q \to H^2(\mathcal{E})$ is an injective contraction.*

Proof. If $\mathcal{H}$ has the form stated above, then

$$S_\mathcal{E}^*(\,Y|\mathcal{K}_\Theta \quad \Gamma_Q\,) = (\,Y|\mathcal{K}_\Theta \quad \Gamma_Q\,)\begin{pmatrix} S_\Theta^* & 0 \\ 0 & U_\mathcal{E}^*|\mathcal{L}_Q \end{pmatrix}.$$

Hence $\mathcal{H}$ is contractively invariant for $S_\mathcal{E}^*$.

For the reverse implication, Theorem I.3.2 of [16], implies there is a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with respect to which $C_\mathcal{H} = C_1 \oplus C_2$ where C_1 is cnu and C_2 is unitary.

Write the embedding map $T : \mathcal{H} \to L^2(\mathcal{E})$ as

$$(\,T_1 \quad T_2\,) : \mathcal{H}_1 \oplus \mathcal{H}_2 \to H^2(\mathcal{E}).$$

The intertwining (1.1) then implies

$$S_{\mathcal{E}}^* T_1 = T_1 C_1,$$
$$S_{\mathcal{E}}^* T_2 = T_2 C_2.$$

Here T_1 and T_2 respectively embed $\mathcal{H}_1$ and $\mathcal{H}_2$ into $H^2(\mathcal{E})$. Hence Theorems 2.1 and 3.1 provide functions of the appropriate type such that

$$\mathcal{H}_1 = \mathcal{M}(Y|\mathcal{K}_\Theta),$$
$$\mathcal{H}_2 = \mathcal{M}(\Gamma_Q).$$

Thus

$$T_1 T_1^* = (Y|\mathcal{K}_\Theta)(Y|\mathcal{K}_\Theta)^*,$$
$$T_2 T_2^* = \Gamma_Q \Gamma_Q^*.$$

This proves $\mathcal{H}$ has the desired form since

$$TT^* = T_1 T_1^* + T_2 T_2^*. \quad \blacksquare$$

Consider now a space $\mathcal{M}(\Gamma_Q)$ where $\mathcal{L}_Q = \overline{QH^2(\mathcal{E})}$ does not necessarily reduce $U_{\mathcal{E}}$. As we saw in Section 2, the space $\mathcal{M}(\Gamma_Q)$ is contractively invariant for $S_{\mathcal{E}}^*$. To represent $\mathcal{M}(\Gamma_Q)$ as in Theorem 4.1, we use Proposition V.4.2 of [16] to provide an outer function A in the unit ball of $H^\infty(\mathbf{L}(\mathcal{E},\mathcal{F}))$ such that

$$A(\lambda)^* A(\lambda) \leqslant Q(\lambda)^2 \qquad \text{for a.e. } \lambda \in \partial\mathbf{D} \tag{4.1}$$

and if $A_1 \in H^\infty(\mathbf{L}(\mathcal{E},\mathcal{F}_1))$ also satisfies (4.1), then $A_1(\lambda)^* A_1(\lambda) \leqslant A(\lambda)^* A(\lambda)$ almost everywhere. The function A is called the *maximal factorable minorant* of Q^2 by J. Ball and T. Kriete ([1]). That A is outer means $\overline{M_A H^2(\mathcal{E})} = H^2(\mathcal{F})$.

Let $R \in L^\infty(\mathbf{L}(\mathcal{E}))$ be the positive function satisfying

$$Q(\lambda)^2 = A(\lambda)^* A(\lambda) + R(\lambda)^2. \tag{4.2}$$

The maximality of A implies that $\mathcal{L}_{\mathcal{R}}$ reduces $U_{\mathcal{E}}$; see the proof of Proposition V.4.2 in [16]. Now let $Y = T_A^* : H^2(\mathcal{F}) \to H^2(\mathcal{E})$. Using (4.2), we have

$$(\,T_A^* \quad \Gamma_R\,)(\,T_A^* \quad \Gamma_R\,)^* = T_{A^*A} + T_{R^2} = T_{Q^2} = \Gamma_Q \Gamma_Q^*$$

so that $\mathcal{M}(\Gamma_Q) = \mathcal{M}((\,T_A^* \quad \Gamma_R\,))$. It can be shown that this is the desired representation.

5. INVARIANCE EMBEDDINGS AND n-CYCLIC OPERATORS

In this section we investigate when a given model space $\mathcal{K}_\Theta$ can be embedded in a given $H^2(\mathcal{E})$ via a map Y as in the statement of Theorem 3.1. To be more precise, we call $Y : \mathcal{H}_\Theta \to H^2(\mathcal{E})$ an *invariance embedding* if

(1) $Y : \mathcal{H}_\Theta \to H^2(\mathcal{E})$ is given by

$$Y = P_+(M_F^* \quad M_G^*)|\mathcal{H}_\Theta$$

for some $F \in H^\infty(\mathbf{L}(\mathcal{E}, \mathcal{G}_*))$ and $G \in L^\infty(\mathbf{L}(\mathcal{E}, \mathcal{G}))$ satisfying $GH^2(\mathcal{E}) \subset \overline{M_\Delta L^2(\mathcal{G})}$; and

(2) $Y|\mathcal{K}_\Theta$ is an injective contraction.

One consequence of this definition is that $U_\Theta Y^* = Y^* S_\mathcal{E}$. When an invariance embedding exists, Theorem 3.1 shows that $\mathcal{M}(Y|\mathcal{K}_\Theta)$ is contractively invariant for $S_\mathcal{E}^*$. Moreover, (2) in the definition gives that the operator $C_\mathcal{H}$ (the adjoint of the action of $S_\mathcal{E}^*$ on $\mathcal{M}(Y|\mathcal{K}_\Theta)$) is unitarily equivalent to the model operator S_Θ.

We first show at least one invariance embedding always exists.

THEOREM 5.1. *If $\mathcal{K}_\Theta$ is any model space, with $\Theta \in H^\infty(\mathbf{L}(\mathcal{G}, \mathcal{G}_*))$, then there is an invariance embedding $Y : \mathcal{H}_\Theta \to H^2(\mathcal{G}_* \oplus \mathcal{G})$.*

The proof is facilitated by the following two lemmas.

LEMMA 5.2. *If $\mathcal{E}$ is a Hilbert space with orthonormal basis $\{e_n\}$ and $f \in H^2(\mathcal{E})$, then $f = \bigoplus_0^\infty \langle f, e_n\rangle_\mathcal{E} e_n$ where $\langle f, e_n\rangle_\mathcal{E}$ denotes the function $\lambda \mapsto \langle f(\lambda), e_n\rangle_\mathcal{E}$.*

Proof. It is obvious that the functions $\langle f, e_n\rangle_\mathcal{E} e_n$ are orthogonal in $H^2(\mathcal{E})$. The inequality

$$\left\| f(\lambda) - \sum_{n=0}^{N} \langle f(\lambda), e_n\rangle_\mathcal{E} e_n \right\|_\mathcal{E} \leqslant \|f(\lambda)\|_\mathcal{E}$$

holds almost everywhere and the left-hand-side goes to zero, as $N \to \infty$, almost everywhere. Hence Lebesgue's Dominated Convergence Theorem yields

$$\left\| f - \sum_{n=0}^{N} \langle f, e_n\rangle_\mathcal{E} e_n \right\|_{L^2(\mathcal{E})} \to 0,$$

giving the lemma. ∎

LEMMA 5.3. *Let x be a function in L^∞ that is non-negative almost everywhere and fails to be* log-*integrable. Define $M_x f = xf$ on $L^2(\mathcal{E})$, for a Hilbert space $\mathcal{E}$. Then $\overline{M_x H^2(\mathcal{E})} = L^2(\mathcal{E})$.*

Proof. The theorem is known in the case that $\mathcal{E} = \mathbb{C}$; it then reduces to the well-known criterion for a function in L^2 to be cyclic for U, see [8].

For the general case, fix $f \in L^2(\mathcal{E})$ and $\varepsilon > 0$. By Lemma 5.2, we can choose $N > 0$ so that

$$\left\| f - \sum_{n=0}^{N} \langle f, e_n \rangle_{\mathcal{E}} e_n \right\|_{L^2(\mathcal{E})} < \varepsilon$$

where $\{e_n\}$ is an orthonormal basis for $\mathcal{E}$. For $n = 0, \dots, N$, choose $g_n \in H^2$ so that

$$\|x g_n - \langle f, e_n \rangle_{\mathcal{E}}\|_2^2 < \frac{\varepsilon^2}{2^n}.$$

Let $g = g_0 e_0 + \cdots + g_N e_N$. Then

$$\begin{aligned} \|f - xg\|_{L^2(\mathcal{E})} &\leqslant \left\| f - \sum_{n=0}^{N} \langle f, e_n \rangle_{\mathcal{E}} e_n \right\|_{L^2(\mathcal{E})} + \left\| xg - \sum_{n=0}^{N} \langle f, e_n \rangle_{\mathcal{E}} e_n \right\|_{L^2(\mathcal{E})} \\ &\leqslant \varepsilon + \left(\sum_{n=0}^{N} \|x g_n - \langle f, e_n \rangle_{\mathcal{E}}\|_2^2 \right)^{\frac{1}{2}} \leqslant \frac{3\varepsilon}{2}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 5.1. Write $f \in H^2(\mathcal{E}) = H^2(\mathcal{G}_* \oplus \mathcal{G})$ as $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ where $f_1 \in H^2(\mathcal{G}_*)$ and $f_2 \in H^2(\mathcal{G})$. Let

$$Y^* f = f_1 \oplus M_\Delta M_x f_2 = (1 \quad 0) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \oplus (0 \quad M_\Delta M_x) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

where x is a function in the closed unit ball of L^∞ that is non-negative almost everywhere and fails to be log-integrable. Clearly $Y^* H^2(\mathcal{E}) \subset \mathcal{H}_\Theta$. In fact, it is dense in $\mathcal{H}_\Theta$. To see this, first note that $Y^*(H^2(\mathcal{G}_*) \oplus 0) = H^2(\mathcal{G}_*) \oplus 0$. By the lemma, $M_x H^2(\mathcal{G})$ is dense in $L^2(\mathcal{G})$, so $M_\Delta M_x H^2(\mathcal{G})$ is dense in $\overline{M_\Delta L^2(\mathcal{G})}$. It follows that $Y^* H^2(\mathcal{E})$ is dense in $\mathcal{H}_\Theta$. Consequently, $P_\Theta Y^* = (Y|\mathcal{K}_\Theta)^*$ has dense range so that $Y|\mathcal{K}_\Theta$ is injective. It is clear that Y is a contraction. We conclude that Y is the desired invariance embedding. ∎

The above embedding result is somewhat crude, particularly with respect to the dimension of $\mathcal{E}$. We wish to find a more explicit connection between properties of S_Θ and the dimension of $\mathcal{E}$. The relevant property of S_Θ is in fact given by the following definition. An operator T on the Hilbert space $\mathcal{K}$ is *n-cyclic*, for a positive integer n, if there are vectors $k_1, \ldots, k_n$ in $\mathcal{K}$ such that

$$\bigvee\{T^i k_j \mid i \in \mathbf{N}, \text{ and } j = 1, \ldots, n\} = \mathcal{K}.$$

A 1-cyclic operator is simply said to be a cyclic operator.

We can now state the main result of this section.

THEOREM 5.4. *Let $\mathcal{K}_\Theta$ be a model space and $\mathcal{E}$ a Hilbert space of dimension $n \in \mathbf{Z}^+ \cup \{\aleph_0\}$. There exists an invariance embedding $Y : \mathcal{H}_\Theta \to H^2(\mathcal{E})$ if and only if S_Θ is n-cyclic.*

Proof. Suppose first that such an embedding exists. Let $\{e_k\}$ be an orthonormal basis for $\mathcal{E}$. Since $Y|\mathcal{K}_\Theta$ is injective, its adjoint, $P_\Theta Y^*$, must have dense range in $\mathcal{K}_\Theta$. That is, $P_\Theta Y^* H^2(\mathcal{E})$ is dense in $\mathcal{K}_\Theta$. The relations $S_\Theta P_\Theta = P_\Theta U_\Theta$ and $U_\Theta Y^* = Y^* S_\mathcal{E}$ give

$$P_\Theta Y^* S_\mathcal{E}^j e_k = S_\Theta^j P_\Theta Y^* e_k.$$

Thus, the fact that the elements $S_\mathcal{E}^j e_k$ span $H^2(\mathcal{E})$, combined with the fact that $P_\Theta Y^* H^2(\mathcal{E})$ is dense in $\mathcal{K}_\Theta$, yields that the elements $S_\Theta^j P_\Theta Y^* e_k$ span $\mathcal{K}_\Theta$. Hence S_Θ is n-cyclic.

The converse requires the following lemma.

LEMMA 5.5. *Suppose $\mathcal{F}$ and $\mathcal{F}_*$ are Hilbert spaces and $u \oplus v \in H^2(\mathcal{F}_*) \oplus L^2(\mathcal{F})$. Then there exist functions $h \in H^2$, $F \in H^\infty(\mathbf{L}(\mathbf{C}, \mathcal{F}_*))$ and $G \in L^\infty(\mathbf{L}(\mathbf{C}, \mathcal{F}))$ such that*

$$u \oplus v = Fh \oplus Gh.$$

Proof of Lemma 5.5. By the theorem of Halmos stated in the introduction, there is a Hilbert space $\mathcal{L}$ and an inner $\Omega \in H^\infty(\mathbf{L}(\mathcal{L}, \mathcal{F}_*))$ such that

$$\bigvee_0^\infty S_{\mathcal{F}_*}^k u = \Omega H^2(\mathcal{L}). \tag{5.1}$$

So $u = \Omega g$ for some $g \in H^2(\mathcal{L})$. The equality (5.1) then implies $S_\mathcal{L}$ is cyclic since Ω is inner. The only cyclic shift is that of multiplicity one ([5]), so we may assume $g \in H^2$.

The function $1 + |g(\lambda)|^2 + \|v(\lambda)\|_\mathcal{F}^2$ is log-integrable, so there is an outer function h in H^2 with modulus satisfying

$$|h(\lambda)|^2 = 1 + |g(\lambda)|^2 + \|v(\lambda)\|_\mathcal{F}^2. \tag{5.2}$$

We define a measurable $\mathbf{L}(\mathbb{C},\mathcal{F})$-valued function by $G(\lambda)h(\lambda) = v(\lambda)$. Note that since h is outer, it is nonzero almost everywhere on $\partial\mathbf{D}$, so there is no difficulty in defining G this way. It follows from (5.2) that

$$\|G(\lambda)\| = \frac{\|v(\lambda)\|_{\mathcal{F}}}{|h(\lambda)|} \leqslant 1.$$

Hence $G \in L^{\infty}(\mathbf{L}(\mathbb{C},\mathcal{F}))$. It also follows from (5.2) that the function $a(z) = g(z)/h(z)$ is in H^{∞}. If we let $F = \Omega a$, then $F \in H^{\infty}(\mathbf{L}(\mathbb{C},\mathcal{F}_*))$ and $Fh = \Omega a h = \Omega g = u$. It is clear that $Gh = v$, so the proof is complete. ∎

To proceed now with the proof of the converse of Theorem 5.4, we first assume that $\dim\mathcal{E} = n$ is finite. Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis for $\mathcal{E}$ and let $\{f_1,\ldots,f_n\}$ be vectors in $\mathcal{K}_\Theta$ such that the $S_\Theta^k f_j$ span $\mathcal{K}_\Theta$. By Lemma 5.5, we can find, for $j = 1,\ldots,n$, functions $h_j \in H^2$, $F_j \in H^{\infty}(\mathbf{L}(\mathbb{C},\mathcal{G}_*))$ and $G_j \in L^{\infty}(\mathbf{L}(\mathbb{C},\mathcal{G}))$ such that $f_j = F_j h_j \oplus G_j h_j$. Writing an element of $H^2(\mathcal{E})$ as $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, define Y^* by

$$Y^* \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1 & \cdots & F_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \oplus \begin{pmatrix} G_1 & \cdots & G_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Note that each $G_j x_j$ is in $\overline{M_\Delta L^2(\mathcal{G})}$, so the operator $\begin{pmatrix} G_1 & \cdots & G_n \end{pmatrix}$ maps $H^2(\mathcal{E})$ into $\overline{M_\Delta L^2(\mathcal{G})}$.

To show Y is an invariance embedding, we need to show that $P_\Theta Y^* \mathcal{K}_\Theta$ is dense in $\mathcal{K}_\Theta$. Let w_j denote the column vector in $H^2(\mathcal{E})$ with h_j in the j-th position of the column and zeroes elsewhere. Then

$$Y^* S_{\mathcal{E}}^k w_j = U_\Theta^k Y^* w_j = U_\Theta^k f_j.$$

So $P_\Theta Y^* \mathcal{K}_\Theta$ contains the vectors

$$P_\Theta U_\Theta^k f_j = S_\Theta^k f_j$$

for $k \in \mathbf{N}$ and $j = 1,\ldots,n$. This gives the desired result. Note that if Y as constructed is not a contraction, we can multiply it by an appropriate constant, so that it becomes a contraction, without loosing any of the desired properties.

In the case that $\dim\mathcal{E}$ is infinite, we may use the invariance embedding provided by Theorem 5.1. ∎

Combining Sz.-Nagy–Foiaş model theory with the results of this section gives the following.

COROLLARY 5.6. *Let $\mathcal{E}$ be a Hilbert space of dimension $n \in \mathbf{Z}^+ \cup \{\aleph_0\}$ and L a cnu-contraction on another Hilbert space. Then L is unitarily equivalent to the action of $S^*_{\mathcal{E}}$ on one of its contractively invariant spaces if and only if L^* is n-cyclic.*

COROLLARY 5.7. *If L is a cnu-contraction with a cyclic adjoint, then there is a Hilbert space $\mathcal{H}$, contractively contained in H^2, that is contractively invariant for S^* and on which S^* is unitarily equivalent to L.*

6. CONTRACTIVELY INVARIANT SPACES IN H^2

Corollary 5.7 suggests that the backward shift S^* on H^2 has quite a variety of contractively invariant spaces. This variety is in contrast with the case for the forward shift, S. For example, we saw in Section 1 that if a space, contractively contained in H^2, is contractively invariant for S, then it must be of the form $\mathcal{M}(T_B)$, where B is in the unit ball of some $H^\infty(\mathbf{L}(\mathcal{F},\mathbf{C}))$ and $T_B = M_B|H^2(\mathcal{F})$. From this it follows that no such space can be unitarily invariant for S. We will see later, however, there is an ample supply of such spaces for S^*.

6.1. DE BRANGES-ROVNYAK SPACES. Perhaps the best known of the spaces we are considering are the de Branges-Rovnyak spaces. For b in the unit ball of H^∞, the de Branges-Rovnyak space for b is the space $\mathcal{H}(b) = \mathcal{H}(T_b) = \mathcal{M}((1 - T_bT_{\bar{b}})^{1/2})$. The details of how S^* acts on these spaces have been worked out by D. Sarason ([14]). In particular the Sz.-Nagy–Foiaş model for the operator $C_{\mathcal{H}(b)}$ is determined. The results depend on whether or not b is an extreme point of the unit ball of H^∞.

Suppose first b is an extreme point. In this case the characteristic function of $C_{\mathcal{H}(b)}$ is simply $\Theta = b$, giving

$$\mathcal{H}_\Theta = H^2 \oplus \overline{M_\Delta L^2}$$

$$\mathcal{K}_\Theta = \{H^2 \oplus \overline{M_\Delta L^2}\} \ominus \begin{pmatrix} M_b \\ M_\Delta \end{pmatrix} H^2$$

where $\Delta = (1 - |b|^2)^{1/2}$. The operator $Y : \mathcal{H}_\Theta \to H^2$ is simply $Y \begin{pmatrix} h \\ g \end{pmatrix} = h$.

For b not an extreme point, we can form the bounded outer function a that is positive at the origin and has modulus $(1-|b|^2)^{1/2}$ almost everywhere on $\partial\mathbb{D}$. The characteristic function of $C_{\mathcal{H}(b)}$ is now the inner function

$$\Theta(z) = \begin{pmatrix} b(z) \\ a(z) \end{pmatrix} : \mathbb{C} \to \mathbb{C}^2.$$

In this case $\mathcal{H}_\Theta = H^2(\mathbb{C}^2)$ and $\mathcal{K}_\Theta = H^2(\mathbb{C}^2) \ominus M_\Theta H^2$. If we represent a function in $H^2(\mathbb{C}^2)$ as $\begin{pmatrix} h \\ g \end{pmatrix}$ for $h, g \in H^2$, then $Y : H^2(\mathbb{C}^2) \to H^2$ is again simply the projection onto the first coordinate.

Analogous results, due to J. Ball and T. Kriete ([1]), are available for vector-valued H^2-spaces. If B is in the unit ball of $H^\infty(\mathbf{L}(\mathcal{G},\mathcal{E}))$, then $\mathcal{H}(B) = \mathcal{H}(T_B) = \mathcal{M}((1-T_B^*T_B)^{1/2})$. Let $A \in L^\infty(\mathbf{L}(\mathcal{G},\mathcal{D}))$ be the maximal factorable minorant of $1 - B^*B$,

$$\Theta(z) = \begin{pmatrix} B(z) \\ A(z) \end{pmatrix} : \mathcal{G} \to \mathcal{E} \oplus \mathcal{D}$$

and $\Delta = (1-\Theta^*\Theta)^{1/2}$.

Now write

$$\mathcal{H}_\Theta = H^2(\mathcal{E}) \oplus H^2(\mathcal{D}) \oplus \overline{M_\Delta L^2(\mathcal{G})}; \quad \text{and}$$

$$\mathcal{K}_\Theta = \mathcal{H}_\Theta \ominus \begin{pmatrix} M_B \\ M_A \\ M_\Delta \end{pmatrix} H^2(\mathcal{G}).$$

In [1], it is shown that if $Y : \mathcal{H}_\Theta \to H^2(\mathcal{E})$ is given by $Y(u \oplus v \oplus w) = u$, then $\mathcal{H}(B) = \mathcal{M}(Y|\mathcal{K}_\Theta)$. This shows how the results given in [10] fit into our scheme.

6.2. THE SPACES $\mathcal{M}(T_{\bar{h}}|\mathcal{H}(\theta))$. Let θ be any inner function in H^∞. Set $\mathcal{H}_\Theta = H^2$ and $\mathcal{K}_\Theta = \mathcal{H}(\theta) = H^2 \ominus \theta H^2$. Choose $h \in H^\infty$ so that $\mathcal{H}(\theta) \cap \ker T_{\bar{h}} = \{0\}$. If we set $Y = T_{\bar{h}}$, then Theorem 3.1 implies that $\mathcal{M}(T_{\bar{h}}|\mathcal{H}(\theta))$ is contractively invariant for S^*. Of course, this is readily apparent from the relations $S^*T_{\bar{h}} = T_{\bar{h}}S^*$ and $S^*\mathcal{H}(\theta) \subset \mathcal{H}(\theta)$.

Similar spaces arise from functions $q \in L^\infty$ that are non-negative and satisfy

$$\int \log q \, d\sigma > -\infty.$$

Such a function is the modulus of an outer h in H^∞. Thus

$$\Gamma_q\Gamma_q^* = T_{q^2} = T_{\bar{h}}T_h$$

so that $\mathcal{M}(\Gamma_q) = \mathcal{M}(T_{\bar{h}})$.

6.3. UNITARILY INVARIANT SPACES IN H^2. Now we give a characterization of the spaces in H^2 that are unitarily invariant for S^*.

THEOREM 6.3.1. *Suppose $\mathcal{H} \prec H^2$. Then $\mathcal{H}$ is unitarily invariant for S^* if and only if $\mathcal{H} = \mathcal{M}(\Gamma_q)$ for some non-negative $q \in L^\infty$ such that*

$$\int_{\partial\mathbf{D}} \log q \, d\sigma = -\infty.$$

Proof. Suppose $\mathcal{H}$ is unitarily invariant for S^*. By Theorem 2.1, it must be that $\mathcal{H} = \mathcal{M}(\Gamma_q)$ for some non-negative $q \in L^\infty$ such that $\mathcal{L}_q = \overline{qH^2}$ reduces $U = U_{\mathbf{C}}$. To determine which q satisfy this condition, we first recall a result concerning the invariant subspaces of the bilateral shift on L^2, a proof of which can be found in [8].

THEOREM. *Let $\mathcal{F}$ be a subspace of L^2. Then $\mathcal{F}$ is a non-trivial invariant subspace for the bilateral shift if and only if either*

(i) $\mathcal{F} = \chi_E L^2$ *for some measurable* $E \subset \partial\mathbf{D}$; *or*

(ii) $\mathcal{F} = uH^2$ *for some* $u \in L^\infty$ *with* $|u| = 1$ *almost everywhere.*

The subspaces of the first type are those that reduce U, while those of the second type contain no subspace which reduces U. So $\mathcal{L}_q$ reduces U, if and only if it is of the form $\chi_E L^2$. Let $|E|$ denote the normalized Lebesgue measure of E. If $|E| = 1$, then $\overline{qH^2} = L^2$, implying q is cyclic for U. This implies that q cannot be log-integrable, i.e., that

$$\int_{\partial\mathbf{D}} \log q \, d\sigma = -\infty.$$

If $|E| < 1$, then q obviously fails to be log-integrable.

On the other hand, if we begin with a $q \geqslant 0$ in L^∞ that is not log-integrable, either $q > 0$ almost everywhere or q is zero on a set of positive measure. In the first case, q is cyclic for U, so $\mathcal{L}_q$ reduces U. In the second case, Beurling's theorem gives that $\mathcal{L}_q = \chi_E L^2$ for some E, so again it reduces U. ∎

In particular, the spaces $\mathcal{M}(\Gamma_{\chi_E})$ for $E \subset \partial\mathbf{D}$ satisfying $0 < |E| < 1$ are unitarily invariant for S^*. These spaces also have the property that their complementary space fails to be invariant for S. Hence the results in [10] mentioned in Section 1 do not apply to these spaces. Moreover, this fact illustrates one way in which contractively contained spaces behave differently than subspaces, since if a subspace is invariant for an operator, then its orthogonal complement is invariant for the adjoint of that operator.

Observe first that the complementary space of $\mathcal{M}(\Gamma_{\chi_E})$ is

$$\mathcal{H}(\Gamma_{\chi_E}) = \mathcal{M}((1 - \Gamma_{\chi_E}\Gamma_{\chi_E}^*)^{1/2}) = \mathcal{M}(T_{1-\chi_E}^{1/2}) = \mathcal{M}(\Gamma_{\chi_{E'}})$$

where $E' = \partial\mathbb{D} \setminus E$. Suppose this space is invariant for S. We know it is invariant for S^* so let A and $C_{\mathcal{M}(\Gamma_{\chi_E})}$ be the operators on $\mathcal{M}(\Gamma_{\chi_E})$ satisfying $S\Gamma_{\chi_{E'}} = \Gamma_{\chi_{E'}}A$ and $S^*\Gamma_{\chi_{E'}} = \Gamma_{\chi_{E'}}C_{\mathcal{M}(\Gamma_{\chi_E})}$. Then

$$\Gamma_{\chi_{E'}} = S^*S\Gamma_{\chi_{E'}} = \Gamma_{\chi_{E'}}C^*_{\mathcal{M}(\Gamma_{\chi_E})}A.$$

Thus $C^*_{\mathcal{M}(\Gamma_{\chi_E})}A = 1$. This implies that A is unitary (because $C_{\mathcal{M}(\Gamma_{\chi_E})}$ is) and thus that $\mathcal{M}(\Gamma_{\chi_{E'}})$ is unitarily invariant for S, a contradiction.

Note also that since $\chi_E L^2 \neq L^2$, it follows that $\mathcal{M}(\Gamma_{\chi_E})$ is proper. To see this, suppose it is not proper. Then

$$H^2 = \Gamma_q\mathcal{L}_q = P_+q\chi_E L^2 \subset P_+\chi_E L^2 \subset H^2.$$

In other words, $P_+\chi_E L^2 = H^2$. This says $P_+|\mathcal{L}_q : \mathcal{L}_q \to H^2$ has closed range. Hence its adjoint, $P_{\mathcal{L}_q}|H^2$, has closed range. The operator $P_{\mathcal{L}_q}$ is just multiplication by χ_E, so we conclude that $\chi_E H^2$ is closed. An application of Beurling's theorem gives that $\chi_E H^2 = \overline{\chi_E H^2} = \chi_E L^2$. In particular then, $\chi_E H^2$ must contain χ_F for any measurable $F \subset E$ satisfying $0 < |F| < |E|$. So $\chi_F = \chi_E h$ for some $h \in H^2$. But this implies that h is zero on $E \setminus F$, a set of positive measure, which is impossible for a non-zero H^2 function. Hence $\mathcal{M}(\Gamma_q)$ is proper. The spaces $\mathcal{M}(\Gamma_Q)$ thus give a large class of spaces that are non-trivial and unitarily invariant for S^*.

6.4. SPACES WITH NORM $\sum \alpha_n|\widehat{f}(n)|^2$. Let $\alpha = \{\alpha_n\}_0^\infty$ be a sequence of numbers satisfying

$$1 \leqslant \alpha_n < \alpha_{n+1} \tag{6.4.1}$$

and

$$\alpha_n \to \infty. \tag{6.4.2}$$

Let

$$\mathcal{K}(\alpha) = \left\{ f \in H^2 \mid \sum_0^\infty \alpha_n|\widehat{f}(n)|^2 < \infty \right\}$$

have the inner product $\langle f, g\rangle_\alpha = \sum\limits_0^\infty \alpha_n \widehat{f}(n)\overline{\widehat{g}(n)}$. Clearly $\mathcal{K}(\alpha)$ is a Hilbert space contractively contained in H^2. The condition (6.4.2) ensures that $\mathcal{K}(\alpha)$ is proper in H^2. One example is the Dirichlet space which has $\alpha_n = (n+1)^{-1}$.

Consider, for a positive integer k and $f \in \mathcal{K}(\alpha)$, the inequalities

$$\|S^{*k}f\|_\alpha^2 = \sum_0^\infty \alpha_n|\widehat{f}(n+k)|^2 = \sum_{n=k}^\infty \alpha_{n-k}|\widehat{f}(n)|^2$$
$$< \sum_{n=k}^\infty \alpha_n|\widehat{f}(n)|^2 \leqslant \|f\|_\alpha < \infty.$$

Several conclusions follow from this. First, $\mathcal{K}(\alpha)$ is contractively invariant for S^*, and second, $\|S^{*k}f\|_\alpha \to 0$ as $k \to \infty$. The second fact, along with Proposition VI.2.1 of [16], gives that the characteristic function of $C_{\mathcal{K}(\alpha)}$ must be inner. A third conclusion is, since $\|S^*f\|_\alpha < \|f\|_\alpha$ for all f in $\mathcal{K}(\alpha)$, that $\ker D_{C^*_{\mathcal{K}(\alpha)}} = \{0\}$ and thus $\mathcal{D}_{C^*} = \mathcal{K}(\alpha)$.

The operator $C_{\mathcal{K}(\alpha)}$ can be computed directly. If $f, g \in \mathcal{K}(\alpha)$, then

$$\begin{aligned}
\langle C_{\mathcal{K}(\alpha)}f, g\rangle_\alpha = \langle f, S^*g\rangle_\alpha &= \sum_{n=0}^{\infty} \alpha_n \widehat{f}(n)\overline{\widehat{g}(n+1)} \\
&= \sum_{n=1}^{\infty} \alpha_{n-1} \widehat{Sf}(n)\overline{\widehat{g}(n)} \\
&= \sum_{n=1}^{\infty} \alpha_n \frac{\alpha_{n-1}}{\alpha_n} \widehat{Sf}(n)\overline{\widehat{g}(n)} = \langle DSf, g\rangle_\alpha
\end{aligned}$$

where D is the diagonal operator on H^2 with respect to the basis $\{\zeta^n\}$ with entries $\{\alpha_{n-1}/\alpha_n\}$. Here $\zeta(z) = z$. The boundedness of D follows from our hypothesis on the sequence α. Thus $C_{\mathcal{K}(\alpha)} = DS$, a weighted shift. A direct computation shows that $\|C_{\mathcal{K}(\alpha)}f\|_\alpha < \|f\|_\alpha$ for all $f \in \mathcal{K}(\alpha)$, so as before, it follows that $\mathcal{D}_C = \mathcal{K}(\alpha)$. Thus the characteristic function of $C_{\mathcal{K}(\alpha)}$ has values that operate between infinite dimensional spaces.

The spaces $\mathcal{K}(\alpha)$ can be used to show that every vector in H^2 is contained in a space that is contractively invariant for S^*. We begin with a lemma.

LEMMA 6.4.1. *Let* $\{c_n\}_0^\infty$ *be a sequence of non-negative numbers such that* $\sum c_n < \infty$. *Then there exits another non-negative sequence* $\{\alpha_n\}$, *satisfying* (6.4.1) *and* (6.4.2), *such that* $\sum \alpha_n c_n < \infty$.

Proof. Choose n_k so that $\sum\limits_{j=n_k}^{\infty} c_n < (k+1)^{-3}$. Let $\alpha_{n_k} = k$ and choose the remaining α_n's so

$$k < \alpha_{n_k+1} < \alpha_{n_k+2} < \cdots < \alpha_{n_{k+1}-1} < k+1.$$

Then

$$\sum \alpha_i c_i = \sum_{i=0}^{\infty} \sum_{j=n_i}^{n_{i+1}-1} \alpha_j c_j < \sum_{i=0}^{\infty} (i+1) \sum_{j=n_i}^{\infty} c_j \leqslant \sum_{i=0}^{\infty} \frac{1}{(i+1)^2} < \infty.$$

This completes the proof since the constructed sequence obviously satisfies (6.4.1) and (6.4.2). ∎

Now let $f \in H^2$. Choose $\{\alpha_n\}$ as in the lemma so $\sum \alpha_n |\widehat{f}(n)|^2 < \infty$. Then $\mathcal{K}(\alpha)$ is the space we are looking for. Another application of the lemma shows that we can also choose a space $\mathcal{K}(\alpha')$ such that

$$f \in \mathcal{K}(\alpha') \subsetneqq \mathcal{K}(\alpha),$$

showing there is no minimal such space.

Shifts on Dirichlet spaces are investigated by S. Richter in [11].

7. GENERAL INVARIANT SPACES

We conclude by considering spaces that are invariant for $S_{\mathcal{E}}^*$, but not necessarily contractively. Our proof is a modification of the proof of an analogous result for $S_{\mathcal{E}}$ that can be found in [10].

First, consider a method to construct an operator-valued function. Suppose $\mathcal{E}$ and $\mathcal{F}$ are Hilbert spaces and $A : H^2(\mathcal{E}) \to H^2(\mathcal{F})$ is an operator. If $e \in \mathcal{E}$, then $Ae \in H^2(\mathcal{F})$. Now define

$$F_A(z)e = (Ae)(z).$$

Then $F_A(z) \in \mathbf{L}(\mathcal{E}, \mathcal{F})$ for all $z \in \mathbb{D}$ and $F_A e \in H^2(\mathcal{F})$ for all $e \in H^2(\mathcal{E})$.

If f is any function defined in $\mathbb{D}$ and $0 < r < 1$, let $(C_r f)(z) = f(rz)$.

THEOREM 7.1. *Suppose $\mathcal{H} \prec H^2(\mathcal{E})$. Then $\mathcal{H}$ is invariant for $S_{\mathcal{E}}^*$ if and only if $\mathcal{H} = \mathcal{M}(X)$ where $X : \mathcal{K}_\Theta \to H^2(\mathcal{E})$ is a contraction, Θ is inner, and there is an $0 < r < 1$ such that*

$$X^* C_r p = P_\Theta F_{X^*} p \tag{7.1}$$

for all $\mathcal{E}$-valued polynomials $p \in H^2(\mathcal{E})$.

Proof. Choose $0 < r < 1$ so that $rS_{\mathcal{E}}^*$ acts on $\mathcal{H}$ as a contraction of norm strictly less than one. Let $C_{\mathcal{H}}$ be the operator on $\mathcal{H}$ whose adjoint is this action of $rS_{\mathcal{E}}^*$. Thus $\|C_{\mathcal{H}}\| < 1$ and so the characteristic function of $C_{\mathcal{H}}$ is inner.

As in the proof of Theorem 3.1, we have a contraction $X : \mathcal{K}_\Theta \to H^2(\mathcal{E})$ which in the present case satisfies $rS_{\mathcal{E}}^* X = X S_\Theta^*$. If $e \in \mathcal{E}$ and $n \geqslant 0$, then

$$\begin{aligned} X^* C_r \zeta^n e &= X^* r^n \zeta^n e = X^* r^n S_{\mathcal{E}}^n e \\ &= S_\Theta^n X^* e = P_\Theta U_\Theta^n X^* e \\ &= P_\Theta(\zeta^n X^* e) = P_\Theta(\zeta^n F_{X^*} e) \\ &= P_\Theta F_{X^*} \zeta^n e \end{aligned}$$

so that (7.1) holds.

On the other hand, if we begin with an X as in the statement of the theorem, one computes that $X^* r S_{\mathcal{E}} = S_\Theta X^*$, which shows that $\mathcal{M}(X)$ is invariant for $S_{\mathcal{E}}^*$. ∎

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