J. OPERATOR THEORY
34(1995), 145–175

STOPPING AND INTEGRATION IN A PRODUCT STRUCTURE

C. BARNETT and S. VOLIOTIS

Communicated by Şerban Strătilă

ABSTRACT. We investigate stochastic integration and random stopping in a non-commutative filtration by embedding our analysis in the von Neumann algebra formed by the product of the algebras of the filtration.

KEYWORDS: *Non-commutative probability, von Neumann algebras, random times.*

AMS SUBJECT CLASSIFICATION: Primary ; Secondary .

1. INTRODUCTION

In two papers, [11] and [8], a non-commutative stochastic integral has been developed which has the property that as function of the integrand, the integral is a homomorphism. In [11] this is achieved by taking as departure point the existence and properties of random times and their associated time projections. From this a stochastic integral is obtained in which the integrands are (identified with) operators in an appropriate L^2 space and it is their action upon points in the L^2 space (with which L^2 bounded martingales are identified) that effects stochastic integration. In [8] a different approach is taken. Here an operator valued belated integral is developed (for a treatment of belated integrals see [9]). This is a 'measure theoretic' integral but it shares features of that developed in [11], in particular, the integral is an operator which when it acts "performs" stochastic integration. This paper gives a variant upon both of the themes explored in the papers noted above. It is novel in that it does so in the context of a product von Neumann algebra and develops the integral "algebraically" in a natural and pleasing way. The reason for looking at the integral in this product structure arose from the realisation that the structure of the algebra of *stochastic integral operators* casts some (general) light upon questions such as martingale representation aside from providing a bridge between operator theory and stochastic analysis.

2. MATHEMATICAL PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$ and $\mathcal{A}$ a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ equipped with a faithful normal state ω. For each $t \in \mathbf{R}^+$ let $\mathcal{A}_t$ be a von Neumann subalgebra of $\mathcal{A}$ with the following properties:

(i) if $t, s \in \mathbf{R}^+$ and $t \leqslant s$ then $\mathcal{A}_t \subseteq \mathcal{A}_s$;

(ii) $\left(\bigcup_{t \geqslant 0} \mathcal{A}_t\right)'' = \mathcal{A}$, i.e. $\mathcal{A}$ is generated by the algebras $\mathcal{A}_t, t \geqslant 0$;

(iii) $\bigcap_{t>s} \mathcal{A}_t = \mathcal{A}_s$. Furthermore, we suppose the existence of a family of conditional expectations $\{\tilde{M}_t : t \in \mathbf{R}^+\}$ with $\tilde{M}_t : \mathcal{A} \to \mathcal{A}_t$ and:

(iv) $\omega \cdot \tilde{M}_t = \omega \; \forall t \in \mathbf{R}^+$;

(v) $\tilde{M}_t(axb) = a\tilde{M}_t(x)b \; \forall a, b \in \mathcal{A}_t, x \in \mathcal{A}$;

(vi) $\tilde{M}_t(I) = I$ for I the identity operator in $\mathcal{A}$.

We can use the state ω to construct G.N.S. spaces $\mathcal{L}^2(\mathcal{A}), \mathcal{L}^2(\mathcal{A}_t)$ in the usual way. The map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{L}^2(\mathcal{A}))$ defined by $\pi(a) \cdot b = ab$ for $b \in \mathcal{A}$, is a well defined normal (isometric) isomorphism. We observe that $\pi(\mathcal{A})$ is a von Neumann algebra, using the fact that the unit ball of $\pi(\mathcal{A})$ is ultra-weakly compact. So $\pi(\mathcal{A})$ becomes an isometric copy of $\mathcal{A}$. Moreover I is a cyclic and separating vector for $\pi(\mathcal{A})$. As seen in [7] we can define

$$M_t : \mathcal{L}^2(\mathcal{A}) \to \mathcal{L}^2(\mathcal{A}_t) \qquad \forall t \in \mathbf{R}^+$$

to be the orthogonal projection onto $\mathcal{L}^2(\mathcal{A}_t)$. We then have:

$$M_t a = \tilde{M}_t a \in \mathcal{A}_t, \quad \forall t \in \mathbf{R}^+, \; \forall a \in \mathcal{A}$$

and M_t lies in the commutant of $\pi(\mathcal{A})$.

DEFINITION 2.1. A *stopping time*, τ, is an increasing family of projections $\{\pi(Q_t)\}_{t \in \overline{\mathbf{R}}^+}$ with Q_t a projection in $\mathcal{A}_t$ and $\tau(0) = 0, \tau(\infty) = I$.

Note that this definition extends the 'usual' (commutative) case. This point is elaborated in [6].

DEFINITION 2.2. A $\pi(\mathcal{A})$ [respectively $\mathcal{L}^2(\mathcal{A})$]-*adapted process* is a family $\{x_t\}_{t\in\overline{\mathbb{R}}^+}$ with $x_t \in \pi(\mathcal{A}_t)$ [respectively $\mathcal{L}^2(\mathcal{A}_t)$] $\forall t \in \mathbb{R}^+$ and $x_\infty \in \pi(\mathcal{A})$ [respectively $\mathcal{L}^2(\mathcal{A})$].

In particular a stopping time is a $\pi(\mathcal{A})$-adapted process. Furthermore we can define the von Neumann algebra

$$\mathcal{U} = \prod_{t\in\overline{\mathbb{R}}^+} \pi(\mathcal{A}_t)$$

(with $\pi(\mathcal{A}_\infty) \equiv \pi(\mathcal{A})$) the product von Neumann algebras for the algebras $\{\pi(\mathcal{A}_t)\}_{t\in\overline{\mathbb{R}}^+}$. See [15], Part 1, Chapter 2, Section 2. We can consequently view $\pi(\mathcal{A})$-adapted processes as elements of $\mathcal{U}$, and a stopping time is a projection in $\mathcal{U}$.

DEFINITION 2.3. (i) For two stopping times $\tau = \{Q_t\}_{t\in\overline{\mathbb{R}}^+}$ and $\sigma = \{Q'_t\}_{t\in\overline{\mathbb{R}}^+}$ we can define an order $\tau \leqslant \sigma \Leftrightarrow Q'_t \leqslant Q_t$, $\forall t \in \mathbb{R}^+$. Note that this order agrees with the order of stopping times in the commutative case, see [8], but is the reverse order for operators in $\mathcal{U}$.

(ii) For $\mathcal{P}$ the set of all finite partitions of $[0,\infty]$ and $T = \{t_0,\ldots,t_n\} \in \mathcal{P}$ we define:

$$M_{\tau(T)} = \sum_{i=0}^{n-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}}.$$

Clearly $M_{\tau(T)}$ is an operator in $\mathcal{B}(\mathcal{L}^2(\mathcal{A}_\infty))$ and as seen in Theorem 2.3 of [7]:

(iii) $M_{\tau(T)}$ is an orthogonal projection;

(iv) for $T, T' \in \mathcal{P}$

$$T' \supseteq T \Rightarrow M_{\tau(T')} \leqslant M_{\tau(T)};$$

(v) if σ is another stopping time with $\sigma \geqslant \tau$ then:

$$M_{\sigma(T)} \geqslant M_{\tau(T)} \qquad \forall T \in \mathcal{P}.$$

In view of these properties and the fact that $\mathcal{P}$ is a directed set with the order of inclusion we note that $\{M_{\tau(T)}\}_{T\in\mathcal{P}}$ becomes a decreasing net of orthogonal projections. Hence there exists a unique orthogonal projection

$$M_\tau \equiv \inf_{T\in\mathcal{P}} M_{\tau(T)}$$

and

$$M_{\tau(T)} \longrightarrow M_\tau$$

in the strong-operator topology as $T \nearrow$. We shall call M_τ the *time projection* for the stopping time τ (Definition 2.4 of [7]).

3. A HOMOMORPHIC INTEGRAL

DEFINITION 3.1. We define a family of von Neumann algebras $\{\mathcal{V}^T\}_{T\in\mathcal{P}}$ by: $\mathcal{V}^T$ is the von Neumann algebra generated by

$$\Big\{M_{\tau(T)} : \tau \text{ is a stopping time }\Big\}.$$

We also define the *time algebra* $\mathcal{V}$ which is generated by:

$$\Big\{M_\tau : \tau \text{ is a stopping time }\Big\}.$$

Note that

$$\mathcal{V} \subseteq \Big(\bigcup_{T\in\mathcal{P}} \mathcal{V}^T\Big)''$$

since

$$M_\tau = \inf_{T\in\mathcal{P}} M_{\tau(T)}.$$

Furthermore for τ a stopping time and $T = \{t_0, \ldots, t_n\} \in \mathcal{P}$, we can define the stopping time

$$\tau^T(s) = \begin{cases} 0 & 0 \leqslant s < t_1 \\ \tau(t_i) & t_i \leqslant s < t_{i+1} \qquad 1 \leqslant i \leqslant n-1 \\ I & s = t_n = \infty \end{cases}$$

and observe that $M_{\tau^T} = M_{\tau(T)}$. Hence $\mathcal{V}^T \subseteq \mathcal{V}\ \forall T \in \mathcal{P}$. So, we conclude that

$$\mathcal{V} = \Big(\bigcup_{T\in\mathcal{P}} \mathcal{V}^T\Big)''.$$

LEMMA 3.2. *$\forall t \in [0,\infty]$ M_t belongs in the centre of $\mathcal{V}$.*

Proof. $M_\infty \equiv I$ and so the result holds trivially in this case.

Suppose now that $t \in [0,\infty)$, τ is a stopping time $\tau = \{\pi(Q_t)\}_{t\in\overline{\mathbb{R}^+}}$ and $T \in \mathcal{P}$ with $T = \{0 = t_0, t_1, \ldots, t_n = \infty\}$. Then:

$$\exists k \in \{0, \ldots, n-1\} \quad \text{with} \quad t_k \leqslant t < t_{k+1}.$$

So

$$
\begin{aligned}
M_{\tau(T)} \cdot M_t &= \left[\sum_{i=0}^{n-1} \pi\big(Q_{t_{i+1}} - Q_{t_i}\big) \cdot M_{t_{i+1}}\right] \cdot M_t \\
&= \sum_{i=0}^{k-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}} \cdot M_t + \sum_{i=k}^{n-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}} \cdot M_t \\
&\quad \text{(with the first sum equal to 0 if } k = 0) \\
&= \sum_{i=0}^{k-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}} + \sum_{i=k}^{n-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_t \\
&\quad \text{(since } M_t\text{'s are orthogonal projections that increase as } t \text{ increases)} \\
&= \sum_{i=0}^{k-1} M_{t_{i+1}} \cdot \pi(Q_{t_{i+1}} - Q_{t_i}) + \pi(I - Q_{t_k}) \cdot M_t \\
&\quad \text{(since } M_s \text{ lies in the commutant of } \pi(\mathcal{A}_s)) \\
&= M_t \cdot \left[\sum_{i=0}^{k-1} M_{t_{i+1}} \cdot \pi(Q_{t_{i+1}} - Q_{t_i})\right] + M_t \cdot \pi(I - Q_{t_k}) \\
&= M_t \cdot \left[\sum_{i=0}^{k-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}} + M_t \cdot \sum_{i=k}^{n-1} M_{t_{i+1}} \cdot \pi(Q_{t_{i+1}} - Q_{t_i})\right] \\
&= M_t \cdot M_{\tau(T)}.
\end{aligned}
$$

Hence, $\forall\, t \in [0, \infty]$, M_t commutes with the generators of $\mathcal{V}$ and as a result $M_t \in \mathcal{V}'$.

If we now consider the stopping time:

$$
\tau(s) = \begin{cases} I & s \geqslant t \\ 0 & s < t \end{cases}
$$

for a fixed t and the partition $T = \{0, t, \infty\}$, then

$$
M_t = M_{\tau(T)} \in \mathcal{V}.
$$

So M_t belongs in the centre of $\mathcal{V}$, as required. ∎

LEMMA 3.3. *For each partition $T = \{t_0, \ldots, t_n\} \in \mathcal{P}$, $\mathcal{V}^T$ is isomorphic to*

$$
\mathbf{C} \cdot M_{t_1} \oplus \pi(\mathcal{A}_{t_1})(M_{t_2} - M_{t_1}) \oplus \cdots \oplus \pi(\mathcal{A}_{t_{n-1}})(I - M_{t_{n-1}})
$$

where the algebras in the direct sum are the reduced algebras ([15], Chapter 2, Section 1).

Proof. For a fixed partition $T = \{t_0, \dots, t_n\} \in \mathcal{P}$ and a fixed stopping time τ we get:

$$\begin{aligned} M_{\tau(T)} &= \sum_{i=0}^{n-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}} \\ &= I - \sum_{i=1}^{n-1} \pi(Q_{t_i})(M_{t_{i+1}} - M_{t_i}) \end{aligned}$$

since $t_n = \infty$ and $\pi(Q_{t_n}) = M_{t_n} = I$ and $Q_{t_0} = 0$. Hence, $M_{\tau(T)}$ for any stopping time τ can be written as a sum of the form

$$\sum_{i=1}^{n-1} \pi(\cdot)(M_{t_{i+1}} - M_{t_i}) + l M_{t_1}.$$

By mutual orthogonality of the projections $(M_{t_{i+1}} - M_{t_i})_{i=1}^{n}$ the same can be said for any element x of the $*$-algebra generated by the $M_{\tau(T)}$, where τ is a stopping time. Furthermore if x_α is a net of such elements with $x_\alpha \to b \in \mathcal{V}^T$ in the strong-operator topology then

$$x_\alpha \cdot (M_{t_{i+1}} - M_{t_i}) \to b(M_{t_{i+1}} - M_{t_i})$$

in the strong-operator topology. But since $\pi(\mathcal{A}_{t_i}) \cdot (M_{t_{i+1}} - M_{t_i})$ forms a (reduced) von Neumann algebra, $\exists\, a_{t_i} \in \mathcal{A}_{t_i}$ such that:

$$\pi(a_{t_i})(M_{t_{i+1}} - M_{t_i}) = b(M_{t_{i+1}} - M_{t_i})$$

and

$$b = \mu M_{t_1} + \sum_{i=1}^{n-1} \pi(a_{t_i})(M_{t_{i+1}} - M_{t_i}), \qquad \mu \in \mathbf{C}.$$

Furthermore, by the mutual orthogonality of the projections $\left(M_{t_{i+1}} - M_{t_i}\right)_{i=1}^{n}$ we observe that this decomposition is unique in the sense that:

$$\begin{aligned} b &= \mu M_{t_1} + \sum_{i=1}^{n-1} \pi(a_{t_i})(M_{t_{i+1}} - M_{t_i}) \\ &= \mu' M_{t_1} + \sum_{i=1}^{n-1} \pi(a'_{t_i})(M_{t_{i+1}} - M_{t_i}) \\ \Rightarrow \mu &= \mu' \end{aligned}$$

and

$$\pi(a_{t_i})(M_{t_{i+1}} - M_{t_i}) = \pi(a'_{t_i})(M_{t_{i+1}} - M_{t_i}), \qquad 1 \leqslant i \leqslant n-1.$$

This shows the existence of a well defined map

$$\Phi : \mathcal{V}^T \to \mathbb{C}M_{t_1} \oplus \pi(\mathcal{A}_{t_1})(M_{t_2} - M_{t_1}) \oplus \cdots \oplus \pi(\mathcal{A}_{t_{n-1}})(I - M_{t_{n-1}}).$$

This map is linear since:

$$\begin{aligned}
\Phi\Big\{lM_{t_1} &+ \sum_{i=1}^{n-1} \pi(a_{t_i})(M_{t_{i+1}} - M_{t_i}) + \mu M_{t_1} + \sum_{i=1}^{n-1} \pi(b_{t_i})(M_{t_{i+1}} - M_{t_i})\Big\} \\
&= \Phi\Big\{(l+\mu)M_{t_1} + \sum_{i=1}^{n-1} \pi(a_{t_i} + b_{t_i})(M_{t_{i+1}} - M_{t_i})\Big\} \\
&= (l+\mu)M_{t_1} \oplus \cdots \oplus \pi(a_{t_{n-1}} + b_{t_{n-1}})(I - M_{t_{n-1}}) \\
&= lM_{t_1} \oplus \cdots \oplus \pi(a_{t_{n-1}})(I - M_{t_{n-1}}) + \mu M_{t_1} \oplus \cdots \oplus \pi(b_{t_{n-1}})(I - M_{t_{n-1}}).
\end{aligned}$$

It is multiplicative since:

$$\begin{aligned}
\Phi\Big\{\Big[lM_{t_1} &+ \sum_{i=1}^{n-1} \pi(a_{t_i})(M_{t_{i+1}} - M_{t_i})\Big]\Big[\mu M_{t_1} + \sum_{i=1}^{n-1} \pi(b_{t_i})(M_{t_{i+1}} - M_{t_i})\Big]\Big\} \\
&= \Phi\Big\{l\mu M_{t_1} + \sum_{i=1}^{n-1} \pi(a_{t_i})\pi(b_{t_i})(M_{t_{i+1}} - M_{t_i})\Big\} \\
&= \Phi\Big\{lM_{t_1} + \sum_{i=1}^{n-1} \pi(a_{t_i})(M_{t_{i+1}} - M_{t_i})\Big\} \cdot \Phi\Big\{\mu M_{t_1} + \sum_{i=1}^{n-1} \pi(b_{t_i})(M_{t_{i+1}} - M_{t_i}\Big\}.
\end{aligned}$$

(by orthogonality and $(M_{t_{i+1}} - M_{t_i}) \in \pi(\mathcal{A}_{t_i})')$. One shows that Φ is adjoint preserving in a similar fashion. Finally, we shall show that the map is an isomorphism. Indeed, consider the stopping time

$$\tau(s) = \begin{cases} 0 & 0 \leqslant s < t_i \\ \pi(Q) & t_i \leqslant s < t_{i+1} \\ I & s \geqslant t_{i+1} \end{cases}$$

with $i \geqslant 1$ and $Q \in \mathcal{A}_{t_i}$. Then

$$M_{\tau(T)} = M_{t_{i+1}} - \pi(Q)(M_{t_{i+1}} - M_{t_i})$$

and for the stopping time

$$\tilde{\tau}(s) = \begin{cases} 0 & s \leqslant t_{i+1} \\ I & s > t_{i+1} \end{cases}$$

we get:

$$M_{t_{i+1}} = M_{\tilde{\tau}(T)} \in \mathcal{V}^T.$$

Since any von Neumann algebra can be generated by strong-operator limits of the span of its projections we deduce that

$$\pi(\mathcal{A}_{t_i})(M_{t_{i+1}} - M_{t_i}) \subseteq \mathcal{V}^T \quad \text{for} \quad i \geqslant 1.$$

Finally, for

$$\tilde{\tilde{\tau}}(s) = \begin{cases} 0 & s \leqslant t_1 \\ I & s > t_1 \end{cases}$$

we observe that

$$M_{t_1} = M_{\tilde{\tilde{\tau}}(T)} \in \mathcal{V}^T$$

and so $\mathbb{C}M_{t_1} \subseteq \mathcal{V}^T$, which shows that Φ is surjective.

For injectivity, suppose that $\Phi(a) = 0$ for

$$a = lM_{t_1} + \sum_{i=1}^{n-1} \pi(a_{t_i})(M_{t_{i+1}} - M_{t_i}).$$

Then,

$$\Phi(a) = lM_{t_1} \oplus \pi(a_{t_1})(M_{t_2} - M_{t_1}) \oplus \cdots \oplus \pi(a_{t_{n-1}})(I - M_{t_{n-1}}) = 0.$$

Hence $l = 0$ and $\pi(a_{t_i})(M_{t_{i+1}} - M_{t_i}) = 0$ for $1 \leqslant i \leqslant n-1$. So Φ is an isomorphism, as required. ∎

COROLLARY 3.4. *In the case when $\mathcal{A}$ admits a finite, faithful, normal trace we can deduce that $\mathcal{V}^T$ is a finite, countably decomposable von Neumann algebra and consequently admits a finite, faithful, normal trace.*

Proof. We have that $\mathcal{A}$ is finite and countably decomposable. Consequently, $\mathbb{C}M_{t_1}, \pi(\mathcal{A}_{t_1})(M_{t_2} - M_{t_1}), \ldots, \pi(\mathcal{A}_{t_{n-1}})(I - M_{t_{n-1}})$ are all finite and countably decomposable and so is their direct sum and hence so is $\mathcal{V}^T$ by Lemma 3.3. ∎

An open and interesting question arises when one considers under what circumstances the algebra $\mathcal{V}$ will admit a finite faithful normal trace. In fact it was just such a question, arising in a somewhat different situation ([11]), that motivated the use of the product construction employed here. The properties of the algebra $\mathcal{V}$ have a bearing upon the relationship between the random times and their time projections and the existence of a cyclic vector for a certain reduced algebra of $\mathcal{V}$ leads directly to a martingale representation theorem. But we will not pursue this here.

Before we proceed with the main results in this section we shall show that $\mathcal{V}$ is *not* the von Neumann algebra $\tilde{\mathcal{V}}$ generated by $\{\pi(\mathcal{A}), M_t : t \in \overline{\mathbb{R}}^+\}$. Indeed, we shall show that I is a tracial vector for $\mathcal{V}$, but not for $\tilde{\mathcal{V}}$. Suppose τ is a stopping time and $T \in \mathcal{P}$ a partition of $[0,\infty]$. Then $\tau(s) = Q_s$ for $s \in [0,\infty]$ and:

$$M_{\tau(T)}(I) = \sum_{i=0}^{n-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}}(I) = \sum_{i=0}^{n-1} \pi(Q_{t_{i+1}} - Q_{t_i})I = I$$

since $Q_\infty = I$, $Q_0 = 0$ for a stopping time. Furthermore, for finite products:

$$P = M_{\tau_1(T_1)} \cdots M_{\tau_m(T_m)} \qquad P(I) = I.$$

Hence,

$$\begin{aligned}\Big(\sum_{i=1}^{n} l_i P_i\Big) \cdot \Big(\sum_{j=1}^{k} \mu_j P_j\Big) I &= \Big(\sum_{i=1}^{n} l_i P_i\Big)\Big(\sum_{j=1}^{k} \mu_j\Big) I = \sum_{i=1}^{n}\sum_{j=1}^{k} l_i \mu_j I \\ &= \Big(\sum_{j=1}^{k} \mu_j P_j\Big)\Big(\sum_{i=1}^{n} l_i P_i\Big) I \quad \text{for } l_i, \mu_j \in \mathbb{C}.\end{aligned}$$

Finally for $A \in \mathcal{V}$ and P_i, as before, finite products of time projections $M_{\tau(T)}$ we have:

$$A \cdot \Big(\sum_{i=1}^{n} l_i P_i\Big) I = \Big(\lim_j A_j\Big) \cdot \Big(\sum_{i=1}^{n} l_i P_i\Big) I$$

where (A_j) is a net of finite linear combinations of finite products of time projections which converges to A in the strong-operator topology

$$= \lim_j \Big(\big(\sum_{i=1}^{n} l_i P_i\big) \cdot A_j I \Big)$$

by above

$$= \Big(\sum_{i=1}^{n} l_i P_i\Big) \cdot \Big(\lim_j A_j I\Big)$$

since $\sum_{i=1}^{n} l_i P_i \in \mathcal{B}(\mathcal{L}^2(\mathcal{A}))$

$$= \Big(\sum_{i=1}^{n} l_i P_i\Big)\Big(\lim_j A_j\Big) I$$

$$= \Big(\sum_{i=1}^{n} l_i P_i\Big) \cdot AI.$$

So we conclude, again by taking strong-operator limits, that for $A, B \in \mathcal{V}$:

$$ABI = BAI.$$

So

$$\langle ABI, I \rangle = \langle BAI, I \rangle$$

which makes I a tracial vector for $\mathcal{V}$.

Suppose now that I was a tracial vector for $\tilde{\mathcal{V}}$. Let $C \in \tilde{\mathcal{V}} \setminus \{0\}$ and $C \geqslant 0$. There exists $a \in \mathcal{A}$ such that $\|a\| = 1$ and $\langle Ca, a \rangle > 0$. This holds since $\mathcal{A}$ is dense in $\mathcal{L}^2(\mathcal{A})$ and the inner-product is continuous on both variables. So now

$$\begin{aligned} \langle \pi(a^*)C\pi(a)I, I \rangle &= \langle Ca, a \rangle > 0 \\ &\Rightarrow \langle C\pi(aa^*)I, I \rangle > 0 \end{aligned}$$

since I is assumed tracial. But,

$$\begin{aligned} \langle CI, I \rangle - \langle C\pi(aa^*)I, I \rangle &= \langle C[I - \pi(aa^*)]I, I \rangle \\ &= \langle C\pi(bb^*)I, I \rangle \\ &\qquad \text{with } b \in \mathcal{A}, \text{ since } \mathcal{A} \text{ is a } C^*\text{-algebra} \\ &\qquad \text{and } I - aa^* \geqslant 0 \\ &= \langle Cb, b \rangle \\ &\qquad \text{as before} \\ &\geqslant 0 \end{aligned}$$

since C is a positive operator. Combining with above we get:

$$\langle CI, I \rangle \geqslant \langle C\pi(aa^*)I, I \rangle > 0$$

and this holds for all positive operators C in $\tilde{\mathcal{V}}$.

Let $C = I - M_t$ for $t \in [0, \infty)$. Then, since $I - M_t > 0$ [otherwise we would be dealing with a trivial filtration] we deduce from above

$$\langle (I - M_t)I, I \rangle > 0.$$

But $(I - M_t)I = 0$, which contradicts our assumption that I is tracial for $\tilde{\mathcal{V}}$, as required. ∎

LEMMA 3.5. *For a net $\{(A_t)_{t\in[0,\infty]}\}_i$ in $\mathcal{U}$ with $i \in I$ a directed set we have the following:*

(i) *if $\{(A_t)_{t\in[0,\infty]}\}_i \to (B_t)_{t\in[0,\infty]}$ in the strong-operator topology, then $(A_t)_i \to B_t$ $\forall t \in [0,\infty]$ in the strong-operator topology in $\pi(\mathcal{A}_t)$;*

(ii) *if $\{(A_t)_{t\in[0,\infty]}\}_i$ are all contained in a bounded set of $\mathcal{U}$, and $(A_t)_i \to B_t$ $\forall t \in [0,\infty]$ in the strong-operator topology in $\pi(\mathcal{A}_t)$, then $\{(A_t)_{t\in[0,\infty]}\}_i \to (B_t)_{t\in[0,\infty]}$ in the strong-operator topology.*

We include the proof of this standard result for those (probabilists?) unfamiliar with the product von Neumann algebra.

Proof. (i) Let $(h_t)_{t\in[0,\infty]}$ be a vector in the Hilbert space $\bigoplus_{t\in\overline{\mathbb{R}}^+} \mathcal{L}^2(\mathcal{A})$, the space on which $\mathcal{U}$ acts. Since

$$\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i \to (B_t)_{t\in\overline{\mathbb{R}}^+}$$

in the strong-operator topology, then

$$\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i (h_t)_{t\in\overline{\mathbb{R}}^+} \to (B_t)_{t\in\overline{\mathbb{R}}^+} (h_t)_{t\in\overline{\mathbb{R}}^+}$$

in $\bigoplus_{t\in\overline{\mathbb{R}}^+} \mathcal{L}^2(\mathcal{A})$ for all such vectors.

Consider now the vector $(h_t)_{t\in\overline{\mathbb{R}}^+}$ with

$$h_t = \begin{cases} 0 & t \neq s \\ h \in \mathcal{L}^2(\mathcal{A}) & t = s \end{cases}$$

for a fixed $s \in [0,\infty]$. Then, by above $(A_s)_i h \to B_s h$ in $\mathcal{L}^2(\mathcal{A})$. Since the choice of h in $\mathcal{L}^2(\mathcal{A})$ and $s \in \overline{\mathbb{R}}^+$ is arbitrary we conclude that $(A_t)_i \to B_t$ in the strong-operator topology.

(ii) Suppose that

$$\sup_i \sup_t \|A_t\| = N < \infty.$$

Choose $(h_t)_{t\in\overline{\mathbb{R}}^+} \in \bigoplus_{t\in\overline{\mathbb{R}}^+} \mathcal{L}^2(\mathcal{A})$. Then

$$\sum_{t\in\overline{\mathbb{R}}^+} \|h_t\|^2 < \infty \Rightarrow \sum_{j=n(\varepsilon,N)}^{\infty} \|h_{t_j}\|^2 < \frac{\varepsilon}{N}$$

for some $n(\varepsilon, N) \in \mathbf{N}$, where $(h_{t_j})_{j\in\mathbf{N}}$ is the countable set of non-zero elements of $(h_t)_{t\in\overline{\mathbf{R}}^+}$. Hence:

$$\begin{aligned}
&\left\|\left\{B_t - (A_t)_i\right\}_{t\in\overline{\mathbf{R}}^+}(h_t)_{t\in\overline{\mathbf{R}}^+}\right\|_2^2 \\
&\quad= \sum_{j=1}^{\infty}\left\|\left[B_{t_j} - (A_{t_j})_i\right]h_{t_j}\right\|_2^2 \\
&\quad\leqslant \sum_{j=1}^{n(\varepsilon,N)-1}\left\|\left[B_{t_j} - (A_{t_j})_i\right]h_{t_j}\right\|_2^2 + \sum_{j=n(\varepsilon,N)}^{\infty}\left\|B_{t_j} - (A_{t_j})_i\right\|^2 \cdot \left\|h_{t_j}\right\|_2^2 \\
&\quad\leqslant \sum_{j=1}^{n(\varepsilon,N)-1}\left\|\left[B_{t_j} - (A_{t_j})_i\right]h_{t_j}\right\|_2^2 + N\frac{\varepsilon}{N}.
\end{aligned}$$

But $(A_t)_i \to B_t \quad \forall t \in \overline{\mathbf{R}}^+$, in the strong-operator topology. Hence for $j \in \{1, 2, \ldots, n(\varepsilon, N) - 1\}$, we can choose $i_0 \in I$ such that:

$$\|(B_{t_j} - (A_{t_j})_{i_0})h_{t_j}\|^2 < \varepsilon.$$

So we get:

$$\|\{B_{t_j} - (A_{t_j})_i\}_{t\in\overline{\mathbf{R}}^+}(h_t)_{t\in\overline{\mathbf{R}}^+}\|^2 < 2\varepsilon, \quad \forall i \geqslant i_0.$$

In other words:

$$\left\{(A_t)_{t\in\overline{\mathbf{R}}^+}\right\}_i \to (B_t)_{t\in\overline{\mathbf{R}}^+}$$

in the strong-operator topology, as required. ∎

DEFINITION 3.6. For each $T = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}$, we define the sub-algebras $\mathcal{U}^T$ of $\mathcal{U}$ as follows:

$$\left\{\pi(a_t)\right\}_{t\in\overline{\mathbf{R}}^+} \in \mathcal{U}^T \iff a_t \in \mathcal{A}_t \quad \forall t \in \overline{\mathbf{R}}^+$$

and

$$a_t = a_{t_j} \text{ for } t_j \leqslant t < t_{j+1} \text{ and } 0 \leqslant j \leqslant n-1.$$

REMARK 3.7. We saw in the previous section that we can view stopping times as operators in the algebra $\mathcal{U}$. We can also, now, regard elements of $\mathcal{U}$ as operator valued functions, i.e.: $\overline{\mathbf{R}}^+ \to \mathcal{B}(\mathcal{L}^2(\mathcal{A}))$. In this context we can view elements of $\mathcal{U}^T$ as "simple functions". These two view points will be useful, the former in the algebraic properties of the "integral" to be defined, the latter in the definition of this "integral".

COROLLARY 3.8. *$\mathcal{U}^T$ is a von Neumann algebra, for each $T \in \mathcal{P}$. Furthermore, there exists an ultra-weakly continuous $*$-homomorphism $T : \mathcal{U} \to \mathcal{U}^T$.*

[There is no ambiguity about the symbol since it will be clear when we are dealing with a map or partition.]

Proof. It is clear by the definition of $\mathcal{U}^T$ that it is a $*$-subalgebra of $\mathcal{U}$. Furthermore, if $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i \to (B_t)_{t\in\overline{\mathbb{R}}^+}$ in the strong-operator topology, with $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i \in \mathcal{U}^T \quad \forall i \in I$ then, by Lemma 3.5 (i), we get $(A_t)_i \to B_t \ \forall t \in \overline{\mathbb{R}}^+$, in the strong-operator topology. So $B_t = B_{t_j}$ for $t_j \leqslant t < t_{j+1}$ and hence $(B_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{U}^T$. This makes $\mathcal{U}^T$ strong-operator closed and hence a von Neumann algebra.

For each partition $T \in \mathcal{P}$ we shall define the map $T : \mathcal{U} \to \mathcal{U}^T$ by:

$$T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\} = \sum_{j=0}^{n-1} A_{t_j}\chi_{[t_j,t_{j+1})}(t) + A_{t_n}\chi_{\{t_n\}}(t)$$

where χ denote the usual characteristic functions. $T(I) = I$ and

$$\begin{aligned} T\big(l(A_t)_{t\in\overline{\mathbb{R}}^+} + (B_t)_{t\in\overline{\mathbb{R}}^+}\big) &= \sum_{j=0}^{n-1}(lA_{t_j} + B_{t_j})\chi_{[t_j,t_{j+1})}(t) + lA_{t_n} + B_{t_n}\chi_{\{t_n\}}(t) \\ &= lT\big((A_t)_{t\in\overline{\mathbb{R}}^+}\big) + T\big((B_t)_{t\in\overline{\mathbb{R}}^+}\big) \end{aligned}$$

with $l \in \mathbb{C}$. Furthermore:

$$\begin{aligned} T\big((A_t)_{t\in\overline{\mathbb{R}}^+} \cdot (B_t)_{t\in\overline{\mathbb{R}}^+}\big) &= \sum_{j=0}^{n-1} A_{t_j} \cdot B_{t_j}\chi_{[t_j,t_{j+1})}(t) + A_{t_n} \cdot B_{t_n}\chi_{\{t_n\}}(t) \\ &= T\big((A_t)_{t\in\overline{\mathbb{R}}^+}\big) \cdot T\big((B_t)_{t\in\overline{\mathbb{R}}^+}\big) \end{aligned}$$

and

$$\begin{aligned} T\big((A_t)_{t\in\overline{\mathbb{R}}^+}\big)^* &= \sum_{j=0}^{n-1} A^*_{t_j}\chi_{[t_j,t_{j+1})}(t) + A^*_{t_n}\chi_{\{t_n\}}(t) \\ &= T\big((A_t)^*_{t\in\overline{\mathbb{R}}^+}\big). \end{aligned}$$

Hence T is a $*$-homomorphism.

If now we have $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i \to (B_t)_{t\in\overline{\mathbb{R}}^+}$ in the strong-operator topology in a norm-bounded subset of $\mathcal{U}$, then, by Lemma 3.5 (i), $(A_t)_i \to B_t \quad \forall t \in \overline{\mathbb{R}}^+$ in the strong operator topology; and by Lemma 3.5 (ii)

$$\{T(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i \to T(B_t)_{t\in\overline{\mathbb{R}}^+}$$

in the strong operator topology, since

$$\sup_{i\in I}\|\{T(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i\| \leqslant \sup_{i\in I}\|\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i\| < \infty.$$

So we deduce that T is normal and by [15], I.4.3 Theorem 2, T becomes ultraweakly continuous as required. ∎

DEFINITION 3.9. For each $T = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}$ we define the map $S^T : \mathcal{U} \to \mathcal{B}(\mathcal{L}^2(\mathcal{A}))$ by:

$$S^T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\} = \sum_{j=1}^{n-1} A_{t_j}(M_{t_{j+1}} - M_{t_j}) + A_o \cdot M_{t_1}.$$

LEMMA 3.10. *S^T is a strong operator continuous $*$-homomorphism.*

Proof.

$$\begin{aligned} S^T(I) &= \sum_{j=1}^{n-1} I(M_{t_{j+1}} - M_{t_j}) + IM_{t_1} \\ &= I. \end{aligned}$$

Furthermore,

$$\begin{aligned} S^T\{l(A_t)_{t\in\overline{\mathbb{R}}^+} + (B_t)_{t\in\overline{\mathbb{R}}^+}\} &= \sum_{j=1}^{n-1}(lA_{t_j} + B_{t_j})(M_{t_{j+1}} - M_{t_j}) + (lA_{t_0} + B_{t_0})M_{t_1} \\ &= lS^T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\} + S^T\{(B_t)_{t\in\overline{\mathbb{R}}^+}\}; \end{aligned}$$

$$\begin{aligned} &S^T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\} \cdot S^T\{(B_t)_{t\in\overline{\mathbb{R}}^+}\} \\ &= \left\{\sum_{j=1}^{n-1} A_{t_j}(M_{t_{j+1}} - M_{t_j}) + A_{t_0} \cdot M_{t_1}\right\} \cdot \left\{\sum_{j'=1}^{n-1} B_{t'_j}(M_{t_{j'+1}} - M_{t'_j}) + B_{t_0}M_{t_1}\right\} \\ &= \sum_{j=1}^{n-1}\sum_{j'=1}^{n-1} A_{t_j}(M_{t_{j+1}} - M_{t_j})(M_{t_{j'+1}} - M_{t'_j})B_{t'_j} + A_{t_0}B_{t_0}M_{t_1} \end{aligned}$$

since $(M_{t_{j'+1}} - M_{t_{j'}})$ lies in the commutant of $\pi(A_{t_{j'}})$ and M_{t_1} commutes with $\pi(A_{t_1})$ and is orthogonal to all projections $(M_{t_{j+1}} - M_{t_j})$ with $j = j'$

$$= \sum_{j=1}^{n-1} A_{t_j}B_{t_j}(M_{t_{j+1}} - M_{t_j}) + A_{t_0}B_{t_0}M_{t_1}$$

since $(M_{t_{j+1}} - M_{t_j})(M_{t_{j'+1}} - M_{t'_j}) = \delta_{jj'}(M_{t_{j+1}} - M_{t_j})$

$$= S^T\{(A_t)_{t\in\overline{\mathbb{R}}^+} \cdot (B_t)_{t\in\overline{\mathbb{R}}^+}\};$$

$$\begin{aligned} S^T\{(a_t)_{t\in\overline{\mathbb{R}}^+}\}^* &= \sum_{j=1}^{n-1}(M_{t_{j+1}} - M_{t_j})\cdot A^*_{t_j} + M_{t_1}A^*_{t_0} \\ &= \sum_{j=1}^{n-1} A^*_{t_j}(M_{t_{j+1}} - M_{t_j}) + A^*_{t_0}M_{t_1} \\ &\qquad \text{since } (M_{t_{j+1}} - M_{t_j}) \text{ lies in the commutant of } \pi(\mathcal{A}_{t_j}) \\ &= S^T\{(A_t)^*_{t\in\overline{\mathbb{R}}^+}\}. \end{aligned}$$

Suppose that $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i \to (B_t)_{t\in\overline{\mathbb{R}}^+}$ in the strong operator topology, in $\mathcal{U}^T$. Then, by Lemma 3.5 (i) $(A_t)_i \to B_t \quad \forall\, t \in [0,\infty]$ in the strong operator topology. Hence, $\forall\, j \in \{0,\dots,n-1\}$

$$(A_{t_j})_i(M_{t_{j+1}} - M_{t_j}) \to B_{t_j}(M_{t_{j+1}} - M_{t_j})$$

in the strong operator topology since this topology is continuous under right multiplication. Then

$$\sum_{j=0}^{n-1}(A_{t_j})_i(M_{t_{j+1}} - M_{t_j}) \to \sum_{j=0}^{n-1} B_{t_j}(M_{t_{j+1}} - M_{t_j})$$

in the strong operator topology. So

$$S^T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_i \to S^T(B_t)_{t\in\overline{\mathbb{R}}^+}$$

in the strong operator topology. ∎

REMARK 3.11. (i) When the algebra $\mathcal{A}_0$ is trivial, that is scalar multiples of the identity, then S^T becomes surjective onto $\mathcal{V}^T$.

(ii) The map S^T is the first step toward the "integral" we wish to define here, and in particular $S^T|\mathcal{U}^T$, the restriction of S^T on $\mathcal{U}^T$, is in fact the integral of the "simple functions" for a partition $T \in \mathcal{P}$. Furthermore, $S^T = S^T|\mathcal{U}^T \cdot T$.

THEOREM 3.12. *Suppose that* $(A_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{U}$ *and that* $\{S^T(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{T\in\mathcal{P}}$ *and* $\{S^T(A^*_t)_{t\in\overline{\mathbb{R}}^+}\}_{T\in\mathcal{P}}$ *are Cauchy nets in the strong operator topology. Then there exists a unique element, which we name* $S(A_t)_{t\in\overline{\mathbb{R}}^+}$, *in* $\mathcal{B}(\mathcal{L}^2(\mathcal{A}))$ *such that* $S^T(A_t)_{t\in\overline{\mathbb{R}}^+} \to S(A_t)_{t\in\overline{\mathbb{R}}^+}$ *as* $T \nearrow$ *in the strong operator topology.*

Furthermore, the set of elements of $\mathcal{U}$ *which have the aforementioned property forms a* C^**-subalgebra of* $\mathcal{U}$ *which we will denote by* $\mathcal{F}$. *Finally, the map* S *above is a well defined* $*$*-homomorphism:* $\mathcal{F} \to \mathcal{B}(\mathcal{L}^2(\mathcal{A}))$.

Proof. We shall first need to use the following property:

$$\forall (A_t)_{t\in\overline{\mathbf{R}}^+} \quad \|S^T(A_t)_{t\in\overline{\mathbf{R}}^+}\| \leqslant \|(A_t)_{t\in\overline{\mathbf{R}}^+}\| = \sup_{t\in\overline{\mathbf{R}}^+} \|A_t\|.$$

This can be shown from first principles, using the orthogonality of the projections involved, but there is a more general result which can be applied in this case since S^T is a $*$-homomorphism between two C^*-algebras. This result states that any $*$-homomorphism between two C^*-algebras is norm reducing ([13], Chapter 8, Proposition 1.11). Turning now to the supposition that $S^T\{(A_t)_{t\in\overline{\mathbf{R}}^+}\}_{T\in\mathcal{P}}$ is a Cauchy net for the strong operator topology in $\mathcal{B}(\mathcal{L}^2(\mathcal{A}))$ we get that

$$\{S^T(A_t)_{t\in\overline{\mathbf{R}}^+} : T \in \mathcal{P}\} \subseteq \{X \in \mathcal{B}(\mathcal{L}^2(\mathcal{A})) : \|X\| \leqslant \|(A_t)_{t\in\overline{\mathbf{R}}^+}\|\}$$

which is a bounded ball in $\mathcal{B}(\mathcal{L}^2(\mathcal{A}))$.

But we know that the unit ball of operators on any Hilbert space is strong-operator complete ([17]). So, we conclude, that there exists an element of $\{X \in \mathcal{B}(\mathcal{L}^2(\mathcal{A})) : \|X\| \leqslant \|(A_t)_{t\in\overline{\mathbf{R}}^+}\|\}$, which we will name $S(A_t)_{t\in\overline{\mathbf{R}}^+}$ which is the strong limit of $S^T\{(A_t)_{t\in\overline{\mathbf{R}}^+}\}_{T\in\mathcal{P}}$. Since the limit in the strong operator topology is unique, we can in fact define the map $S : \mathcal{F} \to B$, with $\mathcal{F} \subseteq \mathcal{U}$, the set of elements in $\mathcal{U}$ with the aforementioned property. Furthermore, $S^T(A_t^*)_{t\in\overline{\mathbf{R}}^+} \to S(A_t^*)_{t\in\overline{\mathbf{R}}^+}$, by definition, as $T \nearrow$ in the strong operator topology, and hence $\mathcal{F}$ is clearly a $*$-closed subset of $\mathcal{U}$.

Suppose now that $(A_t)_{t\in\overline{\mathbf{R}}^+}, (B_t)_{t\in\overline{\mathbf{R}}^+}$ belong in $\mathcal{F}$. Then

$$S^T(A_t)_{t\in\overline{\mathbf{R}}^+} \to S(A_t)_{t\in\overline{\mathbf{R}}^+}$$

and

$$S^T(B_t)_{t\in\overline{\mathbf{R}}^+} \to S(B_t)_{t\in\overline{\mathbf{R}}^+}$$

as $T \nearrow$, in the strong operator topology. Since addition is strong operator continuous, we deduce that

$$S^T\{(A_t)_{t\in\overline{\mathbf{R}}^+} + (B_t)_{t\in\overline{\mathbf{R}}^+}\} \to S(A_t)_{t\in\overline{\mathbf{R}}^+} + S(B_t)_{t\in\overline{\mathbf{R}}^+}.$$

This shows that $\{S^T[(A_t)_{t\in\overline{\mathbf{R}}^+} + (B_t)_{t\in\overline{\mathbf{R}}^+}]\}_{T\in\mathcal{P}}$ is strong operator Cauchy and similarly for $\{S^T[(A_t^*)_{t\in\overline{\mathbf{R}}^+} + (B_t^*)_{t\in\overline{\mathbf{R}}^+}]\}_{T\in\mathcal{P}}$. Hence $(A_t)_{t\in\overline{\mathbf{R}}^+} + (B_t)_{t\in\overline{\mathbf{R}}^+} \in \mathcal{F}$ and furthermore

$$S\{(A_t)_{t\in\overline{\mathbf{R}}^+} + (B_t)_{t\in\overline{\mathbf{R}}^+}\} = S(A_t)_{t\in\overline{\mathbf{R}}^+} + S(B_t)_{t\in\overline{\mathbf{R}}^+}.$$

Furthermore, we shall now use the fact that multiplication is strong operator continuous on $\mathcal{B}_1(\mathcal{L}^2(\mathcal{A})) \times \mathcal{B}(\mathcal{L}^2(\mathcal{A}))$ where $\mathcal{B}_1(\mathcal{L}^2(\mathcal{A}))$ signifies the unit ball of $\mathcal{B}(\mathcal{L}^2(\mathcal{A}))$, ([17]).

Since

$$\{S^T(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{T\in\mathcal{P}} \subseteq \|(A_t)_{t\in\overline{\mathbb{R}}^+}\| \cdot \mathcal{B}_1(\mathcal{L}^2(\mathcal{A}))$$

we deduce that:

$$S^T(A_t)_{t\in\overline{\mathbb{R}}^+} \cdot S^T(B_t)_{t\in\overline{\mathbb{R}}^+} \to S(A_t)_{t\in\overline{\mathbb{R}}^+} \cdot S(B_t)_{t\in\overline{\mathbb{R}}^+}$$

and similarly for $S^T(A_t^*)_{t\in\overline{\mathbb{R}}^+} \cdot S^T(B_t^*)_{t\in\overline{\mathbb{R}}^+}$.

Since S^T is a $*$-homomorphism $\forall T \in \mathcal{P}$ we conclude that $(A_t)_{t\in\overline{\mathbb{R}}^+} \cdot (B_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{F}$ and

$$S\{(A_t)_{t\in\overline{\mathbb{R}}^+} \cdot (B_t)_{t\in\overline{\mathbb{R}}^+}\} = S(A_t)_{t\in\overline{\mathbb{R}}^+} \cdot S(B_t)_{t\in\overline{\mathbb{R}}^+}.$$

Finally, $S^T(A_t)_{t\in\overline{\mathbb{R}}^+} \to S(A_t)_{t\in\overline{\mathbb{R}}^+}$ as $T \nearrow$ in the strong operator topology, so $S^T(A_t)_{t\in\overline{\mathbb{R}}^+} \to S(A_t)_{t\in\overline{\mathbb{R}}^+}$ in the weak operator topology. Thus $S^T(A_t^*)_{t\in\overline{\mathbb{R}}^+} \to \{S(A_t)_{t\in\overline{\mathbb{R}}^+}\}^*$ in the weak operator topology since $*$-operator is continuous in the weak operator topology and S^T is a $*$-homomorphism. But $S^T(A_t^*)_{t\in\overline{\mathbb{R}}^+} \to S(A_t^*)_{t\in\overline{\mathbb{R}}^+}$ in the strong operator topology, by definition, and therefore $S^T(A_t^*)_{t\in\overline{\mathbb{R}}^+} \to S(A_t^*)_{t\in\overline{\mathbb{R}}^+}$ in the weak operator topology. But since the weak operator topology is Hausdorff we combine to get:

$$\{S(A_t)_{t\in\overline{\mathbb{R}}^+}\}^* = S(A_t^*)_{t\in\overline{\mathbb{R}}^+}.$$

So S is a $*$-homomorphism, and $\mathcal{F}$ is a $*$-subalgebra of $\mathcal{U}$. It only remains to be shown that $\mathcal{F}$ is complete with respect to the $\|\cdot\|$.

Indeed, suppose that $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$ which is $\|\cdot\|$-Cauchy. Since $\mathcal{U}$ is a Banach space, $\exists (B_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{U}$ such that $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_n \to (B_t)_{t\in\overline{\mathbb{R}}^+}$ in $\|\cdot\|$.

Fix $\varepsilon > 0, h \in \mathcal{L}^2(\mathcal{A})$. Then $\exists\, n_o \in \mathbb{N}$ such that

$$n \geqslant n_o \Rightarrow \|\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_n - (B_t)_{t\in\overline{\mathbb{R}}^+}\| < \varepsilon.$$

Since $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o} \in \mathcal{F}, \left\{S^T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o}\right\}_{T\in\mathcal{P}}$ is strong operator Cauchy in $\mathcal{B}(\mathcal{L}^2(\mathcal{A}))$ and so $\exists\, T_o \in \mathcal{P}$ such that

$$T, T' \geqslant T_o \Rightarrow \left\|\left\{S^T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o} - S^{T'}\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o}\right\}h\right\|_2 < \varepsilon.$$

So we have for $T, T' \geqslant T_o$

$$\begin{aligned}
&\left\|\{S^T(B_t)_{t\in\overline{\mathbb{R}}^+} - S^{T'}(B_t)_{t\in\overline{\mathbb{R}}^+}\}h\right\|_2 \\
&\quad \leqslant \left\|\{S^T(B_t)_{t\in\overline{\mathbb{R}}^+} - S^T[(A_t)_{t\in\overline{\mathbb{R}}^+}]_{n_o}\}h\right\|_2 \\
&\qquad + \left\|\{S^T\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o} - S^{T'}\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o}\}h\right\| \\
&\qquad + \left\|\{S^{T'}\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o} - S^{T'}(B_t)_{t\in\overline{\mathbb{R}}^+}\}h\right\| \\
&\quad \leqslant \left\|(B_t)_{t\in\overline{\mathbb{R}}^+} - \{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o}\right\| \cdot \|h\|_2 + \varepsilon \\
&\qquad + \left\|(B_t)_{t\in\overline{\mathbb{R}}^+} - \{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{n_o}\right\| \cdot \|h\|_2 \\
&\quad = (2\|h\|_2 + 1) \cdot \varepsilon.
\end{aligned}$$

Hence $\{S^T(B_t)_{t\in\overline{\mathbb{R}}^+}\}_{T\in\mathcal{P}}$ is strong operator Cauchy.

But $\{(A_t)_{t\in\overline{\mathbb{R}}^+}\}_n \to (B_t)_{t\in\overline{\mathbb{R}}^+}$ in $\|\cdot\|$ implies that $\{(A_t^*)_{t\in\overline{\mathbb{R}}^+}\}_n \to (B_t^*)_{t\in\overline{\mathbb{R}}^+}$ in $\|\cdot\|$.

Using a similar argument we deduce that $\{S^T(B_t^*)_{t\in\overline{\mathbb{R}}^+}\}_{T\in\mathcal{P}}$ is also Cauchy in the strong operator topology and hence:

$$(B_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{F}$$

as required. ∎

COROLLARY 3.13.

$$\|S\| = 1.$$

Proof. Indeed, S is a $*$-homomorphism between two C^*-algebras and using the results in [13], Chapter 8, Proposition 1 and $S(I) = I$ we deduce that $\|S\| = 1$. ∎

REMARK 3.14. For $T_o \in \mathcal{P}$ and an element $(A_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{U}^T$ we observe that:

$$T \geqslant T_o \Rightarrow S^T(A_t)_{t\in\overline{\mathbb{R}}^+} = S^{T_o}(A_t)_{t\in\overline{\mathbb{R}}^+}.$$

As a result $\{S^T(A_t)_{t\in\overline{\mathbb{R}}^+}\}_{T\in\mathcal{P}}$ is a Cauchy net in the strong operator topology and so is $\{S^T(A_t^*)_{t\in\overline{\mathbb{R}}^+}\}_{T\in\mathcal{P}}$. Hence $(A_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{F}$ and

$$\begin{aligned}
S(A_t)_{t\in\overline{\mathbb{R}}^+} &= \lim_{T\nearrow} S^T(A_t)_{t\in\overline{\mathbb{R}}^+} \\
&= S^{T_o}(A_t)_{t\in\overline{\mathbb{R}}^+}.
\end{aligned}$$

By viewing S as an integral we see that all "simple functions" are integrable. Furthermore for any "integrable" $(A_t)_{t\in\overline{\mathbb{R}}^+} \in \mathcal{F}$ we have:

$$\begin{aligned} S^T(A_t)_{t\in\overline{\mathbb{R}}^+} &= S^T \cdot T(A_t)_{t\in\overline{\mathbb{R}}^+} \\ &= S \cdot T(A_t)_{t\in\overline{\mathbb{R}}^+}. \end{aligned}$$

This shows that the integral of an integrable process is in fact a limit (in the strong operator topology) of integrals of simple functions.

THEOREM 3.15. (i) *Any process $(A_s)_{s\in\overline{\mathbb{R}}^+} \in \mathcal{U}$ with A_s hermitian $\forall s \in \mathbb{R}^+$ and $A_s \geqslant A_t$ for $s \geqslant t$ belongs in $\mathcal{F}$. Similarly for "decreasing" processes.*

(ii) *If $(A_s)_{s\in\overline{\mathbb{R}}^+} \in \mathcal{U}$ and $s \mapsto A_s$ is norm-continuous on $\overline{\mathbb{R}}^+$ then $(A_s)_{s\in\overline{\mathbb{R}}^+}$ is integrable. If $s \mapsto A_s$ is norm-continuous on $\mathbb{R}$, then $(A_s)_{s\in\overline{\mathbb{R}}^+}$ becomes locally integrable, i.e. $\left(A_s \cdot \chi_{[0,R]}(s)\right)_{s\in\overline{\mathbb{R}}^+} \in \mathcal{F}$, $\forall R \in \mathbb{R}$.*

Proof. (i) Consider the partitions $T, T' \in \mathcal{P}$ with $T \geqslant T'$. Assume first that

$$T' = \{t_o, t_1, \ldots, t_n\}$$

and

$$T = \{t_o, t_1, \ldots, t_{n-2}, s, t_{n-1}, t_n\}.$$

Then

$$\begin{aligned} S^T(A_s)_{s\in\overline{\mathbb{R}}^+} = A_o \cdot M_{t_1} + A_{t_1}(M_{t_2} - M_{t_1}) + \cdots + A_{t_{n-2}}(M_s - M_{t_{n-2}}) + \\ + A_s(M_{t_{n-1}} - M_s) + A_{t_{n-1}}(I - M_{t_{n-1}}) \end{aligned}$$

and

$$\begin{aligned} S^{T'}(A_s)_{s\in\overline{\mathbb{R}}^+} = A_o M_{t_1} + A_{t_1}(M_{t_2} - M_{t_1}) + \cdots + A_{t_{n-2}}(M_s - M_{t_{n-2}}) + \\ + A_{t_{n-2}}(M_{t_{n-1}} - M_s) + A_{t_{n-1}}(I - M_{t_{n-1}}). \end{aligned}$$

Hence

$$\left\{S^T(A_s)_{s\in\overline{\mathbb{R}}^+} - S^{T'}(A_s)_{s\in\overline{\mathbb{R}}^+}\right\} = (A_s - A_{t_{n-2}})(M_{t_{n-1}} - M_s).$$

But $A_s \geqslant A_{t_{n-2}}$ and $(M_{t_{n-1}} - M_s)$ is an orthogonal projection which commutes with both A_s and $A_{t_{n-2}}$. Hence

$$S^T(A_s)_{s\in\overline{\mathbb{R}}^+} \geqslant S^{T'}(A_s)_{s\in\overline{\mathbb{R}}^+}.$$

In the more general case where T contains m more points $s_1, \ldots, s_m$ than T', we can define recursively partitions $T_o = T', T_1, \ldots, T_m = T$, each of which contain one more point s_i than the previous. Using the above result we have:

$$S^{T_{i+1}}(A_s)_{s\in\overline{\mathbf{R}}^+} - S^{T_i}(A_s)_{s\in\overline{\mathbf{R}}^+} \geqslant 0.$$

Hence

$$(S^T - S^{T'})(A_s)_{s\in\overline{\mathbf{R}}^+} = \sum_{i=0}^{m-1}(S^{T_{i+1}}(A_s)_{s\in\overline{\mathbf{R}}^+} - S^{T_i}(A_s)_{s\in\overline{\mathbf{R}}^+}) \geqslant 0.$$

So $S^T(A_s)_{s\in\overline{\mathbf{R}}^+}$ is an increasing net of hermitian operators, and since

$$\|S^T(A_s)_{s\in\overline{\mathbf{R}}^+}\| \leqslant \|(A_s)_{s\in\overline{\mathbf{R}}^+}\|$$

we have

$$S^T(A_s)_{s\in\overline{\mathbf{R}}^+} \leqslant \|(A_s)_{s\in\overline{\mathbf{R}}^+}\| \cdot I.$$

Consequently, $\{S^T(A_s)_{s\in\overline{\mathbf{R}}^+}\}_{T\in\mathcal{P}}$ is strongly Cauchy and so $(A_s)_{s\in\overline{\mathbf{R}}^+} \in \mathcal{F}$ and $S^T(A_s)_{s\in\overline{\mathbf{R}}^+} \to S(A_s)_{s\in\overline{\mathbf{R}}^+}$.

By considering $(-A_s)_{s\in\overline{\mathbf{R}}^+}$ we can deduce the same for decreasing processes.

(ii) $\forall\varepsilon > 0 \quad \exists R \in \mathbf{R}$ such that $\|A_s - A_{s'}\| < \varepsilon$ for $s, s' \geqslant R$, since $s \mapsto A_s$ is assumed norm-continuous on $\overline{\mathbf{R}}^+$. Furthermore, since $[0, R]$ is compact the map $s \mapsto A_s$ on $[0, R]$ becomes uniformly continuous. Hence, $\exists\,\delta > 0$ such that $|s - s'| < \delta \Rightarrow \|A_s - A_{s'}\| < \varepsilon$, for $s, s' \in [0, R]$.

Fix a partition $T_0 = \{s_0, s_1, \ldots, s_n\}$ with $s_{n-1} = R$ and $s_{i+1} - s_i < \delta$ for $0 \leqslant i \leqslant n-2$. If $T = \{t_0, t_1, \ldots, t_m\}$ and $T' = \{t'_0, t'_1, \ldots, t'_{m'}\}$ are two partitions finer than T_0, then there exists a partition $T'' \geqslant T, T'' \geqslant T'$. We then have:

$$\begin{aligned} S^T(A_s)_{s\in\overline{\mathbf{R}}^+} &= S^T \cdot T(A_s)_{s\in\overline{\mathbf{R}}^+} \\ &= S^{T''} \cdot T(A_s)_{s\in\overline{\mathbf{R}}^+} \end{aligned}$$

and

$$\begin{aligned} S^{T'}(A_s)_{s\in\overline{\mathbf{R}}^+} &= S^{T'} \cdot T'(A_s)_{s\in\overline{\mathbf{R}}^+} \\ &= S^{T''} \cdot T'(A_s)_{s\in\overline{\mathbf{R}}^+} \end{aligned}$$

as in Remark 3.14. Hence:

$$\begin{aligned} \|S^T(A_s)_{s\in\overline{\mathbf{R}}^+} - S^{T'}(A_s)_{s\in\overline{\mathbf{R}}^+}\| &= \|S^{T''}(T(A_s)_{s\in\overline{\mathbf{R}}^+} - T'(A_s)_{s\in\overline{\mathbf{R}}^+})\| \\ &\leqslant \|T(A_s)_{s\in\overline{\mathbf{R}}^+} - T'(A_s)_{s\in\overline{\mathbf{R}}^+}\| \end{aligned}$$

as $S^{T''}$ is a $*$-homomorphism between C^*-algebras.

Now fix $s \in \mathbb{R}^+$. There exists an i with $0 \leqslant i \leqslant n-1$ such that $s_i \leqslant s < s_{i+1}$. Then $T(A_s) = A_{t_j}$ for some j with $s_i \leqslant t_j < s_{i+1}$, since $T \geqslant T_0$ and $T'(A_s) = A_{t'_k}$ for some k with $s_i \leqslant t'_k < s_{i+1}$, since $T' \geqslant T_0$. So

$$T(A_s) - T'(A_s) = A_{t_j} - A_{t'_k}$$

with $s_i \leqslant t_j < s_{i+1}$ and $s_i \leqslant t'_k < s_{i+1}$. By above, $\|A_{t_j} - A_{t'_k}\| < \varepsilon$, since T_0 is chosen so that $s_{i+1} - s_i < \delta$ for $0 \leqslant i \leqslant n-2$ and $s_{n-1} = R$.

Finally for $s = \infty$

$$T(A_\infty) = A_\infty = T'A_\infty.$$

Hence:

$$\|T(A_s)_{s\in\overline{\mathbb{R}}^+} - T'(A_s)_{s\in\overline{\mathbb{R}}^+}\| = \sup_{s\in\overline{\mathbb{R}}^+} \|T(A_s) - T'(A_s)\| \leqslant \varepsilon.$$

This makes $S^T(A_s)_{s\in\overline{\mathbb{R}}^+}$ norm-Cauchy and consequently strong operator Cauchy, as required.

If now $s \mapsto A_s$ is continuous on $\mathbb{R}^+$ then to apply the same arguments we have to restrict our attention to compact sets, where the uniform continuity condition holds. We thus say that in this case $(A_s)_{s\in\overline{\mathbb{R}}^+}$ is locally integrable. ∎

COROLLARY 3.16. *Any stopping time τ is integrable. Furthermore*

$$M_\tau = I - S(\tau).$$

Proof. By definition, for a stopping time τ, we have $\tau \in \mathcal{U}$ and $\tau(0) = 0$, $\tau(\infty) = I$, $\tau(s)$ is an orthogonal projection in $\pi(\mathcal{A}_s)$ and τ is increasing. By Theorem 3.15 (i) we deduce that $\tau \in \mathcal{F}$. Furthermore, for $T = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}$ and $\tau(S) = \pi(Q_s) \quad \forall S \in \mathbb{R}^+$,

$$\begin{aligned} M_{\tau(T)} &= \sum_{i=0}^{n-1} \pi(Q_{t_{i+1}} - Q_{t_i}) \cdot M_{t_{i+1}} \\ &= I - \sum_{i=0}^{n-1} \pi(Q_{t_i})(M_{t_{i+1}} - M_{t_i}) \\ &\qquad \text{as seen also in Lemma 3.3} \\ &= I - S^T(\tau). \end{aligned}$$

But $M_{\tau(T)} \to M_\tau$ as $T \nearrow$ in the strong operator topology, as seen in Definition 2.3, and $S^T(\tau) \to S(\tau)$ as $T \nearrow$ in the strong operator topology by Theorem 3.15. So we conclude:

$$M_\tau = I - S(\tau). \quad ∎$$

REMARK 3.17. (i) For *any* projection valued process τ in $\mathcal{F}$ we can define $M_\tau \equiv I - S(\tau)$, which, since τ is a projection in $\mathcal{F}$ and S a homomorphism, means that $S(\tau)$ is a projection and hence so is M_τ. Some of the properties of this correspondence are investigated below.

(ii) For integrable projection valued processes τ, σ we can, following Definition 2.5 of [10], define

$$(\sigma \vee \tau)_t = (\sigma)_t \wedge (\tau)_t$$

and

$$(\sigma \wedge \tau)_t = (\sigma)_t \vee (\tau)_t$$

which extends the order relation for stopping times to integrable projection valued processes in $\mathcal{F}$. Note that if we consider σ, τ as projections in $\mathcal{F}$ then there exists an "operator" definition of $\sigma \wedge \tau$ and $\sigma \vee \tau$ which are the opposite of the ones mentioned above. We shall use the first of the two (as indeed in the case of $\sigma < \tau$), but we will be required to refer to the second. In such cases we shall use the expression "as operators", so that

$$\sigma \leqslant \tau \iff \tau \leqslant \sigma \quad \text{as operators.}$$

THEOREM 3.18. *For two projection valued processes τ, σ in $\mathcal{F}$ and a partition $T \in \mathcal{P}$ the following hold:*

(i) $\tau \leqslant \sigma \Rightarrow M_\tau \leqslant M_\sigma$.

(ii) $M_{\tau(T)} \vee M_{\sigma(T)} = M_{\tau \vee \sigma(T)}$ *and* $M_{\tau(T)} \wedge M_{\sigma(T)} = M_{\tau \wedge \sigma(T)}$.

(iii) *If τ and σ commute then* $M_{\tau \wedge \sigma} = M_\tau \wedge M_\sigma = M_\tau \cdot M_\sigma$ *and* $M_{\tau \vee \sigma} = M_\tau \vee M_\sigma$.

(iv) *For two stopping times τ, σ we also have* $M_{\tau \wedge \sigma} = M_\tau \wedge M_\sigma$ *and for $\mathcal{V}$ finite* $M_{\tau \vee \sigma} = M_\tau \vee M_\sigma$.

Proof. The results (i), (ii) and (iv) have been proved in [10] and [6]. We shall simplify these proofs.

(i)

$$\begin{aligned}\tau \leqslant \sigma \Rightarrow \tau \geqslant \sigma \quad &\text{as operators}\\ \Rightarrow S(\tau) \geqslant S(\sigma)&\\ &\text{since } S \text{ is a } *\text{-homomorphism}\\ \Rightarrow I - M_\tau \geqslant I - M_\sigma&\\ \Rightarrow M_\tau \leqslant M_\sigma.&\end{aligned}$$

(ii)

$$\begin{aligned}
M_{\tau\vee\sigma(T)} &= I - S^T(\tau \vee \sigma) \\
&= I - S^T(\tau \wedge \sigma) \quad \text{as operators} \\
&= I - S^T\{\lim_{k\to\infty}(\tau \cdot \sigma)^k\} \\
&= I - \lim_{k\to\infty} S^T\{(\tau \cdot \sigma)^k\} \\
&\qquad \text{since } S^T \text{ is strong operator continuous by Lemma 3.10} \\
&= I - \lim_{k\to\infty}\{S^T(\tau) \cdot S^T(\sigma)\}^k \\
&\qquad S^T \text{ is a } *\text{-homomorphism} \\
&= I - S^T(\tau) \wedge S^T(\sigma) \\
&= I - \{I - M_{\tau(T)}\} \wedge \{I - M_{\sigma(T)}\} \\
&= M_{\tau(T)} \vee M_{\sigma(T)}.
\end{aligned}$$

Furthermore:

$$\begin{aligned}
M_{\tau\wedge\sigma(T)} &= I - S^T(\tau \wedge \sigma) \\
&= I - S^T(\tau \vee \sigma) \quad \text{as operators} \\
&= I - S^T\{I - (I - \tau) \wedge (I - \sigma)\} \quad \text{as operators} \\
&= I - S^T(I) + S^T(I - \tau) \wedge S^T(I - \sigma) \quad \text{as above} \\
&= [I - S^T(\tau)] \wedge [I - S^T(\sigma)] \\
&= M_{\tau(T)} \wedge M_{\sigma(T)}.
\end{aligned}$$

(iii) Suppose now that τ, σ commute. Then:

$$\begin{aligned}
\tau \vee \sigma &= \tau \wedge \sigma \quad \text{as operators} \\
&= \lim_{k\to\infty}(\tau \cdot \sigma)^k \\
&\qquad \text{strong operator limit} \\
&= \lim_{k\to\infty} \tau \cdot \sigma \\
&\qquad \text{since } \tau, \sigma \text{ commute and are projections in } \mathcal{F} \\
&= \tau \cdot \sigma \quad \text{as operators}
\end{aligned}$$

and $\tau \vee \sigma \in \mathcal{F}$, since $\mathcal{F}$ is a C^*-algebra.

Similarly:

$$\begin{aligned}
\tau \wedge \sigma &= \tau \vee \sigma \quad \text{as operators} \\
&= I - (I - \tau) \wedge (I - \sigma) \\
&= I - (I - \tau) \cdot (I - \sigma) \quad \text{as above} \\
&= \tau + \sigma - \tau \cdot \sigma
\end{aligned}$$

and $\tau \wedge \sigma \in \mathcal{F}$, since $\mathcal{F}$ is a C^*-algebra. Hence:

$$\begin{aligned} S(\tau \vee \sigma) &= S(\tau \wedge \sigma) \quad \text{as operators} \\ &= S(\tau \cdot \sigma) \\ &= S(\tau) \cdot S(\sigma) \\ &\qquad S \text{ is a homomorphism} \\ &= S(\tau) \wedge S(\sigma) \end{aligned}$$

since $S(\tau)$ and $S(\sigma)$ are commuting projections; and

$$\begin{aligned} S(\tau \wedge \sigma) &= S(\tau \vee \sigma) \quad \text{as operators} \\ &= S(\tau + \sigma - \tau \cdot \sigma) \quad \text{as above} \\ &= S(\tau) + S(\tau) - S(\tau) \cdot S(\sigma) \\ &= S(\tau) \vee S(\sigma) \end{aligned}$$

as before. Consequently:

$$\begin{aligned} M_{\tau \vee \sigma} &= I - S(\tau \vee \sigma) = I - S(\tau) \wedge S(\sigma) \\ &= I - (I - M_\tau) \wedge (I - M_\sigma) = M_\tau \vee M_\sigma \end{aligned}$$

and

$$\begin{aligned} M_{\tau \wedge \sigma} &= I - S(\tau \wedge \sigma) = I - S(\tau) \vee S(\sigma) \\ &= I - [I - \{I - S(\tau)\} \wedge \{I - S(\sigma)\}] \\ &= M_\tau \wedge M_\sigma = M_\tau \cdot M_\sigma \end{aligned}$$

since M_τ and M_σ are projections that commute as $S(\tau)$ and $S(\sigma)$ do.

(iv) Suppose now that τ and σ are stopping times. For $t, s \in \mathbf{R}^+$ with $t \leqslant s$ we have:

$$\begin{aligned} (\tau \wedge \sigma)_t &= (\tau)_t \vee (\sigma)_t \\ &\leqslant (\tau)_s \vee (\sigma)_s \\ &\qquad \text{as } \tau, \sigma \text{ are increasing} \\ &= (\tau \wedge \sigma)_s. \end{aligned}$$

Hence $\tau \wedge \sigma$ is also a stopping time, and similarly for $\tau \vee \sigma$. Furthermore, $\tau \wedge \sigma = \tau \vee \sigma$ as operators and since $\tau \vee \sigma \geqslant \tau$ as operators and $\tau \vee \sigma \geqslant \sigma$ as operators we deduce

$$S(\tau \wedge \sigma) \geqslant S(\tau) \vee S(\sigma) \tag{3.1}$$

since S is a $*$-homomorphism and similarly

$$S(\tau \vee \sigma) \leqslant S(\tau) \wedge S(\sigma).$$

By (ii)

$$S^T(\tau \wedge \sigma) = S^T(\tau) \vee S^T(\sigma)$$

so

$$S^T(\tau \wedge \sigma) \leqslant S(\tau) \vee S(\sigma)$$

since $S(\tau) = \sup_{T \in \mathcal{P}} S^T(\tau)$ and $S(\sigma) = \sup_{T \in \mathcal{P}} S^T(\tau)$ or

$$S(\tau \wedge \sigma) = \sup_{T \in \mathcal{P}} S^T(\tau \wedge \sigma) \leqslant S(\tau) \vee S(\sigma). \tag{3.2}$$

Combining (3.1) and (3.2) we deduce:

$$S(\tau \wedge \sigma) = S(\tau) \vee S(\sigma).$$

Hence

$$I - M_{\tau \wedge \sigma} = (I - M_\tau) \vee (I - M_\sigma)$$

so

$$M_{\tau \wedge \sigma} = M_\tau \wedge M_\sigma$$

as required.

Finally suppose that $\mathcal{V}$ is a finite von Neumann algebra and note that for a stopping time τ, $S(\tau) \in \mathcal{V}$. Then:

$$M_\tau \vee M_\sigma = \inf_{T \in \mathcal{P}} \inf_{T' \in \mathcal{P}} \left\{ M_{\tau(T)} \vee M_{\sigma(T')} \right\}$$

since $\mathcal{V}$ is finite which makes the lattice of projection a continuous geometry ([20], Chapter 7). But

$$M_{\tau(T)} \vee M_{\sigma(T')} = M_{\tau(T) \vee \sigma(T')}$$

by applying (ii) for the time $\tau(T)$ and $\sigma(T')$ on the partition $T \cup T'$.

Furthermore since τ, σ are increasing processes $\forall T', T \in \mathcal{P}$:

$$\begin{aligned} & \sigma(T') \geqslant \sigma \\ \Rightarrow \quad & \tau(T) \vee \sigma(T') \geqslant \tau(T) \vee \sigma \\ \Rightarrow \quad & M_{\tau(T) \vee \sigma(T')} \geqslant M_{\tau(T) \vee \sigma} \\ & \qquad \text{by (i)} \\ \Rightarrow \quad & \inf_{T' \in \mathcal{P}} M_{\tau(T) \vee \sigma(T')} \geqslant M_{\tau(T) \vee \sigma}. \end{aligned}$$

Similarly:

$$\inf_{T \in \mathcal{P}} M_{\tau(T) \vee \sigma} \geqslant M_{\tau \vee \sigma}$$

or

$$\begin{aligned} M_\tau \vee M_\sigma &= \inf_{T\in\mathcal{P}} \inf_{T'\in\mathcal{P}} \left\{ M_{\tau(T)} \vee M_{\sigma(T')} \right\} \\ &= \inf_{T\in\mathcal{P}} \inf_{T'\in\mathcal{P}} \left\{ M_{\tau(T)\vee\sigma(T')} \right\} \\ &\geqslant M_{\tau\vee\sigma}. \end{aligned}$$

But we know from (3.1) that:

$$\begin{aligned} & S(\tau\vee\sigma) \leqslant S(\tau)\wedge S(\sigma) \\ \Rightarrow\quad & I - M_{\tau\vee\sigma} \leqslant (I - M_\tau)\wedge(I - M_\sigma) \\ \Rightarrow\quad & M_{\tau\vee\sigma} \geqslant M_\tau \vee M_\sigma. \end{aligned}$$

Combining, we get

$$M_\tau \vee M_\sigma = M_{\tau\vee\sigma}$$

as required. ∎

REMARKS 3.19. (i) We define an $\mathcal{L}^2(\mathcal{A})$ adapted process $(\xi_t)_{t\in\overline{\mathbf{R}}^+}$ to be a martingale if

$$M_s\xi_t = \xi_s \quad \forall s \leqslant t \in \mathbf{R}^+.$$

Note that an $\mathcal{L}^2(\mathcal{A})$ process with the martingale property is always adapted. From Proposition 1.1 in [10] we know that if $(\xi_t)_{t\in\overline{\mathbf{R}}^+}$ is a bounded martingale then $\exists \xi \in \mathcal{L}^2(\mathcal{A})$ such that

$$\xi_t = M_t\xi \quad \forall t \in \mathbf{R}^+.$$

We now observe that, in such a case, for a stopping time τ

$$\xi_\tau \equiv M_\tau(\xi) = \lim_{T\in\mathcal{P}} M_{\tau(T)}(\xi)$$

in $\mathcal{L}^2(\mathcal{A})$, since $M_{\tau(T)} \to M_\tau$ in the strong operator topology. But

$$M_{\tau(T)}(\xi) = \sum_{i=0}^{n-1} \{\tau(t+1) - \tau(t)\} \cdot M_{t+1}(\xi) = \sum_{i=0}^{n-1} (\tau(t+1) - \tau(t))\xi_{t+1}. \tag{3.3}$$

This is in fact the extension of the stopping of a bounded martingale, to the non-commutative set up. Furthermore for a bounded martingale $(M_t\xi)_{t\in\overline{\mathbf{R}}^+}, \xi \in \mathcal{L}^2(\mathcal{A})$, and an integrable process $(A_t)_{t\in\overline{\mathbf{R}}^+}$ we have that $\big(S(A_t)_{t\in\overline{\mathbf{R}}^+}\big)\xi$ can be considered as a stochastic integral of $(A_t)_{t\in\overline{\mathbf{R}}^+}$ with respect to the martingale $(M_t\xi)_{t\in\overline{\mathbf{R}}^+}$.

(ii) The deterministic stopping time $\tilde{t}$ is defined as,

$$\tilde{t}(s) = \begin{cases} 0 & s \leqslant t \\ I & s > t. \end{cases}$$

It is clear that $\tilde{t}$ commutes with any stopping time and in fact $\tilde{t}$ belongs in the commutant of $\mathcal{U}$. Furthermore:

$$M_{\tilde{t}(T)} = M_t$$

if $T \in \mathcal{P}$ and $t \in T$. Consequently:

$$M_{\tilde{t}} = M_t.$$

COROLLARY 3.20. (i) *For a bounded $\mathcal{L}^2(\mathcal{A})$ martingale $(\xi_t)_{t\in\overline{\mathbb{R}}^+}$, and projections τ, σ in $\mathcal{F}$ with $\tau \geqslant \sigma$ as operators, we have:*

$$M_\tau(\xi_\sigma) = \xi_\tau.$$

(ii) *For τ a projection in $\mathcal{F}$ and $(\xi_t)_{t\in\overline{\mathbb{R}}^+}$ a bounded $\mathcal{L}^2(\mathcal{A})$ martingale, $\xi_{\tilde{t}\wedge\tau}$ is a bounded $\mathcal{L}^2(\mathcal{A})$ martingale.*

Proof. (i) From Theorem 3.18 (i) we deduce that $M_\tau \leqslant M_\sigma$. But both are projections. Hence

$$M_\tau \cdot M_\sigma = M_\sigma \cdot M_\tau = M_\tau.$$

So

$$M_\tau(\xi_\sigma) = M_\tau \cdot M_\sigma(\xi) = M_\tau(\xi) = \xi_\tau.$$

(ii) $\tilde{t}$ lies in the commutant of $\mathcal{F}$ and so commutes with any projection valued process τ in $\mathcal{F}$. By Theorem 3.18 (iii)

$$M_{\tilde{t}\wedge\tau} = M_{\tilde{t}} \wedge M_\tau = M_{\tilde{t}} \cdot M_\tau = M_t \cdot M_\tau.$$

Hence:

$$\xi_{\tilde{t}\wedge\tau} = M_{\tilde{t}\wedge\tau}(\xi) = M_t \cdot M_\tau(\xi)$$

i.e. is a martingale. Also,

$$\left\|\xi_{\tilde{t}\wedge\tau}\right\|_2 \leqslant \left\|M_\tau(\xi)\right\|_2 \quad \forall t \in \mathbb{R}^+. \quad \blacksquare$$

REMARK 3.21. We now observe that we can generalise the notion of stopping time. We can consider any projection valued process in $\mathcal{F}$ to be a "stopping time". We can then define the time projection M_σ, for such a process σ, as $M_\sigma = I - S(\sigma)$, which agrees with the definition for stopping time given previously. As we see in Corollary 3.20 these results hold for such projection valued processes. In particular, in (i) we see a version of the Doobs Optimal Stopping Theorem. In (ii) we see a basic property of probabilistic stopping times, that the stopped process obtained from an $\mathcal{L}^2$ martingale is itself, that is the $\mathcal{L}^2$ bounded martingales are stable under stopping.

LEMMA 3.22. *If a process* $(A_t)_{t\in\overline{\mathbf{R}}^+}$ *in* $\mathcal{U}$ *is such that* $\forall\, R \in \mathbf{R}$, $\{t \in [0, R] : A_t \neq 0\}$ *is finite, then* $(A_t)_{t\in\overline{\mathbf{R}}^+} \in \mathcal{F}$ *and* $S(A_t)_{t\in\overline{\mathbf{R}}^+} = 0$.

Proof. From properties (ii) and (iii) of our filtration and the continuity of the state ω we deduce that $M_t \to I$ as $t \to \infty$ in the strong operator topology and $M_s \to M_t$ as $s \searrow t$ in the same topology. See Theorem 5.1 of [4].

Hence, $\forall\, \varepsilon > 0 \quad \xi \in \mathcal{L}^2(\mathcal{A}) \quad \exists\, R \in [0, \infty)$ such that:

$$\left\|(I - M_t)\xi\right\|_2^2 < \frac{\varepsilon}{\sup\limits_{t\in\overline{\mathbf{R}}^+} \|A_t\|} \quad \forall\, t \geqslant R.$$

In $[0, R]$ there exist only finitely many $t_1, \ldots, t_{n(R)}$ for which $A_t \neq 0$. Furthermore, right continuity of M_t implies that $\exists\, \delta$ such that:

$$\left\|(M_{t_i+\delta} - M_{t_i})\xi\right\|_2^2 < \frac{\varepsilon}{\sup\limits_{t\in\overline{\mathbf{R}}^+} \|A_t\| \cdot n(R)} \quad 1 \leqslant i \leqslant n(R).$$

So, for the partition

$$T = \{0, t_1, t_1 + \delta, t_2, t_2 + \delta, \ldots, t_{n(R)}, t_{n(R)} + \delta, \infty\}$$

we have:

$$\begin{aligned}
\left\|S^T(A_t)_{t\in\overline{\mathbf{R}}^+}\xi\right\|_2^2 &= \left\|\sum_{i=1}^{n(R)} A_{t_i}(M_{t_i+\delta} - M_{t_i})\xi + A_{t_i+\delta}(I - M_{t_i+\delta})\xi\right\|_2^2 \\
&= \sum_{i=1}^{n(R)} \left\|A_{t_i}(M_{t_i+\delta} - M_{t_i})\xi\right\|_2^2 + \left\|A_{t_i+\delta}(I - M_{t_i+\delta})\xi\right\|_2 \\
&\qquad \text{since the projections } (M_{t_i+\delta} - M_{t_i}) \text{ are mutually} \\
&\qquad \text{orthogonal and commute with the } A_{t_i} \\
&\leqslant \sup_{t\in\overline{\mathbf{R}}^+} \|A_t\| \cdot \left(n(R) \frac{\varepsilon}{\sup\limits_{t\in\overline{\mathbf{R}}^+} \|A_t\| \cdot n(R)} + \frac{\varepsilon}{\sup\limits_{t\in\overline{\mathbf{R}}^+} \|A_t\|} \right) \\
&= 2\varepsilon.
\end{aligned}$$

Hence $\{S^T(A_t)_{t\in\overline{\mathbf{R}}^+}\}_{T\in\mathcal{P}}$ is strong operator Cauchy and converges to 0, as required. ∎

REMARKS 3.23. (i) The "Doobs Optimal Stopping" theorem tells us that for two stopping times τ, σ:

$$\tau \leqslant \sigma \Rightarrow M_\tau \leqslant M_\sigma .$$

The above lemma can be used to construct two times τ, σ with $M_\tau = M_\sigma$ and τ, σ non-comparable, thus killing off the converse.

(ii) We shall now demonstrate that the integral S is not a "Lebesgue" type integral. Consider the process $(A_t)_{t \in \overline{\mathbb{R}}^+}$ given by:

$$A_t = \begin{cases} 0 & \text{if } t \in \mathbb{R}^+ \setminus \mathbb{Q}^+ \\ I & \text{if } t \in \mathbb{Q}^+ \cup \{\infty\} \end{cases}$$

and the subnets of partitions consisting of rational and irrational points respectively. We then have:

$$S^T (A_t)_{t \in \overline{\mathbb{R}}^+} = I$$

for all rational partitions, and

$$S^T (A_t)_{t \in \overline{\mathbb{R}}^+} = 0$$

for all irrational partitions. So $(A_t)_{t \in \overline{\mathbb{R}}^+}$ cannot be integrable.

(iii) In [11], Definition 4.2 we see that for stopping times σ, τ such that $\sigma \leqslant \tau$ we can define the *stochastic interval* $(\sigma, \tau] \equiv \sigma \cdot (I - \tau)$. Furthermore a stochastic integral, of $(\sigma, \tau]$ with respect to the martingale $(M_t \xi)_{t \in \overline{\mathbb{R}}^+}$, $\xi \in \mathcal{L}^2(\mathcal{A})$, is defined $M_\sigma^\perp(\xi) - M_\tau^\perp(\xi)$.

In view of the previous results, $\sigma \cdot (I - \tau)$ is in fact integrable (Theorem 3.15), and:

$$\begin{aligned} S[\sigma(I - \tau)]\xi &= S(\sigma) \cdot S(I - \tau)\xi \\ &= (I - M_\sigma) \cdot M_\tau \xi \\ &= (M_\tau - M_\sigma)\xi \\ &\qquad \text{since } \sigma \leqslant \tau \Rightarrow M_\sigma \leqslant M_\tau \\ &= (M_\sigma^\perp - M_\tau^\perp)\xi . \end{aligned}$$

So we do not in fact need to make a new definition, but instead compute the integral in question.

Acknowledgements. Dr. Voliotis would like to thank the Science and Engineering Research Council for financial support during the period of his PhD studies. Much of the content of this paper arose from his PhD Thesis ([21]).

REFERENCES

1. C. BARNETT, R.F. STREATER, I.F. WILDE, The Ito-Clifford integral, *J. Funct. Anal.* **48**(1982), 142-212.
2. C. BARNETT, R.F. STREATER, I.F. WILDE, Quasi-free quantum stochastic integrals for the C.A.R. and C.C.R., *J. Funct. Anal.* **52**(1983), 17-47.
3. C. BARNETT, R.F. STREATER, I.F. WILDE, Stochastic integrals in an arbitrary probability gauge space, *Math. Proc. Cambridge Philos. Soc.* **94**(1983), 541-551.
4. C. BARNETT, R.F. STREATER, I.F. WILDE, Quantum stochastic integrals under standing hypothesis, *J. Math. Anal. Appl.* **127**(1987), 181-192.
5. C. BARNETT, T.J. LYONS, Stopping non-commutative processes, *Math. Proc. Cambridge Philos. Soc.* **99**(1986), 151-161.
6. C. BARNETT, B. THAKRAR, Time projections in a von Neumann algebra, *J. Operator Theory* **18**(1987), 19-31.
7. C. BARNETT, B. THAKRAR, A non-commutative random stopping theorem, *J. Funct. Anal.* **88**(1990), 342-350.
8. C. BARNETT, S. VOLIOTIS, A homomorphic integral, *Souchow J. Math.*, to appear.
9. C. BARNETT, I.F. WILDE, Belated integrals, *J. Funct. Anal.* **66**(1986), 283-307.
10. C. BARNETT, I. WILDE, Random times and time projections, *Proc. Amer. Math. Soc.* **110**(1990), 425-440.
11. C. BARNETT, I. WILDE, Random times, predictable processes and stochastic integration in finite von Neumann algebras, *Proc. London Math. Soc.(3)* **67**(1993), 355-383.
12. R.G. BARTLE, A general bilinear vector integral, *Studia Math.* **15**(1956), 337-352.
13. J.B. CONWAY, *A Course in Functional Analysis*, 2nd edition, Graduate Texts in Math., Springer-Verlag, New York Inc., 1990.
14. K.L. CHUNG, R.J. WILLIAMS, *Introduction to stochastic integration*, 2nd edition, Birkhäuser Verlag, Boston 1990.
15. J. DIXMIER, *Von Neumann algebras*, North-Holland Math. Libraray, 1981.
16. A.U. KUSSMAUL, *Stochastic Integration and Generalised Martingales*, Pitman Publishing Limited, 1977.
17. J.R. RINGROSE, *Lecture Notes on von Neumann Algebras*, Department of Mathematics, University of Newcastle upon Tyne, 1966-67 (unpublished lecture notes).
18. M. TAKESAKI, *Theory of Operator Algebras I*, Springer-Verlag, New York Inc., 1973.
19. B. THAKRAR, *Non-commutative stopping times*, Ph.D. Disertation, Imperial College, London, 1988.

20. D.M. TOPPING, *Lectures on von Neumann Algebras*, Van-Nostrand, New York, 1971.
21. S. VOLIOTIS, *Homomorphic operator valued stochastic integrals*, Ph.D. Disertation, Imperial College, 1994.

C. BARNETT and S. VOLIOTIS
Department of Mathematics
Imperial College of Science,
Technology and Medicine
London SW7 2BZ
U.K.

Received August 11, 1994.