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# SPECTRAL ANALYSIS FOR SIMPLY CHARACTERISTIC OPERATORS BY MOURRE'S METHOD. III

GRUIA ARSU

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ABSTRACT. We give a proof of the asymptotic completeness for the wave operators associated to the pair $(h(D),\ h(D)+V_S+V_L)=(H_0,H)$, where $h$ is a simply characteristic symbol, $V_S$ a short-range perturbation and $V_L$ a long-range potential. The proof is done by using a propagation estimate proved in [2] by means of Mourre's theory, a characterisation of the orthogonal complement of the ranges of the wave operators in the space of scattering states of $H$ given in [10] and some technical results.

KEYWORDS: *Simply characteristic operator, propagation estimate, wave operators, asymptotic completeness, bound state.*

AMS SUBJECT CLASSIFICATION: Primary 47A40, 35P25; Secondary 81U05, 47N50, 35P05.

## 1. INTRODUCTION

In [10] the author proved some results for the pair of operators $(h(D), h(D) + V_S + V_L)$ with $h$ as general as posible. Among them there are two results (or rather their proofs) which are particularly important for us because we can use these results to complete the study of spectral and scattering properties of simply characteristic operators proposed in [2]. These results are:

(a) The range of $W_\pm$ is contained in the space of scattering states of $H$.

(b) A characterization of the orthogonal complement of the range of $W_\pm$ in the space of scattering states of $H$.

The purpose of this paper is to use the above characterization and some basic results of [2] to prove the asymptotic completeness of the modified wave operators associated to the pair of operators $(h(D),\ h(D)+V_S+V_L)$ with $h$ a simply characteristic symbol.

The plan of the paper is as follows. In Section 2 we state the main result and we make the reduction of the general case to the case $V_S=0$. The proof of the main result is the object of Section 3 in which we also prove the asymptotic completeness of the modified wave operators defined by means of the exact solution of the Hamilton-Jacobi equation constructed by the methods of [4] and [6]. Let us recall that in [9] and [10] the modified wave operators are defined by means of an approximate solution of the Hamilton-Jacobi equation.

Now we recall some notation. If $A$ is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then $\langle A\rangle$ denotes the operator $(1+|A|^2)^{1/2}$, $\mathcal{H}_{\mathrm{ac}}(A)$ is the absolutely continuous subspace of $A$, $E_{\mathrm{ac}}(A)$ is the orthogonal projection of this subspace, $\mathcal{H}_{\mathrm{sc}}(A)$ is the singularly continuous subspace for $A$ and $\operatorname{Ran} A$ is the range of $A$. Besides we use the following standard notations:

$$\langle x\rangle=(1+|x|^2)^{\frac{1}{2}},\qquad x\in\mathbf{R}^n$$

and $\partial=\partial/\partial x$, $D=-\mathrm{i}\partial$, and for $x=(x_1,\ldots,x_n)$ and $\alpha=(\alpha_1,\ldots,\alpha_n)$

$$x^\alpha=x_1^{\alpha_1}\cdots x^{\alpha_n}\ \text{ and }\ \partial^\alpha=\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

## 2. STATEMENT OF THE RESULT. THE HAMILTONIAN

First we shall recall some definitions and notation we shall use.

Let $F(M)$ denotes the indicator function of the set $M$ and assume that $\mathbf{R}^n$ is divided into unit "cubes" $C_k$, $k\in\mathbf{N}$, so that

$$\mathbf{R}^n=\bigcup_{k\in\mathbf{N}}\overline{C}_k\ \text{ and }\ C_k\cap C_j=\emptyset,\ k\neq j.$$

We say that $f\in c_0(L^p)$, $p\geqslant 1$, if

$$\|f\|_{0,p}:=\sup_{k\in\mathbf{N}}\|F(C_k)f\|_p<\infty\quad\text{and}\quad\lim_{k\to\infty}\|F(C_k)f\|_p=0.$$

Also we say that a function $f$ is quasi-divergent if

$$\lim_{k\to\infty}|C_k\cap B_m|=0,$$

for all $m\in\mathbf{N}$, where $B_m=\{x\in\mathbf{R}^n;\ |f(x)|\leqslant m\}$ and $|M|$ denotes the Lebesgue measure of the measurable set $M$.

We shall work under the following hypotheses.

HYPOTHESES

I. The free hamiltonian $H_0$ is a self-adjoint operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$, with the domain $\mathcal{D}(H_0) = \{u \in \mathcal{H};\ h\widehat{u} \in \mathcal{H}\}$, $H_0 u = \mathcal{F}^{-1} h\widehat{u}$, where $\widehat{u}$ is the Fourier transform of $u$ and $h$ is a real valued function which satisfies:

(i) $h : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

(ii) Let $S_p$ be the set $\{\xi \in \mathbb{R}^n;\ h \text{ is not } C^\infty \text{ in any neighborhood of } \xi\}$, let $C_p$ be the set $\{\xi \in \mathbb{R}^n \setminus S_p;\ \nabla h(\xi) = 0\}$ and let $S = S_p \cup C_p$. Then $\overline{h(S)}$ is a countable subset of $\mathbb{R}$.

(iii) For any compact interval $I \subset \mathbb{R} \setminus \overline{h(S)}$, with $h^{-1}(I) \neq \emptyset$, we have

$$\operatorname{dist}(h^{-1}(I),\ S_p) > 0.$$

(iv) $F(S_p) \in c_0(L^1)$.

(v) $\lim_{\xi \to \infty,\ \xi \notin S_p} (|h(\xi)| + |\nabla h(\xi)|) = \infty$.

(vi) $\sup\{|D^\alpha h(\xi)| \,/\, (1 + |h(\xi)| + |\nabla h(\xi)|);\ \xi \in \mathbb{R}^n \setminus S_p\} < \infty$, for each multi-index $\alpha$ with $|\alpha| \geqslant 2$.

II. (vii) $V_L$ is a $C^\infty$ real valued function which satisfies

$$|D^\alpha V_L(x)| \leqslant c_\alpha \langle x\rangle^{-\delta-|\alpha|}, \quad x \in \mathbb{R}^n,$$

for some $\delta > 0$ and all $\alpha \in \mathbb{N}^n$.

From the hypotheses (iv) and (v) it follows that $h$ is a quasi-divergent function (see Appendix of [2]). Now from Theorem 9 of [3] we obtain that $V_L$ is a symmetric $H_0$-compact operator. We denote by $H_L$ the operator $H_0 + V_L$ with the domain $\mathcal{D}(H_L) = \mathcal{D}(H_0)$.

III. Let $V_S : \mathcal{D}(V_S) \to \mathcal{H}$ be a symmetric operator and let $H$ be a self-adjoint operator on $\mathcal{H}$ such that:

(viii) $\operatorname{Ran} g(H_L) \subset \mathcal{D}(V_S) \cap \mathcal{D}(H)$ for each $g$ in $C_0^\infty(\mathbb{R})$ and $H|X = H_{L|X} + V_{S|X}$, where $X = \bigcup\{\operatorname{Ran} g(H_L); g \in C_0^\infty(\mathbb{R})\}$.

(ix) For some $\varepsilon > 0$ the operator $g(H)V_S g(H_L)\langle X\rangle^{1+\varepsilon}$ has a bounded extension to the whole of $\mathcal{H}$ for each $g$ in $C_0^\infty(\mathbb{R})$.

(x) For any $g$ in $C_0^\infty(\mathbb{R})$ the operator $g(H) - g(H_0)$ is compact.

We can now state the main result of this paper.

THEOREM 2.1. *Assume that the hypotheses* (i)–(x) *are satisfied. Then there exists a* $C^\infty$ *function* $W : \mathbf{R} \times \mathbf{R}^n \setminus S \to \mathbf{R}$ *such that:*

(a) *The modified wave operators*

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}W(t,D)} E_{\mathrm{ac}}(H_0) \tag{2.1}$$

*exist and define partial isometries which intertwine* $H$ *and* $H_0$*;*

(b) $\operatorname{Ran} W_\pm = \mathcal{H}_{\mathrm{ac}}(H)$*;*

(c) $\mathcal{H}_{\mathrm{sc}}(H) = \{0\}$*;*

(d) $\overline{h(S)} \cup \sigma_p(H_L)$ *is a closed countable subset of* $\mathbf{R}$*;*

(e) *The eigenvalues of* $H$ *which are not in* $\overline{h(S)} \cup \sigma_p(H_L)$ *are of finite multiplicity and they can accumulate only at the points of* $\overline{h(S)} \cup \sigma_p(H_L)$.

In [2] we proved the following Theorems 2.2 and 2.3.

THEOREM 2.2. *For any compact interval* $I \subset \mathbf{R} \setminus \overline{h(S)}$ *there is a self-adjoint operator* $A_I$ *such that* $A_I$ *is conjugate to* $H_L$ *on the interval* $I$ *and* $H_L$ *is* $\infty$*-smooth with respect to* $A_I$ *in the sense of Definition* 2.1 *given in* [7]. *Also we have that for any compact interval* $I \subset \mathbf{R} \setminus \overline{h(S)}$ *and any non-negative number* $s$, $\langle A_I \rangle^s \langle X \rangle^{-s}$ *is a bounded operator on* $\mathcal{H}$.

THEOREM 2.3. *Assume that the hypotheses* (i)–(x) *are satisfied. Then*

(a) *The wave operators*

$$W_\pm(H, H_L) = \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_L} E_{\mathrm{ac}}(H_L)$$

*exist and define partial isometries which intertwine* $H$ *and* $H_L$*;*

(b) $\operatorname{Ran} W_\pm(H, H_L) = \mathcal{H}_{\mathrm{ac}}(H)$*;*

(c) $\mathcal{H}_{\mathrm{sc}}(H) = \{0\}$*;*

(d) $\overline{h(S)} \cup \sigma_p(H_L)$ *is a closed countable subset of* $\mathbf{R}$*;*

(e) *The eigenvalues of* $H$ *which are not in* $\overline{h(S)} \cup \sigma_p(H_L)$ *are of finite multiplicity and they can accumulate only at the points of* $\overline{h(S)} \cup \sigma_p(H_L)$.

REMARKS 2.4. (a) Since Theorem 2.3 is true with the condition (ix) replaced by the condition:

(ix)′ For some $\varepsilon > 0$ the operator $g(H)V_S\langle X \rangle^{1+\varepsilon}$ has a bounded extension to the whole of $\mathcal{H}$ for each $g$ in $C_0^\infty(\mathbf{R})$,

it follows that Theorem 2.1 remain true if we replace the condition (ix) by the condition (ix)′.

(b) The condition (ix)′ is always true when $V_S$ is a symmetric $H_0$-compact operator and there is an $\varepsilon > 0$ such that the operator

$$(H_0 + \mathrm{i})^{-1} V_S \langle X \rangle^{1+\epsilon}$$

has a bounded extension.

(c) By using Theorem 2.3 and the chain rule for the wave operators it follows that we can assume that $V_S = 0$, so we proceed directly to the investigation of $H_L$. In what follows we omit the subindex $L$, i.e. we set $H = H_L$ and $V = V_L$.

Finally Theorem 2.2, Theorem 2.10 of [2] combined with a partition of unity in $\mathbf{R} \setminus (\overline{h(S)} \cup \sigma_p(H))$ lead to the following useful estimate:

THEOREM 2.5. *Let $0 \leqslant s' < s$ and let $g \in C_0^\infty(\mathbf{R} \setminus (\overline{h(S)} \cup \sigma_p(H)))$. Then there is a constant $C = C(g, s, s') > 0$ such that*

$$\|\langle X \rangle^{-s} \mathrm{e}^{-\mathrm{i}tH} g(H) \langle X \rangle^{-s}\| \leqslant C \langle t \rangle^{-s'}, \qquad t \in \mathbf{R}. \tag{2.2}$$

This estimate is a basic one for two reasons. The first reason is that it can be used essentially in the proof of the existence of wave operators in Theorem 2.3. The second reason is that we can use it to prove that each element of $\mathcal{H}_{\mathrm{ac}}(H)$ is a bound state for the momentum operator $D$ under the total evolution $\mathrm{e}^{-\mathrm{i}tH}$ ([1]). We shall do this in the next section.

## 3. PROOF OF THE RESULT

Let $\mathcal{F}$ denote the Fourier transform on $\mathcal{S}'(\mathbf{R}^n)$. Let $t$ be a real number and let $a$ be a distribution in $\mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n)$. We define the operator

$$a_t(X, D) : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$$

by

$$\langle a_t(X, D)\varphi, \psi \rangle = (2\pi)^{-\frac{n}{2}} \langle ((1 \otimes \mathcal{F}^{-1})a) \circ T_t,\ \psi \otimes \varphi \rangle, \ \varphi, \psi \in \mathcal{S}(\mathbf{R}^n),$$

where $T_t : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ is the linear map defined by

$$T_t(x, y) = (tx + (1-t)y,\ x - y).$$

Then we have

$$\begin{aligned} a_1(X, D) &= \sum_{|\alpha|<k} \frac{\mathrm{i}^{|\alpha|}}{\alpha!} (\partial_x^\alpha \partial_\xi^\alpha a)_0(X, D) \\ &\quad + k \sum_{|\alpha|=k} \frac{\mathrm{i}^k}{\alpha!} \int_0^1 (1-t)^{k-1} (\partial_x^\alpha \partial_\xi^\alpha a)_t(X, D)\, \mathrm{d}t, \end{aligned} \tag{3.1}$$

with the integral converging weakly.

In particular if $a, b$ are smooth functions such that $a, b$ and all their derivatives are of atmost polynomial growth then (3.1) gives

$$[a(X), b(D)] = \mathrm{i} \sum_{1 \leqslant j \leqslant n} \int_0^1 (\partial_j a \otimes \partial_j b)_t(X, D)\, dt, \tag{3.2}$$

with the integral converging weakly.

Let $m$ be a real number. We define the space of symbols:

$$S^m = \{a \in C^\infty(\mathbf{R}^n);\ \forall \alpha \in \mathbf{N}^n,\ \exists C_\alpha > 0,\ |\partial^\alpha a(x)| \leqslant C_\alpha \langle x \rangle^{m-|\alpha|},\ x \in \mathbf{R}^n\}.$$

Now we can state and prove the following important lemma.

LEMMA 3.1. *Let $a_j \in S^{m_j}$ for $j = 1, 2, 3$ and assume that $m_1 + m_2 + m_3 \leqslant 1$. Let $b$ be a smooth function on $\mathbf{R}^n$ such that* $\operatorname{supp} b'$ *is a compact set. Then for each $r \geqslant 1$ the operator*

$$a_1(X)[a_2(X), b(D/r)]a_3(X)$$

*is bounded on $L^2(\mathbf{R}^n)$ and there exists a positive constant $C = C(a_1, a_2, a_3, b)$ such that*

$$\|a_1(X)[a_2(X), b(D/r)]a_3(X)\|_{B(L^2)} \leqslant C r^{-1},\ r \geqslant 1. \tag{3.3}$$

*Proof.* Taking into account (3.2) it follows that the distribution kernel of the operator $a_1(X)[a_2(X), b(D/r)]a_3(X)$ is

$$K(x, y) = \mathrm{i} r^{-1} \sum_{1 \leqslant j \leqslant n} \int_0^1 a_1(x) a_3(y) r^n (\partial_j a_2 \otimes \widehat{\partial_j b})(tx + (1-t)y, r(y-x))\, dt.$$

Now using the fact that $a_j$ is in $S^{m_j}$ and applying Peetre's inequality twice we obtain

$$\begin{aligned} |K(x, y)| &\leqslant C r^{-1} \langle x \rangle^{m_1 + m_2 + m_3 - 1} r^n \langle r(y-x) \rangle^m \sum_{1 \leqslant j \leqslant n} |\widehat{\partial_j b}(r(y-x))| \\ &\leqslant C r^{-1} \sum_{1 \leqslant j \leqslant n} r^n \langle r(y-x) \rangle^m |\widehat{\partial_j b}(r(y-x))|, \end{aligned}$$

where $m = |m_2 - 1| + |m_3|$.

Hence from Schur's lemma ([5]) it follows that

$$\|a_1(X)[a_2(X), b(D/r)]a_2(X)\|_{B(L^2)} \leqslant C r^{-1} \sum_{1 \leqslant j \leqslant n} \|\langle \cdot \rangle^m \widehat{\partial_j b}\|_{L^1}. \quad \blacksquare$$

THEOREM 3.2. *Let the hypotheses* (i)–(vii) *be satisfied and let* $\chi$ *be a smooth function on* $\mathbb{R}^n$ *such that* $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 0$ *for* $|\xi| \leqslant 1$ *and* $\chi(\xi) = 1$ *for* $|\xi| \geqslant 2$. *Then for each* $u$ *in* $\mathcal{H}_{\mathrm{ac}}(H)$ *we have*

$$\lim_{r\to\infty} \sup_t \|\chi(D/r)\mathrm{e}^{-\mathrm{i}tH}u\| = 0. \tag{3.4}$$

*Proof.* Let $\mathcal{G} = \{g \in C_0^\infty(\mathbb{R});\ \operatorname{supp} g \cap (\overline{h(S)} \cup \sigma_p(H)) = \emptyset\}$. Since $\mathcal{H}_{\mathrm{ac}}(H) = \bigvee_{g\in\mathcal{G}} \operatorname{Ran} g(H)$, it suffices to show that

$$\lim_{r\to\infty} \sup_t \|\chi(D/r)\mathrm{e}^{-\mathrm{i}tH}g(H)u\| = 0 \tag{3.5}$$

for each $u$ in $L^2(\mathbb{R}^n)$. By a density argument we have to prove (3.5) for $u$ in $\mathcal{S}(\mathbb{R}^n)$.

Let $u$ be in $\mathcal{S}(\mathbb{R}^n)$. Then by the fundamental theorem of calculus we have

$$\|\chi(D/r)\mathrm{e}^{-\mathrm{i}tH}g(H)u\|^2 = \|\chi(D/r)g(H)u\|^2 - \mathrm{i}\int_0^t \Big([\chi^2(D/r), V(X)]\mathrm{e}^{-\mathrm{i}sH}g(H)u, \mathrm{e}^{-\mathrm{i}sH}g(H)u\Big)\,\mathrm{d}s.$$

For $\delta$ as in the assumption (vii) we choose $\sigma$ such that $1 < 2\sigma \leqslant 1+\delta$. Then we have the estimate

$$\Big|\Big([\chi^2(D/r), V(X)]\mathrm{e}^{-\mathrm{i}sH}g(H)u, \mathrm{e}^{-\mathrm{i}sH}g(H)u\Big)\Big| \leqslant \|\langle X\rangle^\sigma[\chi^2(D/r), V(X)]\langle X\rangle^\sigma\|\,\|\langle X\rangle^{-\sigma}\mathrm{e}^{-\mathrm{i}sH}g(H)\langle X\rangle^{-\sigma}\|^2\|\langle X\rangle^\sigma u\|^2.$$

Now using Theorem 2.5, Lemma 3.1 and the assumption (vii) we see that

$$\|\langle X\rangle^{-\sigma}\mathrm{e}^{-\mathrm{i}sH}g(H)\langle X\rangle^{-\sigma}\|^2 \leqslant C\langle s\rangle^{-\sigma-\frac{1}{2}},$$

$$\|\langle X\rangle^\sigma[\chi^2(D/r), V(X)]\langle X\rangle^\sigma\|_{\mathcal{B}(L^2)} \leqslant \frac{C}{r}, \quad r \geqslant 1.$$

Summing up we obtain

$$\sup_t \|\chi(D/r)\mathrm{e}^{-\mathrm{i}tH}g(H)u\|^2 \leqslant \|\chi(D/r)g(H)u\|^2 + Cr^{-1}\left(\int \langle s\rangle^{-\sigma-\frac{1}{2}}\,\mathrm{d}s\right)\|\langle X\rangle^\sigma u\|^2, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

Now (3.5) with $u$ in $\mathcal{S}(\mathbb{R}^n)$ is an easy consequence of this estimate. ∎

The next Theorem 3.3 from [10] is an important step in the proof of the asymptotic completeness in Theorem 2.1 (the case $V_S = 0$).

THEOREM 3.3. *Assume that the hypotheses* (i)–(vii) *are satisfied. Then there exists a* $C^\infty$ *function* $W : \mathbf{R} \times \mathbf{R}^n \setminus S \to \mathbf{R}$ *such that* (a)–(e) *are valid.*

(a) *The modified wave operators*

$$W_\pm(H, H_0) = \mathrm{s} - \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\mathrm{e}^{-\mathrm{i}W(t,D)}E_{\mathrm{ac}}(H_0)$$

*exist;*

(b) $W_\pm(H, H_0)$ *are partial isometries;*

(c) $\mathrm{e}^{-\mathrm{i}tH}W_\pm(H, H_0) = W_\pm(H, H_0)\mathrm{e}^{-\mathrm{i}tH_0}$, $t \in \mathbf{R}$;

(d) $\operatorname{Ran} W_\pm(H, H_0) \subset \mathcal{H}_{\mathrm{ac}}(H)$;

(e) *Let* $G = \mathbf{R}^n \setminus S$. *Then*

$$\mathcal{H}_{\mathrm{ac}}(H) \ominus \operatorname{Ran} W_\pm(H, H_0) = \Big\{u \in \mathcal{H}_{\mathrm{ac}}(H);\ \lim_{T\to\infty} \frac{1}{T}\int_0^{\pm T} \|\gamma(D)\mathrm{e}^{-\mathrm{i}tH}u\|\,\mathrm{d}t = 0,\quad \forall \gamma \in C_0^\infty(G)\Big\}.$$

*Proof.* The proof follows the same way as the proof of Theorems 2.1, 2.2, 2.3 of [10]. ∎

THEOREM 3.4. *Assume that the hypotheses* (i)–(vii) *are satisfied. Then* $\operatorname{Ran} W_\pm(H, H_0) = \mathcal{H}_{\mathrm{ac}}(H)$.

*Proof.* Let $u \in \mathcal{H}_{\mathrm{ac}}(H) \ominus \operatorname{Ran} W_\pm(H, H_0)$ and let $g \in C_0^\infty(\mathbf{R})$ such that $\operatorname{supp} g \cap \overline{h(S)} = \emptyset$. Choose $\chi$ a smooth function on $\mathbf{R}^n$ so that $0 \leqslant \chi \leqslant 1$, $\chi(\xi) = 0$ for $|\xi| \leqslant 1$ and $\chi(\xi) = 1$ for $|\xi| \geqslant 2$. Then for all $r \geqslant 1$ we have

$$\begin{aligned}\|g(H)u\| &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \|g(H)\mathrm{e}^{-\mathrm{i}tH}u\|\,\mathrm{d}t \\ &\leqslant \lim_{T\to\infty} \frac{1}{T}\int_0^T \|(g(H) - g(H_0))\mathrm{e}^{-\mathrm{i}tH}u\|\,\mathrm{d}t \\ &\quad + \lim_{T\to\infty} \frac{1}{T}\int_0^T \|g(H_0)(1-\chi)(D/r)\mathrm{e}^{-\mathrm{i}tH}u\|\,\mathrm{d}t \\ &\quad + \|g\|_\infty \sup_t \|\chi(D/r)\mathrm{e}^{-\mathrm{i}tH}u\|.\end{aligned} \tag{3.6}$$

Since $g(H) - g(H_0)$ is a compact operator and $u$ is in $\mathcal{H}_{\mathrm{ac}}(H)$ it follows that the first term in the sum of (3.6) is 0 by the RAGE theorem ([11]). Also the second term is 0 by Theorem 3.3 (e). So we obtain

$$\|g(H)u\| \leqslant \|g\|_\infty \sup_t \|\chi(D/r)\mathrm{e}^{-\mathrm{i}tH}u\|,\quad \forall r \geqslant 1,\ \forall g \in C_0^\infty(\mathbf{R}),\ \operatorname{supp} g \cap \overline{h(S)} = \emptyset.$$

Now using Theorem 3.2 we obtain that $g(H)u = 0$ for each $g$ in $C_0^\infty(\mathbf{R})$, $\operatorname{supp} g \cap \overline{h(S)} = \emptyset$. Since $u$ is in $\mathcal{H}_{\rm ac}(H)$ and $\overline{h(S)}$ is a countable subset of $\mathbf{R}$ we get $u = 0$. Thus $\operatorname{Ran} W_+(H, H_0) = \mathcal{H}_{\rm ac}(H)$.

Similarly $\operatorname{Ran} W_-(H, H_0) = \mathcal{H}_{\rm ac}(H)$. ∎

Now Theorem 2.1 (the general case) is an easy consequence of Theorem 3.4, Theorem 2.3 and Remark 2.4 (c).

As we already mentioned in the introduction, the function $W$, used in the Definition 2.1 of the modified wave operators, is an approximate solution of the Hamilton-Jacobi equation ([9], [10]). In [4] and [6] the function $W$ which defines the modified free evolution is an exact solution of this equation. The next theorem implies that the asymptotic completeness in one case implies the asymptotic completeness in the other case.

THEOREM 3.5. *Assume that the limits* (2.1) *exist as well as the corresponding limits* $\widetilde{W}_\pm$ *when* $W$ *is replaced by* $\widetilde{W}$. *Then the following are equivalent:*

(a) $\operatorname{Ran} \widetilde{W}_\pm \subset \operatorname{Ran} W_\pm$ *(same sign);*

(b) *There exist two measurable functions* $F_\pm$ *such that for every compact set* $K \subset G$ *and every* $\varepsilon > 0$

$$\lim_{t\to\pm\infty} \left| \left\{ \xi \in K;\ |e^{i(W(t,\xi)-\widetilde{W}(t,\xi))} - F_\pm(\xi)| > \varepsilon \right\} \right| = 0;$$

(c) $\operatorname{Ran} \widetilde{W}_\pm = \operatorname{Ran} W_\pm$ *(same sign).*

The theorem is an easy consequence of the definition and of the following elementary results:

*Result 1.* Let $V$ be a partial isometry on the Hilbert space $\mathcal{H}$ and let $u$ be in $\mathcal{H}$. Then $\|V^* u\| = \|u\|$ if and only if $u$ is in the range of $V$.

*Result 2.* Let $(M, \mu)$ be a $\sigma$-finite measurable space, let $\{f_t\}_{t>0}$ be a family of bounded measurable functions on $M$ and let $T$ be a bounded operator on $L^2(M)$ such that

$$T = \mathrm{w} - \lim_{t\to\infty} M_{f_t}.$$

Then there is a bounded measurable function $f$ on $M$ such that $T = M_f$ and $\|f\|_\infty \leqslant \varliminf_{t\to\infty} \|f_t\|_\infty$.

Here $M_g$ denotes the multiplication operator by the measurable function $g$.

*Result 3.* Let $G$ be an open subset of $\mathbf{R}^n$, let $f$ be a bounded measurable function on $G$ and let $\{f_t\}_{t>0}$ be a family of measurable functions on $G$ such that for every $t > 0$, $|f_t| = 1$ a.e. in $G$. Then the following statements are equivalent:

(a) $M_f = \mathrm{s} - \lim_{t\to\infty} M_{f_t}$;

(b) For every compact set $K \subset G$ and every $\varepsilon > 0$

$$\lim_{t\to\infty} |\{\xi \in K;\ |f_t(\xi) - f(\xi)| > \varepsilon\}| = 0.$$

If one of the two conditions are satisfied then $|f| = 1$ a.e. and

$$M_{\bar f} = \mathrm{s} - \lim_{t\to\infty} M_{\bar f_t}.$$

*Proof of Theorem* 3.5. Using the first result we obtain that the condition (a) is equivalent to the condition

$$W_+^* \widetilde{W}_+ = \mathrm{s} - \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}(W(t,D)-\widetilde{W}(t,D))} E_{\mathrm{ac}}(H_0).$$

Now using the next two results we obtain that the condition (a) is equivalent to the condition (b).

Since it follows that $|F_+| = 1$ a.e. we obtain that the condition (b) is symmetric in $W$ and $\widetilde{W}$ so this condition is equivalent to the condition (c). ∎

REMARK 3.6. As a consequence of this theorem we have that all the conclusions of Theorem 2.1 remain valid if in the Definition 2.1 the function $W$ is the exact solution of the Hamilton-Jacobi equation constructed following the method of [4] and [6].

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*GRUIA ARSU*
*Institute of Mathematics*
*P.O. Box 1-764*
*RO-70700 Bucharest*
*ROMANIA*

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