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A PARAMETRIZATION OF CANONICALLY KOSZUL INVERTIBLE PAIRS

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ABSTRACT. Let $T = (T_1, T_2)$ be a commuting pair of operators on a Hilbert space $\mathcal{H}$, and let $T_i = V_i P_i$, $i = 1, 2$, be the polar decompositions of T_1 and T_2. The pair T is called canonically Koszul invertible if the Koszul complex $\mathrm{K}(T, \mathcal{H})$ admits a C^*-split, i.e., if $[(D^0)^* D^0]^{-1}(D^0)^*$ and $(D^1)^*[D^1(D^1)^*]^{-1}$ are the boundary maps of a Koszul complex, where D^0 and D^1 are the boundaries of $\mathrm{K}(T, \mathcal{H})$. We find a parametrization of canonically Koszul invertible pairs in terms of the factors V_1, P_1, V_2 and P_2. In addition, we obtain a new characterization of the commutant spectrum of T.

KEYWORDS: *Operators on Hilbert space, Koszul complex, commutant spectrum.*

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Let $\mathcal{H}$ be a Hilbert space and let $T = (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$.

Let $\mathrm{K}(T, \mathcal{H})$ be the Koszul complex associated to T on $\mathcal{H}$, given by

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H} \arrow[r, "D^0=\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}"] & \mathcal{H} \oplus \mathcal{H} \arrow[r, "D^1=\begin{pmatrix} -T_2 & T_1 \end{pmatrix}"] & \mathcal{H} \arrow[r] & 0. \end{tikzcd}$$

DEFINITION 1. We say that the pair (T_1, T_2) is *invertible* if there exist operators S_1, S_2, S_1' and S_2' on $\mathcal{H}$ such that

$$S_1 T_1 + S_2 T_2 = I = T_1 S_1' + T_2 S_2'$$

and

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \begin{pmatrix} S_1 & S_2 \end{pmatrix} + \begin{pmatrix} -S_2' \\ S_1' \end{pmatrix} \begin{pmatrix} -T_2 & T_1 \end{pmatrix} = I.$$

REMARK 2. Definition 1 implies trivially that $\mathrm{K}(T, \mathcal{H})$ is an exact complex.

Conversely, if $\mathrm{K}(T, \mathcal{H})$ is exact, then T is invertible. For, using the Open Mapping Theorem, it can be shown that $(D^0)^* D^0$ and $D^1(D^1)^*$ are invertible and therefore, if $H := [(D^0)^* D^0]^{-1}$ and if $K := [D^1(D^1)^*]^{-1}$, then $S_i := HT_i^*$ and $S_i' := T_i^* K$ $(i = 1, 2)$ satisfy the equations in Definition 1 (cf. [1], p. 664, Remarks).

Now if the pair (T_1, T_2) is invertible, then one can ask whether the operators S_i, S_i' $(i = 1, 2)$ can be chosen to come from the Koszul complex of another commuting pair of operators; in other words, whether one can have $S_1 = S_1'$, $S_2 = S_2'$, and S_1 commuting with S_2.

DEFINITION 3. The invertible pair (T_1, T_2) is *Koszul invertible* (denoted K.i.) if in Definition 1, $S_1 = S_1'$, $S_2 = S_2'$ and $S_1 S_2 = S_2 S_1$. The corresponding joint spectrum is

$$\sigma_{\mathrm{K.i.}}(T) := \{\lambda \in \mathbf{C}^2 : T - \lambda \text{ is not K.i.}\}.$$

REMARK 4. The pair (T_1, T_2) is K.i. if and only if there exist operators S_1, S_2 on $\mathcal{H}$ such that

$$S_1 T_1 + S_2 T_2 = I$$

and the 4-tuple (T_1, T_2, S_1, S_2) is commuting.

From this it follows at once that if T is K.i. then T is invertible with respect to its commutant $(T)'$; that is, $\sigma'(T) \subseteq \sigma_{\mathrm{K.i.}}(T)$. The following result shows that this containment is actually an equality and, in particular, that $\sigma_{\mathrm{K.i.}}(T)$ is always compact.

THEOREM 5. *Let $T = (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$. Then*

$$\sigma'(T) = \sigma_{\mathrm{K.i.}}(T).$$

Proof. By Remark 4, it suffices to prove that $\sigma_{\mathrm{K.i.}}(T) \subseteq \sigma'(T)$. Assume now that $0 \notin \sigma'(T)$. Then there exist $S_1, S_2 \in (T)'$ such that $S_1 T_1 + S_2 T_2 = I$. Hence

$$S_1 T_1 S_1 S_2 + S_2 T_2 S_1 S_2 = S_1 S_2.$$

But $S_1 T_1 S_1 S_2 = S_2 S_1 T_1 S_1$. For S_2 commutes with $S_1 T_1$ since

$$S_2 S_1 T_1 = S_2(I - S_2 T_2) = (I - S_2 T_2) S_2 = S_1 T_1 S_2,$$

thus

$$S_1T_1S_1S_2 = S_1(S_1T_1S_2) = S_1(S_2S_1T_1) = (S_1T_1S_2)S_1 = S_2S_1T_1S_1.$$

Therefore

$$S_1S_2 = S_2S_1T_1S_1 + S_2S_1T_2S_2 = S_2S_1(T_1S_1 + T_2S_2) = S_2S_1,$$

so that $0 \notin \sigma_{\mathrm{K.i.}}(T)$. ∎

It is rather surprising that by assuming that $S_1T_1 + S_2T_2 = I$ with $S_1, S_2 \in (T)'$, one automatically gets $S_1S_2 = S_2S_1$, and this is precisely the key feature of Theorem 5. At the same time, we now have many examples of Koszul invertible pairs. Before we look at some of them, we need to introduce a subclass of Koszul invertible pairs, the canonically Koszul invertible pairs.

DEFINITION 6. (i) The complex $\mathrm{K}(T, \mathcal{H})$ *splits as a Koszul complex* if there exist operators S_1 and S_2 on $\mathcal{H}$, S_1 commuting with S_2, such that the complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{H} \arrow[r, "\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}"] & \mathcal{H} \oplus \mathcal{H} \arrow[r, "\begin{pmatrix} -T_2 & T_1 \end{pmatrix}"] \arrow[l, bend left, "\begin{pmatrix} S_1 & S_2 \end{pmatrix}"] & \mathcal{H} \arrow[r] \arrow[l, bend left, "\begin{pmatrix} -S_2 \\ S_1 \end{pmatrix}"] & 0,
\end{tikzcd}$$

are exact. The complex $\mathrm{K}(S, \mathcal{H})$ is called a *co-Koszul complex*.

(ii) The complex $\mathrm{K}(T, \mathcal{H})$ *C^*-splits* if it splits as a Koszul complex and moreover

$$\begin{pmatrix} S_1 & S_2 \end{pmatrix} = [(D^0)^* D^0]^{-1} (D^0)^*$$

and

$$\begin{pmatrix} -S_2 \\ S_1 \end{pmatrix} = (D^1)^* [D^1 (D^1)^*]^{-1}.$$

(iii) We say that T is *canonically Koszul invertible* (abbreviated c.K.i.) if $\mathrm{K}(T, \mathcal{H})$ C^*-splits.

REMARK 7. (i) Let $T = (T_1, T_2)$ be an invertible pair. Since $(D^0)^* D^0$ and $D^1(D^1)^*$ are invertible (see Remark 2), then

$$[(D^0)^* D^0]^{-1} (D^0)^* = \begin{pmatrix} HT_1^* & HT_2^* \end{pmatrix}$$

and

$$(D^1)^* [D^1 (D^1)^*]^{-1} = \begin{pmatrix} -T_2^* K \\ T_1^* K \end{pmatrix}.$$

Therefore T is c.K.i. if and only if $HT_1^* = T_1^* K$ and $HT_2^* = T_2^* K$. In other words, an invertible pair T is c.K.i. if in Definition 1,

$$S_i = HT_i^*, \quad S_i' = T_i^* K \quad \text{and} \quad S_i = S_i' \quad \text{for} \quad i = 1, 2.$$

(ii) We had previously announced that c.K.i. pairs would form a subclass of the K.i. pairs, but this is not explicitly formulated in Definition 6 (ii). However, if T is c.K.i. and $S_i = S_i' = HT_i^*$ $(i = 1, 2)$, then

$$S_1 S_2 = HT_1^* HT_2^* = HT_1^* T_2^* K = HT_2^* T_1^* K = HT_2^* HT_1^* = S_2 S_1.$$

Therefore, T is K.i.. Incidentally, notice how easily the commutativity of S_1 and S_2 can be established in this case (as compared with the K.i. situation, described by Theorem 5).

Our main result, Theorem 13, establishes that a commuting pair T is c.K.i. if and only if T can be written as a direct sum $(A_1, 0) \oplus (0, B_2) \oplus (V_1 P_1, V_2 P_2)$ with respect to the orthogonal decomposition $\mathrm{N}(T_1) \oplus \mathrm{N}(T_2) \oplus \mathcal{M}$ where A_1 and B_2 are invertible operators, V_1 and V_2 are unitary operators, P_1 commutes with P_2, $P_1^2 + P_2^2$ is invertible, $V_2^* V_1$ commutes with P_1 and P_2, and V_2 commutes with $V_1 P_1 P_2 H$.

Also we will see later in Example 9 (ii) that $(A_1, 0)$ and $(0, B_2)$ are trivially c.K.i. and therefore the interesting subspace in the decomposition is $\mathcal{M}$.

The proof of Theorem 13 rests on a lemma, a proposition, and some examples.

The following proposition gives an alternative characterization of c.K.i. pairs.

PROPOSITION 8. *Let $T = (T_1, T_2)$ be an invertible pair of operators on $\mathcal{H}$. Then T is* c.K.i. *if and only if*

$$T_1 T_2^* T_2 = T_2 T_2^* T_1 \quad \text{and} \quad T_2 T_1^* T_1 = T_1 T_1^* T_2.$$

Proof. If $\mathrm{K}(T, \mathcal{H})$ admits a splitting Koszul complex given by

$$(HT_1^*, HT_2^*) = (T_1^* K, T_2^* K),$$

then $T_1 H = K T_1$, so

$$(T_1 T_1^* + T_2 T_2^*) T_1 = T_1 (T_1^* T_1 + T_2^* T_2),$$

thus $T_2 T_2^* T_1 = T_1 T_2^* T_2$. Similarly, $T_2 H = K T_2$ implies $T_1 T_1^* T_2 = T_2 T_1^* T_1$.

For the sufficiency, just reverse the previous argument. ∎

EXAMPLES 9. (i) *Let A be an operator on $\mathcal{H}$, and let f and g be two functions analytic in a neighborhood of the spectrum of A, $\sigma(A)$. Assume that $\{(f(\lambda), g(\lambda)) : \lambda \in \sigma(A)\}$ does not contain the origin. Then $(f(A), g(A))$ is* K.i..

(ii) *Let A be an invertible operator on $\mathcal{H}$. Then $(A, 0)$ and $(0, A)$ are* c.K.i..

(iii) *Let (T_1, T_2) be* K.i. *(respectively* c.K.i.*). Then (T_1^*, T_2^*) is* K.i. *(respectively* c.K.i.*).*

(iv) *Let T and S be invertible pairs. Then $T \oplus S$ is* K.i. *(respectively* c.K.i.*) if and only if T and S are* K.i. *(repsectively* c.K.i.*).*

(v) *Let (T_φ, T_ψ) be a pair of Toeplitz operators on $H^2(\mathbf{T})$, $\varphi, \psi \in H^\infty(\mathbf{T})$. Then:*

(1) *(T_φ, T_ψ) is* K.i. *if and only if there exists $\varepsilon > 0$ such that $|\varphi| + |\psi| \geqslant \varepsilon$ on $\mathbb{D}$. For $\sigma_T(T_\varphi, T_\psi) = (\varphi, \psi)(\mathbb{D})^-$, see* [3], *Corollary* 2.13, *and since $\sigma_T(T_\varphi, T_\psi) = \sigma'(T_\varphi, T_\psi)$ we get $\sigma'(T_\varphi, T_\psi) = (\varphi, \psi)(\mathbb{D})^-$, and therefore by Theorem* 5,

$$\sigma_{\mathrm{K.i.}}(T_\varphi, T_\psi) = (\varphi, \psi)(\mathbb{D})^- .$$

Thus the pair (T_φ, T_ψ) is K.i. *if and only if $0 \notin (\varphi, \psi)(\mathbb{D})^-$.*

(2) *(T_φ, T_ψ) is* c.K.i. *if and only if there exist $\varepsilon > 0$ and $\lambda, \mu \in \mathbb{C}$ such that $|\varphi| + |\psi| \geqslant \varepsilon$ on $\mathbb{D}$ and $\lambda\varphi = \mu\psi$.*

Proof. Sufficiency follows from Proposition 8 and (v) (1). To prove necessity, we assume that (T_φ, T_ψ) is c.K.i.. Then

$$T_\varphi T_\psi^* T_\psi = T_\psi T_\psi^* T_\varphi,$$

by Proposition 8, thus

$$T_\varphi T_{\overline{\psi}\psi} = T_\psi T_{\overline{\psi}\varphi}.$$

It follows that $P\varphi P\overline{\psi}\psi = P\psi P\overline{\psi}\varphi$, where P is the projection from $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$. Therefore

$$\varphi P\overline{\psi}\psi = \psi P\overline{\psi}\varphi,$$

and then

$$\varphi(0)(P\overline{\psi}\psi)(0) = \psi(0)(P\overline{\psi}\varphi)(0),$$

hence

$$\varphi(0)\langle\psi, \psi\rangle = \psi(0)\langle\varphi, \psi\rangle.$$

Similarly, by using the other equality in Proposition 8, we get

$$\psi(0)\langle\varphi, \varphi\rangle = \varphi(0)\langle\psi, \varphi\rangle.$$

Let $\lambda = \psi(0)$ and $\mu = \varphi(0)$. Then $\mu\langle\psi,\psi\rangle = \lambda\langle\varphi,\psi\rangle$ and $\lambda\langle\varphi,\varphi\rangle = \mu\langle\psi,\varphi\rangle$. It follows that

$$\begin{aligned}||\lambda\varphi - \mu\psi||^2 &= \langle\lambda\varphi,\lambda\varphi\rangle + \langle\mu\psi,\mu\psi\rangle - \langle\lambda\varphi,\mu\psi\rangle - \langle\mu\psi,\lambda\varphi\rangle\\ &= |\lambda|^2\langle\varphi,\varphi\rangle + |\mu|^2\langle\psi,\psi\rangle - \lambda\overline{\mu}\langle\varphi,\psi\rangle - \mu\overline{\lambda}\langle\psi,\varphi\rangle\\ &= \overline{\lambda}\mu\langle\psi,\varphi\rangle + \overline{\mu}\lambda\langle\varphi,\psi\rangle - \lambda\overline{\mu}\langle\varphi,\psi\rangle - \mu\overline{\lambda}\langle\psi,\varphi\rangle\\ &= 0.\end{aligned}$$

This shows that $\lambda\varphi = \mu\psi$. The remaining condition follows from the example given above. ∎

A trivial consequence of the above examples is that the class of c.K.i. pairs is properly contained in that of K.i. pairs. Our main result provides a complete parametrization of c.K.i. pairs. First, we need two lemmas and a proposition.

LEMMA 10. *Let $T = (T_1, T_2)$ be a* c.K.i. *pair of operators on $\mathcal{H}$. Then*

(i) $\mathrm{N}(T_i) = \mathrm{N}(T_i^*)$ *for* $i = 1, 2$,
(ii) $\mathrm{N}(T_i)$ *reduces* T *for* $i = 1, 2$, *and*
(iii) $\mathrm{N}(T_1)$ *and* $\mathrm{N}(T_2)$ *are orthogonal.*

Proof. (i) Let $x \in \mathrm{N}(T_1)$, then $0 = T_2T_1^*T_1x = T_1T_1^*T_2x$ by Proposition 8, thus $T_2x \in \mathrm{N}(T_1T_1^*)$, hence $T_2x \in \mathrm{N}(T_1^*)$ and therefore

$$T_2\mathrm{N}(T_1) \subseteq \mathrm{N}(T_1^*).$$

Now, let $y \in \mathrm{N}(T_1)$, then $0 = T_1y = T_2 0$, so that $(0, y) \in \mathrm{N}(-T_2\ T_1)$. It follows that $(0, y) \in \mathrm{R}\begin{pmatrix}T_1\\T_2\end{pmatrix}$ by the exactness of $\mathrm{K}(T, \mathcal{H})$ at the middle stage, so that there exists $x \in \mathcal{H}$ such that $T_1x = 0$ and $T_2x = y$; thus $y \in T_2\mathrm{N}(T_1)$, showing that $\mathrm{N}(T_1) \subseteq T_2\mathrm{N}(T_1)$. Therefore $\mathrm{N}(T_1) \subseteq \mathrm{N}(T_1^*)$. Since T^* is also c.K.i., we get $\mathrm{N}(T_1^*) \subseteq \mathrm{N}(T_1)$. In a completely analogous manner one shows that $\mathrm{N}(T_2) = \mathrm{N}(T_2^*)$.

(ii) The proof follows from (i) together with the fact that the kernel of an operator is an hyperinvariant subspace.

(iii) Let $x \in \mathrm{N}(T_1)$ and $y \in \mathrm{N}(T_2)$; then, since $T_1x = T_2y = 0$, it follows that $(y, x) \in \mathrm{N}(-T_2\ T_1)$, so that $(y, x) \in \mathrm{R}\begin{pmatrix}T_1\\T_2\end{pmatrix}$ by the exactness of $\mathrm{K}(T, \mathcal{H})$ at the middle stage. Thus, there exists $z \in \mathcal{H}$ such that $T_1z = y$ and $T_2z = x$, hence

$$\langle x, y\rangle = \langle x, T_1z\rangle = \langle T_1^*x, z\rangle = 0$$

by using part (i). ∎

LEMMA 11. *Let $T = (T_1, T_2)$ be a* c.K.i. *pair of operators on $\mathcal{H}$. Assume that T_i is injective for $i = 1, 2$, and let $T_i = V_i P_i$ be the left polar decomposition of T_i, for $i = 1, 2$. Then V_1 and V_2 are unitary, $P_1 P_2 = P_2 P_1$ and $P_1^2 + P_2^2$ is invertible.*

Proof. Since T is c.K.i., $\mathrm{N}(T_i) = \mathrm{N}(T_i^*)$ and therefore $\mathrm{N}(T_i^*) = 0$ because T_i is injective ($i = 1, 2$). Thus $\overline{\mathrm{R}(T_i)} = \mathcal{H}$, hence T_i is a quasiaffinity ($i = 1, 2$). Now, since $\mathrm{R}(T_i) \subseteq \mathrm{R}(V_i)$ and $\mathrm{N}(T_i) = \mathrm{N}(V_i)$, it follows that V_i is unitary ($i = 1, 2$). Moreover,

$$P_1^2 P_2^2 = T_1^* T_1 T_2^* T_2 = T_2^* T_1 T_1^* T_2 = T_2^* T_2 T_1^* T_1 = P_2^2 P_1^2,$$

from which it follows that $P_1 P_2 = P_2 P_1$. Finally, the invertibility of T implies that $(D^0)^* D^0$ is invertible, and therefore $P_1^2 + P_2^2$ is invertible. ∎

PROPOSITION 12. *Let $T = (T_1, T_2)$ be a commuting pair of operators on $\mathcal{H}$, and let $T_i = V_i P_i$ be the left polar decomposition of T_i ($i = 1, 2$). Assume that T_i is injective for $i = 1, 2$. Then T is* c.K.i. *if and only if*

(i) *V_i is unitary for $i = 1, 2$,*
(ii) *$P_1 P_2 = P_2 P_1$ and $P_1^2 + P_2^2$ is invertible,*
(iii) *$V_1^* V_2$ commutes with P_1 and P_2, and*
(iv) *V_2 commutes with $V_1 P_1 P_2 H$.*

Proof. ($\Rightarrow$) By Lemma 11, we only need to show (iii) and (iv). Now,

$$\begin{aligned}(V_1^* V_2 P_1^2 - P_1^2 V_1^* V_2) P_2 &= V_1^* V_2 P_2 P_1^2 - P_1^2 V_1^* V_2 P_2 \\ &= V_1^* T_2 P_1^2 - P_1^2 V_1^* T_2 = V_1^*(T_2 T_1^* T_1) - P_1^2 V_1^* T_2 \\ &= V_1^*(T_1 T_1^* T_2) - P_1^2 V_1^* T_2 \qquad \text{(by Proposition 8)} \\ &= V_1^* V_1 P_1^2 V_1^* T_2 - P_1^2 V_1^* T_2 = 0,\end{aligned}$$

showing that $V_1^* V_2 P_1^2 = P_1^2 V_1^* V_2$ on $\mathrm{R}(P_2)$. Since P_2 is injective, it follows that $V_1^* V_2 P_1^2 = P_1^2 V_1^* V_2$ and, a fortiori, that $V_1^* V_2 P_1 = P_1 V_1^* V_2$. In a similar way, we show that $V_1^* V_2$ commutes with P_2, thus establishing (iii). As for (iv), we recall that $T_1 H T_2^* = H T_2^* T_1$ (Remarks 4 and 7 (i)), so that

$$\begin{aligned}V_1 P_1 P_2 H V_2^* &= V_1 P_1 H P_2 V_2^* = T_1 H T_2^* \\ &= H T_2^* T_1 = H P_2 V_2^* V_1 P_1 \\ &= V_2^* V_1 P_1 P_2 H \qquad \text{(using (ii) and (iii))}.\end{aligned}$$

Therefore, $V_2 V_1 P_1 P_2 H = V_1 P_1 P_2 H V_2$, as desired.

($\Leftarrow$) Since $P_1^2 + P_2^2$ is invertible, it follows that (T_1, T_2) is left invertible. Moreover, (T_1, T_2) is right invertible. For,

$$\begin{aligned} \mathrm{R}(T_1) + \mathrm{R}(T_2) &= V_2 V_2^* V_1 \mathrm{R}(P_1) + V_2 \mathrm{R}(P_2) = V_2 \mathrm{R}(V_2^* V_1 P_1) + V_2 \mathrm{R}(P_2) \\ &= V_2(\mathrm{R}(P_1 V_2^* V_1) + \mathrm{R}(P_2)) = V_2(\mathrm{R}(P_1) + \mathrm{R}(P_2)) = \mathcal{H}, \end{aligned}$$

since (P_1, P_2) is invertible, V_1 and V_2 are unitary, and $V_2^* V_1$ commutes with P_1.

We will use [2], Theorem 6.12 to show the invertibility of (T_1, T_2), and therefore we must verify the condition $T_1 H T_2^* = T_2^* K T_1$.

Since $V_1^* V_2$ commutes with P_1, we have

$$P_1 V_1^* T_2 = P_1 V_1^* V_2 P_2 = V_1^* V_2 P_1 P_2 = V_1^* T_2 P_1,$$

and similarly

$$P_2 V_2^* T_1 = V_2^* T_1 P_2.$$

Therefore,

$$T_1 T_1^* T_2 = V_1 P_1^2 V_1^* T_2 = T_2 P_1^2 = T_2 T_1^* T_1,$$

and thus,

$$(T_1 T_1^* + T_2 T_2^*) T_2 = T_2 (T_1^* T_1 + T_2^* T_2).$$

It follows that $K T_2 = T_2 H$, hence $T_2^* K = H T_2^*$, then $T_2^* K T_1 = H T_2^* T_1$. A straightforward calculation shows that the condition

$$V_2 V_1 P_1 P_2 H = V_1 P_1 P_2 H V_2$$

implies that $T_1 H T_2^* = H T_2^* T_1$. We then have $T_1 H T_2^* = T_2^* K T_1$, as desired. Therefore (T_1, T_2) is invertible, and we already proved that $T_1 T_1^* T_2 = T_2 T_1^* T_1$. In a similar manner we can show the other condition in Proposition 8, thus (T_1, T_2) is c.K.i.. ∎

We can now state the main result.

THEOREM 13. *Let* $T = (T_1, T_2)$ *be a commuting pair of operators on* $\mathcal{H}$. *Then* T *is* c.K.i. *if and only if*

$$T = (A_1, 0) \oplus (0, B_2) \oplus (V_1 P_1, V_2 P_2)$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathrm{N}(T_1) \oplus \mathrm{N}(T_2) \oplus \mathcal{M}$, *where*

(i) A_1 *and* B_2 *are invertible operators,*

(ii$_a$) V_1 *and* V_2 *are unitary operators,*

(ii$_b$) P_1 *commutes with* P_2 *and* $P_1^2 + P_2^2$ *is invertible,* (ii$_c$) $V_2^*V_1$ *commutes with* P_1 *and* P_2, *and*

(ii$_d$) V_2 *commutes with* $V_1P_1P_2H$.

Proof. To prove necessity combine Proposition 12 with Lemma 10. For the sufficiency use Proposition 12 together with Examples 9 (ii) and (iv).

REFERENCES

1. R. CURTO, Spectral permanence for joint spectra, *Trans. Amer. Math. Soc.* **270**(1982), 659–665.
2. R. CURTO, Applications of several complex variables to multiparameter spectral theory, in *Surveys of Some Recent Results in Operator Theory*, vol. II (J.B. Conway and B.B. Morrel, editors), Pitman Res. Notes Math. Ser., vol. 192, Longman Sci. Techn., London, 1988, pp. 25–90.
3. R. CURTO, L. FIALKOW, Elementary operators with H^∞-symbols, *Integral Equation Operator Theory* **10**(1987), 707–720.

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