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ON THE STABILITY OF SEMI-FREDHOLM OPERATORS

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ABSTRACT. We give some stability results for the "nullity" and "deficiency" of semi-Fredholm operators. We also give characterizations of the operators that are bounded from below (resp. surjective) in terms of the stability of the "nullity" (resp. the "deficiency"), as well as a generalization of the "punctured neighbourhood theorem".

KEYWORDS: *Semi-Fredholm operator, s-regular, generalized kernel, generalized range, perturbation.*

AMS SUBJECT CLASSIFICATION: Primary 47A53; Secondary 47A55.

0. INTRODUCTION AND NOTATION

If a bounded linear operator $A \in B(X)$ on a Banach space X is semi-Fredholm then the "punctured neighbourhood theorem" says that there is $k_A > 0$ for which

$$n(A - \lambda I) \text{ is constant } (0 < |\lambda| < k_A) \text{ if } A \in \Phi_+(X) \tag{0.1}$$

and

$$d(A - \lambda I) \text{ is constant } (0 < |\lambda| < k_A) \text{ if } A \in \Phi_-(X). \tag{0.2}$$

Here the nullity and deficiency of A are

$$n(A) = \dim N(A) \text{ and } d(A) = \operatorname{codim} R(A)$$

where $N(A)$ and $R(A)$ denote respectively the kernel and the range of A;

$$\Phi_+(X) = \{A \in B(X) : n(A) < \infty \text{ and } R(A) \text{ closed}\}$$

is the set of upper semi-Fredholm, and

$$\Phi_-(X) = \{A \in B(X) : d(A) < \infty \text{ and } R(A) \text{ closed}\}$$

the set of lower semi-Fredholm operators on X. We write

$$\Phi_\pm(X) = \Phi_+(X) \cup \Phi_-(X) \text{ and } \Phi(X) = \Phi_+(X) \cap \Phi_-(X)$$

the semi-Fredholm and the Fredholm operators on X. If $A \in \Phi_\pm(X)$ we write

$$\text{(0.3)} \qquad \operatorname{ind}(A) = n(A) - d(A)$$

for the index of A. By the punctured neighbourhood theorem we can define $j(A)$, the "jump" of A, by setting

$$\text{(0.4)} \qquad j(A) = n(A) - n(A - \lambda I) \text{ if } 0 < |\lambda| < k_A$$

if $A \in \Phi_+(X)$, and

$$\text{(0.5)} \qquad j(A) = d(A) - d(A - \lambda I) \text{ if } 0 < |\lambda| < k_A$$

if $A \in \Phi_-(X)$. If in particular $A \in \Phi(X)$ then by the continuity of the index

$$\text{(0.6)} \qquad j(A) = n(A) - n(A - \lambda I) = d(A) - d(A - \lambda I) \text{ if } 0 < |\lambda| < k_A.$$

We call $A \in B(X)$ s-regular ("semi-regular") ([5], [7], [8]) if

$$\text{(0.7)} \qquad R(A) \text{ is closed and } N^\infty(A) \subseteq R^\infty(A)$$

where

$$\text{(0.8)} \qquad N^\infty(A) = \bigcup_{n \geqslant 0} N(A^n) \text{ and } R^\infty(A) = \bigcap_{n \geqslant 0} R(A^n)$$

are respectively the generalized kernel and generalized range of A.

(0.9) If $A \in \Phi_\pm(X)$ then A is s-regular if and only if $j(A) = 0$

(see [3], [5] and [10], Corollaire 2.3).

In this note we show (Theorem 2.5) that the nullity or the deficiency of a semi-Fredholm operator A remains constant under small "A-s-regular" (Definition 2.1) perturbations, we show (Theorems 2.6 and 2.7) that this constancy holds under small arbitrary perturbations precisely when A is surjective or bounded from below, and extend (Theorem 3.1) the punctured neighbourhood theorem by relaxing both the scalarity and the invertibility of the perturbations.

1. ALGEBRAIC PRELIMINARIES

Suppose X is a vector space and A a linear operator from X to itself.

DEFINITION 1.1. A is said to be of *type* n if $N(A^n) \subseteq R(A)$, and of *type* ∞ if this is so for all $n \in \mathbb{N}$. We denote these classes by $\mathrm{Typ}_n(X)$ and $\mathrm{Typ}_\infty(X)$ respectively.

There are various equivalent forms of these conditions:

LEMMA 1.2. *If $n \in \mathbb{N}$, the following conditions are equivalent:*
(i) *A is of type n;*
(ii) *$N(A^k) \subseteq R(A^j)$ for all $1 \leqslant j+k \leqslant n+1$;*
(iii) *$N(A^k) = A^j(N(A^{j+k}))$ for all $1 \leqslant j+k \leqslant n+1$.*

Proof. For the equivalence between (i) and (ii) suppose $U : W \to X$, $T : X \to Y$ and $V : Y \to Z$ are linear between vector spaces and note ([2], Lemma 1)

$$N(V) \subseteq R(TU),\ N(T) \subseteq R(U) \Rightarrow N(VT) \subseteq R(U) \tag{1.1}$$

and

$$N(VT) \subseteq R(U),\ N(V) \subseteq R(T) \Rightarrow N(V) \subseteq R(TU). \tag{1.2}$$

For the implication (ii) $\Rightarrow$ (iii) note

$$N(V) \subseteq R(T) \Rightarrow N(V) = TN(VT). \tag{1.3}$$ ∎

It is clear that the generalized range and the generalized kernel are "invariant subspaces" for an operator:

$$A(M) \subseteq M \text{ if } M = N^\infty(A) \text{ or } M = R^\infty(A). \tag{1.4}$$

Conversely

$$A^{-1}M \subseteq M \text{ if } M = N^\infty(A). \tag{1.5}$$

PROPOSITION 1.3. *If $TA = AT$ then*
(i) $N(A-T) \cap N^\infty(A) \subseteq N^\infty(T)$.
If A is of type ∞ then
(ii) $AR^\infty(A) = R^\infty(A)$
and
(iii) $AN^\infty(A) = N^\infty(A)$.

Proof. If $Ax = Tx$ and $AT = TA$ then $A^n x = T^n x$ for each $n \in \mathbf{N}$, so that if $A^d x = 0$ then also $T^d x = 0$, giving (i). If $x \in R^\infty(A)$ then there is $(v_n)_{n \geqslant 0}$ in X for which

$$x = Av_0 = A^{n+1} v_{n+1}.$$

It follows that $v_0 - A^n v_{n+1} \in N(A) \subseteq R(A^n)$ and hence $v_0 \in R(A^n)$ for each n, so that $x = Av_0$ with $v_0 \in R^\infty(A)$. This gives (ii); for (iii) note that if A is of type ∞ then $N^\infty(A) \subseteq R^\infty(A)$. Now, using (1.5),

$$x \in N^\infty(A) \Rightarrow x = Aw \in N^\infty(A) \Rightarrow w \in N^\infty(A). \quad \blacksquare$$

2. s-REGULAR SEMI-FREDHOLM OPERATORS

Suppose X is a Banach space: $A \in B(X)$ is "s-regular" in the sense of (0.7) if and only if it is of type ∞, with closed range.

DEFINITION 2.1. $T \in B(X)$ will be called *A-s-regular* if there exists a closed subspace $M \subseteq X$ for which

$$N(A) \subseteq M = A(M) \text{ and } T(M) \subseteq M. \tag{2.1}$$

Notice that if there exist a subspace $M \subseteq X$ satisfying (2.1) then the operator A must be of type ∞, although not necessarily with closed range. When A also has closed range then we are in the situation of Definition 3.1 of [8] (see also Definition 4.1 of [9]). In this case however Definition 2.1 is not changed if we relax the requirement that the subspace M of (2.1) be closed: we claim that if A has closed range and (2.1) holds then also

$$A(\overline{M}) = \overline{M}. \tag{2.2}$$

This is because it is true that ([3], Lemma 331), when $R(A)$ is closed,

$$N(A) \subseteq \overline{M} \Rightarrow A(\overline{M}) \text{ closed}. \tag{2.3}$$

Two special kinds of s-regular operators are the *bounded from below* operators and the *surjective* operators: for such A every operator $T \in B(X)$ is A-s-regular. If A is s-regular then

$$TA = AT \Rightarrow T \text{ is } A\text{-s-regular}. \tag{2.4}$$

The adjoint T^* of an A-s-regular operator T is s-regular relative to the adjoint of A.

LEMMA 2.2. *If $A \in B(X)$ is s-regular and if $T \in B(X)$ is A-s-regular then T^* is A^*-s-regular.*

Proof. Proposition 2.3 of [7] shows that the adjoint A^* of an s-regular operator A is s-regular, and Lemme 3.2 of [8] shows that if M satisfies (2.1) then

$$N^\infty(A) \subseteq M \subseteq R^\infty(A).$$

It follows that $R^\infty(A)^\perp \subseteq M^\perp \subseteq N^\infty(A)^\perp$, where $M^\perp$ is the usual annihilator $\{f \in X^* : M \subseteq f^{-1}(0)\}$ of M. Using Lemme 1.2 of [8], this gives

$$N(A^*) \subseteq N^\infty(A^*) \subseteq M^\perp \subseteq R^\infty(A^*).$$

Now since $A(M) \subseteq M$ and $T(M) \subseteq M$ it follows that $A^*(M^\perp) \subseteq M^\perp$ and $T^*(M^\perp) \subseteq M^\perp$; then it remains only to show that

$$M^\perp \subseteq A^*(M^\perp).$$

But if $f \in M^\perp \subseteq R^\infty(A^*)$ then there is $g \in R^\infty(A^*)$ for which $f = A^*g$, and we claim that $g \in M^\perp$. If $x \in M = A(M)$ then there is $w \in M$ for which $x = Aw$, giving $g(x) = g(Aw) = (A^*g)(w) = f(w) = 0$. ∎

Corollaries 3.6 and 3.7 of [8] (see also [9], Section 4) give the following stability result, which we state without proof.

THEOREM 2.3. *If A is s-regular then there is $\delta > 0$ for which if $T \in B(X)$ is A-s-regular with $\|T\| < \delta$ then*

(i) *$A - T$ is s-regular;*
(ii) *$R^\infty(A-T) = R^\infty(A)$;*
(iii) *$\overline{N^\infty(A-T)} = \overline{N^\infty(A)}$.*

Theorem 2.3 shows in particular that the bounded from below operators and the surjective operators are stable under arbitrary small perturbations; further, the generalized range of a bounded from below operator and the closure of the generalized kernel of a surjective operator are unchanged by small perturbations.

We shall write

$$\text{SReg}(A) = \{T \in B(X) : A - T \text{ is s-regular}\} \tag{2.5}$$

and

$$\text{biSReg}(A) = \text{SReg}(A) \cap \text{bicomm}(A), \tag{2.6}$$

where

$$\text{bicomm}(A) = \{T \in B(X) : SA = AS \Rightarrow TS = ST\} \tag{2.7}$$

is the usual bicommutant of A. It is familiar that $\text{bicomm}(A)$ is always a closed commutative subalgebra of $B(X)$.

THEOREM 2.4. *If $A \in B(X)$ is s-regular then* biSReg(A) *is open in* bicomm(A). *The mappings $T \mapsto R^\infty(A-T)$ and $T \mapsto \overline{N^\infty(A-T)}$ are constant on each connected component of* biSReg(A).

Proof. If $T_0 \in$ biSReg(A) then $T - T_0$ commutes with $A - T_0$ for arbitrary $T \in$ bicomm(A), and hence by (2.4) $T - T_0$ is $A - T_0$-s-regular. Theorem 2.3 applied with $A - T_0$ in place of A gives local constancy for the generalized range and the closure of the generalized kernel. ∎

We can now see stability of the nullity and deficiency.

THEOREM 2.5. *If $A \in B(X)$ is s-regular then there is $\delta > 0$ such that if $T \in B(X)$ is A-s-regular with $\|T\| < \delta$ then we have the following implications:*

(i) $A \in \Phi_+(X) \Rightarrow n(A-T) = n(A)$;

(ii) $A \in \Phi_-(X) \Rightarrow d(A-T) = d(A)$.

Proof. By Theorem 2.3 there is $\delta > 0$ such that $N(A-T) \subseteq R^\infty(A)$ whenever T is A-s-regular with $\|T\| < \delta$. It follows that

$$N(A-T) = N(A-T) \cap R^\infty(A) = N(A-T)^\wedge,$$

where we write $S^\wedge$ for the restriction to $R^\infty(A)$ of operators S leaving $R^\infty(A)$ invariant. Now $(A-T)^\wedge$ and $A^\wedge$ are both surjective; by the continuity of the index it follows that

$$n(A-T) = n((A-T)^\wedge) = \mathrm{ind}((A-T)^\wedge) = \mathrm{ind}(A^\wedge). \tag{2.8}$$

The right hand side of (2.8) is independent of T giving (i), and (ii) follows by duality. ∎

Now let Y also be a Banach space, and let $B(X,Y)$ be the space of all bounded operators from X into Y.

One might attempt to generalize the notion of "jump" (see (0.4) and (0.5)) in the following manner:

$$j(A) = \lim_{T\to 0}(n(A) - n(A-T)) \text{ if } A \in \Phi_+(X,Y),$$

and

$$j(A) = \lim_{T\to 0}(d(A) - d(A-T)) \text{ if } A \in \Phi_-(X,Y).$$

But generally, these two limits do not exist. The following results characterize those operators for which these limits do exist.

THEOREM 2.6. *If $A \in B(X,Y)$, then the following conditions are equivalent:*

(i) *A is bounded from below;*

(ii) *$A \in \Phi_{\pm}(X,Y)$ with $\operatorname{ind}(A) \leqslant 0$ and the limit*

$$\lim_{T\to 0}(n(A) - n(A-T))$$

exists;

(iii) *$A \in \Phi_{\pm}(X,Y)$ with $\operatorname{ind}(A) \leqslant 0$ and there is a $\delta > 0$ such that for every $T \in B(X,Y)$ with $\|T\| < \delta$ one has*

$$n(A-T) = n(A);$$

(iv) *$A \in \operatorname{int}\{$injective operators$\}$, where $\operatorname{int}\{M\}$ denotes the interior of the set M.*

Proof. Since the set of operators that are bounded from below is open in $B(X,Y)$, it is clear that (i) implies (ii), (iii) and (iv).

We prove that (ii) implies (iii). By the definition of the limit, there are $\delta > 0$ and $d \in \mathbb{N}$ such that for every $T \in B(X,Y)$ with $\|T\| < \delta$ one has $0 \leqslant n(A) - n(A-T) - d < 1$. Hence $n(A-T) = n(A) - d$ for all $T \in B(X,Y)$ with $\|T\| < \delta$. If $T = (\delta/2\|A\|)A$, then $\|T\| < \delta$. Thus $n((1-(\delta/2\|A\|))A) = n(A) - d$, and consequently, $d = 0$. Hence $n(A-T) = n(A)$ for all $T \in B(X,Y)$ with $\|T\| < \delta$.

(iii) $\Rightarrow$ (iv). By [6], Corollaire 3 (2), for every $\varepsilon > 0$ there is an $A_\varepsilon \in B(X,Y)$ that is bounded from below such that $\|A - A_\varepsilon\| < \varepsilon$. Let $\varepsilon = \delta$ and $T_\varepsilon = A - A_\varepsilon$. Then $\|T_\varepsilon\| < \delta$, so we have $n(A) = n(A - T_\varepsilon) = n(A_\varepsilon) = 0$. Thus for every $T \in B(X,Y)$ with $\|T\| < \delta$, we obtain $n(A-T) = 0$, and consequently, $A \in \operatorname{int}\{$injective operators$\}$.

(iv) $\Rightarrow$ (i). Assume that A is not bounded from below. Then there is a sequence $\{x_n\} \subset X$ with $\|x_n\| = 1$ and $Ax_n \to 0$ as $n \to \infty$. By the Hahn-Banach theorem, we can find a sequence $\{f_n\} \subset X^*$ (where X^* is the topological dual of X) such that $f_n(x_n) = 1 = \|f_n\|$.

Let $T_n = f_n \otimes Ax_n \in B(X,Y)$ be given by $T_n(x) = f_n(x)Ax_n$ for all $x \in X$. Then $T_n(x_n) = f_n(x_n)Ax_n = Ax_n$. Hence $x_n \in N(A - T_n)$.

But since

$$\|A - (A - T_n)\| = \|T_n\| = \|Ax_n\| \to 0 \quad (n \to \infty),$$

this show that $A \notin \operatorname{int}\{$injective operators$\}$. The implication (iv) $\Rightarrow$ (i) follows. ∎

By duality, we have the following result.

THEOREM 2.7. *If $A \in B(X,Y)$, then the following conditions are equivalent:*

(i) *A is surjective;*

(ii) *$A \in \Phi_{\pm}(X,Y)$ with $\mathrm{ind}(A) \geqslant 0$, and the limit*

$$\lim_{T\to 0}(d(A) - d(A-T))$$

exists;

(iii) *$A \in \Phi_{\pm}(X,Y)$ $\mathrm{ind}(A) \geqslant 0$ and there is a $\delta > 0$ such that for every $T \in B(X,Y)$ with $\|T\| < \delta$ we have*

$$d(A-T) = d(A);$$

(iv) *$A \in \mathrm{int}\{$operators with dense range$\}$.*

We introduce the following notation:

$$CR(X,Y) = \{T \in B(X,Y);\ T \text{ has closed range}\},$$

and

$$M(X,Y) = \{T \in B(X,Y);\ T \text{ is surjective or bounded from below }\}.$$

An operator $A \in M(X,Y)$ will be called *monojective.*

REMARK 2.8. The condition "$A \in \Phi_{\pm}(X,Y)$" in Theorem 2.6 (ii), (iii), Theorem 2.7 (ii) and (iii) may be replaced by the condition "$A \in \mathrm{int}\{CR(X,Y)\}$". Indeed, using [1], Theorem V.2.6, it is easy to deduce that

$$\mathrm{int}\{CR(X,Y)\} = \Phi_{\pm}(X,Y).$$

For every $n \in \mathbb{N}^* \cup \{\infty\}$ we define the sets

$$\mathrm{S\,Reg}_n(X) = \mathrm{Typ}_n(X) \cap CR(X)$$

(cf. Definition 1.1). Then we have the following inclusions :

$$M(X,X) = M(X) \subseteq \mathrm{S\,Reg}_\infty(X) \subseteq \cdots \subseteq \mathrm{S\,Reg}_n(X) \subseteq \cdots \subseteq \mathrm{S\,Reg}_1(X).$$

PROPOSITION 2.9. *For all $n \in \mathbb{N}^* \cup \{\infty\}$*

$$\mathrm{int}\{\mathrm{S\,Reg}_n(X)\} = M(X).$$

Proof. It suffices to show that

$$\mathrm{int}\{\mathrm{S\,Reg}_1(X)\} \subset M(X).$$

Assume that $A \in \mathrm{int}\{\mathrm{Typ}_1(X)\}$ and $N(A) \neq 0$, $R(A) \neq X$. Then there exist $u \in N(A)$ with $\|u\| = 1$ and $z \notin R(A)$. By the Hahn-Banach theorem, there exists $f \in X^*$ such that $f(u) = 1 = \|f\|$ and $f(z) = 0$. For $\varepsilon > 0$, let $A_\varepsilon = \varepsilon f \otimes Az$ be given by $A_\varepsilon x = \varepsilon f(x)Az$ for all $x \in X$. We also set $w_\varepsilon = \varepsilon^{-1}u + z$. Then $w_\varepsilon \in N(A - A_\varepsilon) \setminus R(A)$. On the other hand, $R(A - A_\varepsilon) \subseteq R(A) + R(A_\varepsilon) \subset R(A)$. Consequently, $A - A_\varepsilon \notin \mathrm{Typ}_1(X)$, which is a contradiction. ∎

REMARK 2.10. Proposition 2.9 generalizes [4], Théorème 6.5 to Banach spaces.

COROLLARY 2.11.

(i) $\operatorname{int}\{T \in \Phi_{\pm}(X);\ j(T) = 0\} = M(X)$.

(ii) *If $A \in B(X)$ then the conditions of Theorem 2.6 (with $X = Y$) are equivalent to:*

$$A \in \operatorname{int}\{T \in \Phi_{\pm}(X);\ \operatorname{ind}(T) \leqslant 0 \quad \textit{and} \quad j(T) = 0\}.$$

(iii) *$A \in B(X)$ then the conditions of Theorem 2.7 (with $X = Y$) are equivalent to :*

$$A \in \operatorname{int}\{T \in \Phi_{\pm}(X);\ \operatorname{ind}(T) \geqslant 0 \quad \textit{and} \quad j(T) = 0\}.$$

(iv) $\operatorname{int}\{T \in \Phi_{\pm}(X);\ \operatorname{ind}(T) = 0 = j(T)\} = \mathrm{GL}(X)$ *where* $\mathrm{GL}(X)$ *denotes the group of invertible operators of* $B(X)$.

Proof. By (0.9), $\{T \in \Phi_{\pm}(X);\ j(T) = 0\} \subset \operatorname{S Reg}_n(X)$ for all $n \in \mathbb{N} \cup \{\infty\}$. But since $M(X) \subset \{T \in \Phi_{\pm}(X);\ j(T) = 0\} \subset \operatorname{S Reg}_n(X)$, it follows from Proposition 2.9 that (i) holds.

It is clear that Theorem 2.6 (i) implies $A \in \operatorname{int}\{T \in \Phi_{\pm}(X);\ \operatorname{ind}(T) \leqslant 0 \text{ and } j(T) = 0\}$.

Assume that $A \in \operatorname{int}\{T \in \Phi_{\pm}(X);\ \operatorname{ind}(T) \leqslant 0 \text{ and } j(T) = 0\}$. Then $A \in M(X)$ and $\operatorname{ind}(A) \leqslant 0$. This proves (ii), and (iii) follows by duality.

The equality (iv) is immediate from (ii) and (iii). ∎

3. THE PUNCTURED NEIGHBOURHOOD THEOREM

Theorems 2.6 and 2.7 show that a definition of "jump" involving all perturbations of A necessarily leads to the jump being 0. On the other hand, the classical theory shows that if one considers only perturbations subject to certain restrictions, e.g. λI, or T where T is invertible and commutes with A, or λB where B is a fixed bounded operator (see [1], Corollary V.1.7), then positive jumps may occur. In the following result we consider another class of perturbations.

We say that the operator T is *dense* if its range $R(T)$ is a dense subset of X, or equivalently if the adjoint T^* is injective.

THEOREM 3.1. *If $A \in B(X)$ then there is $\delta > 0$ for which if T and T' in* bicomm(A) *satisfy* $\max(\|T\|, \|T'\|) < \delta$ *then we have the following implications:*

(i) $A \in \Phi_+(X)$ *and* T, T' *injective* $\Rightarrow n(A-T) = n(A-T')$;

(ii) $A \in \Phi_-(X)$ *and* T, T' *dense* $\Rightarrow d(A-T) = d(A-T')$.

Proof. This proceeds via the *Kato decomposition* ([3], [10]) of A:

$$X = X_1 \oplus X_0, \tag{3.1}$$

where X_1 and X_0 are invariant subspaces for A, X_0 is finite dimensional, the restriction A_1 of A to X_1 is s-regular and the restriction A_0 of A to X_0 is nilpotent. Towards the proof of (i), we claim that if $T \in$ bicomm(A) with $\|T\| < \delta$ then

$$N(A-T) \subseteq X_1. \tag{3.2}$$

Since T commutes with the induced projection it leaves both X_1 and X_0 invariant; now if $x = x_1 + x_0 \in N(A-T)$ with $x_1 \in X_1$ and $x_0 \in X_0$ then $(A-T)x_1 + (A-T)x_0 = (A-T)x = 0$ giving

$$(A-T)x_1 = -(A-T)x_0 \in X_1 \cap X_0 = \{0\},$$

so that $(A-T)x_0 = 0$. Now, recalling Proposition 1.3 (i), if T is also injective then

$$x_0 \in N(A-T) \cap X_0 \subseteq N(A-T) \cap N^\infty(A) = \{0\}.$$

This means that $x_0 = 0$ and hence $x = x_1 \in X_1$ giving (3.2). Thus

$$N(A-T) = N(A-T) \cap X_1 = N(A_1 - T_1),$$

writing T_1 and T_0 for the restrictions of T to X_1 and X_0. Since A_1 is s-regular and commutes with T_1, Theorem 2.4 gives $\delta > 0$ for which $A_1 - T_1$ is s-regular with

$$R^\infty(A_1 - T_1) = R^\infty(A_1) = R^\infty(A).$$

Thus $N(A_1 - T_1) \subseteq R^\infty(A)$, and the restriction of $A_1 - T_1$ to $R^\infty(A)$ is surjective. Once more the continuity of the index gives

$$n(A_1 - T_1) = n((A_1 - T_1)^\wedge) = \mathrm{ind}(A_1^\wedge) = \mathrm{ind}(A^\wedge).$$

It follows that

$$n(A-T) = n(A_1 - T_1) = \mathrm{ind}(A^\wedge),$$

independent of T. This proves (i), and (ii) follows by duality. ∎

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