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A CLASS OF STRONGLY IRREDUCIBLE OPERATORS WITH NICE PROPERTY

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ABSTRACT. A bounded linear operator T on Hilbert space H is strongly irreducible if T does not commute with any non-trivial idempotent. T is said to have nice property if either $\mathcal{A}'(T)$ or $\mathcal{A}'(T^*)$ is a commutative strictly cyclic operator algebra. This paper uses the multiplication operators on Sobolev space to construct a class of strongly irreducible operators with nice property and proves that the set of operators similar to orthogonal direct sums of finitely many strongly irreducible operators with nice property is dense in $\mathcal{L}(H)$. The paper also proves that for each essentially normal operator T with connected $\sigma_e(T)$ and $\operatorname{ind}(T-\lambda) = 0$ $(\lambda \in \rho_{S-F}(T))$, there exists a compact operator K such that $T+K$ is strongly irreducible.

KEYWORDS: *Strongly irreducible operator, Sobolev space, spectrum, strictly cyclic operator algebra.*

AMS SUBJECT CLASSIFICATION: Primary 47A, 47B, 47C; Secondary 46B, 46C, 46H.

1. INTRODUCTION

It is well known that in finite dimensions, and $n \times n$ matrix T is similar to a block diagonal matrix, i.e.

$$T = W \begin{pmatrix} J_1 & & 0 & \\ & J_2 & & \\ & & \ddots & \\ & 0 & & \\ & & & J_k \end{pmatrix} W^{-1},$$

where W is an invertible matrix and J_i is a Jordan block

$$J_i = \begin{pmatrix} \lambda_i & & & & \\ 1 & \lambda_i & & 0 & \\ & \ddots & \ddots & & \\ & 0 & & & \\ & & & 1 & \lambda_i \end{pmatrix} \qquad (i = 1, 2, \ldots, k).$$

What is the analogue of the Jordan block in infinite-dimensional Hilbert space? C. Apostol, R. Douglas and C. Foiaş ([2]) showed that nilpotent operators can be classified up to quasisimilarity by their "Jordan forms". K.R. Davidson and D.A. Herrero ([8]) extended this result to bitriangular operators. The "Jordan operators" they considered are all operators which are direct sums of the basic building blocks $\lambda I_n + J_n$, where J_n is the $n \times n$ Jordan nilpotent matrix.

Recall that a bounded linear operator T acting on Hilbert space H is said to be strongly irreducible if T does not commute with any non-trivial idempotent ([13], [21]). Z.J. Jiang ([21]) thinks that the strongly irreducible operator can be considered as the natural generalization of Jordan blocks in infinite dimensional Hilbert space. D.A. Herrero (personal communication) believed that in order for a class of operators to be considered as the analogue of Jordan blocks, the class needs to include all strongly irreducible operators which have an additional "nice property" in the following sense.

An operator T is said to have the *nice property* if either $\mathcal{A}'(T)$, the commutant of T, or $\mathcal{A}'(T^*)$, the commutant of the adjoint of T, is a commutative strictly cyclic operator algebra.

A weakly closed operator algebra $\mathcal{A}$ on Hilbert space H is said to be *strictly cyclic* if there exists a vector $e \in H$ such that

$$\mathcal{A}e := \{Ae : A \in \mathcal{A}\} = H.$$

The aim of this paper is to vindicate Herrero's belief by showing that similarities of direct sums of strongly irreducible operators with a nice property can be used to model arbitrary Hilbert space operators up to small norm or compact perturbation.

Recall that each $k \times k$ Jordan block

$$J = \begin{pmatrix} \lambda & & & 0 \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda \end{pmatrix}$$

has the following properties:

(i) J is strongly irreducible on $\mathbf{C}^k$;

(ii) $\mathcal{A}'(J)$ is strictly cyclic;

(iii)

$$\mathcal{A}'(J) = \begin{pmatrix} a_1 & & & 0 \\ a_2 & \ddots & & \\ & \ddots & \ddots & \\ a_k & & a_2 & a_1 \end{pmatrix},$$

where $a_i \in \mathbf{C}$, $i = 1, 2, \ldots, k$.

(iv) $\mathcal{A}^\alpha(J)$, the algebra generated by the rational functions of J with poles outside $\sigma(J) = \{\lambda\}$, is an algebra of strict multiplicity k.

An operator algebra $\mathcal{A}$ on Hilbert space H is an algebra of strict multiplicity k if there exist k vectors $e_1, \ldots, e_k$ in H such that

$$\sum_{i=1}^{k} \mathcal{A}e_i = H,$$

and there are no $k-1$ vectors satisfying this property ([17], [18]).

In the second section of this paper we use the multiplication operators on some Sobolev space to construct a class of operators $\{M_k(\Omega)\}$. Each $\{M_k(\Omega)\}$ is associate with a Cauchy region Ω with the cone property in the complex plane and satisfies:

(i) $\{M_k(\Omega)\}$ is strongly irreducible;

(ii) $\mathcal{A}'(M_k(\Omega))$ is strictly cyclic;

(iii)

$$\mathcal{A}'(M_k(\Omega)) = \begin{pmatrix} M_{f_1}(\Omega) & & & 0 \\ M_{f_2}(\Omega) & \ddots & & \\ & \ddots & \ddots & \\ M_{f_k}(\Omega) & & M_{f_2}(\Omega) & M_{f_1}(\Omega) \end{pmatrix},$$

where $M_{f_i}(\Omega)$ is "multiplication by function f_i" on a subspace of Sobolev space.

(iv) $\mathcal{A}^\alpha(M_k(\Omega))$ is an operator algebra of strict multiplicity k.

Mainly using this class of operators, and by also using a result about Apostol–Morrel simple models ([4]) and the Similarity Orbit Theorem ([3]), we prove that the operators similar to orthogonal direct sums of finitely many strongly irreducible operators with nice property are dense in $\mathcal{L}(H)$, the algebra of all bounded linear operators acting on Hilbert space H.

In the third section, for an arbitrary compact subset Γ of C, we construct a strongly irreducible operator with the nice property, whose spectrum is Γ. We also prove that if T is an essentially normal operator such that $\sigma_e(T)$ is connected and $\operatorname{ind}(T-\lambda)=0$, for all $\lambda \in \rho_{S-F}(T)$, then T unitarily equivalent to a compact perturbation of some strongly irreducible operator with the nice property.

2. APPROXIMATE JORDAN THEOREM

Recall that a Cauchy domain Ω is a non-empty bounded open set of $\mathbf{C}$ whose boundary consists of finitely many pairwise disjoint rectifiable Jordan curves. A connected Cauchy domain is called a *Cauchy region*.

Let Ω denote a Cauchy region with the cone property (see [1]) and let $W^{2,2}(\Omega)$ denote the Sobolev space:

$$W^{2,2}(\Omega) = \{f \in L^2(\Omega, \mathrm{d}m) : \text{the distributional partial derivatives of first and second order of } f \text{ belong to } L^2(\Omega, \mathrm{d}m)\},$$

where $\mathrm{d}m$ denotes the planar Lebesgue measure.

It is well-known from the Sobolev embedding theorem ([1]) that $W^{2,2}(\Omega)$ is the Hilbert space of continuous functions on $\overline{\Omega}$, the closure of Ω, under the norm

$$\|f\| = \Big(\int_{\Omega} \sum_{|\alpha|\leqslant 2} |D^\alpha f|^2 \mathrm{d}m\Big)^{\frac{1}{2}},$$

where α is a 2-index and D^α is one of the differential operators of order $|\alpha|$. Furthermore, $W^{2,2}(\Omega)$ is a regular Banach algebra with identity (under pointwise multiplication and an equivalent norm) whose maximal ideal space can be naturally identified with $\overline{\Omega}$ via "point evaluation".

Set

$$W(\Omega) = \{M_f : f \in W^{2,2}(\Omega)\},$$

where $M_f =$ "multiplication by f" on $W^{2,2}(\Omega)$, i.e., $M_f \in \mathcal{L}(W^{2,2}(\Omega))$ and $M_f g = fg$ for each $g \in W^{2,2}(\Omega)$. It is obvious that $W(\Omega)$ is a strictly cyclic operator algebra with strictly cyclic vector e ($e(s,t)=1$).

PROPOSITION 2.1. *Let M_λ be the multiplication by the independent variable λ on $W^{2,2}(\Omega)$, then*

(i) $\sigma(M_\lambda) = \sigma_e(M_\lambda) = \overline{\Omega}$, *where $\sigma_e(T)$ denotes the essential spectrum of T,* $\operatorname{nul}(M_\lambda - \lambda_0) = 0$ *and* $\operatorname{nul}(M_\lambda - \lambda_0)^* = 1$ *for all $\lambda_0 \in \overline{\Omega}$;*

(ii) $\mathcal{A}'(M_\lambda) = W(\Omega)$, *i.e. $\mathcal{A}'(M_\lambda)$ is strictly cyclic;*

(iii) *M_λ is strongly irreducible.*

Proof. (i) It is easy to see that $\sigma(M_\lambda) = \overline{\Omega}$, $\mathrm{nul}(M_\lambda - \lambda_0) = 0$ and $\mathrm{nul}(M_\lambda - \lambda_0)^* = 1$ for all $\lambda_0 \in \overline{\Omega}$.

Set

$$f(s,t) = [(s-s_0)^2 + (t-t_0)^2]^{\frac{2}{3}} = |\lambda - \lambda_0|^{\frac{4}{3}},$$

where $\lambda_0 = s_0 + it_0 \in \Omega$. Calculations indicate that $f \in W^{2,2}(\Omega)$ and $(\lambda-\lambda_0)^{-1} f \notin W^{2,2}(\Omega)$, i.e. $f \notin \mathrm{Im}(M_\lambda - \lambda_0)$. Since $f(\lambda_0) = 0$, f is in the closure of $\mathrm{Im}(M_\lambda - \lambda_0)$, i.e. $\mathrm{Im}(M_\lambda - \lambda_0)$ is not closed and $\lambda_0 \in \sigma_e(M_\lambda)$. Therefore $\sigma_e(M_\lambda) = \overline{\Omega}$.

(ii) Since $W(\Omega)$ is a strictly cyclic algebra,

$$W^*(\Omega) = \{k^* : k^*(M_f) = (f,k)_{W^{2,2}(\Omega)}, \quad k \text{ in } W^{2,2}(\Omega)\},$$

where $W^*(\Omega)$ denotes the dual of Banach space $W(\Omega)$ ([23]). For each $\mu \in \overline{\Omega}$, if k_μ^* denotes the multiplication functional on $W(\Omega) : k_\mu^*(M_f) = f(\mu)$, then there exists a $k_\mu \in W^{2,2}(\Omega)$ such that $f(\mu) = (f, k_\mu)$.

Given $T \in \mathcal{A}'(M_\lambda)$, we have $M_\lambda^* T^* k_\mu = T^* M_\lambda^* k_\mu = \overline{\mu} T^* k_\mu$. Since $\mathrm{nul}(M_\lambda - \mu)^* = 1$, $T^* k_\mu = \overline{t(\mu)} k_\mu$ for some $t(\mu) \in \mathbf{C}$. Therefore, for each $\lambda \in \Omega$ and each $f \in W^{2,2}(\Omega)$,

$$(Tf)(\lambda) = (Tf, k_\lambda) = (f, T^* k_\lambda) = (f, \overline{t(\lambda)} k_\mu) = t(\lambda) f(\lambda).$$

This implies that $t(\lambda) = (Te)(\lambda) \in W^{2,2}(\Omega)$ and $T = M_t \in W(\Omega)$.

(iii) Assume that P is an idempotent and commutes with M_λ. From (ii), $P = M_g$ for some $g \in W^{2,2}(\Omega)$. Since Ω is connected, $g^2 = g$ implies that $g = 0$ or $g = e$, i.e. $P = 0$ or $P = 1$. Thus M_λ is strongly irreducible and the proof is complete. ∎

Let $\mathcal{A}^\alpha(M_\lambda)$ be the algebra generated by rational functions of M_λ with poles outside $\overline{\Omega}$ and denote $R(\Omega) = \mathcal{A}^\alpha(M_\lambda)e$. Then it is obvious that $R(\Omega)$ is the subspace of $W^{2,2}(\Omega)$ generated by rational functions with poles outside $\overline{\Omega}$ and each f in $R(\Omega)$ has an analytic continuation to $\overline{\Omega}^0$, the interior of $\overline{\Omega}$, and $R(\Omega)$ is invariant under M_λ. Denote $M_\lambda(\Omega) = M_\lambda|_{R(\Omega)}$. Then $M_\lambda(\Omega)$ is a rationally strictly cyclic operator on Hilbert space $R(\Omega)$ with strictly cyclic vector e.

Proposition 2.2. (i) *For all* $\lambda_0 \in \overline{\Omega}^0$,

$$\sigma(M_\lambda(\Omega)) = \overline{\Omega};$$

$$\sigma_e(M_\lambda(\Omega)) = \overline{\Omega} \setminus \overline{\Omega}^0,$$

$$\mathrm{nul}(M_\lambda(\Omega) - \lambda_0) = 0$$

and

$$\mathrm{ind}(M_\lambda(\Omega) - \lambda_0) = 1.$$

(ii) $\mathcal{A}'(M_\lambda(\Omega)) = \mathcal{A}^\alpha(M_\lambda(\Omega)) = \{M_f(\Omega) : f \in R(\Omega)\}$, *where* $M_f(\Omega) = M_f|_{R(\Omega)}$. *Thus* $\mathcal{A}'(M_\lambda(\Omega))$ *is strictly cyclic.*

(iii) $M_\lambda(\Omega)$ *is a strongly irreducible operator.*

Proof. (i) It is obvious that $\sigma(M_\lambda(\Omega)) \subset \overline{\Omega}$. For each $\lambda_0 \in \overline{\Omega}^0$, assume that $[M_\lambda(\Omega) - \lambda_0]f_n = F_n \overset{W^{2,2}(\Omega)}{\longrightarrow} F$, where $f_n \in R(\Omega)$. By Sobolev's imbedding theorem, F_n and F are analytic in $\overline{\Omega}^0$ and continuous to $\partial\Omega$. Since $F(\lambda_0) = 0$, there exist a number $\delta > 0$ and a function $\overline{f}_1$ analytic in $\Gamma = \{\lambda \in \mathbf{C} : |\lambda - \lambda_0| < \delta\}$ such that $\overline{\Gamma} = \{\lambda \in \mathbf{C} : |\lambda - \lambda_0| \leqslant \delta\} \subset \overline{\Omega}^0$ and $F(\lambda) = (\lambda - \lambda_0)\overline{f}_1(\lambda)$ if $\lambda \in \Gamma$. Denote $\Sigma = \overline{\Omega}^0 \setminus \frac{1}{3}\overline{\Gamma} = \Omega \setminus \{\lambda \in \mathbf{C} : |\lambda - \lambda_0| \leqslant \frac{\delta}{3}\}$; then $F(\lambda)$ is analytic in Σ. Define

$$f(\lambda) = \begin{cases} \overline{f}_1(\lambda), & \lambda \in \Gamma; \\ \frac{F(\lambda)}{\lambda - \lambda_0}, & \lambda \in \Sigma; \end{cases}$$

then $f(\lambda)$ is an analytic extension of $\overline{f}_1$ satisfying $F(\lambda) = (\lambda - \lambda_0)f(\lambda)$ for $\lambda \in \overline{\Omega}^0$. Note that

$$\begin{aligned} \int_\Sigma |f_n - f|^2 \, dm &\leqslant \frac{9}{\delta^2} \int_\Sigma |(\lambda - \lambda_0)(f_n(\lambda) - f(\lambda))|^2 \, dm \\ &\leqslant \|F_n - F\|_{W^{2,2}(\Omega)} \to 0, \quad (n \to \infty) \end{aligned}$$

and

$$\frac{\partial}{\partial s}F_n(\lambda) = f_n(\lambda) + (\lambda - \lambda_0)f_n'(\lambda),$$

$$\frac{\partial}{\partial s}F(\lambda) = f(\lambda) + (\lambda - \lambda_0)f'(\lambda).$$

From $\frac{\partial}{\partial s}F_n \overset{L^2(\Sigma)}{\longrightarrow} \frac{\partial}{\partial s}F$ and $f_n \overset{L^2(\Sigma)}{\longrightarrow} f$, we get

$$(\lambda - \lambda_0)f_n'(\lambda) \overset{L^2(\Sigma)}{\longrightarrow} (\lambda - \lambda_0)f'.$$

These convergences mean that the restrictions of the functions to Σ converge in $L^2(\Sigma)$. Thus

$$\int_\Sigma |f_n' - f'| \, dm \leqslant \frac{9}{\delta^2} \int_\Sigma |(\lambda - \lambda_0)(f_n' - f')|^2 \, dm \to 0 \quad (n \to \infty).$$

Similarily, from $\frac{\partial^2}{\partial s^2}F_n \overset{L^2(\Sigma)}{\longrightarrow} \frac{\partial^2}{\partial s^2}F$ and $f_n' \overset{L^2(\Sigma)}{\longrightarrow} f'$, we have

$$(\lambda - \lambda_0)f_n''(\lambda) \overset{L^2(\Sigma)}{\longrightarrow} (\lambda - \lambda_0)f'',$$

therefore, $f_n''(\lambda) \xrightarrow{L^2(\Sigma)} f''$. Finally we get $f_n \xrightarrow{W^{2,2}(\Sigma)} f$. From Sobolev's imbedding theorem, f_n converge to f uniformly on $r = \partial\Gamma$. Then for $\lambda \in \frac{2}{3}\Gamma$ and $0 \leqslant k \leqslant 2$,

$$|f_n^{(k)}(\lambda) - f^{(k)}(\lambda)| \leqslant \frac{1}{2\pi} \int_r \frac{|f_n(\xi) - f(\xi)|}{|\xi - \lambda|^{k+1}} \, d|\xi| \leqslant \frac{3^{k+1}}{\delta^k} \max_{\xi \in r} |f_n(\xi) - f(\xi)|.$$

Therefore, $f_n \xrightarrow{W^{2,2}(\frac{2}{3}\Gamma)} f$. Note that $f_n \xrightarrow{W^{2,2}(\Sigma)} f$ and $f_n \xrightarrow{W^{2,2}(\frac{2}{3}\Gamma)} f$ imply $f_n \xrightarrow{W^{2,2}(\Omega)} f$ and $f \in R(\Omega)$. Thus the range of $M_\lambda(\Omega) - \lambda_0$ is closed. The remainder of (i) is obvious.

(ii), (iii). By the same arguments used in the proof of Proposition 2.1 (ii) and (iii), we can prove (ii) and (iii). ∎

Given A, B in $\mathcal{L}(H)$, the mapping $\tau_{A,B} : \mathcal{L}(H) \to \mathcal{L}(H)$ is defined by $\tau_{A,B}(X) = AX - XB$ for all $X \in \mathcal{L}(H)$.

LEMMA 2.3. $[\mathrm{Im}\, \tau_{M_\lambda(\Omega), M_\lambda(\Omega)}] \cap [\mathrm{Ker}\, \tau_{M_\lambda(\Omega), M_\lambda(\Omega)}] = \{0\}$, *i.e. if there exists* $X \in \mathcal{L}(R(\Omega))$ *satisfying*

$$XM_\lambda(\Omega) = M_\lambda(\Omega)X$$

and $M_\lambda(\Omega)Y - YM_\lambda(\Omega) = X$ *for some* $Y = \mathcal{L}(R(\Omega))$, *then* $X = 0$.

Proof. Without loss of generality, we can assume that $\Omega \subset S = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$. Otherwise, consider $M_\lambda(\Omega')$, where $\Omega' = \left\{\frac{\lambda - \lambda_0}{a} : \lambda \in \Omega\right\}$, λ_0 is a fixed point in Ω and $0 < a =$ diameter of Ω.

From Proposition 2.2, $X = Mg(\Omega)$ for some g in $R(\Omega)$. Thus $M_\lambda(\Omega)Yf - YM_\lambda(\Omega)f = gf$ for all $f \in R(\Omega)$; $f = e$ implies

$$\lambda Y(e) - Y(\lambda) = g,$$

or

$$Y(\lambda) = \lambda h - g, \quad \text{where } h = Y(e) \in R(\Omega).$$

Now, $f = \lambda$ implies

$$\lambda Y(\lambda) - Y(\lambda^2) = \lambda g,$$

or

$$Y(\lambda^2) = \lambda(\lambda h - g) - \lambda g = \lambda^2 h - 2\lambda g, \cdots.$$

In general, $f = \lambda^{n-1}$ implies

$$\lambda Y(\lambda^{n-1}) - Y(\lambda^n) = \lambda^{n-1} g,$$

or

$$Y(\lambda^n) \doteq \lambda^n h - n\lambda^{n-1} g \quad (n = 1, 2, \dots).$$

Since $|\lambda|^{2n} \leqslant |\lambda|^{2(n-1)}$ ($\lambda \in \Omega$ and $n = 1, 2, \dots$),

$$\frac{n^2 \int_\Omega |\lambda|^{2(n-1)}\, dm}{\int_\Omega |\lambda|^{2n}\, dm} \to \infty \quad (n \to \infty).$$

Let a_n and b_n denote $n^2 \int_\Omega |\lambda|^{2(n-1)}\, dm$ and $\int_\Omega |\lambda|^{2n}\, dm$ respectively ($n = 1, 2, \dots$), then for each positive number M there exists N such that $a_n/b_n > M$ if $n > N$. Therefore, when $n > N + 2$,

$$\begin{aligned} \frac{\|n\lambda^{n-1}\|^2_{W^{2,2}(\Omega)}}{\|\lambda^n\|^2_{W^{2,2}(\Omega)}} &= \frac{\int_\Omega [n^2|\lambda|^{2(n-1)} + 2n^2(n-1)^2|\lambda|^{2(n-2)}]\, dm}{\int_\Omega [|\lambda|^{2n} + 2n^2|\lambda|^{2(n-1)} + 3n^2(n-1)^2|\lambda|^{2(n-2)}]\, dm} \\ &\quad + \frac{\int_\Omega [3n^2(n-1)^2(n-2)^2|\lambda|^{2(n-3)}]\, dm}{\int_\Omega [|\lambda|^{2n} + 2n^2|\lambda|^{2(n-1)} + 3n^2(n-1)|\lambda|^{2(n-2)}]\, dm} \\ &= \frac{a_n + 2n^2 a_{n-1} + 3n^2(n-1)^2 a_{n-2}}{b_n + 2n^2 b_{n-1} + 3n^2(n-1)^2 b_{n-2}} \\ &> \frac{M b_n + 2n^2 M b_{n-1} + 3n^2(n-1)^2 M b_{n-2}}{b_n + 2n^2 b_{n-1} + 3n^2(n-1)^2 b_{n-2}} = M, \end{aligned}$$

i.e.

$$\frac{\|n\lambda^{n-1}\|^2_{W^{2,2}(\Omega)}}{\|\lambda^n\|^2_{W^{2,2}(\Omega)}} \to \infty \quad (n \to \infty).$$

This implies that if $g \neq 0$, Y is unbounded. ∎

For each n, $1 \leqslant n < \infty$, we define

$$M_n(\Omega) = \begin{pmatrix} M_\lambda(\Omega) & & & 0 \\ I & M_\lambda(\Omega) & & \\ & \ddots & \ddots & \\ 0 & & I & M_\lambda(\Omega) \end{pmatrix}$$

with respect to the orthogonal direct sum $R(\Omega)^{(n)}$ of n copies of $R(\Omega)$.

PROPOSITION 2.4. *Let $M_n(\Omega)$ be defined as above, then*

(i)

$$\mathcal{A}'(M_n(\Omega)) = \begin{pmatrix} M_{f_1}(\Omega) & & & 0 \\ M_{f_2}(\Omega) & M_{f_1}(\Omega) & & \\ \vdots & \ddots & \ddots & \\ M_{f_n}(\Omega) & \cdots & M_{f_2}(\Omega) & M_{f_1}(\Omega) \end{pmatrix},$$

where $f_i \in R(\Omega)$, $i = 1, 2, \ldots, n$. *Furthermore,* $\mathcal{A}'(M_n(\Omega))$ *is strictly cyclic and its cyclic vector is* $e \oplus e \oplus \cdots \oplus e$;

(ii) $M_n(\Omega)$ *is strongly irreducible;*

(iii) $\sigma(M_n(\Omega)) = \overline{\Omega}$, $\sigma_e(M_n(\Omega)) = \overline{\Omega} \setminus \overline{\Omega}^0$, $\mathrm{nul}(M_n(\Omega) - \lambda_0) = 0$ *and* $\mathrm{ind}(M_n(\Omega) - \lambda_0) = -n$ *for* $\lambda_0 \in \overline{\Omega}^0$;

(iv) $\mathcal{A}^\alpha(M_n(\Omega))$ *is an operator algebra of strict multiplicity* n.

Proof. (i) Assume that

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \in \mathcal{A}'(M_n(\Omega)),$$

i.e.

$$\begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} M_\lambda(\Omega) & & & 0 \\ I & \ddots & & \\ & \ddots & \ddots & \\ 0 & & I & M_\lambda(\Omega) \end{pmatrix}$$
$$= \begin{pmatrix} M_\lambda(\Omega) & & & 0 \\ I & \ddots & & \\ & \ddots & \ddots & \\ 0 & & I & M_\lambda(\Omega) \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \cdots & \cdots & \cdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}.$$

At $(1, n)$, $(1, n-1)$ entries,

$$A_{1,n} M_\lambda(\Omega) = M_\lambda(\Omega) A_{1,n}$$

and

$$A_{1,n-1} M_\lambda(\Omega) + A_{1,n} = M_\lambda(\Omega) A_{1,n-1}.$$

By Lemma 2.3, $A_{1,n} = 0$ and $A_{1,n-1} \in \mathcal{A}'(M_\lambda(\Omega))$. Similarly, we can conclude that

$$A_{ij} = 0, \quad (1 \leqslant i < j \leqslant n).$$

The (i, i) entry indicates that

$$A_{ii} \in \mathcal{A}'(M_\lambda(\Omega)), \quad 1 \leqslant i \leqslant n.$$

At $(i+1,i)$ entry $(1 \leqslant i \leqslant n-1)$,

$$A_{i+1,i}M_\lambda(\Omega) + A_{i+1,i+1} = A_{ii} + M_\lambda(\Omega)A_{i+1,i}.$$

By Lemma 2.3, $A_{i+1,i+1} = A_{ii}$ and $A_{i+1,i} \in \mathcal{A}'(M_\lambda(\Omega))$. At $(i+2,i)$ entry $(1 \leqslant i \leqslant n-2)$,

$$A_{i+2,i}M_\lambda(\Omega) + A_{i+2,i+1} = A_{i+1,i} + M_\lambda(\Omega)A_{i+2,i}.$$

Thus $A_{i+2,i+1} = A_{i+1,i}$ and $A_{i+2,i} \in \mathcal{A}'(M_\lambda(\Omega))$. Using this argument repeatedly, we get the general form of A.

(ii) Take an idempotent P commuting with $M_n(\Omega)$. From (i), P has the form indicated in (i), since $P^2 = P$, $f_1^2 = f_1$. Since Ω is connected, either $f_1 \equiv 1$ or $f_1 \equiv 0$. In the either cases, $f_2 = f_3 = \cdots = f_n = 0$, i.e. either $P = I$ or $P = 0$.

(iii) Given $\lambda_0 \in \overline{\Omega}^0$, since $\ker(M_n(\Omega) - \lambda_0) = 0$ and since $\lambda_0 \in \rho_{\text{S-F}}(M_\lambda(\Omega))$, $\operatorname{Im}(M_n(\Omega) - \lambda_0)^* = R(\Omega)$. Thus, calculations show that $\operatorname{Im}(M_n(\Omega) - \lambda_0)^* = R^n(\Omega)$, i.e. $\lambda_0 \in \rho_{\text{S-F}}(M_\lambda(\Omega))$. It is not difficult to show that $\operatorname{nul}(M_n(\Omega) - \lambda_0)^* = n$ and $\operatorname{nul}(M_n(\Omega) - \lambda_0) = 0$.

(iv) Computations show that

$$\mathcal{A}^\alpha(M_n(\Omega))\overline{e}_1 + \mathcal{A}^\alpha(M_n(\Omega))\overline{e}_2 + \cdots + \mathcal{A}^\alpha(M_n(\Omega))\overline{e}_n = R^{(n)}(\Omega),$$

where $\overline{e}_i$ is the vector in $R^{(n)}(\Omega)$ whose i-th coordinate is e and j-th coordinate is 0 $(j \neq i)$. Thus the strict multiplicity of $\mathcal{A}^\alpha(M_n(\Omega))$ is $\leqslant n$. Since $\operatorname{ind}(M_n(\Omega) - \lambda_0) = -n$ for all $\lambda_0 \in \Omega$, $M_n(\Omega)$ rationally n-strictly cyclic ([14], [3]). ∎

Given a Cauchy region with the cone property, let $L^2(\partial\Omega)$ be the Hilbert space of (equivalence classes of) complex functions on $\partial\Omega$ which are square integrable with respect to $(\frac{1}{2\pi})$-times the arc-length measure on $\partial\Omega$. The subspace $H^2(\partial\Omega)$ spanned by the rational functions with poles outside $\overline{\Omega}$ is invariant under M, where M will stand for the "multiplication by λ" operator acting on $L^2(\partial\Omega)$. By $M_+(\partial\Omega)$ and $M_-(\partial\Omega)$ we shall denote the restriction of M to $H^2(\partial\Omega)$ and its compression to $L^2(\partial\Omega) \ominus H^2(\partial\Omega)$, i.e.

$$M = \begin{pmatrix} M_+(\partial\Omega) & * \\ 0 & M_-(\partial\Omega) \end{pmatrix} \begin{matrix} H^2(\partial\Omega) \\ L^2(\partial\Omega) \ominus H^2(\partial\Omega). \end{matrix}$$

It is well-known ([5], [9], [10], [15]) that

$$\begin{aligned} &\sigma(M) = \sigma_e(M) = \sigma_e(M_+(\partial\Omega)) = \sigma_e(M_-(\partial\Omega)) = \partial\Omega, \\ &\sigma_e(M_+(\partial\Omega)) = \sigma(M_-(\partial\Omega)) = \overline{\Omega}, \\ &\operatorname{nul}(M_+(\partial\Omega) - \lambda_0) = \operatorname{nul}(M_-(\partial\Omega) - \lambda_0)^* = 0, \\ &\operatorname{ind}(M_+(\partial\Omega) - \lambda_0) = \operatorname{ind}(M_-(\partial\Omega) - \lambda_0)^* = -1, \text{ for } \lambda_0 \in \Omega. \end{aligned}$$

LEMMA 2.5. *Given a region Ω with the cone property and given $\varepsilon > 0$, there exists a strongly irreducible operator $M_\infty(\Omega)$ with the nice property such that $\sigma(M_\infty(\Omega)| \subset \Omega$, $\mathrm{nul}(M_\infty(\Omega) - \lambda_0) = 0$ and $\mathrm{ind}(M_\infty(\Omega) - \lambda_0)^* = -\infty$ for all $\lambda_0 \in \rho_{\mathrm{S-F}}(M_\infty(\Omega))$, and $\|M_+(\partial\Omega)^{(\infty)} - M_\infty(\Omega)\| < \varepsilon$.*

Proof. From Similarity Orbit Theorem ([3]),

$$M_+(\partial\Omega)^{(\infty)} \in \overline{S}(M_+(\partial\Omega)^{(\infty)} \oplus N),$$

where N is normal and $\sigma(N) = \sigma_e(N) = \partial\Omega$, $\overline{S}(A)$ denotes the closure of similarity orbit $S(A)$ of operator A, $S(A) = \{WAW^{-1} : W \text{ is invertible}\}$. Thus there exists W invertible such that

$$\|M_+(\partial\Omega)^{(\infty)} - W(M_+(\partial\Omega)^{(\infty)} \oplus N)W^{-1}\| < \frac{\varepsilon}{3}.$$

Let Γ be an analytic region such that $\Gamma \subset \Omega \cap (\partial\Omega)_\delta$ and $\partial\Gamma \supset \partial\Omega$, where $\delta = \frac{\varepsilon}{6}\|W\|\,\|W^{-1}\|$ and for a subset F of $\mathbb{C}$, $F_\varepsilon := \{\lambda \in \mathbb{C} : \mathrm{dist}(\lambda, F) < \varepsilon\}$. Let Q' be a normal operator such that $\sigma(Q') = \sigma_e(Q') = \overline{\Gamma}$; then $\|N - UQ'U^{-1}\| < 2\delta$ for some unitary U ([15], Lemma 5.4) and

$$\|W(M_+(\partial\Omega)^{(\infty)} \oplus N)W^{-1} - W(M_+(\partial\Omega)^{(\infty)} \oplus Q)W^{-1}\| < \frac{\varepsilon}{3},$$

where $Q = UQ'U^{-1}$.

Let Σ be an analytic Cauchy region satisfying:

(i) $\partial\Sigma \subset \Gamma$ and $\partial\Sigma$ meets each component of Γ;

(ii) there is a $\mu > 0$ such that $\bigcup\{\partial\Sigma + r\mu : 0 \leqslant r \leqslant 1\} \subset \Gamma$.

Let L be the operator given by D.A. Herrero ([16], also see [19]), which satisfies:

(i) $\sigma_{\mathrm{le}}(L) = \bigcup\{\partial\Sigma + r\mu : 0 \leqslant r \leqslant 1\}$;

(ii) $\mathrm{nul}(L - \lambda_0) = 0$ and $\mathrm{ind}(L - \lambda_0) = -\infty$ for all $\lambda_0 \in \Sigma \setminus \sigma_{\mathrm{le}}(L)$;

(iii) $\mathcal{A}'(L)$ is strictly cyclic;

(iv) L is strongly irreducible.

The Similarity Orbit Theorem implies that

$$M_+(\partial\Omega)^{(\infty)} \oplus Q \in \overline{S}(L).$$

Thus there exists a V invertible such that

$$\|W(M_+(\partial\Omega)^{(\infty)} \oplus Q)W^{-1} - WVLW^{-1}W^{-1}\| < \frac{\varepsilon}{3}.$$

Denote $WVLV^{-1}W^{-1} = M_\infty(\Omega)$. Since strong irreducibility and strict cyclicity are invariant under similarity, the proof is complete. ∎

PROPOSITION 2.6. *The set of operator similar to orthogonal direct sums of finitely many strongly irreducible operators with the nice property is dense in $\mathcal{L}(H)$.*

Proof. Given $T \in \mathcal{L}(H)$ and $\varepsilon > 0$, from a result of C. Apostol and B.B. Morrel ([4], [15], Theorem 6.1), there exist operator T', S_+, S_- and N satisfying:

(i)

$$S_+ \simeq \bigoplus_{i=1}^{m} M_+(\partial\Omega)^{(k_i)}, \quad S_- \simeq \bigoplus_{i=1}^{n} M_-(\partial\Phi_i)^{(h_j)},$$

where $\{\Omega_i\}_{i=1}^m$ ($m < \infty$) and $\{\Phi_j\}_{j=1}^n$ ($n < \infty$) are two families of disjoint analytic Cauchy regions and $\overline{\Omega}_i^0 = \Omega_i$, $\overline{\Phi}_j^0 = \Phi_j$, ($i = 1, \ldots, m$; $j = 1, \ldots, n$) and

$$\bigcup_{i=1}^{m} \Omega_i \subset \rho_{\mathrm{S-F}}^-(T) \subset \Big(\bigcup_{i=1}^{m} \Omega_i\Big)_\varepsilon,$$

$$\bigcup_{i=1}^{n} \Phi_i \subset \rho_{\mathrm{S-F}}^-(T) \subset \Big(\bigcup_{i=1}^{n} \Phi_i\Big)_\varepsilon,$$

and each k_i (or h_j) equals to the index of $T - \lambda_0$ when $\lambda_0 \in \Omega_i$ (or Φ_j);

(ii) N is normal with finite spectrum and $\sigma(N) \subset \sigma(T)_\varepsilon$;

(iii) $T' \sim S_+ \oplus N \oplus S$ and $\|T - T'\| < \varepsilon$.

The Similarity Orbit Theorem implies that when $k_i < \infty$ (or $h_j < \infty$), $M_+(\partial\Omega_i)^{(k_i)} \in \overline{S}(M_k(\Omega_i))$ (or $M_-(\partial\Phi_i)^{(h_i)} \in \overline{S}(M_{h_i}(\Phi_i^*)^*)$, where $\Phi_j^* := \{\lambda \in \mathbf{C} : \overline{\lambda} \in \Phi_j\}$.

When $k_i = \infty$ (or $h_j = \infty$), from Lemma 2.5, we can find an $M_\infty(\Omega_i)$ (or $M_\infty^*(\Phi_j^*)$) to approximate $M_+(\partial\Omega_i)^\infty$ (or $M_-(\partial\Omega_j)^\infty$).

Assume that $N = \sum_{i=1}^{k} \oplus\lambda_i I_i$, where I_i is the identity on subspace H_i ($i = 1, 2, \ldots, k$). If H_i is infinite dimensional, $\lambda_i I_i \in \overline{S}(A + \lambda_i)$, where A is the (forward) weighted shift with weight sequence $\{\frac{1}{n}\}$, i.e., $Ae_n = \frac{1}{n}e_{n+1}$ if $\{e_n\}_{n=1}^\infty$ is the orthonormal basis of H_i. It is well-known ([24]) that $\mathcal{A}'(A + \lambda_i)$ is strictly cyclic and A is compact, and it is not difficult to show that $A + \lambda_i$ is strongly irreducible. Thus N can be approximated by orthogonal direct sums of finitely many strongly irreducible operators with the nice property. The proof of the proposition is now complete. ∎

3. STRONGLY IRREDUCIBLE OPERATORS WITH THE NICE PROPERTY AND ARBITRARY SPECTRUM

Given a compact subset Γ of $\mathbf{C}$ and assume that $\Gamma \subset (a,b)^2 = D$. Set

$$m_0(\Gamma) = \{f \in W_0^{2,2}(D) : f|_\Gamma \equiv 0\},$$

where $W_0^{2,2}(D) = \{f \in W^{2,2}(D) : f|_{\partial D} = 0\}$. It is clear that $W_0^{2,2}(D)$ is a subspace of $W^{2,2}(D)$. For each $f \in W^{2,2}(D)$, denote

$$M_f^0 = \text{"multiplication by } f\text{" on } W_0^{2,2}(D).$$

Then

$$M_f^0 = \begin{pmatrix} M_f^0|m_0(\Gamma) & * \\ 0 & M_f^0(\Gamma) \end{pmatrix}$$

with respect to the decomposition $W_0^{2,2}(D) = m_0(\Gamma) \oplus [W_0^{2,2}(D) \ominus m_0(\Gamma)]$. Denote $W_0(\Gamma) = \{M_f^0(\Gamma) : f \in W^{2,2}(D)\}$; then it is easy to see that $W_0(\Gamma)$ is a strictly cyclic operator algebra with stricty cyclic vector $e_0(\Gamma)$ = orthogonal projection of e onto $W_0^{2,2}(D) \ominus m_0(\Gamma)$.

PROPOSITION 3.1. (i) $\sigma(M_\lambda^0(\Gamma)) = \sigma_e(M_\lambda^0(\Gamma)) = \Gamma$, $\operatorname{nul}(M_\lambda^0(\Gamma) - \lambda_0) = 0$ *and* $\operatorname{nul}(M_\lambda^0(\Gamma) - \lambda_0)^* = 1$ *for* $\lambda_0 \in \Gamma$;

(ii) $\mathcal{A}'(M_\lambda^0(\Gamma)) = W_0(\Gamma)$, *thus* $\mathcal{A}'(M_\lambda^0(\Gamma))$ *is strictly cyclic;*

(iii) *If* Γ *is connected and consists of more than one point, then* $M_\lambda^0(\Gamma)$ *is a strongly irreducible operator.*

The proof is similar to that of Proposition 2.1, therefore it is omitted.

Define a mapping $\Delta : W_0^{2,2}(D) \to L^2(D, \mathrm{d}m)$ by

$$\Delta f = \left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}\right) f, \quad f \in W_0^{2,2}(D).$$

Since $W_0^{2,2}(D)$ contains no non-zero harmonic function, Δ is injective. Furthermore, for each $g \in L^2(D, \mathrm{d}m)$, assume that

$$g(s,t) = \sum_{m,n=1}^{\infty} C_{mn} \sin \frac{\pi}{l} m(s-a) \sin \frac{\pi}{l} n(t-a),$$

where $l = b - a$, C_{mm}'s are the Fourier coefficients of g and

$$\sum_{m,n=1}^{\infty} |C_{mn}|^2 < \infty.$$

Set

$$u_k(s,t) = \left(\frac{l}{\pi}\right)^2 \sum_{m+n\leqslant k} \frac{-C_{mn}}{m^2+n^2} \sin\frac{\pi}{l}m(s-a)\sin\frac{\pi}{l}n(t-a).$$

Since $\{D^\alpha u_k\}_{n=1}^\infty$ ($|\alpha| \leqslant 2$) is a Cauchy sequence in $L^2(D, \mathrm{d}m)$, there is a function f in $W_0^{2,2}(D)$ such that

$$f(s,t) = \sum_{m,n}^{\infty} - \left(\frac{1}{\pi}\right)^2 \frac{C_{mn}}{m^2+n^2} \sin\frac{\pi}{l}m(s-a)\sin\frac{\pi}{l}(t-a),$$

and $\Delta f = g$. Thus Δ is surjective. It is easy to see that the $W^{2,2}(D)$-norm, restricted to $W_0^{2,2}(D)$ is actually equivalent to norm

$$\|f\|_0 = \left[\int_D |\Delta f|^2 \,\mathrm{d}m\right]^{\frac{1}{2}}.$$

Thus Δ is an isometric isomorphism from the Hilbert space $W_0^{2,2}(D)$ onto the Hilbert space $L^2(D, \mathrm{d}m)$

PROPOSITION 3.2. (i) $M_\lambda^0 \simeq m_\lambda + K$, *where* m_λ *is the normal operator "multiplication by* λ*" on* $L^2(D, \mathrm{d}m)$ *and* K *is compact in Schatten 3-class* C^3.

(ii) $M_\lambda^0(\Gamma) \simeq \text{Normal} + \text{Compact}$ and $[M_\lambda^0(\Gamma)^*, M_\lambda^0(\Gamma)] \in C^3$ *where* $[A,B] = AB - BA$ *for operators* A *and* B.

Proof. (i) Consider the orthonormal basis

$$\left\{e_{mn}(s,t) = \frac{2}{l}\sin\frac{\pi}{l}m(s-a)\sin\frac{\pi}{l}n(t-a)\right\}_{m,n=1}^\infty$$

of $L^2(D, \mathrm{d}m)$ and the corresponding basis

$$\left\{\Delta^{-1}e_{mn} = -\left(\frac{l}{\pi}\right)^2 \frac{l}{m^2+n^2} e_{mn}\right\}_{m,n=1}^\infty$$

of $W_0^{2,2}(D)$. Note that

$$\begin{aligned}\Delta M_s^0 \Delta^{-1} e_{mn} &= -\left(\frac{l}{\pi}\right)^2 (m^2+n^2)^{-1} \Delta M_s^0 e_{mn} \\ &= -\left(\frac{l}{\pi}\right)^2 (m^2+n^2)^{-1} \Delta\left(\frac{2}{l}s \cdot \sin\frac{\pi}{l}m(s-a)\sin\frac{\pi}{l}n(t-a)\right) \\ &= se_{mn} - \frac{l}{\pi}2m(m^2+n^2)^{-1} f_{mn}, \quad m,n = 1,2,\ldots,\end{aligned}$$

where

$$\left\{f_{mm} = \frac{2}{l}\cos\frac{\pi}{l}m(s-a)\sin\frac{\pi}{l}n(t-a)\right\}_{m,n=1}^{\infty}$$

is another orthonormal basis of $L^2(D, \mathrm{d}m)$.

Thus

$$M_s^0 \simeq m_s + \sum \frac{-2lm}{\pi(m^2+n^2)} f_{mn} \otimes e_{mn}^* = m_s + K_s,$$

where m_s is the self-adjoint operator "multiplication by s" on $L^2(D, \mathrm{d}m)$ and computation shows that K_s is in $\mathcal{C}^3$.

A similar argument shows that

$$M_t^0 \simeq m_t + \sum \frac{-2ln}{\pi(m^2+n^2)} g_{mn} \otimes e_{mn}^* = m_t + K_t,$$

where m_t is the self-adjoint operator "multiplication by t" on $L^2(D, \mathrm{d}m)$, $\{g_{mn} = \frac{2}{l}\sin\frac{\pi}{l}m(s-a)\cos\frac{\pi}{l}n(t-a)\}_{m,n=1}^{\infty}$ is another orthogonal basis of $L^2(D, \mathrm{d}m)$ and $K_t \in \mathcal{C}^3$.

Thus $M_\lambda^0 \simeq m_\lambda + K$, $K \in \mathcal{C}^3$.

(ii) Since M_s^0 and M_t^0 have compact imaginary parts, $M_s^0(\Gamma)$ and $M_t^0(\Gamma)$ also have compact imaginary parts. A simple computation shows that

$$[M_\lambda^0(\Gamma)^*, M_\lambda^0(\Gamma)] \in \mathcal{C}^3.$$

From Proposition 3.1 (i) and Brown–Douglas–Fillmore Theorem ([6]), $M_\lambda^0(\Gamma) =$ Normal + Compact. ∎

COROLLARY 3.3. *If T is an essentially normal operator on H such that $\sigma_e(T)$ is connected and* $\mathrm{ind}(T-\lambda) = 0$ *for all $\lambda \in \rho_{S-F}(T)$, then $T \simeq M + K$, where M is a strongly irreducible operator with the nice property and K is compact.*

Proof. After a compact perturbation, $\sigma_0(T+K_1) = 0$. By Brown–Douglas–Fillmore Theorem there is a compact operator K_2 such that

$$\begin{aligned} T + K_1 + K_2 &\simeq M_\lambda^0(\Gamma), \quad \text{if } \Gamma = \sigma_e(T) \text{ has more than one point;} \\ T + K_1 + K_2 &\simeq \lambda_0 + A, \quad \text{if } \sigma_e(T) = \{\lambda_0\} \end{aligned}$$

where A is the weighted shift used in the proof of Proposition 2.6. ∎

Similar to analytic Cauchy region with the cone property case, we can consider $M_\lambda^0(R,\Gamma) := M_\lambda^0(\Gamma)|r_0(\Gamma)$ on subspace $r_0(\Gamma)$ of $W_0^{2,2}(D) \ominus m_0(\Gamma)$, where $r_0(\Gamma) = \mathcal{A}^\alpha(M_\lambda^0(\Gamma))e_0(\Gamma)$, and we have:

PROPOSITION 3.4. (i) $\sigma(M^0_\lambda(R,\Gamma)) = \Gamma$, $\mathrm{nul}(M^0_\lambda(R,\Gamma) - \lambda_0) = 0$ *and* $\mathrm{nul}(M^0_\lambda(R,\Gamma) - \lambda_0)^* = -1$ *for* $\lambda_0 \in \Gamma$;

(ii)

$$\mathcal{A}^a(M^0_\lambda(R,\Gamma)) = \mathcal{A}'(M^0_\lambda(R,\Gamma)) = \{M^0_f(\Gamma)|r_0(\Gamma) : f \in r_0(\Gamma)\},$$

$\mathcal{A}'(M^0_\lambda(R,\Gamma))$ *is strictly cyclic with cyclic vector* $e_0(\Gamma)$;

(iii) $M^0_\lambda(R,\Gamma)$ *is strongly irreducible if* Γ *is connected.*

The proof of Proposition 3.4 is omitted.

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