

THE EQUIVARIANT BRAUER GROUP OF PRINCIPAL BUNDLES

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ABSTRACT. We study the structure of the equivariant Brauer group $\text{Br}_G(T)$ of a principal G/N -bundle by exhibiting a filtration of $\text{Br}_G(T)$ predicted by a spectral sequence of Grothendieck's in the case of finite G . The first ingredient $M(A, \alpha)$ is the Mackey obstruction to implementing $\alpha|_N$ by a unitary group. The kernel of M can be identified with an equivariant cohomology group $H_G^2(T, \mathcal{S})$, and our main theorem makes four nontrivial assertions about this group. Our constructions extend to higher dimensional groups $H_G^n(T, \mathcal{S})$, and we show that there is a long exact sequence involving these groups generalizing the usual Gysin sequence associated to a principal circle bundle.

KEYWORDS: *Brauer group, continuous trace, Dixmier-Douady class.*

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The previous article ([2]) introduced and studied an equivariant Brauer group $\text{Br}_G(T)$ associated to an action of a locally compact group G on a locally compact space T . The objects in $\text{Br}_G(T)$ are Morita equivalence classes of dynamical systems (A, G, α) in which A is a continuous-trace C^* -algebra with spectrum T and α is an action of G on A inducing the given action on $T = \widehat{A}$. It was proved in [2], Theorem 3.6 that $\text{Br}_G(T)$ is a group with respect to the operation $[A, \alpha] \cdot [B, \beta] := [A \otimes_{C(T)} B, \alpha \otimes_{C(T)} \beta]$. For discrete G , Kumjian ([9]) had previously studied $\text{Br}_G(T)$, and had shown it to be isomorphic to the equivariant sheaf

cohomology group $H^2(T, G; \mathcal{S})$ of Grothendieck ([7]), where \mathcal{S} is the sheaf of germs of continuous circle-valued functions.

The main result of [2] is a structure theorem for $\text{Br}_G(T)$, which was motivated by Kumjian's theorem. Grothendieck had proved that there is a spectral sequence with $E_2^{p,q} = H^p(G, H^q(T, \mathcal{S}))$ (the group cohomology of G with coefficients in the sheaf cohomology of T) converging to $H^{p+q}(T, G; \mathcal{S})$. Although for technical reasons we do not expect to have a similar spectral sequence for locally compact G , Theorem 5.1 of [2] provides all the information about $\text{Br}_G(T)$ one could hope to get from such a spectral sequence converging to $H^2(T, G; \mathcal{S}) \cong \text{Br}_G(T)$.

There is another spectral sequence F in [7] which converges to $H^{p+q}(T, G; \mathcal{S})$. At least for finite G , Grothendieck identifies $F_2^{p,q}$ in terms of the cohomology of a sheaf $\mathcal{H}^q(G_t, \mathbb{T})$ over T/G , in which the stalk over $G \cdot t$ is the group cohomology $H^q(G_t, \mathbb{T})$ of the stabiliser G_t of t . This construction is not obviously available to us, because both the orbit space T/G and the stabiliser map $t \mapsto G_t$ are notoriously badly-behaved for general locally compact transformation groups (T, G) . Nevertheless, these problems do not arise when G is abelian and T is a principal bundle for some quotient G/N of G : the stabiliser of every point is N , and the orbit space $Z = T/G$ is locally compact and Hausdorff. Our present main theorem gives a filtration for $\text{Br}_G(T)$ in this case, in terms of cohomology groups of the form $H^p(T/G, \mathcal{H}^q(N, \mathbb{T}))$, which gives the sort of information one would expect from a full spectral sequence converging to $H^2(T, G; \mathcal{S})$.

Quite a bit is already known about dynamical systems (A, G, α) in which $\hat{A} \rightarrow \hat{A}/G$ is a principal bundle: they have various topological and algebraic invariants which interact in fascinating and nontrivial ways. One key topological invariant is the generalized Dixmier-Douady class studied in [24], which takes values in a cocycle-based cohomology group $H_G^2(T, \mathcal{S})$. (For lack of a better name, this was called an equivariant cohomology group, though it can be strictly smaller than Grothendieck's when they both make sense; crudely speaking, it does not admit classes coming from the cohomology of the fibre.) We have observed before that some of the cocycle calculations in [24] and [25] help establish an exact sequence relating this group to ordinary sheaf cohomology, which reduces in the case of a circle bundle and $G = \mathbf{R}$ to the usual Gysin sequence of the bundle. Similar calculations are involved in the proof of our main theorem, so we have included a quick discussion of these groups, and go on to derive the general Gysin sequence.

We begin with a detailed statement of our main theorem, which describes the equivariant Brauer group $\text{Br}_G(T)$ of a principal G/N -bundle. The first ingredient $M(A, \alpha)$ is the Mackey obstruction to implementing $\alpha|_N$ by a unitary group. The Dixmier-Douady invariant of [24] identifies the kernel of M with the equivariant

cohomology group $H_G^2(T, \mathcal{S})$, and the theorem makes four nontrivial assertions about this group. Two of these have effectively been dealt with in [23] and [24]. For the remaining two, the arguments work in any dimension. Hence we shall digress in Section 2 to introduce the higher-dimensional equivariant analogues of H_G^2 , and then prove the main theorem in Section 3, giving the new arguments in full generality. In Section 4, we complete the proof of the general Gysin sequence.

Our last section contains some examples and applications. Our theorem gives a complete description of $\text{Br}_G(T)$ when G acts trivially, and more generally when G is a direct product. It is then tempting to relate $\text{Br}_G(T)$ to $\text{Br}_N(T)$, where the group acts trivially, and $\text{Br}_{G/N}(T)$, which we know from [2] is isomorphic to $\text{Br}(Z)$. Our result is most striking for $G = \mathbf{R}^k$ and $N = \mathbf{Z}^k$, when the filtration of the Brauer group $\text{Br}_{\mathbf{R}^k}(T)$ of a principal \mathbf{T}^k -bundle over Z involves only $\text{Br}_{\mathbf{Z}^k}(T)$ and the ordinary Brauer group $\text{Br}(Z) \cong H^2(Z, \mathcal{S})$. We close by showing that in the case $G = \mathbf{R}$, $N = \mathbf{Z}$, our general Gysin sequence reduces to the usual one. This is surprisingly tricky, and amounts to proving that the equivariant groups $H_{\mathbf{R}}^r(T, \mathcal{S})$ of a circle bundle are naturally isomorphic to the usual sheaf cohomology groups $H^r(T, \mathcal{S})$.

NOTATION. We shall use the notation of [2] without comment, and we shall try to be consistent with it. In particular, we shall try to distinguish between systems (A, α) in the set $\mathfrak{Bt}_G(T)$ and their classes $[A, \alpha]$ in $\text{Br}_G(T)$. We shall sometimes use the notation $\mathbf{H}^2(G, M)$ or $\mathbf{Z}^2(G, M)$ to stress that we are considering Moore cohomology groups as topological groups; \mathbf{Z}^2 is actually different as a set from Z^2 (cocycles equal almost everywhere have been identified), but $\mathbf{H}^2(G, M)$ has $H^2(G, M)$ as underlying set ([14], Theorem 5). Many of our results require that the natural map $G \rightarrow G/N$ has local sections, which is equivalent to asking that $G \rightarrow G/N$ is a locally trivial principal N -bundle. This holds in most cases of interest, and is automatic if G is discrete, or if N is a Lie group (cf. [24], Section 1). For example, if G is an elementary group — that is, G has the form $\mathbf{R}^k \times \mathbf{T}^m \times \mathbf{Z}^n \times F$ for some finite group F — then $G \rightarrow G/N$ and $\widehat{G} \rightarrow \widehat{N}$ have local sections for any closed subgroup N . Therefore our main Theorem 1.2 holds for any elementary G . If G is a locally compact abelian group, we denote by \mathcal{G} the sheaf of germs of continuous G -valued functions; the underlying space should be clear by context. This convention also applies to the dual group \widehat{G} yielding a sheaf $\widehat{\mathcal{G}}$.

1. THE MAIN THEOREM

Suppose G is a second countable compactly generated abelian group, N is a closed subgroup of G , and $p : T \rightarrow Z$ is a (locally trivial) principal G/N -bundle over a second countable locally compact space Z . Let $\mathbf{H}^2(N, \mathbf{T})$ denote the Moore cohomology group $H^2(N, \mathbf{T})$ with the topology inherited from the Polish topology on $\mathbf{Z}^2(N, \mathbf{T})$ as in [14]. For $(A, \alpha) \in \mathfrak{Br}_G(T)$ we define $\mu(A, \alpha) : T \rightarrow \mathbf{H}^2(N, \mathbf{T})$ by taking $\mu(A, \alpha)(x)$ to be the Mackey obstruction to implementing $\alpha|_N$ by a unitary representation in a representation π_x of A corresponding to $x \in T = \widehat{A}$. Notice that $\mu(A, \alpha)$ is a continuous function of T into $\mathbf{H}^2(N, \mathbf{T})$ by [17], Lemma 3.3. If $\pi_x : A \rightarrow B(H_x)$ is a representative for $x \in T$, then $\pi_x \circ \alpha_s^{-1}$ is a representative for $s \cdot x \in T$, and because G is abelian, any multiplier representation $U : N \rightarrow U(H_x)$ which implements $\alpha|_N$ in the representation π_x also implements $\alpha|_N$ in $\pi_x \circ \alpha_s^{-1}$. Because the Mackey obstruction (i.e., the class in $\mathbf{H}^2(N, \mathbf{T})$) is independent of the choice of representative π_x and multiplier representation U , $\mu(A, \alpha)$ gives a well-defined continuous function of $Z = T/G$ into $\mathbf{H}^2(N, \mathbf{T})$, and it similarly follows from [2], 3.1 that this function depends only on the class of (A, α) in $\mathfrak{Br}_G(T)$. We write M (for Mackey) for the resulting homomorphism

$$M : \mathfrak{Br}_G(T) \rightarrow C(Z, \mathbf{H}^2(N, \mathbf{T})).$$

A class $[A, \alpha]$ is in the kernel of M precisely when $\alpha|_N$ is pointwise unitary, and hence locally unitary by [27], Theorem 2.1. Thus such a system (A, α) is N -principal ([24], Definition 4.4), and $(A \rtimes_\alpha G)^\wedge$ is a principal \widehat{N} -bundle over Z with respect to the dual action of $\widehat{N} = G/N^\perp$ and the quasi-orbit map $q : (A \rtimes_\alpha G)^\wedge \rightarrow \widehat{A}/G = Z$ ([21], Theorem 2.2). Since Morita equivalent systems have Morita equivalent dual systems, the class $[q]$ depends only on $[A, \alpha]$, and gives a well-defined map S (for Spectrum)

$$S : \ker M \rightarrow H^1(Z, \widehat{N}).$$

(If (X, \mathfrak{u}) is an (A, α) - (B, β) -imprimitivity bimodule, then $\widehat{u}_\gamma(z)(s) = \gamma(s)z(s)$ defines an action of \widehat{G} on $C_c(G, X)$ which extends to an action of \widehat{G} on the bimodule $X \rtimes_{\mathfrak{u}} G$ implementing the equivalence of $A \rtimes_\alpha G$ and $B \rtimes_\beta G$.)

We defer the proof of the following lemma till the end of the section.

LEMMA 1.1. *For any transformation group (G, T) as above, S is a homomorphism of $\ker M$ into $H^1(Z, \widehat{\mathcal{N}})$.*

Next we define a map P (for Pull-back)

$$P : H^2(Z, \mathcal{S}) \rightarrow \text{Br}_G(T)$$

by $P([A]) := [p^*A, p^*\text{id}]$, where we use the Dixmier-Douady Theorem to identify $H^2(Z, \mathcal{S})$ with the Brauer group $\text{Br}(Z)$. Then

$$p^*A \rtimes_{p^*\text{id}} G = (C_0(T) \otimes_{C(Z)} A) \rtimes_{\tau \otimes_{C(Z)} \text{id}} G \cong (C_0(T) \rtimes_{\tau} G) \otimes_{C(Z)} A.$$

Since the spectrum of $C_0(T) \rtimes_{\tau} G$ is known to be \widehat{N} -homeomorphic to $T/G \times \widehat{N} = Z \times \widehat{N}$ ([31]), and $\widehat{A} = Z$, this implies that $(p^*A \rtimes_{p^*\text{id}} G)^\wedge \cong Z \times \widehat{N}$, and we have $S \circ P = 0$. That the converse also holds is part of our main Theorem.

THEOREM 1.2. *Let G be a second countable compactly generated locally compact abelian group such that $G \rightarrow G/N$ and $\widehat{G} \rightarrow \widehat{N} = \widehat{G}/N^\perp$ have local sections, and let $p : T \rightarrow Z$ be a locally trivial principal G/N -bundle over a Polish space Z . Let $M : \text{Br}_G(T) \rightarrow C(Z, \mathbf{H}^2(N, \mathbf{T}))$, $S : \ker M \rightarrow H^1(Z, \widehat{\mathcal{N}})$ and $P : H^2(Z, \mathcal{S}) \rightarrow \text{Br}_G(T)$ be the homomorphisms discussed above. Then $\text{im } P = \ker S$, and there are homomorphisms*

$$\begin{aligned} d'_2 &: H^1(Z, \widehat{\mathcal{N}}) \rightarrow H^3(Z, \mathcal{S}) \\ d''_2 &: C(Z, \widehat{\mathcal{N}}) \rightarrow H^2(Z, \mathcal{S}) \end{aligned}$$

such that $\text{im } S = \ker d'_2$ and $\ker P = \text{im } d''_2$.

Even though this theorem has substantially stronger hypotheses than the structure theorem of [2], and in some situations gives substantially more information, it is not as complete as one would like. The spectral sequence for finite G on which it is modeled suggests that there should also be homomorphisms

$$d_2 : C(Z, \mathbf{H}^2(N, \mathbf{T})) \rightarrow H^2(Z, \widehat{\mathcal{N}}),$$

and

$$d_3 : \ker d_2 \rightarrow H^3(Z, \mathcal{S})/\text{im } d'_2$$

such that $\text{im } M = \ker d_3$. We have so far been unable to find such homomorphisms.

On the other hand, the homomorphisms d'_2 and d''_2 are both easy to define. Since $G \rightarrow G/N$ has local sections, there is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N} \longrightarrow 0,$$

and an associated long exact sequence

$$\dots \longrightarrow H^r(Z, \mathcal{N}) \longrightarrow H^r(Z, \mathcal{G}) \longrightarrow H^r(Z, \mathcal{G}/\mathcal{N}) \xrightarrow{\partial_G} H^{r+1}(Z, \mathcal{N}) \longrightarrow \dots$$

in sheaf cohomology. Then both d'_2 and d''_2 are given by taking cup product with the class $\partial_G([p])$ in $H^2(Z, \mathcal{N})$ (recall that $C(Z, \widehat{N}) = H^0(Z, \widehat{N})$).

To prove this theorem, we shall want to convert everything to statements in sheaf cohomology. We showed in [24] that N -principal systems $(A, \alpha) \in \mathfrak{Pr}_G(T)$ are characterized by a Dixmier-Douady class $\delta(A, \alpha)$ which lies in an *equivariant sheaf cohomology group* $H_G^2(T, \mathcal{S})$ (whose precise definition is in the next section). We begin by checking that the map $(A, \alpha) \mapsto \delta(A, \alpha)$ induces an isomorphism of $\ker M$ onto $H_G^2(T, \mathcal{S})$.

LEMMA 1.3. *Suppose that G is a locally compact abelian group, and N is a closed subgroup such that $\widehat{G} \rightarrow \widehat{G}/N^\perp = \widehat{N}$ has local sections. If $p : T \rightarrow Z$ is a locally trivial principal G/N -bundle over a paracompact space Z , then the Dixmier-Douady invariant of [24], Section 6, induces an isomorphism of $\ker M$ onto $H_G^2(T, \mathcal{S})$.*

Proof. Proposition 4.9 of [24] implies that equivariant line bundles over (T, G) are locally trivial, so Theorem 6.3 of [24] applies. Corollary 4.5 of [24] identifies the systems considered there as the N -principal systems, i.e. as those in $\ker M$, and hence we can deduce from [24], Theorem 6.3 that δ induces a bijection of $\ker M$ onto $H_G^2(T, \mathcal{S})$. To see that $\delta(A \otimes_{C(T)} B, \alpha \otimes_{C(T)} \beta) = \delta(A, \alpha)\delta(B, \beta)$, one has to go into the construction of [24], Lemmas 6.1 and 6.2. But since we know the classes are independent of the choices made there ([24], Lemma 6.2 (4)), we can choose local equivalences (X_i, u^i) and (Y_i, v^i) of (A, α) and (B, β) with $(C_0(T), \tau)$ relative to the same G -invariant cover $\{F_i\}$ of T , and then $(X_i \otimes_{C(F_i)} Y_i, u^i \otimes_{C(F_i)} v^i)$ are local equivalences of $(A \otimes_{C(T)} B, \alpha \otimes_{C(T)} \beta)$ with $(C_0(T), \tau)$ (cf. [2], Lemma 3.2). Now one can compute that the resulting cocycle representing $\delta(A \otimes_{C(T)} B, \alpha \otimes_{C(T)} \beta)$ is the product of those representing $\delta(A, \alpha)$ and $\delta(B, \beta)$. (One just has to keep track of the actions in the argument of [2], Proposition 2.2.) ■

Proof of Lemma 1.1. By [24], Lemma 7.2, the map S is the composition of the homomorphisms $\delta : \ker M \rightarrow H_G^2(G, \mathcal{S})$ from Lemma 1.3 and the homomorphism $b : H_G^2(T, \mathcal{S}) \rightarrow H^1(Z, \widehat{N})$ of [24] Lemma 7.1. ■

2. THE EQUIVARIANT COHOMOLOGY GROUPS

Throughout this section, G will be a locally compact group and T a locally compact G -space with orbit map $p : T \rightarrow T/G$. For the moment, fix an open cover $\mathfrak{A} = \{N_i\}_{i \in A}$ of T/G and let $M_i = p^{-1}(N_i)$ for each $i \in A$. (Equivalently, $\{M_i\}$ is an arbitrary cover of T by G -invariant open sets.)

For each $r \geq 1$, $C^r(\mathfrak{A}, \mathcal{S})$ will denote the group consisting of pairs (ν, λ) in which ν is an r -cochain in $C^r(\{M_i\}_{i \in A}, \mathcal{S})$ and $\lambda \in C^{r-1}(\{M_i \times G\}_{i \in A}, \mathcal{S})$ satisfies

$$(2.1) \quad \lambda_J(x, st) = \lambda_J(x, s)\lambda_J(s^{-1} \cdot x, t)$$

for each multi-index $J = (j_0, \dots, j_{r-1})$, $x \in M_J$, and $s, t \in G$. It will be convenient to let $C_G^0(\mathfrak{A}, \mathcal{S})$ be the collection of pairs $(\nu, 1)$ where ν is a standard 0-cochain and 1 denotes the function which is identically 1 on $T \times G$.

Our coboundary homomorphism $\Delta^r : C_G^r(\mathfrak{A}, \mathcal{S}) \rightarrow C_G^{r+1}(\mathfrak{A}, \mathcal{S})$ will be given by

$$\Delta^r(\nu, \lambda) = (\partial^r \nu, \delta(\nu, \lambda)),$$

where ∂^r is the usual Čech coboundary operator, and δ is defined in terms of $\partial^{r-1} : C^{r-1} \rightarrow C^r$ by

$$(2.2) \quad \delta(\nu, \lambda)_K(x, s) = \nu_K(x)^{-1} \nu_K(s^{-1} \cdot x) (\partial(\lambda(\cdot, s)))_K(x)^{-1}$$

for each $(r+1)$ -multi-index K . Since it will be crucial in our calculations, we note that if L is the $(r+2)$ -multi-index (l_0, \dots, l_{r+1}) , and L_i is the $(r+1)$ -multi-index obtained by deleting l_i , then

$$(2.3) \quad (\partial \nu)_L(x) = \nu_{L_{r+1}}(x) \nu_{L_r}(x)^{-1} \dots = \prod_{i=0}^{r+1} \nu_{L_{r+1-i}}(x)^{(-1)^i}.$$

(The formula in (2.3) differs slightly from that in [30], Section 5.33 and [24], [25] — the two definitions are intertwined by sending $(l_0, l_1, \dots, l_r, l_{r+1})$ to $(l_{r+1}, l_r, \dots, l_1, l_0)$. The above definition avoids some powers of -1 in our formulas.) Of course, we know that $\partial^2 = 0$, and a quick calculation using (2.2) and (2.3) shows that $\delta(\partial \mu, \delta(\mu, \sigma)) = 1$. Thus $\Delta^2 = 0$, and we can define the groups $H_G^r(\mathfrak{A}, \mathcal{S})$ in the standard way:

$$H_G^r(\mathfrak{A}, \mathcal{S}) = \frac{\ker \Delta^r}{\text{Im } \Delta^{r-1}} := \frac{Z_G^r(\mathfrak{A}, \mathcal{S})}{B_G^r(\mathfrak{A}, \mathcal{S})}$$

for all $r \geq 1$. (We bothered to define $C_G^0(\mathfrak{A}, \mathcal{S})$ so that $\text{Im } \Delta^0$ is defined in the case $r = 1$.)

As usual, the crunch comes when we consider refinements of \mathfrak{A} . Let $\mathfrak{B} = \{N'_\beta\}_{\beta \in B}$ be a refinement of \mathfrak{A} , with refining map $\iota : B \rightarrow A$ such that $N'_\beta \subseteq N_{\iota(\beta)}$. Then we obtain a map $\iota : Z_G^r(\mathfrak{A}, \mathcal{S}) \rightarrow Z_G^r(\mathfrak{B}, \mathcal{S})$ by setting $\iota(\nu, \lambda) = (\iota(\nu), \iota(\lambda))$ where

$$\iota(\nu)_K(x) = \nu_{\iota(K)}(x) \quad \text{and} \quad \iota(\lambda)_J(x, s) = \lambda_{\iota(J)}(x, s).$$

(If $K = (k_0, \dots, k_r)$, then $\iota(K) = (\iota(k_0), \dots, \iota(k_r))$.) Note that $\iota(B_G^r(\mathfrak{A}, \mathcal{S})) \subseteq B_G^r(\mathfrak{B}, \mathcal{S})$. We claim that the induced map ι^* on H_G^r is independent of the choice of refining map ι .

Let τ be another refining map, and put $(\tilde{\nu}, \tilde{\lambda}) = (\iota(\nu), \iota(\lambda))(\tau(\nu), \tau(\lambda))^{-1}$. Following [30], Section 5.33, we define $h_r : C^r(\mathfrak{A}, \mathcal{S}) \rightarrow C^{r-1}(\mathfrak{B}, \mathcal{S})$ by

$$h_r(\nu)_J(x) = \prod_{i=0}^{r-1} \nu_{\tilde{j}_{r-1-i}}(x)^{(-1)^i},$$

where $J = (j_0, \dots, j_{r-1})$ is a r -multi-index and \tilde{J}_i is the $(r+1)$ -multi-index $(\iota(j_0), \dots, \iota(j_i), \tau(j_i), \dots, \tau(j_{r-1}))$. The crucial observation is that h is a homotopy operator: for any $\sigma \in C^r(\mathfrak{A}, \mathcal{S})$, we have

$$(2.4) \quad h_{r+1}(\partial(\sigma)) \partial(h_r(\sigma)) = \iota(\sigma)\tau(\sigma)^{-1}$$

(cf. equation (10) of [30], Section 5.33). Notice that if κ is in $Z^r(\mathfrak{A}, \mathcal{S})$, so that $\partial\kappa \equiv 1$, then $\partial(h_r(\kappa)) = \iota(\kappa)\tau(\kappa)^{-1}$. This suggests that we define $(\mu, \sigma) \in C_G^{r-1}(\mathfrak{B}, \mathcal{S})$ by

$$(\mu, \sigma) = (h_r(\nu), h_{r-1}(\lambda)^{-1}).$$

Then $\Delta(\mu, \sigma) = (\partial\mu, \delta(\mu, \sigma)) = (\tilde{\nu}, \delta(\mu, \sigma))$. Moreover

$$\begin{aligned} \delta(\mu, \sigma)_J(x, s) &= \mu_J(x)^{-1} \mu_J(\tau^{-1} \cdot x) [\partial(\sigma(\cdot, s))_J(x)]^{-1} \\ &= h_r(\nu^{-1}s \cdot \nu)_J(x) [\partial(h_{r-1}(\lambda(\cdot, s)))_J(x)]^{-1} \\ &= h_r(\nu^{-1}s \cdot \nu)_J(x) [h_r(\partial(\lambda(\cdot, s)))_J(x)^{-1} \iota(\lambda)_J(x) \tau(\lambda)_J(x)^{-1}] \\ &= \iota(\lambda)_J(x) \tau(\lambda)_J(x)^{-1}, \end{aligned}$$

where the third equation holds because of (2.4) and the last equation because h_r is a homomorphism and (ν, λ) is a cocycle. This shows that $(\tilde{\nu}, \tilde{\lambda})$ is in $B_B^r(\mathfrak{B}, \mathcal{S})$, and hence that $\iota^* = \tau^*$ as claimed.

We may now regard $\{H_G^r(\mathfrak{A}, \mathcal{S})\}_{\mathfrak{A}}$, where \mathfrak{A} runs over all open covers of T/G , as a system of abelian groups directed by refinement. We define $H_G^r(T, \mathcal{S})$ to be the direct limit.

The group $H_G^1(T, \mathcal{S})$ is an interesting special case: it classifies certain equivariant line bundles over T . To explain, we need to recall some ideas from [24], Section 4. An *equivariant line bundle over a transformation group* (T, G) is a Hermitian line bundle $\pi : L \rightarrow T$ with a unitary action u of G satisfying $\pi(u_s(\ell)) = s \cdot \pi(\ell)$. The equivariant line bundle L is *locally trivial over T/G* if each point of T has a G -invariant neighborhood F such that L is trivial over F as a line bundle. An equivariant line bundle can fail to be locally trivial over T/G : see Example 4.12 in [24]. (Note that the trivialization is *not* required to be equivariant. On the other hand, when we say that two equivariant line bundles over (T, G) are isomorphic we mean there is a G -equivariant isomorphism between them.)

PROPOSITION 2.1. *The set $H_G^1(T, \mathcal{S})$ is in one-to-one correspondence with the isomorphism classes of equivariant line bundles over (T, G) which are locally trivial over T/G .*

Proof. It is well-known that the isomorphism classes of Hermitian line bundles over T are in one-to-one correspondence with the elements of the Čech group $H^1(T, \mathcal{S})$: the extra structure is exactly what is needed to keep track of the group action. If $\pi : L \rightarrow T$ a locally trivial equivariant line bundle over (T, G) , then there are an open cover $\{N_i\}$ of T/G , Hermitian line bundle isomorphisms

$$\varphi_i : \pi^{-1}(p^{-1}(N_i)) \rightarrow p^{-1}(N_i) \times \mathbb{C},$$

and continuous functions $\nu_{ij} : p^{-1}(N_{ij}) \rightarrow \mathbb{T}$ such that

$$(2.5) \quad \varphi_i^{-1}(x, \omega) = \varphi_j^{-1}(x, \omega \nu_{ij}(x))$$

for all $x \in p^{-1}(N_{ij})$ and $\omega \in \mathbb{C}$. If τ_s denotes the unitary action on the trivial bundle given by $\tau_s(x, \omega) = (s \cdot x, \omega)$, then $\tau_s \circ \varphi_i \circ u_s^{-1} \circ \varphi_i^{-1}$ is an isomorphism of $p^{-1}(N_i) \times \mathbb{C}$, and so there are continuous functions $\lambda_i : p^{-1}(N_i) \times G \rightarrow \mathbb{T}$ satisfying

$$(2.6) \quad u_s^{-1}(\varphi_i^{-1}(x, \omega)) = \varphi_i^{-1}(s^{-1} \cdot x, \omega \lambda_i(x, s)).$$

A straightforward calculation using (2.5) and (2.6) shows that $(\nu, \lambda) \in Z_G^1(\{N_i\}, \mathcal{S})$. Routine arguments show that the class of (ν, λ) in $H_G^1(T, \mathcal{S})$ depends only on the G -isomorphism class $[L]$ of L .

On the other hand, if $(\nu, \lambda) \in Z_G^1(\mathfrak{A}, \mathcal{S})$ for some open cover $\mathfrak{A} = \{N_i\}$ of T/G , we can form a Hermitian line bundle

$$\coprod_i p^{-1}(N_i) \times \mathbb{C} / \sim$$

where \sim identifies (i, x, ω) with $(j, x, \omega \nu_{ij}(x))$, and the formula

$$u_s([i, s, \omega]) = [i, s \cdot x, \omega \lambda_i(x, s^{-1})]$$

gives a well-defined unitary action of G on L . Another routine argument shows that cohomologous cocycles give G -isomorphic bundles, and the result follows. ■

REMARK 2.2. If (T, G) is such that equivariant line bundles are automatically locally trivial over T/G (cf. Definition 4.5 in [24]), then $H_G^1(T, \mathcal{S})$ parameterizes the G -isomorphism classes of all equivariant line bundles over (T, G) . We showed in [24], Proposition 4.8 and Lemma 4.9 that this is the case if G is compact and abelian, or if $p : T \rightarrow T/G$ is a principal G/N -bundle for some quotient G/N of G .

3. THE PROOF OF THEOREM 1.2

Suppose first that $[A, \alpha] \in \text{Br}_G(T)$ belongs to $\ker M$, so that (A, α) is N -principal, and that $[A, \alpha] \in \ker S$, so that the \widehat{N} -bundle $q : (A \rtimes_{\alpha} G)^{\wedge} \rightarrow Z$ is trivial. Theorem 7.2 of [23] implies that α is given on N by a Green twisting map, and the stabilisation trick (as in, for example, [8] or [4]) that (A, α) is Morita equivalent to a system $(B, \text{Inf } \beta)$ inflated from some $(B, \beta) \in \mathfrak{B}_{G/N}(T)$. Because the pull-back construction gives an isomorphism of $\text{Br}(T)$ onto $\text{Br}_{G/N}(T)$ ([2], Section 6.2), $(B, \beta) \sim (p^*C, p^*\text{id})$ for some $C \in \text{Br}(T)$, and we then have $P([C]) = [A, \alpha]$. Since we have already seen that $\text{im } P \subset \ker S$, this proves that $\text{im } P = \ker S$.

That $\text{im } S = \ker d_2'$ is precisely the content of [25], Theorem 3.1: for $[q] \in H^1(Z, \widehat{\mathcal{N}})$, the cup product $[q] \cup \partial_G([p]) \in H^3(Z, \mathcal{S})$ coincides with the pairing $\langle [p], [q] \rangle_G$ used in [25] (cf. [25], Definition 1.1). (If we use Lemma 1.3 to identify $\ker M$ with $H_G^2(T, \mathcal{S})$, then S is carried into the homomorphism $b : H_G^2(T, \mathcal{S}) \rightarrow H^1(Z, \widehat{\mathcal{N}})$; in the next section we shall give a direct proof of the statement $\text{im } b = \ker (\cup \partial_G([p]))$ based on the proof of Proposition 3.3 in [25].)

It remains to show that $\ker P = \text{im } d_2''$. In view of Lemma 1.3, and the usual identification of $\text{Br}(Z)$ with $H^2(Z, \mathcal{S})$, it is enough to prove that

$$C(Z, \widehat{\mathcal{N}}) = H^0(Z, \widehat{\mathcal{N}}) \cup_{\partial_G([p])} H^2(Z, \mathcal{S}) \xrightarrow{p^*} H_G^3(T, \mathcal{S})$$

is exact, where p^* is defined on cocycles $\mu \in Z^2(Z, \mathcal{S})$ by $p^*(\mu) := (\mu \circ p, 1)$. We shall prove that

$$(3.1) \quad H^{r-1}(Z, \widehat{\mathcal{N}}) \cup_{\partial_G([p])} H^{r+1}(Z, \mathcal{S}) \xrightarrow{p^*} H_G^{r+1}(T, \mathcal{S})$$

is exact for all $r \geq 1$. Recall that if $s_{ij} : N_{ij} \rightarrow G$ are continuous functions such that $s_{ij}N$ are transition functions for $p : T \rightarrow Z$ defined relative to a cover $\{N_i\}$, then $\partial_G([p])$ is the class of the cocycle $\{n_{ijk}\}$ defined by $s_{ik}n_{ijk} = s_{ij}s_{jk}$. If $\{\gamma\} \in Z^{r-1}(Z, \widehat{N})$ is defined relative to the same cover, then $[\gamma] \cup \partial_G([p])$ is the class of the cocycle $\sigma \in Z^{r+1}(\{N_i\}, \mathcal{S})$ defined by

$$(3.2) \quad \sigma_L(z) = \lambda_{l_0 \dots l_{r-1}}(z)(n_{l_{r-1}l_r l_{r+1}}(z)) \quad \text{for } z \in N_L \text{ and } |L| = r + 2.$$

As in [25], Lemma 1.2, one can show that $[\gamma] \cup \partial_G([p]) = \partial_{\widehat{G}}([\gamma]) \cup [p]$, so we could equally well define a representative for $[\gamma] \cup \partial_G([p])$ by pushing $\{\gamma_J\} \in Z^{r-1}(Z, \widehat{N})$ forward to $Z^r(Z, \widehat{G/\widehat{N}})$ and pairing with $\{s_{ij}N\}$.

To prove the exactness of (3.1), we first have to show that $p^* \circ (\cup \partial_G([p])) = 0$. Since the cup product is natural, we have

$$p^*([\gamma] \cup \partial_G([p])) = p^*([\gamma]) \cup p^*(\partial_G([p])) = p^*([\gamma]) \cup \partial_G(p^*[p]).$$

Now the bundle $p^*(p) : p^*T \rightarrow T$ is trivial ($t \mapsto (t, t)$ is a global section), and hence if μ represents $[\gamma] \cup \partial_G([p])$, then $[p^*\mu] = 0$ in $H^p(T, \mathcal{S})$. But we need to show $p^*\mu = \partial \eta$ for some cochain $\eta \in C^{r-1}(T, \mathcal{S})$ which is defined on a G -invariant cover $\{p^{-1}(N_i)\}$, and which extends to an element (η, σ) of $C_G^{r-1}(T, \mathcal{S})$ satisfying $\Delta(\eta, \sigma) = (p^*\mu, 1)$. Thus we shall need to look more closely at the above reasoning.

For our present purposes, we use a representative for $\partial_{\widehat{G}}([\gamma]) \cup [p]$ rather than $[\gamma] \cup \partial_G([p])$. So we assume $\gamma_I : N_I \rightarrow \widehat{N}$ has the form $\gamma_I = \xi_I|N$ for some $\xi_I : N_I \rightarrow \widehat{G}$, let $\tau_J = (\partial \xi)_J$, and observe that $\{\tau_J\}$ is a $(r-1)$ -cocycle with values in N^\perp which represents $\partial_{\widehat{G}}([\gamma])$. Thus, if $\{t_{ij}\}$ are transition functions for $p : T \rightarrow Z$, our cocycle μ is given by

$$\mu_K(z) = \tau_{k_0 \dots k_{r-1}}(z)(t_{k_{r-1}k_r}(z)).$$

If $h_i : p^{-1}(N_i) \rightarrow N_i \times G/N$ are local trivializations such that $(z, sN) = h_i \circ h_j^{-1}(z, st_{ij}(z))$, then a cochain trivializing $p^*(\{t_{ij}\}) = \{t_{ij} \circ p\}$ is given by the corresponding G -equivariant projections $w_i : p^{-1}(N_i) \rightarrow G/N$: since

$$h_j^{-1}(p(x), w_j(x)) = x = h_i^{-1}(p(x), w_i(x)) = h_j^{-1}(p(x), w_i(x)t_{ij}(p(x))),$$

we have $t_{ij} \circ p = w_i^{-1}w_j$. It is now routine to check that

$$\eta_J(x) = \tau_{j_0 \dots j_{r-1}}(p(x))(w_{j_{r-1}}(x))^{-1}$$

defines a cochain in $C^{r-1}(\{p^{-1}(N_i)\}, \mathcal{S})$ with $\partial\eta = p^*(\mu)$. Further, if we define $\sigma_I : p^{-1}(N_I) \times G \rightarrow \mathbb{T}$ by

$$\sigma_I(x, s) = \xi_I(p(x))(s)^{-1},$$

then $(\eta, \sigma) \in C_G^{r-1}(\{N_i\}, \mathcal{S})$, and another calculation shows $\Delta(\eta, \sigma) = (p^*\mu, 1)$. Thus $p^* \circ (\cup \partial_G([\hat{p}])([\gamma]))([\gamma]) = [p^*\mu, 1]$ vanishes in $H_G^r(T, \mathcal{S})$, and we have shown that $p^* \circ (\cup \partial_G([\hat{p}])) = 0$.

Now suppose that $[\mu] \in H^r(Z, \mathcal{S})$ and $p^*([\mu]) = 0$ in $H_G^r(T, \mathcal{S})$: we have to find $\gamma \in Z^{r-2}(Z, \hat{N})$ such that $[\gamma] \cup \partial_G([\hat{p}]) = [\mu]$. Saying $p^*([\mu]) = 0$ means (after possibly refining the cover $\{N_i\}$) that there are $\eta_J : p^{-1}(N_J) \rightarrow \mathbb{T}$ and $\sigma_I : p^{-1}(N_I) \times G \rightarrow \mathbb{T}$ such that

$$(3.3) \quad \sigma_I(x, st) = \sigma_I(x, s)\sigma_I(s^{-1} \cdot x, t)$$

$$(3.4) \quad \mu_K(p(x)) = (\partial\eta)_K(x)$$

$$(3.5) \quad 1 = \eta_J(x)\eta_J(s^{-1} \cdot x)^{-1}(\partial\sigma(\cdot, s))_J(x)^{-1}.$$

We define $\gamma_I : N_I \rightarrow \hat{N}$ by $\gamma_I(z)(n) = \sigma_I(x, n)^{-1}$ for any $x \in p^{-1}(z)$; (3.3) implies that σ_I is well-defined with values in \hat{N} , and (3.5) that $\{\gamma_I\}$ is a cocycle. We have to show there are maps $\theta_J : N_J \rightarrow \mathbb{T}$ such that

$$\gamma_{k_0 \dots k_{r-2}}(\cdot)(n_{k_{r-2}k_{r-1}k_r}(\cdot)) \cdot (\partial\theta)_K = \mu_K.$$

We use the local trivializations $h_k : p^{-1}(N_k) \rightarrow N_k \times G/N$ to define sections $c_k : N_k \rightarrow T$ by $c_k(z) = h_k^{-1}(z, N)$, so that $c_i(z) = s_{ij}(z)c_j(z)$. By (3.4) applied to $x = c_{k_r}(z)$,

$$(3.6) \quad \mu_K(z) = (\partial\eta)_K(c_{k_r}(z)).$$

For each multi-index $J = (j_0, \dots, j_{r-1})$, define $\varphi_J := \eta_J(c_{j_{r-1}}(z))$. Then by expanding the right-hand side of (3.6) using (2.3), we compute that

$$\begin{aligned} \mu_K(z) &= \eta_{k_0 \dots k_{r-1}}(c_{k_r}(z))\eta_{k_0 \dots k_{r-1}}(c_{k_{r-1}}(z))^{-1}(\partial\varphi)_K(z), \\ &= \eta_{k_0 \dots k_{r-1}}(c_{k_r}(z))\eta_{k_0 \dots k_{r-1}}(s_{k_{r-1}k_r}(z)c_{k_r}(z))^{-1}(\partial\varphi)_K(z), \end{aligned}$$

which by (3.5)

$$= (\partial\sigma(\cdot, s))_{k_0 \dots k_{r-1}}(c_{k_r}(z)) \Big|_{s=s_{k_{r-1}k_r}(z)^{-1}} \cdot (\partial\varphi)_K(z).$$

Again expanding the right-hand side using (2.3), we obtain

$$\begin{aligned} \mu_K(z) &= [\sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), s_{k_{r-1}k_r}(z)^{-1}) \sigma_{k_0 \dots k_{r-3}k_{r-1}}(c_{k_r}(z), s_{k_{r-1}k_r}(z)^{-1})^{-1} \dots \\ &\quad \cdot \sigma_{k_1 \dots k_{r-1}}(c_{k_r}(z), s_{k_{r-1}k_r}(z)^{-1})^{(-1)^r}] (\partial \varphi)_K(z). \end{aligned}$$

Thus if $J = (j_0, \dots, j_{r-1})$ and $\psi_J(z) := \sigma_{j_0 \dots j_{r-2}}(c_{j_{r-1}}(z), s_{j_{r-2}j_{r-1}}(z)^{-1})^{-1}$, then

$$\begin{aligned} \mu_K(z) &= \sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), s_{k_{r-1}k_r}(z)^{-1}) \sigma_{k_0 \dots k_{r-2}}(c_{k_{r-1}}(z), s_{k_{r-2}k_{r-1}}(z)^{-1}) \cdot \\ &\quad \cdot \sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), s_{k_{r-2}k_r}(z)^{-1})^{-1} (\partial \psi)_K(z) (\partial \varphi)_K(z) \\ &= \sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), s_{k_{r-1}k_r}(z)^{-1} s_{k_{r-2}k_{r-1}}(z)^{-1}) \cdot \\ &\quad \cdot \sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), s_{k_{r-2}k_r}(z)^{-1})^{-1} \partial(\psi\varphi)_K(z) \\ &= \sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), (s_{k_{r-2}k_r}(z) n_{k_{r-2}k_{r-1}k_r}(z))^{-1}) \cdot \\ &\quad \cdot \sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), s_{k_{r-2}k_r}(z)^{-1})^{-1} \partial(\psi\varphi)_K(z) \\ &= \sigma_{k_0 \dots k_{r-2}}(c_{k_r}(z), n_{k_{r-2}k_{r-1}k_r}(z))^{-1} \partial(\psi\varphi)_K(z), \end{aligned}$$

which, by our definition of γ_J above, is

$$= \gamma_{k_0 \dots k_{r-2}}(z) (n_{k_{r-2}k_{r-1}k_r}(z)) \partial(\psi\varphi)_K(z).$$

It follows that $\theta_J = \psi_J \varphi_J$ have the required property.

This completes the proof of exactness of (3.1), and hence also of Theorem 1.2.

4. THE GENERAL GYSIN SEQUENCE

THEOREM 4.1. *Let G be a locally compact abelian group, and N a closed subgroup such that $G \rightarrow G/N$ and $\widehat{G} \rightarrow \widehat{G}/N^\perp$ have local sections. Then for any principal G/N bundle $p: T \rightarrow Z$, there is an exact sequence*

$$\dots \longrightarrow H^r(Z, \mathcal{S}) \xrightarrow{p_G^*} H_G^r(T, \mathcal{S}) \xrightarrow{b} H^{r-1}(Z, \widehat{\mathcal{N}}) \cup_{\partial_{\mathcal{G}}([p])} H^{r+1}(Z, \mathcal{S}) \xrightarrow{p_G^*} \dots$$

The sequence starts with $H^1(Z, \mathcal{S})$, and then we claim p_G^* is one-to-one.

We begin by defining the homomorphisms in the sequence. For $\mu \in Z^r(\{N_i\}, \mathcal{S})$, the class $p_G^*([\mu])$ is represented by the pair $(\mu \circ p, 1) = (p^*(\mu), 1)$ in $Z_G^r(\{N_i\}, \mathcal{S})$; we have added the subscript G to stress that this is not just the usual pull-back homomorphism $p^*: H^r(Z) \rightarrow H^r(T)$. If $(\nu, \lambda) \in Z_G^r(\{N_i\}, \mathcal{S})$, then the cocycle identity (2.1) for $\lambda_J: p^{-1}(N_J) \times G \rightarrow \mathbf{T}$ implies that, for $n \in N$, $\lambda_J(\cdot, n)$ is constant on G -orbits and multiplicative in n ; $b([\nu, \lambda])$ is by definition the class of the cocycle $\gamma_J: N_J \rightarrow \widehat{N}$ such that

$$\gamma_J(p(x))(n) = \lambda_J(x, n) \quad \text{for } n \in N, x \in p^{-1}(N_J).$$

The homomorphism $\cup \partial_G ([p])$ is the operation of cup product with the fixed class $\partial_G ([p])$, and was described in the previous section, where we also showed exactness at $H^r(Z, \mathcal{S})$ provided $r \geq 2$ (see the discussion following (3.1)).

Since it is clear that $b \circ p_G^* = 0$, to show exactness at $H_G^r(T, \mathcal{S})$ for $r \geq 1$, we need to see that if $(\nu, \lambda) \in Z_G^r(T, \mathcal{S})$ and $b([\nu, \lambda]) = 0$ in $H^{r-1}(Z, \widehat{N})$, then there exists $\mu \in Z^r(Z, \mathcal{S})$ such that (ν, λ) is equivalent to $(p^*(\mu), 1)$ in Z_G^r . Saying $b([\nu, \lambda]) = 0$ means that we can refine the cover and find $\xi_J : N_J \rightarrow \widehat{N}$ such that $(\partial \xi)_J = \gamma_J$ in all J with $|J| = r$; by the argument of [3], 10.7.11, we may also suppose there are maps $\tilde{\xi}_J : N_J \rightarrow \widehat{G}$ such that $\xi_J = \tilde{\xi}_J|_N$. (In case $r = 1$, $b([\nu, \lambda]) = 0$ means $\gamma_J^* \equiv 1$ and we can take $\tilde{\xi}_J \equiv 1$.) For each J , we define $\zeta_J : p^{-1}(N_J) \times G/N \rightarrow \mathbf{T}$ by

$$\zeta_J(x, sN) = (\partial \tilde{\xi})_J(p(x))(s)^{-1} \lambda_J(x, s);$$

the right-hand side is well-defined on $p^{-1}(N_J) \times G/N$ because

$$(\partial \tilde{\xi})_J(p(x)) = \gamma_J(p(x)) = \lambda_J(x, \cdot) \quad \text{on } N.$$

Note that since $(\partial \tilde{\xi})_J$ takes values in \widehat{G} and $\lambda_J(x, s)$ is a cocycle in s , ζ_J is also a cocycle: i.e.,

$$\zeta_J(x, stN) = \zeta_J(x, sN) \zeta_J(s^{-1} \cdot x, tN).$$

As usual, we let $w_i : p^{-1}(N_i) \rightarrow G/N$ be the G -equivariant projections on the fibres, and define

$$\begin{aligned} \eta_J : p^{-1}(N_J) &\rightarrow \mathbf{T} \quad \text{by } \eta_J(x) = \zeta_J(x, w_{j_{r-1}}(x)) \text{ for } |J| = r, \\ \sigma_I : p^{-1}(N_I) \times G &\rightarrow \mathbf{T} \quad \text{by } \sigma_I(x, s) = \tilde{\xi}_I(p(x))(s)^{-1} \text{ for } |I| = r - 1. \end{aligned}$$

Since σ_I is a homomorphism in the second variable, the pair (η, σ) is a cochain belonging to $C_G^{r-1}(\{N_i\}, \mathcal{S})$. We claim there is a cocycle $\mu \in Z^r(\{N_i\}, \mathcal{S})$ such that

$$(\nu, \lambda) = \Delta(\eta, \sigma)(\mu \circ p, 1).$$

We first prove that $\nu^{-1}(\partial \eta)$ is constant on G -orbits, and hence has the form $\mu \circ p$ for some cocycle $\mu \in Z^r(\{N_i\}, \mathcal{S})$. Suppose $|K| = r + 1$ and $x \in p^{-1}(N_K)$. Then

$$(\partial \eta)_K(x) = \zeta_{k_0 \dots k_{r-1}}(x, w_{k_{r-1}}(x)) \zeta_{k_0 \dots k_{r-1}}(x, w_{k_r}(x))^{-1} (\partial \zeta(\cdot, sN))_K(x) \Big|_{sN=w_{k_r}(x)}.$$

Now since $(\nu, \lambda) \in Z_G^r$,

$$\begin{aligned} (\partial \zeta(\cdot, sN))_K(x) &= \partial(\partial \tilde{\xi})_K(p(x))(s)^{-1} (\partial \lambda(\cdot, s))_K(x) \\ &= 1 \cdot \nu_K(x) \nu_K(s^{-1} \cdot x)^{-1}. \end{aligned}$$

For any cocycle ζ , $\zeta(x, t)\zeta(x, g)^{-1} = \zeta(t^{-1} \cdot x, t^{-1}g)^{-1}$, so

$$\begin{aligned} (\partial\eta)_K(x) &= \zeta_{k_0 \dots k_{r-1}}(w_{k_{r-1}}(x)^{-1} \cdot x, w_{k_{r-1}}(x)^{-1} w_{k_r}(x)^{-1}) \nu_K(x) \nu_K(w_{k_r}(x)^{-1} \cdot x) \\ &= \nu_K(x) \mu_K(p(x))^{-1}, \end{aligned}$$

where $\mu_K : N_K \rightarrow \mathbf{T}$ is defined by

$$\mu_K(p(x)) = \nu_K(w_{k_r}(x)^{-1} \cdot x) \zeta_{k_0 \dots k_{r-1}}(w_{k_{r-1}}(x)^{-1} \cdot x, w_{k_{r-1}}(x)^{-1} w_{k_r}(x)).$$

Note that since each $w_i(x)$ is G -equivariant, the right-hand side of this last equation is constant on G -orbits, so μ is well defined on N_K ; it is a cocycle because ν and $\partial\eta$ are.

We have chosen μ so that $\nu = (\partial\eta)(\mu \circ p)$, and to verify $(\nu, \lambda) = \Delta(\eta, \sigma)(\mu \circ p, 1)$, we also have to check the second coordinates agree:

$$\begin{aligned} \eta_J(x) \eta_J(s^{-1}x)^{-1} (\partial\sigma(\cdot, s))_J(x)^{-1} \\ &= \zeta_J(x, w_{j_{r-1}}(x)) \zeta_J(s^{-1} \cdot x, s^{-1} \cdot w_{j_{r-1}}(x))^{-1} (\partial\tilde{\xi})_J(p(x))(s) \\ &= \zeta_J(x, sN) (\partial\tilde{\xi})_J(p(x))(s) = \lambda_J(x, s). \end{aligned}$$

This justifies our claim, and we have proved exactness at H_G^r .

Next we consider exactness at $H^{r-1}(Z, \widehat{N})$ for $r \geq 1$. We present the main construction in a separate lemma, which will apply also to certain nonabelian groups G . A special case was also a key ingredient in [25], Proposition 3.3.

LEMMA 4.2. *Suppose N is a closed central subgroup of a locally compact (possibly nonabelian) group G , $p : T \rightarrow Z$ is a principal G/N -bundle, and $\{N_i\}$ is an open cover of Z for which there are continuous G/N -equivariant maps $w_i : p^{-1}(N_i) \rightarrow G/N$, and $s_{ij} : N_{ij} \rightarrow G$ satisfying $w_i(x) s_{ij}(p(x)) N = w_j(x)$. Define $n_{ijk} : N_{ijk} \rightarrow N$ by $s_{ij} s_{jk} = s_{ik} n_{ijk}$. Suppose that for each multi-index J with $|J| = r$, there is a continuous function $\lambda_J : N_J \times G \rightarrow \mathbf{T}$ satisfying*

- (a) $\lambda_J(x, st) = \lambda_J(x, s) \lambda_J(s^{-1} \cdot x, t)$;
- (b) $(\partial\lambda(\cdot, n))_K = 1$ for $n \in N$, $|K| = r + 1$.

We define $\chi_K : N_K \times G/N \rightarrow \mathbf{T}$ by $\chi_K(\cdot, s) = (\partial\lambda(\cdot, s))_K$, and $\gamma_J : N_J \rightarrow \widehat{N}$ by $\gamma_J(p(x))(n) = \lambda_J(x, n)$.

- (i) *If $\mu \in C^r(\{N_j\}, \mathcal{S})$ satisfies*

$$(4.1) \quad (\partial\mu)_L(z) = \gamma_{l_0 \dots l_{r-1}}(z) (n_{l_{r-1} l_r l_{r+1}}(z)),$$

and we define $\nu_K : p^{-1}(N_J) \rightarrow \mathbf{T}$ by

$$\nu_K(x) = \chi_K(x, w_{k_r}(x)) \lambda_{k_0 \dots k_{r-1}}(w_{k_{r-1}}(x)^{-1} \cdot x, s_{k_{r-1} k_r}(p(x)))^{-1} \mu_K(p(x)),$$

then ν is a cocycle such that $(\nu, \lambda) \in Z_G^r(\{N_i\}, \mathcal{S})$.

(ii) Conversely, if $\{\nu_K\}$ is a cocycle such that $(\nu, \lambda) \in Z_G^r$, then

$$(4.2) \quad \mu_K(p(x)) = \chi_K(x, w_{k_r}(x))^{-1} \lambda_{k_0 \dots k_{r-1}}(w_{k_{r-1}}(x)^{-1} \cdot x, s_{k_{r-1}k_r}(p(x))) \nu_K(x)$$

defines a cochain μ such that (4.1) holds.

Proof. We begin by observing that, because $\lambda_J(\chi, sn) = \lambda_J(x, s)\lambda_J(\chi, n)$, and $(\partial \lambda(\cdot, n))_K = 1$, χ_K is indeed well-defined on $N_K \times G/N$ rather than $N_K \times G$; further, each χ_K satisfies the cocycle identity

$$(4.3) \quad \chi_K(x, stN) = \chi_K(x, sN)\chi_K(s^{-1} \cdot x, tN).$$

Next, we set $\rho_K(x) = \chi_K(x, w_{k_r}(x))$, and compute $\partial \rho$ using (4.3)

$$(4.4) \quad \begin{aligned} (\partial \rho)_L(x) &= \chi_{l_0 \dots l_r}(x, w_{l_r}(x)) \chi_{l_0 \dots l_r}(x, w_{l_{r+1}}(x))^{-1} \partial(\chi(\cdot, s))_L(x) \Big|_{s=w_{l_{r+1}}(x)} \\ &= \chi_{l_0 \dots l_r}(w_{l_r}(x)^{-1} \cdot x, w_{l_r}(x)^{-1} w_{l_{r+1}}(x))^{-1}. \end{aligned}$$

We now define

$$\theta_K(p(x)) = \lambda_{k_0 \dots k_{r-1}}(w_{k_{r-1}}(x)^{-1} \cdot x, s_{k_{r-1}k_r}(x))$$

and compute

$$(4.5) \quad \begin{aligned} (\partial \theta)_L(p(x)) &= \lambda_{l_0 \dots l_{r-1}}(w_{l_{r-1}}(x)^{-1} \cdot x, s_{l_{r-1}l_r}(x)) \lambda_{l_0 \dots l_{r-1}}(w_{l_{r-1}}(x)^{-1} \cdot x, s_{l_{r-1}l_{r+1}}(x))^{-1} \\ &\quad \cdot \lambda_{l_0 \dots l_{r-1}}(w_{l_r}(x)^{-1} \cdot x, s_{l_r l_{r+1}}(x)) [(\partial \lambda(\cdot, s))_{l_0 \dots l_r}(w_{l_r}(x)^{-1} \cdot x) \Big|_{s=s_{l_r l_{r+1}}(x)}] \\ &= \lambda_{l_0 \dots l_{r-1}}(w_{l_{r-1}}(x)^{-1} \cdot x, s_{l_{r-1}l_{r+1}}(x))^{-1} \\ &\quad \cdot \lambda_{l_0 \dots l_{r-1}}(w_{l_{r-1}}(x)^{-1} \cdot x, s_{l_{r-1}l_r}(x) s_{l_r l_{r+1}}(x)) \chi_{l_0 \dots l_r}(w_{l_r}(x), s_{l_r l_{r+1}}(x))^{-1} \\ &= \lambda_{l_0 \dots l_{r-1}}(w_{l_{r-1}}(x)^{-1} \cdot x, n_{l_{r-1}l_r l_{r+1}}(x)) (\partial \rho)_L(x). \end{aligned}$$

It follows immediately that $(\partial \nu)_L = 1$. Also, using (4.3),

$$\begin{aligned} \nu_K(x) \nu_K(t^{-1} \cdot x)^{-1} &= \chi_K(x, w_{k_r}(x)) \chi_K(t^{-1} \cdot x, t^{-1} \cdot w_{k_r}(x))^{-1} \\ &= \chi_K(x, t) = (\partial \lambda(\cdot, t))_K(x), \end{aligned}$$

so that $(\nu, \lambda) \in Z_G^r(\{N_i\}, \mathcal{S})$. This gives (i).

To establish (ii), we notice that, since ν is a cocycle, the calculations (4.4) and (4.5) immediately give (4.1), and the problem is to show that the right-hand side of (4.2) is constant on orbits. But

$$\begin{aligned} \chi_K(t^{-1} \cdot x, w_{k_r}(t^{-1} \cdot x))^{-1} \nu_K(t^{-1} \cdot x) &= \chi_K(t^{-1} \cdot x, t^{-1} \cdot w_{k_r}(x))^{-1} (\partial \lambda(\cdot, t))_K(x)^{-1} \nu_K(x) \\ &= \chi_K(t^{-1} \cdot x, t^{-1} \cdot w_{k_r}(x))^{-1} \chi_K(x, t)^{-1} \nu_K(x) \\ &= \chi_K(x, w_{k_r}(x))^{-1} \nu_K(x), \end{aligned}$$

by (4.3). Multiplying this by the expression $\theta_K(p(x))$, which is obviously constant on orbits, gives (ii), and the lemma is proved. ■

Since the right-hand side of (4.1) represents $[\gamma] \cup \partial_G([p])$, part (ii) of the Lemma 4.2 immediately implies that $(\cup \partial_G([p])) \circ b = 0$; that is, $\text{im}(b) \subseteq \ker(\cup \partial_G)$. To see the reverse containment, we need to verify that if $\gamma \in Z^{r-1}(Z, \widehat{N})$ satisfies $[\gamma] \cup \partial_G([p]) = 0$, then there exists $(\nu, \lambda) \in Z_G^r(T, \mathcal{S})$ such that γ represents $b(\nu, \lambda)$. Because the map $\widehat{G} \rightarrow \widehat{G}/N^\perp = \widehat{N}$ has local sections, we can by refining the cover $\{N_i\}$ suppose there are functions $\tilde{\gamma}_J : N_J \rightarrow \widehat{G}$ such that $\gamma_J = \tilde{\gamma}_J|N$, and define $\lambda_J : p^{-1}(N_J) \times G \rightarrow \mathbf{T}$ by

$$(4.6) \quad \lambda_J(x, s) = \tilde{\gamma}_J(p(x))(s).$$

Then λ_J satisfies condition (a) in Lemma 4.2, because each $\tilde{\gamma}_J(z)$ is a homomorphism, and condition (b) because $\{\gamma_J\}$ is a cocycle. Since the cocycle on the right of (4.1) represents $[\gamma] \cup \partial_G([p])$, the hypothesis on γ means we can find a cochain μ satisfying (4.1). Thus the Lemma gives us a cocycle $\nu \in Z^r(\{N_i\}, \mathcal{S})$ such that $(\nu, \lambda) \in Z_G^r(\{N_i\}, \mathcal{S})$, and (4.6) then says that $\{\gamma_J\}$ represents $b([\nu, \lambda])$. This proves the exactness at $H^{r-1}(Z, \widehat{N})$.

Proof of Theorem 4.1. We have now proved everything except the last comment in the theorem. If $[\mu] \in H^1(Z, \mathcal{S})$ is in the kernel of p_G^* , there exist \mathfrak{A} and $(\nu, 1) \in C^0(\mathfrak{A}, \mathcal{S})$ such that $\Delta(\nu, 1) = (p^*(\mu), 1)$. But $\delta(\nu, 1) = 1$ says precisely that each $\nu_i : p^{-1}(N_i) \rightarrow \mathbf{T}$ is constant on orbits, hence has the form $\tau_i \circ p$ for some $\tau_i : N_i \rightarrow \mathbf{T}$. Now $\partial(p^*\tau) = p^*(\mu)$ implies that $[\mu] = 0$ in $H^1(Z, \mathcal{S})$. This completes the proof of the theorem. ■

5. EXAMPLES AND APPLICATIONS

5.1. DIRECT PRODUCT GROUPS. When the extension $0 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 0$ splits, G is isomorphic to the direct product of N and $Q := G/N$, and the coboundary map ∂_G is zero. Thus $\partial_G([p]) = 0$ for any Q -bundle $p : T \rightarrow Z$, and the Gysin sequence of Theorem 4.1 gives a short exact sequence

$$0 \longrightarrow H^2(Z, \mathcal{S}) \xrightarrow{p^*} H_G^2(T, \mathcal{S}) \xrightarrow{b} H^1(Z, \widehat{\mathcal{N}}) \longrightarrow 0.$$

This sequence is also split: if $\{\gamma_{ij}\} \in Z^1(\{N_i\}, \widehat{\mathcal{N}})$, then $\lambda_{ij}(x, (n, q)) := \gamma_{ij}(p(x))(n)$ defines a cocycle $(1, \lambda) \in Z_G^2(\{p^{-1}(N_i)\}, \mathcal{S})$ satisfying $b([1, \lambda]) = [\gamma_{ij}]$. Thus we can deduce from Lemma 1.3 that

$$\ker M \cong H^2(Z, \mathcal{S}) \oplus H^1(Z, \widehat{\mathcal{N}}).$$

If N is discrete, then the canonical map of $Z^1(N, \mathbf{T})$ onto $\mathbf{H}^1(N, \mathbf{T})$ has a splitting θ ([12], p. 82), which induces a splitting θ_* for the natural map of $H^2(N, C(Z, \mathbf{T}))$ into $C(Z, \mathbf{H}^2(N, \mathbf{T}))$ (see the proof of [18], Theorem 1.1). Since there is a natural homomorphism ξ of $H^2(N, C(Z, \mathbf{T}))$ into $\text{Br}_N(Z)$ (as in [2]), we obtain a splitting $\xi \circ \theta_*$ for $M : \text{Br}_N(Z) \rightarrow C(Z, \mathbf{H}^2(N, \mathbf{T}))$ such that each $\xi \circ \theta_*(f)$ is realized by a system $(C_0(Z, \mathcal{K}), \beta^f)$. We can use this to define a splitting φ for $M : \text{Br}_G(Z) \rightarrow C(Z, \mathbf{H}^2(N, \mathbf{T}))$: realize $C_0(T, \mathcal{K})$ as $C_0(T) \otimes_{C(Z)} C_0(Z, \mathcal{K})$, and take $\varphi(f)$ to be $[C_0(T, \mathcal{K}), \alpha^f]$ where $\alpha_{(n, q)}^f := (\text{id} \otimes_{C(Z)} \beta_n^f) \circ (\tau_q \otimes_{C(Z)} \text{id})$. We conclude from this and the previous paragraph that, for any Q -bundle $p : T \rightarrow Z$,

$$(5.1) \quad \text{Br}_{N \times Q}(T) \cong H^2(Z, \mathcal{S}) \oplus H^1(Z, \widehat{\mathcal{N}}) \oplus C(Z, \mathbf{H}^2(N, \mathbf{T})).$$

This generalizes Corollary 2.6 of [18], at least for compactly generated N and Q .

To extend this analysis to nondiscrete N , we need a splitting of $Z^2 \rightarrow \mathbf{H}^2$. Our approach to this problem uses structure theory, so we first consider some special cases. The following description of $H^2(G, \mathbf{T})$ is essentially known (for example, in [13], Section 10, much of it is stated without proof).

For $d \geq 2$, let $\text{LT}_d(\mathbf{R})$ denote the set of strictly lower triangular real matrices; as an additive group, $\text{LT}_d(\mathbf{R}) \cong \mathbf{R}^{d(d-1)/2}$. For $A \in \text{LT}_d(\mathbf{R})$ define $\omega_A : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{T}$ by $\omega_A(x, y) := \exp(2\pi i \cdot x^t A y)$. Then $A \mapsto \omega_A$ is a continuous homomorphism of $\mathbf{R}^{d(d-1)/2} \cong \text{LT}_d(\mathbf{R})$ into $\chi(\mathbf{R}^d, \mathbf{T})$, the group of bicharacters on \mathbf{R}^d with the compact-open topology. Since $\chi(\mathbf{R}^d, \mathbf{T})$ embeds continuously in $Z^2(\mathbf{R}^d, \mathbf{T})$ (convergence in the compact-open topology certainly implies convergence in measure), we have a continuous map $A \mapsto [\omega_A]$ of $\mathbf{R}^{d(d-1)/2}$ into $\mathbf{H}^2(\mathbf{R}^d, \mathbf{T})$, which is bijective by, for example [29], Theorem 10.38. We know from [15], Theorem 7 that $\mathbf{H}^2(\mathbf{R}^d, \mathbf{T})$ is

Polish, so by the Open Mapping Theorem $A \mapsto [\omega_A]$ is an isomorphism of topological groups. Similar arguments and the results of [1] show that $A \mapsto [\omega_A | \mathbf{Z}^d \times \mathbf{Z}^d]$ induces a topological isomorphism of $\mathbf{T}^{d(d-1)/2}$ onto $\mathbf{H}^2(\mathbf{Z}^d, \mathbf{T})$, for $d \geq 2$. In both cases, ω_A gives an explicit parametrisation of \mathbf{H}^2 by bicharacters, which is continuous in the compact-open topology. (Note that ω_A is unchanged on $\mathbf{Z}^d \times \mathbf{Z}^d$ if we add an integer to any entry in A , so the parametrisation is continuous on $\mathbf{T}^{d(d-1)/2}$.)

LEMMA 5.1. *Suppose G_1, G_2 , are second countable locally compact abelian groups such that there are continuous homomorphisms $\varphi_i : \mathbf{H}^2(G_i, \mathbf{T}) \rightarrow \chi(G_i, \mathbf{T})$ which split the canonical map (i.e., which satisfy $[\varphi_i([\omega])] = [\omega]$). Then there is a continuous homomorphism $\varphi : \mathbf{H}^2(G_1 \times G_2, \mathbf{T}) \rightarrow \chi(G_1 \times G_2, \mathbf{T})$ such that $[\varphi([\omega])] = [\omega]$.*

Proof. We define a continuous homomorphism $\psi : \text{Hom}(G_1, \widehat{G}_2) \rightarrow \chi(G_1 \times G_2, \mathbf{T})$ by

$$\psi(\gamma)((s_1, s_2), (t_1, t_2)) := \gamma(s_1)(t_2).$$

Let Res_i denote the restriction map of $H^2(G_1 \times G_2, \mathbf{T})$ into $H^2(G_i, \mathbf{T})$. Then there is an isomorphism $d : \ker(\text{Res}_1 \times \text{Res}_2) \rightarrow \text{Hom}(G_1, \widehat{G}_2)$: this result goes back to Mackey, but is an easy special case of the discussion in [19], Appendix 2, which also makes it clear that γ splits d . We know that $\mathbf{H}^2(G_1 \times G_2, \mathbf{T})$ is Polish ([15], Theorem 7), and $\ker(\text{Res}_1 \times \text{Res}_2)$ is a closed subgroup because Res_i is continuous ([14], Proposition 27). Since ψ is continuous for the compact-open topology on $\text{Hom}(G_1, \widehat{G}_2)$ and the Polish topology on $\mathbf{Z}^2(G_1 \times G_2, \mathbf{T})$, it is a continuous isomorphism of $\text{Hom}(G_1, \widehat{G}_2)$ onto the Polish group $\ker(\text{Res}_1 \times \text{Res}_2)$. Thus the Open Mapping Theorem implies that the inverse d is continuous. We now define φ by

$$(5.2) \quad \begin{aligned} \varphi([\omega]) := & [\varphi_1(\text{Res}_1([\omega])) \\ & \times \varphi_2(\text{Res}_2([\omega]))] [\psi \circ d([\omega] \text{Inf}_1 \circ \text{Res}_1([\omega])^{-1} \text{Inf}_2 \circ \text{Res}_2([\omega])^{-1})], \end{aligned}$$

where $\text{Inf}_i : H^2(G_i, \mathbf{T}) \rightarrow H^2(G_1 \times G_2, \mathbf{T})$ are the inflation maps. ■

COROLLARY 5.2. *For any elementary locally compact abelian group G there is a homomorphism φ of $\mathbf{H}^2(G, \mathbf{T})$ into $\chi(G, \mathbf{T})$ which is continuous for the compact-open topology on the group χ of bicharacters, and satisfies $[\varphi([\omega])] = [\omega]$.*

Proof. We may suppose $G = \mathbf{R}^k \times \mathbf{Z}^m \times \mathbf{T}^n \times F$, with F finite. Since we have verified the hypotheses of the Lemma 4.2 for \mathbf{R}^k and \mathbf{Z}^m , $H^2(\mathbf{T}^n, \mathbf{T}) = 0$ ([11], Proposition 2.1), and $H^2(F, \mathbf{T})$ is finite, the Lemma 4.2 applies. ■

Now suppose $p : T \rightarrow Z$ is a principal Q -bundle, and N is an elementary locally compact abelian group. Given $f \in C(T, \mathbf{H}^2(N, \mathbf{T}))$, the Corollary gives a lifting $g : T \rightarrow \chi(N, \mathbf{T})$ which is continuous for the compact-open topology on $\chi(N, \mathbf{T}) \subset C(N \times N, \mathbf{T})$. If we give everything the compact-open topology, $C(T, C(N \times N, \mathbf{T}))$ is naturally topologically isomorphic to $C(N \times N, C(T, \mathbf{T}))$. Thus the lifting gives a continuous map of $N \times N$ into $C(T, \mathbf{T})$ which is multiplicative in either variable, and hence is a cocycle in $Z^2(N, C(T, \mathbf{T}))$. Thus we can extend the above analysis for discrete N to obtain:

THEOREM 5.3. *Suppose that N and Q are second countable locally compact abelian groups, and that N is elementary. Then for any principal Q -bundle $p : T \rightarrow Z$, we have*

$$\mathrm{Br}_{N \times Q}(T) \cong H^2(Z, S) \oplus H^1(Z, \widehat{N}) \oplus C(Z, \mathbf{H}^2(N, \mathbf{T})).$$

REMARK 5.4. An amusing consequence of this Theorem and the corresponding result for trivial Q is that the map $(A, \alpha) \mapsto (C_0(T) \otimes_{C(Z)} A, (\mathrm{id} \otimes \alpha) \times (\tau \otimes \mathrm{id}))$ induces an isomorphism of $\mathrm{Br}_N(Z)$ onto $\mathrm{Br}_{N \times Q}(T)$. This is nonobvious, but is known to be true for arbitrary N and Q ; it is the content of Proposition 7 of [10].

5.2. RESTRICTION AND INFLATION. Associated to any normal subgroup N of G is a natural restriction map $\mathrm{Res} : \mathrm{Br}_G(T) \rightarrow \mathrm{Br}_N(T)$ such that $\mathrm{Res}([A, \alpha]) = [A, \alpha|_N]$; if N acts trivially on T , there is also an inflation map $\mathrm{Inf} : \mathrm{Br}_{G/N}(T) \rightarrow \mathrm{Br}_G(T)$ such that $\mathrm{Inf}([A, \alpha]) = [A, \mathrm{Inf} \alpha]$, where $\mathrm{Inf} \alpha$ is the composition of $\alpha : G/N \rightarrow \mathrm{Aut} A$ with the quotient map of G onto G/N . We trivially have $\mathrm{Res} \circ \mathrm{Inf} = 0$, and anyone familiar with group cohomology will immediately wonder if one can describe $\ker(\mathrm{Res})/\mathrm{im}(\mathrm{Inf})$. First we shall apply our main theorem to this problem, next we shall give an alternative answer in terms of a group-cohomological invariant, and then we shall compare the two answers using the theory of [23].

COROLLARY 5.5. *Suppose G , N and $p : T \rightarrow Z$ satisfy the hypotheses of Theorem 1.2. If $(A, \alpha) \in \mathfrak{B}r_G(T)$ and $\mathrm{Res}([A, \alpha]) = 0$ in $\mathrm{Br}_N(T)$, then $[A, \alpha] \in \ker M$. The class $S([A, \alpha])$ in $H^1(Z, \widehat{N})$ vanishes if and only if there is a system of the form $(C_0(T, \mathcal{K}), \beta) \in \mathfrak{B}r_{G/N}(T)$ such that $[A, \alpha] = \mathrm{Inf}[C_0(T, \mathcal{K}), \beta]$.*

Proof. Since $M(A, \alpha) = M(A, \alpha|_N)$, it is clear that $M([A, \alpha]) = 0$. Next, suppose that $(C_0(T, \mathcal{K}), \beta) \in \mathfrak{B}r_{G/N}(T)$, and let π denote the quotient map of G onto G/N , so that $\mathrm{Inf}(C_0(T, \mathcal{K}), \beta) = (C_0(T, \mathcal{K}), \beta \circ \pi)$. Then by a theorem of Olesen and Pedersen ([16]),

$$C_0(T, \mathcal{K}) \rtimes_{\beta \circ \pi} G \cong \mathrm{Ind}_{N^\perp}^{\widehat{G}} (C_0(T, \mathcal{K}) \rtimes_{\beta} G/N, \widehat{\beta}).$$

Since G/N acts freely and properly on $T = C_0(T, \mathcal{K})^\wedge$, $C_0(T, \mathcal{K}) \rtimes G/N$ is a continuous-trace algebra with spectrum $Z = T/(G/N)$ ([6]), and the induced algebra has spectrum \widehat{G} -homeomorphic to $\widehat{G}/N^\perp \times Z = \widehat{N} \times Z$ because $N^\perp = (G/N)^\wedge$ acts trivially on $(C_0(T, \mathcal{K}) \rtimes G/N)^\wedge$ (e.g., [22], Proposition 3.1). (Alternatively, we could use the isomorphism of $\text{Br}_{G/N}(T)$ with $\text{Br}(Z)$ to write $[C_0(T, \mathcal{K}), \beta] = [p^*B, p^*\text{id}]$ for some $B \in \mathfrak{Br}(Z)$, and then compute

$$p^*B \rtimes_{p^*\text{id}} G \cong (C_0(T) \otimes_{C(Z)} B) \rtimes_{\tau \otimes_{C(Z)} \text{id}} G \cong (C_0(T) \rtimes G) \otimes_{C(Z)} B.$$

Conversely, if $S([A, \alpha]) = 0$, then our main theorem says there exists $B \in \mathfrak{Br}(Z)$ such that $[A, \alpha] = P([B]) = [p^*B, p^*\text{id}]$. However, $[A, \alpha|N] = \text{Res}([A, \alpha]) = 0$ implies $\delta(A) = 0$, so that p^*B must be stably $C_0(T)$ -isomorphic to $C_0(T, \mathcal{K})$. Since the action $p^*\text{id}$ of G on p^*B is inflated from G/N , so is the corresponding action on $C_0(T, \mathcal{K})$. ■

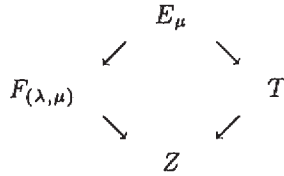
REMARK 5.6. Because pulling-back induces an isomorphism of $\text{Br}(Z)$ onto $\text{Br}_{G/N}(T)$, we can alternatively say that $S([A, \alpha]) = 0$ if and only if there exists $B \in \mathfrak{Br}(Z)$ such that $p^*(\delta(B)) = 0$ and $[A, \alpha] = [p^*B, p^*\text{id}]$.

PROPOSITION 5.7. *Suppose G, N and $p : T \rightarrow Z$ satisfy the hypotheses of Theorem 1.2. If $(A, \alpha) \in \mathfrak{Br}_G(T)$ and $\text{Res}([A, \alpha]) = 0$ in $\text{Br}_N(T)$, then there is a class $d_1(\alpha)$ in $H^1(G/N, \text{Hom}(N, C(T, \mathbb{T})))$ which vanishes if and only if there is a system of the form $(C_0(T, \mathcal{K}), \beta) \in \mathfrak{Br}_{G/N}(T)$ such that $[A, \alpha] = \text{Inf}[C_0(T, \mathcal{K}), \beta]$.*

Proof. It follows from [2], Lemma 3.1 that $\text{Res}(A, \alpha)$ is trivial if and only if A is stably $C_0(T)$ -isomorphic to $C_0(T, \mathcal{K})$ and the action $\alpha|N \otimes \text{id}$ of N on $A \otimes \mathcal{K}$ is outer conjugate over T to the trivial action; equivalently, $\alpha|N \otimes \text{id} = \text{Ad } u$ for some strictly continuous homomorphism $u : N \rightarrow \text{UM}(A \otimes \mathcal{K})$. So we may as well suppose that $(A, \alpha|N) = (C_0(T, \mathcal{K}), \text{Ad } u)$. In the notation of [23], Section 5, comparing $\alpha_s(u_n)$ with u_n gives a cocycle $(\lambda, \mu) \in Z(G, N; C(T, \mathbb{T}))$ which is trivial in the relative cohomology group $\Lambda(G, N; C(T, \mathbb{T}))$ if and only if we can adjust u by scalars to obtain a Green twisting map for α on N ([23], Proposition 5.4). Since u is already a homomorphism, the 2-cocycle μ in $Z^2(N, C(T, \mathbb{T}))$ is identically 1, and the first component λ belongs to $Z^1(G/N, \text{Hom}(N, C(T, \mathbb{T})))$ (see formulas (5.7), (5.8) and (5.9) in [23]). An arbitrary cocycle $(\lambda, \mu) \in Z(G, N; M)$ is trivial if and only if there is a Borel map $\rho : N \rightarrow M$ such that $\mu = \delta\rho$ and $\lambda(s, n) = s \cdot \rho(n)\rho(n)^{-1}$. If $\mu = 1$, ρ has to be a homomorphism, and hence our $(\lambda, 1)$ is trivial if and only if there is a (Borel, hence continuous) homomorphism $\rho : N \rightarrow C(T, \mathbb{T})$ such that $\lambda(s, n) = s \cdot \rho(n)\rho(n)^{-1}$, i.e., if and only if the cocycle λ in $Z^1(G/N, \text{Hom}(N, C(T, \mathbb{T})))$ is a coboundary. Thus $\lambda \mapsto (\lambda, 1)$ embeds $H^1(G/N, \text{Hom}(N, C(T, \mathbb{T})))$ in $\Lambda(G, N; C(T, \mathbb{T}))$, and we can define $d_1(\alpha) := [\lambda]$.

If α is inflated from an action of G/N , then we can take $u = 1$ in the construction of the previous paragraph; since the class $[\lambda, \mu]$ is independent of the choice of u , it follows that $[\lambda, 1] = 0$ and $d_1(\alpha) = 0$. Conversely, if $d_1(\alpha) = 0$, then α is implemented by a Green twisting map over N . From the stabilization trick (the version of [8] or [4]) we deduce that $(C_0(T, \mathcal{K}), \alpha)$ is Morita equivalent to a system in which the action of G is inflated from an action of G/N . Since every stable algebra which is Morita equivalent over T to $C_0(T, \mathcal{K})$ is $C_0(T)$ -isomorphic to $C_0(T, \mathcal{K})$ (cf., the end of the proof of Lemma 2.3, [25]) we can deduce that $(C_0(T, \mathcal{K}), \alpha) \sim (C_0(T, \mathcal{K}), \text{Inf } \beta)$ for some action β of G/N on $C_0(T, \mathcal{K})$. Finally, because this equivalence respects the actions of $C_0(T)$, β must induce the given action of G/N on T , so that $(C_0(T, \mathcal{K}), \beta) \in \mathfrak{B}\tau_{G/N}(T)$. ■

The theory of [23] connects these two viewpoints. Associated to each cocycle $(\lambda, \mu) \in Z(G, N; C(T, \mathbf{T}))$ is a commutative diamond



of principal bundles, in which the southeast arrows are \widehat{N} -bundles and the southwest arrows are G/N -bundles ([23], Proposition 6.3). If $[\lambda, \mu]$ is the obstruction to realizing $\alpha : G \rightarrow \text{Aut } A$ by a Green twist on N ([23], Section 5) and the cocycle $\mu \in Z^2(N, C(T, \mathbf{T}))$ is pointwise trivial, then the system (A, α) is N -principal and $F_{(\lambda, \mu)}$ is \widehat{N} -isomorphic to $(A \rtimes_\alpha G)^\wedge$ ([23], Proposition 7.1). When $\text{Res}([A, \alpha]) = 0$, μ is trivial, so $E_\mu \cong p^* F_{(\lambda, \mu)}$ is trivial. From Theorem 7.3 of [23] we deduce that $(\lambda, 1) \mapsto F_{(\lambda, 1)}$ induces a bijection between $[\lambda] \in H^1(G/N, \text{Hom}(N, C(T, \mathbf{T})))$ and the (classes of) \widehat{N} -bundles F over Z such that $p^* F$ is trivial.

It is interesting to note that, even though the group-theoretic invariant $d_1(\alpha)$ is ostensibly more tractable than the topological invariant $S(A, \alpha)$, our topological theory currently yields more information. Thus, for example, it seems hard to see directly what the range of d_1 is, whereas we have an explicit criterion $[q] \cup \partial_G([p]) = 0$ for the realizability of an \widehat{N} -bundle $q : F \rightarrow Z$ as $S(A, \alpha)$.

EXAMPLE 5.8. (Showing that d_1 is not in general surjective) First note that since $p^*[p] = 0$, $p : T \rightarrow Z$ is realisable as $F_{(\lambda, 1)}$ for some $\lambda \in Z^1(G/N, \text{Hom}(N, C(T, \mathbf{T})))$. Thus we just need to see that $p : T \rightarrow Z$ is not necessarily realisable as $S(A, \alpha)$ for some N -principal system $(A, \alpha) \in \ker M \subset \mathfrak{B}\tau_G(T)$. For a concrete example, we take $G = \mathbf{R}$, $N = \mathbf{Z}$, and $p : S^{2n+1} \rightarrow \mathbf{P}_n(\mathbf{C})$ to be the

canonical circle bundle over complex projective space $\mathbf{P}_n(\mathbf{C})$ for $n \geq 2$. The image of $[q] \cup \partial_{\mathbf{R}}([p]) \in H^3(Z, \mathcal{S})$ in $H^4(Z, \mathbf{Z})$ is $\partial_{\mathbf{R}}([q]) \cup \partial_{\mathbf{R}}([p])$, which is the product of $\partial_{\mathbf{R}}([q])$ and $\partial_{\mathbf{R}}([p])$ in the cohomology ring $H^*(Z, \mathbf{Z})$. Since $H^*(\mathbf{P}_n(\mathbf{C}), \mathbf{Z})$ is a truncated polynomial ring with generator $\partial_{\mathbf{R}}([p])$ (e.g., [28], Theorem 5.8.5), we certainly have $\partial_{\mathbf{R}}([p])^2 \neq 0$ in $H^4(\mathbf{P}_n(\mathbf{C}), \mathbf{Z}) \cong \mathbf{Z}$. Thus the class $[p]$ is not realisable as $S([A, \alpha])$, and the corresponding $[\lambda] \in H^1(G/N, \text{Hom}(N, C(T, \mathbf{T})))$ is not realisable as $d_1(\alpha)$.

5.3. THE CASE $G = \mathbf{R}^k, N = \mathbf{Z}^k$. We shall show that the natural map of $H_{\mathbf{R}^k}^r(T, \mathcal{S})$ into $H^r(T, \mathcal{S})$ is injective (Proposition 5.10 below). This will lead to the following striking improvement of Corollary 5.5.

COROLLARY 5.9. *Suppose $p : T \rightarrow Z$ is a principal \mathbf{T}^k -bundle. If $(A, \alpha) \in \mathfrak{B}_{\mathbf{R}^k}(T)$ satisfies $\text{Res}([A, \alpha]) = 0$ in $\text{Br}_{\mathbf{Z}^k}(T)$, then there is a system of the form $(C_0(T, \mathcal{K}), \beta) \in \mathfrak{B}_{\mathbf{T}^k}(T)$ such that $[A, \alpha] = \text{Inf}[C_0(T, \mathcal{K}), \beta]$.*

Proof. Since we certainly have $M([A, \alpha]) = 0$, Corollary 5.5 says it is enough to prove that $S([A, \alpha]) = 0$, and since $S([A, \alpha]) = b(\delta(A, \alpha))$, it is enough to prove $\delta(A, \alpha) = 0$. But $[A, \alpha]_{\mathbf{Z}^k} = 0$ also implies $\delta(A) = 0$, and because

$$\begin{array}{ccc} \text{Br}_{\mathbf{R}^k}(T, \mathcal{S}) & \xrightarrow{\delta} & H_{\mathbf{R}^k}^2(T, \mathcal{S}) \\ F \downarrow & & \downarrow \\ \text{Br}(T) & \xrightarrow{\delta} & H^2(T, \mathcal{S}) \end{array}$$

commutes, Proposition 5.10 forces $\delta(A, \alpha) = 0$. ■

PROPOSITION 5.10. *Suppose that $p : T \rightarrow Z$ is a principal \mathbf{T}^k -bundle. If $r \geq 0$ and $(1, \lambda) \in Z_{\mathbf{R}^k}^{r+1}(\{N_i\}, \mathcal{S})$ and $\{N_i\}$ is locally finite, then for each multi-index J with $|J| = r$ there is a continuous function $\sigma_J : p^{-1}(N_J) \times \mathbf{R}^k \rightarrow \mathbf{T}$ such that*

$$\sigma_J(x, \theta + \varphi) = \sigma_J(x, \theta) \sigma_J(e^{-2\pi i \theta} \cdot x, \varphi)$$

and

$$(\partial \sigma(\cdot, \theta))_K(x) = \lambda_K(x, \theta) \quad \text{for } |K| = r + 1, x \in p^{-1}(N_K), \theta \in \mathbf{R}^k.$$

In other words, $[1, \lambda] = 0$ in $H_{\mathbf{R}^k}^{r+1}(\{N_i\}, \mathcal{S})$, and the natural map of $H_{\mathbf{R}^k}^{r+1}(T, \mathcal{S})$ into $H^{r+1}(T, \mathcal{S})$ is injective.

Proof. The cocycle identity for λ_K implies that $\lambda_K(x, 0) = 1$ for all x , and since the map $f_t : p^{-1}(z) \times \mathbf{R}^k \times [0, 1] \rightarrow \mathbf{T}$ given by $f_t(x, \theta) = \lambda_K(x, t\theta)$ is a

homotopy joining $\lambda_K = f_1$ to $\lambda_K(\cdot, 0) = f_0$, it follows that for each $z \in N_K$ there is a continuous function $\alpha_K^z : p^{-1}(z) \times \mathbf{R}^k \rightarrow \mathbf{R}$ such that $\alpha_K^z(x, 0) = 0$ and

$$\lambda_K(x, \theta) = \exp(2\pi i \alpha_K^z(x, \theta)) \quad \text{for } x \in p^{-1}(z), \theta \in \mathbf{R};$$

further, since the kernel of $\exp(2\pi i \cdot) : \mathbf{R} \rightarrow \mathbf{T}$ is discrete, any other lifting β defined on a connected set of the form $p^{-1}(z) \times B$, and satisfying $\beta(x, 0) = 0$, must agree with α^z throughout that neighborhood. The same argument applied to the function

$$\alpha_K^z(x, \theta + \varphi) - \alpha_K^z(x, \theta) - \alpha_K^z(e^{-2\pi i \theta} \cdot x, \varphi)$$

shows that α_K^z is a cocycle in θ . Thus the function $\alpha_K : p^{-1}(N_K) \times \mathbf{R}^k \rightarrow \mathbf{R}$ defined by

$$\alpha_K(x, \theta) = \alpha_K^{p(x)}(x, \theta)$$

is a cocycle in θ , and we claim that it is also continuous. For if $z \in N_K$, we can use the local triviality of p over N_K to extend α_K^z to a function R_K^z defined on $p^{-1}(M) \times \mathbf{R}^k$ for some neighborhood M of z , and then $\lambda_K \exp(-2\pi i R_K^z)$ is a continuous function on $p^{-1}(M) \times \mathbf{R}^k$ which is identically 1 on $p^{-1}(z) \times \mathbf{R}^k$. If B is any compact neighborhood of 0 in \mathbf{R}^k , we can use a standard compactness argument to find a smaller neighborhood N of z such that

$$|\lambda_K(x, \theta) \exp(-2\pi i R_K^z(x, \theta)) - 1| < 2 \quad \text{for } (x, \theta) \in p^{-1}(N) \times B,$$

and then take logs to find a continuous function $\beta : p^{-1}(N) \times B \rightarrow \mathbf{R}$ such that $\beta(x, 0) = 0$ and $\lambda_K = \exp(2\pi i \beta)$ throughout $p^{-1}(N) \times B$. The uniqueness of the lifting on each fibre $p^{-1}(z) \times B$ implies that $\alpha_K = \beta$ throughout $p^{-1}(N) \times B$, and hence α_K itself must be continuous there. But z was an arbitrary point of $p^{-1}(N_K)$ and B an arbitrary neighborhood of 0, and hence α_K is continuous on all of $p^{-1}(N_K) \times \mathbf{R}^k$.

Since $(1, \lambda)$ is in $Z_{\mathbf{R}^k}^{r+1}$, we know that $(\partial \lambda(\cdot, \theta))_L = 1$ for all L with $|L| = r + 2$, and hence

$$\exp(2\pi i (\partial \alpha(\cdot, \theta))_L) = \partial(\exp(2\pi i \alpha(\cdot, \theta)))_L = (\partial \lambda(\cdot, \theta))_L = 1;$$

thus $(\partial \alpha(\cdot, \theta))_L(x)$ is a continuous \mathbf{Z} -valued function of (x, θ) satisfying $(\partial \alpha(\cdot, 0))_L(x) = 0$, and is therefore identically 0. Since the sheaf \mathcal{R} of \mathbf{R} -valued functions is fine, $H^{r+1}(T, \mathcal{R}) = 0$, and $\alpha_K(\cdot, \theta)$ is equivalent to a coboundary, and

in fact can be realized as a coboundary relative to the same cover $\{p^{-1}(N_i)\}$: if $\{\rho_i\}$ is a partition of unity subordinate to $\{N_i\}$, just take

$$\beta_J(x, \theta) = \sum_i \rho_i(p(x)) \alpha_{iJ}(x, \theta),$$

where iJ denotes the $r+1$ -multi-index (i, j_0, \dots, j_{r-1}) . (Since ρ_i vanishes on ∂N_i , extending each summand to be 0 on $N_J \setminus N_{iJ}$ gives a continuous function on N_J .) It is easy to verify that we then have $\partial \beta(\cdot, \theta) = \alpha$, and β_J is a cocycle in θ since each α_K is. Thus if we write

$$\sigma_J(x, \theta) = \exp(2\pi i \beta_J(x, \theta)),$$

then we have $(1, \sigma) \in C_{\mathbb{R}^k}^r(\{N_i\}, \mathcal{S})$ and $\Delta(1, \sigma) = (1, \lambda)$, as required. ■

EXAMPLE 5.11. In view of Corollary 5.9, one might guess that $(C_0(T, \mathcal{K}) \rtimes_{\alpha} G)^{\wedge}$ is homeomorphic to $Z \times \widehat{N}$ whenever $\alpha|_N$ is unitary. This is not true: there is something special about the case $G = \mathbb{R}^k$, $N = \mathbb{Z}^k$. For an example, take $G = \mathbb{Z}_{n^2} := \mathbb{Z}/n^2\mathbb{Z}$, and N the subgroup isomorphic to \mathbb{Z}_n , so that $G/N \cong \widehat{N} \cong \mathbb{Z}_n$ also. For p and q we take the n -fold covering $z \mapsto z^n : \mathbb{T} \rightarrow \mathbb{T}$. Since $H^4(\mathbb{T}, \mathbb{Z}) = 0$, the class $[q] \cup \partial_G([p])$ trivially vanishes, so there is a system $(A, \alpha) \in \mathfrak{Br}_{\mathbb{Z}_{n^2}}(\mathbb{T})$ with $(A \rtimes_{\alpha} \mathbb{Z}_{n^2})^{\wedge} \cong \mathbb{T}$ as principal \mathbb{Z}_n -bundles. Because $p^*[q] = p^*[p] = 0$, we know that $\text{Res} : (A \rtimes_{\alpha} \mathbb{Z}_n)^{\wedge} \rightarrow \widehat{A}$ is trivial, and α is unitary on $N = \mathbb{Z}_n$ by [20], Proposition 2.5. Notice that $\delta(A)$ also trivially vanishes, so we may take $A = C_0(T, \mathcal{K})$.

5.4. THE CASE $G = \mathbb{R}$, $N = \mathbb{Z}$: THE USUAL GYSIN SEQUENCE. To recover the usual Gysin sequence, we need to identify the equivariant groups $H_{\mathbb{R}}^r(T, \mathcal{S})$ with $H^r(T, \mathcal{S})$. There is a natural homomorphism taking $[\nu, \lambda]$ to $[\nu]$, and we have already seen that it is injective. To prove it is surjective, we have to know that every class in $H^r(T, \mathcal{S})$ can be realised by a cocycle defined on \mathbb{T} -invariant covers. This is not true for higher-dimensional torus bundles (consider the trivial bundle over a point), but for classes which can be realised this way our argument works.

PROPOSITION 5.12. *Suppose that $p : T \rightarrow Z$ is a principal \mathbb{T} -bundle. Then every class in $H^r(T, \mathcal{S})$ can be realized by a cocycle defined on a cover of the form $\{p^{-1}(N_i)\}$, for some open cover $\{N_i\}$ of T/\mathbb{R} .*

LEMMA 5.13. *Suppose that $p : T \rightarrow Z$ is a locally trivial fibre bundle with a paracompact base space Z and compact fibre F . If $z \in Z$ is fixed and $\{\lambda_K\} \in Z^r(\{M_j\}, \mathcal{S})$ is a cycle such that $[\lambda_K|_{M_K \cap p^{-1}(z)}]$ is trivial in $H^r(p^{-1}(z), \mathcal{S})$,*

then there is a neighborhood N of z such that $[\lambda_K |_{M_K \cap p^{-1}(N)}]$ is trivial in $H^r(p^{-1}(N), \mathcal{S})$.

Proof. By refining $\{M_i\}$, we may assume it is locally finite, and that there are $\mu_K : M_K \cap p^{-1}(z) \rightarrow \mathbb{T}$ such that $\partial \mu = \lambda |_{p^{-1}(z)}$. We choose a neighborhood M of z such that $p^{-1}(M)$ meets only finitely many M_j ; and $p^{-1}(M)$ is homeomorphic to $M \times F$, and extend $\{\mu_K\}$ to a cochain on $p^{-1}(M) = M \times F$ by taking it to be constant on M -slices. Then $\lambda(\partial \mu)^{-1}$ is a cocycle on $p^{-1}(M)$ which is identically 1 on $p^{-1}(z)$. Since there are only finitely many functions involved, we can find a smaller neighborhood N of z such that

$$|\lambda_K(x)(\partial \mu)_K^{-1}(x) - 1| < \left| \exp\left(\frac{\pi i}{r+1}\right) - 1 \right| \quad \text{for } x \in M_K \cap p^{-1}(N).$$

Now $\log(\lambda(\partial \mu)^{-1})$ is a cocycle with values in the fine sheaf \mathcal{R} of \mathbf{R} -valued functions, and hence equal to $\partial \nu$ for some cochain $\nu \in C^{r-1}(\{M_i \cap p^{-1}(N)\}, \mathcal{R})$. But then

$$\lambda_K |_{p^{-1}(N)} = (\partial \mu)_K \exp(\partial \nu)_K = \partial(\mu \exp \nu)_K$$

is a coboundary, and the result follows. ■

Proof of Proposition 5.12. We apply [5], Theorem II.4.17.1, which asserts that if \mathcal{H}^q is the sheaf generated by the presheaf $U \mapsto H^q(p^{-1}(U), \mathcal{S})$, then there is a spectral sequence with $E_2^{r,q} = H^r(Z, \mathcal{H}^q)$ which converges to $H^{r+q}(T, \mathcal{S})$. By definition, the stalks of the sheaf \mathcal{H}^q are

$$\mathcal{H}^q(z) = \varinjlim \{H^q(p^{-1}(U), \mathcal{S}) : U \text{ is an open neighborhood of } z\}.$$

Since $p^{-1}(z)$ is homeomorphic to \mathbb{T} , and $H^q(\mathbb{T}, \mathcal{S}) \cong H^{q+1}(\mathbb{T}, \mathbf{Z}) = 0$ for $q \geq 1$, Lemma 5.13 implies that if U is a neighborhood of z , then every element of $H^q(p^{-1}(U), \mathcal{S})$ vanishes in $H^q(p^{-1}(N), \mathcal{S})$ for some smaller neighborhood N of z — or, equivalently, that the direct limit $\mathcal{H}^q(z)$ is trivial. Thus for $q \geq 1$, all sections of the sheaf \mathcal{H}^q are trivial, $E_2^{r,q} = H^r(Z, \mathcal{H}^q) = 0$, and the statement that $\{E^{r,q}\}$ converges to H^{r+q} just says that $E_\infty^{r,0} = E_2^{r,0} = H^r(Z, \mathcal{H}^0)$ is isomorphic to $H^r(T, \mathcal{S})$. But by definition $\mathcal{H}^0(U) = H^0(p^{-1}(U), \mathcal{S}) = C(p^{-1}(U), \mathbb{T})$, and hence $H^r(Z, \mathcal{H}^0)$ is the subgroup of $H^r(T, \mathcal{S})$ consisting of cocycles which are realizable on covers of the form $\{p^{-1}(N_i)\}$. ■

PROPOSITION 5.14. *Suppose $p : T \rightarrow Z$ is a principal \mathbf{T}^k -bundle and c is a class in $H^r(T, \mathcal{S})$ with a representative defined with respect to a cover by \mathbf{T}^k -invariant sets. Then there is a cover $\{p^{-1}(N_i)\}$ of T for which there exists a cocycle $\nu_K : p^{-1}(N_K) \rightarrow \mathbf{T}$ with $c = [\nu]$, and functions $\lambda_J : p^{-1}(N_J) \times \mathbf{R}^k \rightarrow \mathbf{T}$ such that (ν, λ) is in $Z_{\mathbf{R}^k}^r(\{N_i\}, \mathcal{S})$.*

Suppose that $\nu_K : p^{-1}(N_K) \rightarrow \mathbf{T}$ is a cocycle representing the class c . We want to show that $\{\nu_K\}$ is equivalent to a cocycle of a standard form, and for this we will need some notation.

We recall that there is an isomorphism

$$\text{deg} : [\mathbf{T}^k, \mathbf{T}] = \frac{C(\mathbf{T}^k, \mathbf{T})}{\exp(2\pi i C(\mathbf{T}^k, \mathbf{R}))} \rightarrow \mathbf{Z}^k,$$

which is uniquely characterized by insisting that, if $m = (m_1, \dots, m_k) \in \mathbf{Z}^k$ and w^m denotes the function $(w_1, \dots, w_k) \mapsto \prod_i w_i^{m_i}$, then $\text{deg}(w^m) = m$. If $\zeta \in \mathbf{T}^k$ and $f : \mathbf{T}^k \rightarrow \mathbf{T}$, then the function $\sigma_\zeta(f)$ given by $w \mapsto f(w\zeta)$ has the same degree as f : choose a path ζ_t joining $\zeta = \zeta_1$ to $\zeta_0 = 1$, and then σ_{ζ_t} is a homotopy joining $\sigma_\zeta(f)$ to f . As in Section 2, we let $h_i : p^{-1}(N_i) \rightarrow N_i \times \mathbf{T}^k$ be local trivializations, so that the transition functions $t_{ij} : N_{ij} \rightarrow \mathbf{T}$ satisfy

$$h_j \circ h_i^{-1}(z, w) = (z, wt_{ij}(z)).$$

Then the above observations about deg imply that, if $z \in N_{ij}$ and $f \in C(p^{-1}(z), \mathbf{T})$, we have

$$\text{deg}(w \mapsto f \circ h_i^{-1}(z, w)) = \text{deg}(w \mapsto f \circ h_j^{-1}(z, w)),$$

and hence we have a well-defined family of homomorphisms

$$\text{deg}_z : [p^{-1}(z), \mathbf{T}] \rightarrow \mathbf{Z}^k \quad \text{for } z \in Z.$$

(If $k = 1$, we have just defined an orientation on the circle bundle $p : T \rightarrow Z$.)

We now define $m_K : N_K \rightarrow \mathbf{Z}^k$ by $m_K(z) = \text{deg}_z(\nu_K | p^{-1}(z))$; note that $\{m_K\}$ is in $Z^r(\{N_i\}, \mathbf{Z}^k)$ because $\{\nu_K\}$ is a cocycle and deg is a homomorphism. The local trivializations h_i define \mathbf{T}^k -equivariant maps $w_i : p^{-1}(N_i) \rightarrow \mathbf{T}^k$ by $h_i(p(x), w_i(x))$, and we can now give our standard form for $\{\nu_K\}$.

LEMMA 5.15. *We can refine the cover $\{p^{-1}(N_i)\}$ to ensure there is a cochain μ in $C^r(\{N_i\}, \mathcal{S})$ such that ν is equivalent to the cocycle in $Z^r(\{p^{-1}(N_i)\}, \mathcal{S})$ given by*

$$w_{k_r}(x)^{m_K(p(x))} \mu_K(p(x)) \quad \text{for } x \in p^{-1}(N_K).$$

Proof. If $z \in N_K$, the cocycle defined by

$$(\nu_K w_{k_r}^{-m_K})(x) = \nu_K(x) w_{k_r}(x)^{-m_K(p(x))}$$

has degree 0 on the fibre $p^{-1}(z)$, and hence equals $\exp(2\pi i \beta_K)$ for some continuous function $\beta_K : p^{-1}(z) \rightarrow \mathbf{R}$. We can extend each β_K to $p^{-1}(N_K)$, and then $\nu_K w_{k_r}^{-m_K} \exp(-2\pi i \beta_K)$ is identically 1 on $p^{-1}(z)$. Reasoning as in the proof of Lemma 5.13, we can find a neighborhood N of z and a continuous function $\alpha : p^{-1}(N) \rightarrow \mathbf{R}$ satisfying

$$\nu_K w_{k_r}^{-m_K} = \exp(2\pi i \alpha) \quad \text{throughout } p^{-1}(N).$$

We can now refine the cover $\{p^{-1}(N_i)\}$, using the argument of [3], 10.7.11, to ensure that there are continuous functions $\alpha_K : p^{-1}(N_K) \rightarrow \mathbf{R}$ such that

$$\nu_K(x) w_{k_r}(x)^{-m_K(p(x))} = \exp(2\pi i \alpha_K(x)) \quad \text{for } x \in p^{-1}(N_K).$$

If $|L| = r + 2$, then

$$\begin{aligned} \exp(2\pi i (\partial \alpha)_L(x)) &= w_{k_r}(x)^{-m_{k_0 \dots k_r}(p(x))} w_{k_{r+1}}(x)^{m_{k_0 \dots k_{r-1} k_{r+1}}(p(x))} \\ &\quad \cdot w_{k_{r+1}}(x)^{-m_{k_0 \dots k_{r-2} k_r k_{r+1}}(p(x))} \dots \\ &= [w_{k_r}(x)^{-1} w_{k_{r+1}}(x)]^{m_{k_0 \dots k_r}(p(x))} w_{k_{r+1}}(x)^{(\partial m)_L(p(x))} \\ &= t_{k_r k_{r+1}}(p(x))^{m_{k_0 \dots k_r}(p(x))}, \end{aligned}$$

since $\{m_K\}$ is a cocycle and $w_i^{-1} w_j = t_{ij} \circ p$. Since this last formula depends only on $p(x)$, we can average the left-hand side over \mathbf{T}^k -orbits without changing the right-hand side. Thus we set

$$\mu_K(p(x)) = \exp \left[2\pi i \left(\int_0^1 \dots \int_0^1 \alpha_K(e^{2\pi i \theta} \cdot x) \, d\theta_1 \dots d\theta_k \right) \right],$$

observe that the coboundary operator pulls through the combination of exponential and integral, and deduce that

$$(5.3) \quad (\partial \mu)_L(p(x)) = \exp(2\pi i (\partial \alpha)_L(x)) = \partial(\exp(2\pi i \alpha))_L(x).$$

To prove the lemma, it is enough to show that $\{\exp(2\pi i\alpha_K)\}$ differs from $\{\mu_K\}$ by a coboundary. If we set

$$s_K(x) = \alpha_K(x) - \int_0^1 \cdots \int_0^1 \alpha_K(e^{2\pi i\theta} \cdot x) d\theta_1 \cdots d\theta_k,$$

then $\exp(2\pi i s_K) = (\exp(2\pi i\alpha_K))\mu_K^{-1}$, and (5.3) says that $\exp(2\pi i(\partial s)_L) = 1$; thus $(\partial s)_L$ is constant on connected components. On the other hand,

$$(\partial s)_L(x) = (\partial \alpha)_L(x) - \int_0^1 \cdots \int_0^1 (\partial \alpha)_L(e^{2\pi i\theta} \cdot x) d\theta_1 \cdots d\theta_k$$

vanishes at some point of each fibre, and because the fibres are connected, we deduce that $(\partial s)_L$ is identically 0. Thus $\{s_K\}$ is a cocycle in $Z^r(T, \mathcal{R})$, and is therefore equivalent to a coboundary. Thus $\{\exp(2\pi i s_K)\}$ is a coboundary, $\{\exp(2\pi i\alpha_K)\}$ differs from $\{\mu_K\}$ by a coboundary, and the lemma is proved. ■

Proof of Proposition 5.14. We replace $\{\nu_K\}$ by the equivalent cocycle $\{w_{k_r}^{m_K} \mu_K\}$. Although $\{m_K\}$ may be non-trivial as a cocycle in $Z^r(Z, \mathbf{Z}^k)$, it is equivalent in $Z^r(Z, \mathcal{R}^k)$ to a coboundary, and, by refining, we may suppose $m_K = (\partial \eta)_K$ for some cochain $\eta \in C^{r-1}(\{N_i\}, \mathcal{R}^k)$. We now have

$$\begin{aligned} \nu_K(e^{-2\pi i\theta} \cdot x) &= \exp(-2\pi i(\theta \cdot m_K(p(x)))) w_{k_r}(x)^{m_K(p(x))} \mu_K(p(x)) \\ &= \exp(-2\pi i(\theta \cdot m_K(p(x)))) \nu_K(x) \\ &= \exp(-2\pi i(\theta \cdot (\partial \eta)_K(p(x)))) \nu_K(x). \end{aligned}$$

If we define $\lambda_J(x, \theta) = \exp(-2\pi i(\theta \cdot \eta_J(p(x))))$, then the above says precisely that $\partial \lambda(\cdot, \theta) = \nu \cdot (\theta \cdot \nu)^{-1}$, and since λ_J is clearly a homomorphism in θ , we have found λ such that (ν, λ) is in $Z_{\mathbf{R}^k}^r(T, \mathcal{S})$. ■

REMARK 5.16. Since $H_{\mathbf{T}^k}^r(T, \mathcal{S})$ need not map onto $H^r(T, \mathcal{S})$, it is instructive to note where it was crucial to take $G = \mathbf{R}^k$. If we had required λ_J well-defined on $p^{-1}(N_J) \times \mathbf{T}^k$ rather than $p^{-1}(N_J) \times \mathbf{R}^k$, we would need $\exp(2\pi i\theta \cdot \eta_J)$ to depend only on $\exp(2\pi i\theta) \in \mathbf{T}^k$, and hence η_J would have to take values in \mathbf{Z}^k rather than \mathbf{R}^k . So it is because $\{m_K\}$ need not be a coboundary in $Z^r(T, \mathbf{Z}^k)$ that we may not be able to construct the family $\{\lambda_J\}$. If the cocycle $\{m_K\}$ is trivializable in $Z^r(T, \mathbf{Z}^k/q\mathbf{Z}^k)$ for some $q \in \mathbf{Z}^k$, then $\lambda_J(\cdot, \theta)$ will be well-defined for $\theta \in \mathbf{T}^k = \mathbf{R}^k/q\mathbf{Z}^k$, and (ν, λ) will belong to $Z_{\mathbf{T}^k}^r(T, \mathcal{Z})$, where the action of \mathbf{T}^k on T is defined in terms of the given action by $z \star x = z^q \cdot x$.

THEOREM 5.17. *If $p : T \rightarrow Z$ is a principal \mathbb{T} -bundle, then the map $[\nu, \lambda] \mapsto [\nu]$ is an isomorphism of $H_{\mathbb{R}}^r(T, \mathcal{S})$ onto $H^r(T, \mathcal{S})$ for all $r \geq 1$.*

Proof. If $p : T \rightarrow Z$ is a circle bundle, Propositions 5.12 and 5.14 imply that $[\nu, \lambda] \mapsto [\nu]$ maps $H_{\mathbb{R}}^r(T, \mathcal{S})$ onto $H^r(T, \mathcal{S})$, and Proposition 5.10 implies that this map is an injection. (To apply Proposition 5.10, one needs to observe that if $\nu_L = \partial(\varphi_K)$, then $(\nu, \lambda) \sim (1, \tilde{\lambda})$ where $\tilde{\lambda}_K(x, s) = \lambda_K(x, s)\varphi_K(x)\overline{\varphi_K(s^{-1} \cdot x)}$.) Thus the theorem is proved. ■

For higher-dimensional torus bundles, Propositions 5.14 and 5.10 immediately give

COROLLARY 5.18. *Suppose that $p : T \rightarrow Z$ is a principal \mathbb{T}^k -bundle. Then the map $[\nu, \lambda] \mapsto [\nu]$ is an isomorphism of $H_{\mathbb{R}}^r(T, \mathcal{S})$ onto the subgroup of $H^r(T, \mathcal{S})$ consisting of classes which are realizable by cocycles defined on covers by invariant sets.*

COROLLARY 5.19. *(The Gysin sequence for a circle bundle) Suppose that $p : T \rightarrow Z$ is a principal \mathbb{T} -bundle over a locally compact paracompact space Z . Then there is an exact sequence*

$$\dots \rightarrow H^r(Z, \mathbb{Z}) \xrightarrow{p^*} H^r(T, \mathbb{Z}) \xrightarrow{p_!} H^{r-1}(Z, \mathbb{Z}) \xrightarrow{\cup \partial([p])} H^{r+1}(Z, \mathbb{Z}) \rightarrow \dots$$

where $\partial([p])$ denotes the class of the bundle in $H^2(Z, \mathbb{Z})$.

Proof. Theorem 4.1, Theorem 5.17, and the natural isomorphism $\partial = \partial_{\mathbb{R}}$ of $H^r(\cdot, \mathcal{S})$ onto $H^{r+1}(\cdot, \mathbb{Z})$ give us a diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & H^r(Z, \mathcal{S}) & \xrightarrow{p_{\mathbb{R}}^*} & H_{\mathbb{R}}^r(T, \mathcal{S}) & \xrightarrow{b} & H^{r-1}(Z, \mathcal{S}) & \xrightarrow{\cup \partial_{\mathbb{R}}([p])} & H^{r+1}(Z, \mathcal{S}) & \rightarrow & \dots \\ & & \downarrow \partial & & \downarrow & & \downarrow \partial & & \downarrow \partial & & \\ \dots & \rightarrow & H^{r+1}(Z, \mathbb{Z}) & \xrightarrow{p^*} & H^{r+1}(T, \mathbb{Z}) & \xrightarrow{p_!} & H^r(Z, \mathbb{Z}) & \xrightarrow{\cup \partial([p])} & H^{r+2}(Z, \mathbb{Z}) & \rightarrow & \dots \end{array}$$

in which the vertical arrows are isomorphisms, and the top row is exact. Thus all we have to prove is that the diagram commutes. Since $p_{\mathbb{R}}^*(\mu) = (p^*\mu, 1)$ and the isomorphism of Theorem 5.17 takes $[\nu, \lambda]$ to $[\nu]$, the commutativity of the first square follows from the naturality of ∂ . That of the third square follows from the realization of the cup product in Čech cohomology in terms of cocycles (3.2). It therefore remains to identify the homomorphism b with $p_!$.

The map $p_!$, often known as *integration over the fibres of p* , can be realized on $H^r(T, \mathcal{S}) \cong H^{r+1}(T, \mathbb{Z})$ as follows. We fix an orientation for the bundle $p : T \rightarrow Z$ — that is, a family of homomorphisms $\text{deg}_z : [p^{-1}(z), \mathbb{T}] \rightarrow \mathbb{Z}$ (see the

discussion preceding Lemma 5.15). If we realize a class in $H^{r+1}(T, \mathbf{Z})$ via a cocycle $\nu \in Z^r(T, \mathcal{S})$, then $p_!([\nu])$ is the class of the cocycle

$$m_K(z) = \deg_z(\nu_K | p^{-1}(z))$$

in $H^r(Z, \mathbf{Z})$. On the other hand, if we extend ν to an element (ν, λ) of $Z_{\mathbb{R}}^r(T, \mathcal{S})$, then a cocycle $\zeta \in Z^r(Z, \mathbf{Z})$ representing the image of $b(\nu, \lambda) \in H^{r-1}(Z, \widehat{N}) = H^{r-1}(Z, \mathcal{S})$ in $H^r(Z, \mathbf{Z})$ is obtained by writing $\gamma_J(z) = \exp(2\pi\tilde{\gamma}_J(z))$ for some $\tilde{\gamma}_J : N_J \rightarrow \mathbb{R}$, and taking

$$\xi_K(z) = (\partial \tilde{\gamma})_K(z) \in \mathbf{Z} = N.$$

If we examine our construction of λ compatible with ν (see the proof of Proposition 5.14), we find γ_J has a natural extension $\tilde{\gamma}_J : N_J \rightarrow \mathbb{R}$ satisfying $(\partial \tilde{\gamma})_K = m_K$, and hence we do have $b(\nu, \lambda) = p_!(\nu)$, as required. ■

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REFERENCES

1. N.B. BACKHOUSE, Projective representations of space groups, II: Factor systems, *Quart. J. Math. Oxford Ser. (2)* **21**(1970), 277–295.
2. D. CROCKER, A. KUMJIAN, I. RAEBURN, D.P. WILLIAMS, An equivariant Brauer group and actions of groups on C^* -algebras, *J. Funct. Anal.*, 1996, in press.
3. J. DIXMIER, *C^* -Algebras*, North-Holland, New York, 1977.
4. S. ECHTERHOFF, Morita equivalent actions and a new version of the Packer-Raeburn stabilization trick, *J. London Math. Soc. (2)* **50**(1994), 170–186.
5. R. GODEMENT, *Topologie algébrique et théorie des faisceaux*, Hermann, Paris, 1964.
6. P. GREEN, C^* -algebras of transformation groups with smooth orbit space, *Pacific J. Math.* **72**(1977), 71–97.
7. A. GROTHENDIECK, Sur quelques points d'algèbre homologique, *Tôhoku Math. J. (2)* **9**(1957), 119–221.
8. S. KALISZEWSKI, A note on Morita equivalence of twisted C^* -dynamical systems, *Proc. Amer. Math. Soc.* **123**(1995), 1737–1740.

9. A. KUMJIAN, On equivariant sheaf cohomology and elementary C^* -bundles, *J. Operator Theory* **20**(1988), 207–240.
10. A. KUMJIAN, I. RAEBURN, D.P. WILLIAMS, The equivariant Brauer groups of commuting free and proper actions are isomorphic, *Proc. Amer. Math. Soc.* **124**(1996), 809–817.
11. C.C. MOORE, Extensions and low dimensional cohomology theory of locally compact groups. I, *Trans. Amer. Math. Soc.* **113**(1964), 40–63.
12. C.C. MOORE, Extensions and low dimensional cohomology theory of locally compact groups. II, *Trans. Amer. Math. Soc.* **113**(1964), 64–86.
13. C.C. MOORE, Restrictions of unitary representations to subgroups and ergodic theory: group extensions and group cohomology, in *Group representations in mathematics and physics*, Battelle Seattle 1969 Rencontres, New York, Lecture Notes in Phys., vol. 6, Springer-Verlag, New York, 1970, pp. 1–35.
14. C.C. MOORE, Group extensions and cohomology for locally compact groups. III, *Trans. Amer. Math. Soc.* **221**(1976), 1–33.
15. C.C. MOORE, Group extensions and cohomology for locally compact groups. IV, *Trans. Amer. Math. Soc.* **221**(1976), 34–58.
16. D. OLESEN, G.K. PEDERSEN, Partially inner C^* -dynamical systems, *J. Funct. Anal.* **66**(1986), 263–281.
17. D. OLESEN, I. RAEBURN, Pointwise unitary automorphism groups, *J. Funct. Anal.* **93**(1990), 278–309.
18. J.A. PACKER, Moore cohomology and central twisted crossed product C^* -algebras, *Canad. J. Math.* **48**(1996), 159–174.
19. J.A. PACKER, I. RAEBURN, On the structure of twisted group C^* -algebras, *Trans. Amer. Math. Soc.* **334**(1992), 685–718.
20. J. PHILLIPS, I. RAEBURN, Crossed products by locally unitary automorphism groups and principal bundles, *J. Operator Theory* **11**(1984), 215–241.
21. I. RAEBURN, J. ROSENBERG, Crossed products of continuous-trace C^* -algebras by smooth actions, *Trans. Amer. Math. Soc.* **305**(1988), 1–45.
22. I. RAEBURN, D.P. WILLIAMS, Pull-backs of C^* -algebras and crossed products by certain diagonal actions, *Trans. Amer. Math. Soc.* **287**(1985), 755–777.
23. I. RAEBURN, D.P. WILLIAMS, Moore cohomology, principal bundles, and actions of groups on C^* -algebras, *Indiana Univ. Math. J.* **40**(1991), 707–740.
24. I. RAEBURN, D.P. WILLIAMS, Dixmier-Douady classes of dynamical systems and crossed products, *Canad. J. Math.* **45**(1993), 1032–1066.
25. I. RAEBURN, D.P. WILLIAMS, Topological invariants associated to the spectrum of crossed product C^* -algebras, *J. Funct. Anal.* **116**(1993), 245–276.
26. I. RAEBURN, D.P. WILLIAMS, Equivariant cohomology and a Gysin sequence for principal bundles, Tech. Report PMA-TR95-184, Dartmouth College, 1995.
27. J. ROSENBERG, Some results on cohomology with Borel cochains, with applications to group actions on operator algebras, in *Oper. Theory Adv. Appl.*, vol. 17, Birkhäuser-Verlag, Basel, 1986, pp. 301–330.
28. E.H. SPANIER, *Algebraic Topology*, McGraw-Hill, 1966.
29. V.S. VARADARAJAN, *Geometry of Quantum Theory*, vol. II, Van Nostrand-Reinhold, New York 1970.
30. F.W. WARNER, *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foresman and Company, Glenview, Illinois 1971.

31. D.P. WILLIAMS, The topology on the primitive ideal space of transformation group C^* -algebras and CCR transformation group C^* -algebras, *Trans. Amer. Math. Soc.* **266**(1981), 335-359.

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