J. OPERATOR THEORY
38(1997), 43–66

NONCOMMUTATIVE ARZELA-ASCOLI THEOREMS

DON HADWIN

Communicated by Norberto Salinas

ABSTRACT. We study a type of reflexivity in the C^*-algebra of bounded sequences of operators on a separable Hilbert space H modulo the ideal of null sequences. Our results lead to a non-commutative notion of equicontinuity that relates certain reflexivity results to non-commutative analogues of the Arzela-Ascoli theorem.

KEYWORDS: *C^*-algebra, reflexive algebra, approximate reflexivity.*

AMS SUBJECT CLASSIFICATION: Primary 46L05, 47D25; Secondary 46L85.

Suppose H is a Hilbert space, $B(H)$ is the C^*-algebra of all operators on H, and $\mathcal{K}(H)$ is the ideal of all compact operators. Let $\ell^\infty(B(H))$ denote the C^*-direct product of countably many copies of $B(H)$, and let $c_0(B(H))$ denote the C^*-direct sum of countably many copies of $B(H)$. In other words, $\ell^\infty(B(H))$ is the C^*-algebra of bounded sequences of operators in $B(H)$ with the supremum norm, and $c_0(B(H))$ is the ideal of all null sequences of operators in $B(H)$. Let $\mathcal{Q}(H)$ be the quotient $\ell^\infty(B(H))/c_0(B(H))$. Note that $\ell^\infty(B(H))$ is the multiplier C^*-algebra of $c_0(B(H))$; thus $\mathcal{Q}(H)$ is a so-called *corona C^*-algebra* ([21]). Another example of a corona C^*-algebra is the Calkin algebra $B(H)/\mathcal{K}(H)$.

We let η denote the natural quotient map from $\ell^\infty(B(H))$ to $\mathcal{Q}(H)$. There is a natural isometric $*$-homomorphism $\pi : B(H) \to \mathcal{Q}(H)$ defined by $\pi(T) = \eta(T, T, T, \ldots)$.

It is well known that every unitary element in $\mathcal{Q}(H)$ can be lifted to a unitary in $\ell^\infty(B(H))$, every projection in $\mathcal{Q}(H)$ can be lifted to a projection in $\ell^\infty(B(H))$, and every invertible element in $Q(H)$ can be lifted to an invertible element in $\ell^\infty(B(H))$. These facts lead to the following relationship between concepts in what is sometimes called "approximate" operator theory.

Suppose $S, T \in B(H)$. The operators S and T are *approximately equivalent* if there is a sequence $\{U_n\}$ of unitary operators such that $\|U_n S U_n^* - T\| \to 0$. We say S and T are *approximately similar* if there is a sequence $\{A_n\}$ of invertible operators such that $\sup_n \|A_n\| \, \|A_n^{-1}\| < \infty$ and $\|A_n S A_n^{-1} - T\| \to 0$. It is easily seen that S and T are approximately equivalent (resp., approximately similar) if and only if $\pi(S)$ and $\pi(T)$ are unitarily equivalent (resp., similar) in $Q(H)$. If $\mathcal{S}$ is a norm-separable subspace of $B(H)$, then $\operatorname{appr} \operatorname{Alg} \operatorname{Lat} (\mathcal{S})$ is the set of all operators T such that $\|(1 - P_n)TP_n\| \to 0$ whenever $\{P_n\}$ is a sequence of projections such that, for every S in $\mathcal{S}$, $\|(1 - P_n)TP_n\| \to 0$. Similarly, $\operatorname{appr} (\mathcal{S})''$, the approximate double commutant of $\mathcal{S}$, is defined to be the set of all operators T such that $\|A_n T - TA_n\| \to 0$ whenever $\{A_n\}$ is a bounded sequence such that, for every S in $\mathcal{S}$, $\|A_n S - SA_n\| \to 0$. It is a simple exercise to show that $\operatorname{appr} \operatorname{Alg} \operatorname{Lat} (\mathcal{S}) = \pi^{-1}(\operatorname{Alg} \operatorname{Lat} \pi(\mathcal{S}))$ and $\operatorname{appr} (\mathcal{S})'' = \pi^{-1}(\pi(\mathcal{S})'')$.

For simplicity of notation, we define $\operatorname{appr} \operatorname{Lat} \mathcal{S}$ to be the class of all nets $\{P_\lambda\}$ of projections such that, for every S in $\mathcal{S}$, $\|(1 - P_\lambda)SP_\lambda\| \to 0$. Note that if $\mathcal{S} = \mathcal{S}^*$, then $\operatorname{appr} \operatorname{Lat} \mathcal{S}$ is th class of all nets $\{P_\lambda\}$ of projections such that $\|SP_\lambda - P_\lambda S\| \to 0$ for every S in $\mathcal{S}$.

In [13] the author proved the "approximate" version of von Neumann's double commutant theorem: For any norm-separable subset $\mathcal{S}$ of $B(H)$ such that $\mathcal{S} = \mathcal{S}^*$, $\operatorname{appr} \operatorname{Alg} \operatorname{Lat} (\mathcal{S}) = \operatorname{appr} (\mathcal{S})'' = C^*(\mathcal{S})$.

In [15] the author extended the preceding result with a distance estimate. Note that if $\mathcal{G}$ is a C^*-subalgebra of $B(H)$, $T \in B(H)$, and $\{P_\lambda\}$ is a net in $\operatorname{appr} \operatorname{Lat} \mathcal{G}$, then

$$\limsup_\lambda \|TP_\lambda - P_\lambda T\| \leqslant \inf\{\limsup_\lambda \|(T - A)P_\lambda(T - A)\| : A \in \mathcal{G}\} \leqslant \operatorname{dist}(T, \mathcal{G}).$$

PROPOSITION 1. ([15]) *If $\mathcal{G}$ is a unital C^*-subalgebra of $B(H)$ and $T \in B(H)$, then there is a net $\{P_\lambda\}$ of projections such that:*

(i) $\|SP_\lambda - P_\lambda S\| \to 0$ *for every S in $\mathcal{G}$;*

(ii) $\lim_\lambda \|TP_\lambda - P_\lambda T\| \geqslant (\frac{1}{29}) \operatorname{dist}(T, \mathcal{G})$.

In this paper we wish to consider reflexivity results in $Q(H)$. More precisely, we wish to consider the problem of determining $\operatorname{Alg} \operatorname{Lat} \mathcal{G}$ when $\mathcal{G}$ is a separable unital C^*-algebra of $Q(H)$. It is clear that $\mathcal{G} \subset \mathcal{G}'' \subset \operatorname{Alg} \operatorname{Lat} \mathcal{G}$ always holds when $\mathcal{G}$ is a C^*-algebra. Unlike $B(H)$, the algebra $Q(H)$ has a non-trivial center, namely the image of the center of $\ell^\infty(B(H))$, which is the set of bounded sequences of scalars. Let $\mathcal{Z}$ denote the center of $Q(H)$. We then have

$$C^*(\mathcal{G} \cup \mathcal{Z}) \subset \mathcal{G}'' \subset \operatorname{Alg} \operatorname{Lat} \mathcal{G}.$$

It is natural to ask if equality holds when $\mathcal{G}$ is separable and unital. This would be an analog for $Q(H)$ of D. Voiculescu's reflexivity theorem for norm closed unital separable subalgebras of the Calkin algebra ([28]) (also see [1]).

QUESTION A. If $\mathcal{G}$ is a separable unital C^*-subalgebra of $Q(H)$, must $\operatorname{Alg} \operatorname{Lat} \mathcal{G} = C^*(\mathcal{G} \cup \mathcal{Z})$?

We will not completely answer the above question, but in a few cases we give an affirmative answer, which leads to questions whose positive answers may be considered non-commutative generalizations of the Arzela-Ascoli theorem.

We first look at the problem lifted to the algebra $\ell^\infty(B(H))$. Suppose $\mathcal{G}$ is a separable C^*-subalgebra of $Q(H)$. Then there is a separable unital C^*-subalgebra $\mathcal{B}$ of $\ell^\infty(B(H))$ such that $\eta(\mathcal{B}) = \mathcal{G}$. If $T \in \ell^\infty(B(H))$, we write

$$T = (T(1), T(2), \ldots).$$

For each positive integer n, let $\mathcal{B}_n$ denote the C^*-algebra $\{T(n) : T \in \mathcal{B}\}$. Then $\mathcal{B} \subset \prod_n \mathcal{B}_n$, but the containment is always proper, since the product is never separable.

At this time the following theorem is the most we can say in the general situation.

THEOREM 2. *Suppose $\mathcal{B}$ is a separable unital C^*-subalgebra of $\ell^\infty(B(H))$. Then* $\operatorname{Alg} \operatorname{Lat} \eta(\mathcal{B}) \subset \eta\Big(\prod_n \mathcal{B}_n\Big)$.

Proof. Suppose $S \in \ell^\infty(B(H))$ and $\eta(S) \in \operatorname{Alg} \operatorname{Lat} \eta(\mathcal{B})$. Choose a dense sequence $\{T_n\}$ in $\mathcal{B}$. Using the distance formula in Proposition 1, we can choose a sequence $\{P_n\}$ of projections in $B(H)$ such that, for each n,

$$\|T_k(n)P_n - P_nT_k(n)\| < \frac{1}{n} \quad \text{for } 1 \leqslant k \leqslant n,$$

and

$$\|S(n)P_n - P_nS(n)\| \geqslant \left(\frac{1}{30}\right) \operatorname{dist}(S(n), \mathcal{B}_n).$$

Let $P = (P_1, P_2, \ldots)$. It follows that $\eta(P) \in \operatorname{Lat} \eta(\mathcal{B})$, which implies that $0 = \eta(P)\eta(S) - \eta(S)\eta(P) = \eta(PS - SP)$. Hence $P_nS(n) - S(n)P_n \to 0$. Therefore $\operatorname{dist}(S(n), \mathcal{B}_n) \to 0$. Hence $\eta(S) \in \eta\Big(\prod_n \mathcal{B}_n\Big)$. ∎

COROLLARY 3. *If $\mathcal{G}$ is a finite-dimensional C^*-subalgebra of $\mathcal{Q}(H)$, then* $\operatorname{Alg}\operatorname{Lat}\mathcal{G} = C^*(\mathcal{G} \cup \mathcal{Z})$.

Proof. We can choose a finite-dimensional C^*-algebra $\mathcal{B}$ of $\ell^\infty(B(H))$ such that $\eta(\mathcal{B}) = \mathcal{G}$. Since $\mathcal{B}$ is finite-dimensional, $\prod_n \mathcal{B}_n = C^*(\mathcal{B} \cup \ell^\infty(\mathbf{C})) = \eta^{-1}(C^*(\mathcal{G} \cup \mathcal{Z}))$. It follows from the theorem that $\operatorname{Alg}\operatorname{Lat}\mathcal{G} \subset C^*(\mathcal{G} \cup \mathcal{Z})$. The reverse inclusion always holds. ∎

We now consider the case in which $\mathcal{G} = \pi(\mathcal{D})$ for some separable unital C^*-subalgebra $\mathcal{D}$ of $B(H)$. We can take $\mathcal{B} = \{(T, T, \ldots) : T \in \mathcal{D}\}$. We would like to show that $\operatorname{Alg}\operatorname{Lat}\mathcal{G} = C^*(\mathcal{G} \cup \mathcal{Z})$. Note that $\eta^{-1}(C^*(\mathcal{G} \cup \mathcal{Z})) = C^*(\mathcal{B} \cup \ell^\infty(\mathbf{C}))$. Hence we neeed to know more about $C^*(\mathcal{B} \cup \ell^\infty(\mathbf{C}))$ and its relationship to $\prod_n \mathcal{B}_n$.

LEMMA 4. *Suppose $\mathcal{D}$ is a unital C^*-subalgebra of $B(H)$, and $\mathcal{B} = \{(T, T, \ldots) : T \in \mathcal{D}\}$. Then $C^*(\mathcal{B} \cup \ell^\infty(\mathbf{C})) = \eta^{-1}(C^*(\pi(\mathcal{D}) \cup \mathcal{Z})) = \{(T_1, T_2, \ldots) \in \prod_n \mathcal{B}_n : \{T_1, T_2, \ldots\}$ is a totally bounded subset of $\mathcal{D}\}$.*

Total boundedness in a commutative C^*-algebra is characterized by the Arzela-Ascoli theorem ([26]), which can be used to give the following affirmative result.

THEOREM 5. *Suppose $\mathcal{D}$ is a separable commutative unital C^*-subalgebra of $B(H)$. Then* $\operatorname{Alg}\operatorname{Lat}\pi(\mathcal{D}) = C^*(\pi(\mathcal{D}) \cup \mathcal{Z})$.

Proof. The maximal ideal space of $\mathcal{D}$ is a compact metric space (X, d). Let $\rho : C(X) \to \mathcal{D}$ be the inverse of the Gelfand map. Then ρ extends to a $*$-homomorphism, which we shall call ρ, from the set $\operatorname{Bor}(X)$ of bounded Borel functions on X into $B(H)$. Suppose $s \in \operatorname{Alg}\operatorname{Lat}\pi(\mathcal{D})$. Then, by Theorem 2, there is an S in $\ell^\infty(B(H))$ such that $\eta(S) = s$ and, for every $n \geqslant 1$, $S(n) \in \mathcal{D}$. For each $n \geqslant 1$, choose f_n in $C(X)$ such that $\rho(f_n) = S(n)$. To prove that $s \in C^*(\pi(\mathcal{D}) \cup \mathcal{Z})$, it suffices, by Lemma 4, to show that $\{S(1), S(2), \ldots\}$ is totally bounded. Since ρ is isometric on $C(X)$, it suffices to show that $\{f_1, f_2, \ldots\}$ is totally bounded in $C(X)$.

Assume, via contradiction, that $\{f_1, f_2, \ldots\}$ is not totally bounded. It follows that there is an $\varepsilon > 0$, and an increasing sequence $\{n_k\}$ of positive integers, and sequences $\{x_k\}$, $\{y_k\}$ in X such that $d(x_k, y_k) \to 0$ and $|f_{n_k}(x_k) - f_{n_k}(y_k)| \geqslant \varepsilon$ for $k \geqslant 1$. By choosing an appropriate subsequence, if necessary, we can assume that $x_k \to x$ and $y_k \to x$ for some x in X. For each k, choose disjoint open sets U_k and V_k with diameter less than $1/k$ such that $x_k \in U_k$, $y_k \in V_k$, and the diameters of $f_{n_k}(U_k)$ and $f_{n_k}(V_k)$ are less than ε. For each $k \geqslant 1$, choose vectors $u_k \in \operatorname{ran}(\rho(\chi_{U_k}))$ and $v_k \in \operatorname{ran}(\rho(\chi_{V_k}))$ such that $\|u_k\| = \|v_k\| = \sqrt{2}/2$, and define the rank-one projection $Q_k = (u_k + v_k) \otimes (u_k + v_k)$. Define a projection P in $\ell^\infty(B(H))$ by $P(n_k) = Q_k$ for $k \geqslant 1$, and $P(n) = 0$ otherwise.

Suppose $f \in C(X)$. It follows from continuity at x that $\|(f - f(x))\chi_{U_k}\| \to 0$ and $\|(f - f(x))\chi_{V_k}\| \to 0$. It follows that $\|(\rho(f) - f(x))Q_k\| \to 0$ and $\|Q_k(\rho(f) - f(x))\| \to 0$. In particular, we have $\|\rho(f)Q_k - Q_k\rho(f)\| \to 0$. It follows that $\pi(\rho(f))$ commutes with $\eta(P)$, so $\eta(P)$ commutes with s. We must therefore have $\|\rho(f_{n_k})Q_k - Q_k\rho(f_{n_k})\| \to 0$. However, it follows that $\|(f_{n_k} - f_{n_k}(x_k))\chi_{U_k}\| \to 0$ and $\|(f_{n_k} - f_{n_k}(y_k))\chi_{V_k}\| \to 0$. Hence $\|[\rho(f_{n_k})Q_k - Q_k\rho(f_{n_k})] - (f_{n_k}(x_k) - f_{n_k}(y_k))(u_k \otimes v_k - v_k \otimes u_k)\| \to 0$. It follows that $\|(f_{n_k}(x_k) - f_{n_k}(y_k))(u_k \otimes v_k - v_k \otimes u_k)\| \to 0$. However,

$$\|(f_{n_k}(x_k) - f_{n_k}(y_k))(u_k \otimes v_k - v_k \otimes u_k)\| = \left(\frac{1}{2}\right) |f_{n_k}(x_k) - f_{n_k}(y_k)| \geqslant \frac{\varepsilon}{2}$$

for each k. This contradiction completes the proof. ∎

We now turn to a non-commutative version of equicontinuity in $B(H)$ that is equivalent to Theorem 5 for non-commutative C^*-algebras. We call a subset $\mathcal{S}$ of $B(H)$, P-*equicontinuous* if $\mathcal{S}$ is bounded and, for every net $\{P_\lambda\}$ of projections such that $\|P_\lambda S - SP_\lambda\| \to 0$ for each S in $\mathcal{S}$, we have $\lim_\lambda \sup\{\|P_\lambda S - SP_\lambda\| : S \in \mathcal{S}\} = 0$ (i.e., $\|P_\lambda S - SP_\lambda\| \to 0$ uniformly on $\mathcal{S}$). It is clear that every totally bounded subset of $B(H)$ is P-equicontinuous; one of our main problems is determining whether the converse is true.

More generally, suppose $\mathcal{G}$ is a unital C^*-subalgebra of $B(H)$ and $\mathcal{S}$ is a norm-closed bounded subset of $\mathcal{G}$. We say that $\mathcal{S}$ is *relatively P-equicontinuous in* $\mathcal{G}$, if, for every net $\{P_\lambda\}$ in $\operatorname{appr Lat} \mathcal{G}$, we have $\lim_\lambda \sup\{\|P_\lambda S - SP_\lambda\| : S \in \mathcal{S}\} = 0$. We call the C^*-subalgebra $\mathcal{G}$ of $B(H)$ an *Arzela-Ascoli algebra* if every relatively P-equicontinuous subset of $\mathcal{G}$ is totally bounded in norm.

We state our two main questions concerning P-equicontinuity.

QUESTION B. Is every P-equicontinuous subset of $B(H)$ totally bounded?

QUESTION C. Is every unital C^*-subalgebra of $B(H)$ an Arzela-Ascoli algebra?

It is clear that a norm separable subset $\mathcal{S}$ of $B(H)$ is P-equicontinuous if and only if $\mathcal{S}$ is relatively P-equicontinuous in $C^*(\mathcal{S})$. We shall see that every P-equicontinuous subset of $B(H)$ is norm separable. The following lemma is obvious (consider the contrapositive).

LEMMA 6. *A bounded subset $\mathcal{S}$ of $B(H)$ is* P*-equicontinuous if and only if, for every $\varepsilon > 0$, there is a $\delta > 0$ and a finite subset F of $\mathcal{S}$ such that, for every projection P such that* $\max\{\|PT - TP\| : T \in F\} < \delta$ *implies* $\sup\{\|PS - SP\| : S \in \mathcal{S}\} < \varepsilon$.

COROLLARY 7. *Every norm closed bounded* P*-equicontinuous subset of $B(H)$ is norm separable.*

Proof. If in the preceding lemma we choose $\varepsilon = 1/n$, we obtain a finite subset F_n of $\mathcal{S}$ and a $\delta_n > 0$ such that

$$\max\{\|PT - TP\| : T \in F_n\} < \delta_n \Rightarrow \sup\{\|PS - SP\| : S \in \mathcal{S}\} < \frac{1}{n}.$$

It follows from Proposition 1 that, for every $S \in \mathcal{S}$, $\mathrm{dist}(S, C^*(F_n)) < 29/n$. It follows that if F is the union of the F_n's, then F is countable and $\mathcal{S} \subset C^*(F)$. Hence $\mathcal{S}$ is norm separable. ∎

COROLLARY 8. *If every unital C^*-subalgebra of $B(H)$ is an Arzela-Ascoli algebra, then every* P*-equicontinuous subset of $B(H)$ is totally bounded.*

The relationship between P-equicontinuity and Theorem 5 is contained in the following straightforward consequence of Theorem 2.

LEMMA 9. *Suppose $\mathcal{G}$ is a separable unital C^*-subalgebra of $B(H)$. The following are equivalent:*

(i) $\mathrm{Alg\,Lat}\,(\pi(\mathcal{G})) = C^*(\pi(\mathcal{G}) \cup \mathcal{Z})$.

(ii) *$\mathcal{G}$ is an Arzela-Ascoli algebra.*

Proof. (i) $\Rightarrow$ (ii). Suppose $\mathcal{G}$ is not an Arzela-Ascoli algebra. Then there is a bounded subset $\mathcal{S}$ of $\mathcal{G}$ that is not totally bounded, but is relatively P-equicontinuous in $\mathcal{G}$. Since $\mathcal{G}$ is norm separable, so is $\mathcal{S}$, and we can choose a norm-dense sequence $\{S_n\}$ is $\mathcal{S}$. Let s be the image of $\{S_n\}$ in $Q(H)$. Since $\mathcal{S}$ is relatively P-equicontinuous in $\mathcal{G}$, $s \in \mathrm{Alg\,Lat}\,\pi(\mathcal{G})$. Since $\{S_1, S_2, \ldots\}$ is not totally bounded, s is not in $C^*(\pi(\mathcal{G}) \cup \mathcal{Z})$, which violates (i).

(ii) $\Rightarrow$ (i). Suppose $\mathcal{G}$ is an Arzela-Ascoli algebra and $\eta(T_1, T_2, \ldots) \in \mathrm{Alg\,Lat}\,\pi(\mathcal{G})$. It follows from Theorem 2 that we can assume $T_n \in \mathcal{G}$ for each n. The fact that $\eta(T_1, T_2, \ldots) \in \mathrm{Alg\,Lat}\,\pi(\mathcal{G})$ implies that the set $\{T_1, T_2, \ldots\}$ is P-equicontinuous relative to $\mathcal{G}$. Since $\mathcal{G}$ is an Arzela-Ascoli algebra, $\{T_1, T_2, \ldots\}$ is totally bounded, which, by Lemma 4, implies $\eta(T_1, T_2, \ldots) \in C^*(\pi(\mathcal{G}) \cup \mathcal{Z})$. ∎

We see that the answers to Questions B and C are related not only to each other, but to questions of reflexivity of sets of the form $\pi(\mathcal{S})$ with $\mathcal{S}$ a separable subset of $B(H)$.

THEOREM 10. *The following are equivalent:*

(i) *If* $\{A_n\}$ *is a bounded sequence in* $B(H)$ *such that* $\inf\{\|A_n - A_m\| : m \neq n\} > 0$, *then there is a positive number* r *such that, for every* $\varepsilon > 0$ *and every finite subset* $\mathcal{F}$ *of* $B(H)$, *there is a projection* P *such that* $\max\{\|PT-TP\| : T \in \mathcal{F}\} < \varepsilon$ *and* $\sup\{\|PA_n - A_nP\| : n \geqslant 1\} \geqslant r$.

(ii) *Every separable* C^**-subalgebra of* $B(H)$ *is an Arzela-Ascoli algebra.*

(iii) $B(H)$ *is an Arzela-Ascoli algebra.*

(iv) *For every separable unital* C^**-subalgebra* $\mathcal{G}$ *of* $B(H)$, $\operatorname{Alg}\operatorname{Lat}\pi(\mathcal{G}) = C^*(\pi(\mathcal{G}) \cup \mathcal{Z})$.

(v) *For every separable subset* $\mathcal{S}$ *of* $B(H)$, $\operatorname{Alg}\operatorname{Lat}\pi(\mathcal{S})$ *is the norm closed algebra generated by* $\pi(\operatorname{appr}\operatorname{Alg}\operatorname{Lat}\mathcal{S}) \cup \mathcal{Z}$.

Proof. The implications (ii) $\Leftrightarrow$ (iv) follow from Lemma 9. The implication (v) $\Rightarrow$ (iv) follows from Proposition 1, and (iii) $\Rightarrow$ (ii) is obvious.

(iv) $\Rightarrow$ (v). Suppose (iv) is true and $\mathcal{S}$ is a separable subset of $B(H)$, and let $\mathcal{G} = C^*(\mathcal{S})$. Let $\eta(T_1, T_2, \ldots) \in \operatorname{Alg}\operatorname{Lat}\pi(\mathcal{S}) \subset \operatorname{Alg}\operatorname{Lat}\pi(\mathcal{G}) = C^*(\pi(\mathcal{G}) \cup \mathcal{Z})$. From Lemma 4 and $(T_1, T_2, \ldots) \in \eta^{-1}(C^*(\pi(\mathcal{G}) \cup \mathcal{Z}))$ it follows that $\{T_1, T_2, \ldots\}$ is totally bounded. To prove that $\eta(T_1, T_2, \ldots)$ belongs to the norm-closed algebra generated by $\pi(\operatorname{appr}\operatorname{Alg}\operatorname{Lat}\mathcal{S}) \cup \mathcal{Z}$, it suffices to show that $\operatorname{dist}(T_n, \operatorname{appr}\operatorname{Alg}\operatorname{Lat}\mathcal{S}) \to 0$ as $n \to \infty$. Since $\{T_1, T_2, \ldots\}$ is totally bounded, we need only show that every subsequential limit of $\{T_n\}$ is in $\operatorname{appr}\operatorname{Alg}\operatorname{Lat}\mathcal{S}$. However, if $\{T_{n_k}\}$ is a subsequence of $\{T_n\}$, it is clear that $\eta(T_{n_1}, T_{n_2}, \ldots) \in \operatorname{Alg}\operatorname{Lat}\pi(\mathcal{S})$, and if $\|T_{n_k} - T\| \to 0$, then $\pi(T) = \eta(T_{n_1}, T_{n_2}, \ldots) \in \operatorname{Alg}\operatorname{Lat}\pi(\mathcal{S})$. Thus $T \in \operatorname{appr}\operatorname{Alg}\operatorname{Lat}(\mathcal{S})$. This proves (v).

(ii) $\Rightarrow$ (i). Suppose (i) is false. Then there is a bounded sequence $\{A_n\}$ in $B(H)$ such that $\inf\{\|S_n - S_m\| : m \neq n\} > 0$ and, every $r > 0$, there is an $\varepsilon_r > 0$ and a finite subset F_r of $B(H)$ with the property that whenever P is a projection with $\max\{\|PT - TP\| : T \in \mathcal{F}_r\} < \varepsilon_r$, then $\sup\{\|PA_n - A_nP\| : n \geqslant 1\} < r$. Suppose $\mathcal{G}$ is the C^*-algebra generated by the A_n's and the $\mathcal{F}_{1/n}$'s. Suppose $\{P_n\}$ is a sequence in $\operatorname{appr}\operatorname{Lat}\mathcal{G}$. Then for every positive integer m, there is an integer N such that $k \geqslant N \Rightarrow \max\{\|P_kT - TP_k\| : T \in \mathcal{F}_{1/m}\} < \varepsilon_{1/m}$, which implies $\sup\{\|P_kA_n - A_nP_k\| : n \geqslant 1\} < 1/m$. Therefore

$$\lim_{k\to\infty} \sup\{\|P_kA_n - A_nP_k\| : n \geqslant 1\} = 0.$$

Hence $\{A_1, A_2, \ldots\}$ is relatively P-equicontinuous in $\mathcal{G}$, but not totally bounded. Thus (ii) is false.

(i) $\Rightarrow$ (iii). Suppose $\mathcal{S}$ is a bounded subset of $B(H)$ that is not totally bounded. Then there is a sequence $\{A_n\}$ in $\mathcal{S}$ such that $\inf\{\|S_n - S_m\| : m \neq$

$n\} > 0$. Let r be given as in (i), and let Λ be the set of pairs $(\varepsilon, \mathcal{F})$ with $\varepsilon > 0$ and $\mathcal{F}$ a finite subset of $B(H)$, partially ordered by $>$ in the first coordinate and $\subset$ in the second coordinate. For each $\lambda = (\varepsilon, \mathcal{F}) \in \Lambda$, we can, by (i), choose a projection P_λ so that $\max\{\|P_\lambda - T - TP_\lambda\| : T \in \mathcal{F}\} < \varepsilon$ and $\sup\{\|P_\lambda A_n - A_n P_\lambda\| : n \geqslant 1\} \geqslant r$. It follows that the net $\{P_\lambda\} \in \operatorname{appr} \operatorname{Lat} B(H)$ and, for every λ in Λ, $\sup\{\|P_\lambda A_n - A_n P_\lambda\| : n \geqslant 1\} \geqslant r$. Thus $\mathcal{S}$ is not P-equicontinuous relative to $B(H)$. Hence $B(H)$ is an Arzela-Ascoli algebra. ∎

The proof of (iv) ⇒ (v) is the preceding theorem actually yields the following "hereditary" result.

COROLLARY 11. *Suppose $\mathcal{S}$ is a separable subset of $B(H)$ and* $\operatorname{Alg} \operatorname{Lat} \pi(\mathcal{S})$ *is the norm closed algebra generated by* $\pi(\operatorname{appr} \operatorname{Alg} \operatorname{Lat} \mathcal{S}) \cup \mathcal{Z}$. *Then, for every subset $\mathcal{W} \subset$* $\operatorname{appr} \operatorname{Alg} \operatorname{Lat} \mathcal{S}$, *we have* $\operatorname{Alg} \operatorname{Lat} \pi(\mathcal{W})$ *is the norm closed algebra generated by* $\pi(\operatorname{appr} \operatorname{Alg} \operatorname{Lat} \mathcal{W}) \cup \mathcal{Z}$.

It is reasonable to ask, assuming statement (i) in Theorem 10 is true, whether r only depends on $\inf\{\|A_n - A_m\| : m \neq n\}$, if we assume that $\|A_n\| \leqslant 1$ for $n \geqslant 1$. This leads to a stronger notion of Arzela-Ascoli algebra.

A subset $\mathcal{S}$ of $\operatorname{ball} \mathcal{G}$ (the closed unit ball of $\mathcal{G}$) is called *t-separated* $(t > 0)$ if $\|S - T\| \geqslant t$ whenever $S, T \in \mathcal{S}$ and $S \neq T$. A C^*-subalgebra $\mathcal{G}$ of $B(H)$ is *uniformly Arzela-Ascoli* if there is a function $\rho : (0, 2] \to (0, 1]$ such that if $0 < t \leqslant 2$ and $\mathcal{S}$ is an infinite t-separated subset of $\operatorname{ball} \mathcal{G}$, then there is a net $\{P_\lambda\}$ in $\operatorname{appr} \operatorname{Lat} \mathcal{G}$ such that, for every λ, $\sup\{\|P_\lambda S - SP_\lambda\| : S \in \mathcal{S}\} \geqslant \rho(t)$. We call a function ρ described above a *uniform-equicontinuity function* for $\mathcal{G}$. Note that if ρ is a uniform-equicontinuity function for $\mathcal{G}$, then any smaller positive function is also a uniform-equicontinuity function for $\mathcal{G}$. We define $\Gamma_{\mathcal{G}}(t)$ to be the supremum of all the uniform-equicontinuity functions for $\mathcal{G}$ when $\mathcal{G}$ is a uniformly Arzela-Ascoli algebra; otherwise, we define $\Gamma_{\mathcal{G}} = 0$. It is clear that if $0 < \rho < \Gamma_{\mathcal{G}}$, then ρ is a uniform-equicontinuity function for $\mathcal{G}$. Note that when $\mathcal{G}$ is finite-dimensional, $\mathcal{G}$ is vacuously uniformly Arzela-Ascoli and any positive function ρ is a uniform-equicontinuity function for $\mathcal{G}$.

Uniform-equicontinuity functions for separable C^*-subalgebras of $B(H)$ are related to distance formulas in $Q(H)$.

THEOREM 12. *Suppose $\mathcal{G}$ is a separable unital C^*-subalgebra of $B(H)$. The following are equivalent:*

(i) *$\mathcal{G}$ is a uniformly Arzela-Ascoli algebra and $t/\Gamma_{\mathcal{G}}(t)$ is bounded on* $(0, 2]$, *(i.e., there is an $M > 0$ so that $\rho(t) = t/M$ is a uniform-equicontinuity function for $\mathcal{G}$).*

(ii) *There is a number* $\kappa \geqslant 1$ *such that, for every* s *in* $\mathcal{Q}(H)$ *there is a projection* p *in* $\operatorname{Lat}\pi(\mathcal{G})$ *such that* $\operatorname{dist}(s, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \leqslant \kappa\|ps - sp\|$.

Proof. (i) $\Rightarrow$ (ii). Suppose ρ is a uniform-equicontinuity function for $\mathcal{G}$, $M > 0$, and $\rho(t)/t \geqslant 1/M$ for $t \in (0, 2]$. Suppose $s = \eta(S_1, S_2, \ldots) \in \mathcal{Q}(H)$. We can assume that $\|s\| = 1$. Let $d = \sup\{\|ps - sp\| : p \in \operatorname{Lat}(\pi(\mathcal{G}))\}$. It follows from the proof of Theorem 2 that there is an $a = \eta(A_1, A_2, \ldots)$ with each A_n in $\mathcal{G}$ such that $\|a - s\| \leqslant 29d$. Note that

$$\operatorname{dist}(s, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \leqslant \|a - s\| + \operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \leqslant 29d + \operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})).$$

Suppose $r > \operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z}))$. Let $\mathcal{S}$ be a maximal r-separated subset of $\{A_1, A_2, \ldots\}$. Then every A_n is within a distance r to some point in $\mathcal{S}$. Thus there is a sequence $\{B_n\}$ in $\mathcal{S}$ such that $\|(A_1, A_2, \ldots) - (B_1, B_2, \ldots)\| \leqslant r$. If $\mathcal{S}$ were finite, then $b = \eta(B_1, B_2, \ldots) \in C^*(\pi(\mathcal{G}) \cup \mathcal{Z})$, which would violate the choice of r. Hence $\mathcal{S}$ is infinite. Since ρ is a uniform-equicontinuity function for $\mathcal{G}$, there is a sequence $\{P_n\}$ in $\operatorname{appr}\operatorname{Lat}\mathcal{G}$ such that

$$\sup\{\|P_n A_k - A_k P_n\| : k \geqslant 1\} \geqslant \rho(r) \geqslant \frac{r}{M} \geqslant \left(\frac{1}{M}\right) \operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})).$$

Let $n_1 = 1$, and choose $k_1 \geqslant 1$ such that $\|P_{n_1} A_{k_1} - A_{k_1} P_{n_1}\| \geqslant r/M - 1/2$. Next choose $n_2 > n_1$ so that $\|P_{n_2} A_k - A_k P_{n_2}\| < r/M - 1/2$ for $1 \leqslant k \leqslant k_1$. Thus there is a $k_2 > k_1$ such that $\|P_{n_2} A_{k_2} - A_{k_2} P_{n_2}\| \geqslant r/M - 1/4$. Proceeding inductively, we choose increasing sequences $\{n_m\}$ and $\{k_m\}$ of positive integers such that, for each m, $\|P_{n_m} A_{k_m} - A_{k_m} P_{n_m}\| \geqslant r/M - 1/2^m$. We define a sequence $\{Q_j\}$ of projections by $Q_{k_m} = P_{n_m}$, for $m \geqslant 1$, and $Q_j = 1$ otherwise. Since $\{P_n\} \in \operatorname{appr}\operatorname{Lat}\mathcal{G}$, it is clear that $\{Q_j\} \in \operatorname{appr}\operatorname{Lat}\mathcal{G}$. Let $p = \eta(Q_1, Q_2, \ldots)$. Then $p \in \operatorname{Lat}\pi(\mathcal{G})$ and

$$\begin{aligned}\|ap - pa\| &= \limsup_{n\to\infty} \|A_n Q_n - Q_n A_n\| \geqslant \limsup_{m\to\infty} \|P_{n_m} A_{k_m} - A_{k_m} P_{n_m}\| \\ &\geqslant \limsup_{m\to\infty} \left[\frac{r}{M} - \frac{1}{2^m}\right] = \frac{r}{M}.\end{aligned}$$

But

$$\|ap - pa\| = \max\{\|(1-p)ap\|, \|pa(1-p)\|\} \leqslant \|a - s\| + \|sp - ps\| \leqslant 29d + d = 30d.$$

Thus $r/M \leqslant 30d$, which implies $M \operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \leqslant 30d$. Thus $\operatorname{dist}(s, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \leqslant \|s - a\| + \operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \leqslant 29d + 30dM$. If we choose $\kappa > 29 + 30M$, we see that (ii) holds.

(ii) $\Rightarrow$ (i). Suppose (ii) holds and $\{A_1, A_2, \ldots\}$ is a subset of ball $\mathcal{G}$ such that $\|A_i - A_j\| \geqslant t > 0$ when $1 \leqslant i \neq j < \infty$. Let $a = \eta(A_1, A_2, \ldots)$. Since $\eta^{-1}(C^*(\pi(\mathcal{G}) \cup \mathcal{Z}))$ consists of the totally bounded sequences in $\mathcal{G}$, which is the closure of the set of sequences in $\mathcal{G}$ with finite range, it follows that $\operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \geqslant t/2$. However, by (ii), there is a $p = \eta(Q_1, Q_2, \ldots)$ in $\operatorname{Lat} \pi(\mathcal{G})$ such that $\operatorname{dist}(a, C^*(\pi(\mathcal{G}) \cup \mathcal{Z})) \leqslant \kappa\|ap - pa\| = \kappa \limsup_{n\to\infty} \|A_n P_n - P_n A_n\|$. Thus $t/2\kappa \leqslant \limsup_{n\to\infty} \|A_n P_n - P_n A_n\|$. Suppose $M > 2\kappa$. Hence there is a subsequence $\{P_{n_k}\}$ of $\{P_n\}$ such that $\|A_{n_k} P_{n_k} - P_{n_k} A_{n_k}\| > t/M$ for $k \geqslant 1$. Since $\{P_{n_k}\} \in \operatorname{appr} \operatorname{Lat} \mathcal{G}$, it follows that $\rho(t) = t/M$ defines a uniform-equicontinuity function for $\mathcal{G}$. ∎

We now attack the problem of finding sufficient conditions for an algebra to be an Arzela-Ascoli algebra. The next result lists elementary facts that simplify the proofs of subsequent results.

LEMMA 13. *The following are true.*

(i) *Every subalgebra of an Arzela-Ascoli algebra is an Arzela-Ascoli algebra. Moreover, if $\mathcal{B} \subset \mathcal{G}$, then $\Gamma_{\mathcal{B}} \geqslant \Gamma_{\mathcal{G}}$.*

(ii) *A finite direct sum of algebras is an Arzela-Ascoli algebra if and only if each summand is. Moreover, if $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_n$, then $\Gamma_{\mathcal{G}} \geqslant \min_{1\leqslant i\leqslant n} \Gamma_{\mathcal{G}_i}$.*

(iii) *Suppose n is a positive integer and $\mathcal{G}$ is a C^*-subalgebra of $B(H)$. Then $\mathcal{G}$ is an Arzela-Ascoli algebra if and only if $\mathfrak{M}_n(\mathcal{G})$ is an Arzela-Ascoli algebra in $B(H \oplus \cdots \oplus H)$.*

(iv) *If $\sigma : \mathcal{G} \to B(H)$ is a unital representation that is approximately equivalent to the identity representation, then $\sigma(\mathcal{G})$ is an Arzela-Ascoli algebra if and only if $\mathcal{G}$ is an Arzela-Ascoli algebra. Moreover, $\Gamma_{\mathcal{G}} = \Gamma_{\sigma(\mathcal{G})}$.*

(v) *If M is a separating reducing subspace for $\mathcal{G}$ and $\mathcal{G}|M$ is an Arzela-Ascoli algebra, then so is $\mathcal{G}$. Moreover, $\Gamma_{\mathcal{G}} \geqslant \Gamma_{\mathcal{G}|M}$.*

(vi) *Suppose $\mathcal{B}$ is a C^*-subalgebra of $B(H)$ and P is a projection in $\mathcal{B}$. Let $M = P(H)$, and $\mathcal{G} = \{PB|M : B \in \mathcal{B}\} \subset B(H)$. If $\mathcal{B}$ is an Arzela-Ascoli algebra, then so is $\mathcal{G}$. Moreover, $\Gamma_{\mathcal{G}} \geqslant \Gamma_{\mathcal{B}}$.*

THEOREM 14. *Every* AF *C^*-subalgebra $\mathcal{G}$ of $B(H)$ is a uniformly Arzela-Ascoli algebra with $\Gamma_{\mathcal{G}}(t) \geqslant t/8$. If $\mathcal{G}$ is commutative, then $\Gamma_{\mathcal{G}}(t) \geqslant t/4$.*

Proof. Suppose $\{\mathcal{G}_\lambda : \lambda \in \Lambda\}$ is an increasing sequence of finite-dimensional C^*-algebras whose union is dense in $\mathcal{G}$. Suppose $t > 0$ and $\mathcal{S}$ is an infinite t-separated subset of ball $\mathcal{G}$. Suppose $\lambda \in \Lambda$. Since ball $\mathcal{G}$ is compact, there is an S_λ in $\mathcal{S}$ such that $\operatorname{dist}(S_\lambda, \mathcal{G}_\lambda) \geqslant t/2$. Since $\mathcal{G}_\lambda$ is hyperreflexive with constant of hyperreflexivity at most 4 (see [25]), there is a projection P_λ in the commutant of

$\mathcal{G}_\lambda$ such that $\|P_\lambda S_\lambda - S_\lambda P_\lambda\| \geqslant (1/4)\mathrm{dist}(S_\lambda, \mathcal{G}_\lambda) \geqslant t/8$. It is clear that the net $\{P_\lambda\}$ is in $\mathrm{appr\,Lat}\,\mathcal{G}$. Thus $\rho(t) = t/8$ is a uniform-equicontinuity function for $\mathcal{G}$. The result for commutative $\mathcal{G}$ follows from the fact [25] that a commutative von Neumann algebra has a hyperreflexivity constant at most 2. ∎

We can use the preceding theorem to recapture Theorem 5.

COROLLARY 15. *Every commutative C^*-algebra $\mathcal{G} \subset B(H)$ is a uniform Arzela-Ascoli algebra with $\Gamma_{\mathcal{G}}(t) \geqslant t/4$.*

Proof. The von Neumann algebra generated by $\mathcal{G}$ is a commutative AF-algebra, and $\Gamma_{\mathcal{G}} \geqslant \Gamma_{\mathcal{G}''}$. ∎

THEOREM 16. *Suppose $\{\mathcal{G}_\beta\}$ is an increasing net of uniformly Arzela-Ascoli C^*-subalgebras of $B(H)$ whose union is dense in $\mathcal{G}$. Suppose also, there is a positive function $\rho : (0,2] \to (0,1]$ such that ρ is a uniform-equicontinuity function for each $\mathcal{G}_\beta$. Then $\mathcal{G}$ is a uniformly Arzela-Ascoli algebra and $\rho'(t) = \min(\rho(3t/4)/240, t/480)$ is a uniform-equicontinuity function for $\mathcal{G}$. In other words, $\Gamma_{\mathcal{G}}(t) \geqslant \inf_\beta \min(\Gamma_{\mathcal{G}_\beta}(3t/4)/240, t/480)$.*

Proof. Let $\rho'(t) = \min(\rho(3t/4)/240, t/480)$. Suppose $t > 0$ and $\mathcal{S}$ is an infinite t-separated subset of $\mathrm{ball}\,\mathcal{G}$. Let Λ be the collection of all pairs $(\mathcal{F}, \varepsilon)$, with $\varepsilon > 0$ and $\mathcal{F}$ a finite subset of the union of the $\mathcal{G}_\beta$'s, and order Λ by $(\subset, \geqslant)$. Suppose $\lambda = (\mathcal{F}, \varepsilon) \in \Lambda$. Choose a β so that $\mathcal{F} \subset \mathcal{G}_\beta$. Let $s = \min(\rho(3t/4)/4, t/8)$.

Case 1. $\sup\{\mathrm{dist}(S, \mathrm{ball}\,\mathcal{G}_\beta) : S \in \mathcal{S}\} < s$. In this case we have, for each S in $\mathcal{S}$, an element A_S in $\mathrm{ball}\,\mathcal{G}_\beta$ such that $\|S - A_S\| < s$. It follows that if $S, T \in \mathcal{S}$ and $S \neq T$, then $\|A_S - A_T\| \geqslant t - 2s \geqslant t - 2(t/8) = 3t/4$. Since ρ is a uniform-equicontinuity function for $\mathcal{G}_\beta$, there is a projection P_λ such that $\|P_\lambda A - AP_\lambda\| < \varepsilon$ for every A in $\mathcal{F}$, and such that $\sup\{\|P_\lambda A_S - A_S P_\lambda\| : S \in \mathcal{S}\} \geqslant \rho(3t/4)$. It follows that

$$\sup\{\|P_\lambda S - SP_\lambda\| : S \in \mathcal{S}\} \geqslant \rho(3t/4) - 2s \geqslant \rho(3t/4) - 2\frac{\rho(3t/4)}{4} \geqslant \rho'(t).$$

Case 2. $\sup\{\mathrm{dist}(S, \mathrm{ball}\,\mathcal{G}_\beta); S \in \mathcal{S}\} \geqslant s$. Then $\sup\{\mathrm{dist}(S, \mathcal{G}_\beta); S \in \mathcal{S}\} \geqslant s/2$. Thus, by Proposition 1, there is a projection P_λ such that $\|P_\lambda A - AP_\lambda\| < \varepsilon$ for every A in $\mathcal{F}$, and such that

$$\sup\{\|P_\lambda S - SP_\lambda\| : S \in \mathcal{S}\} \geqslant \frac{s/2}{30} \geqslant \rho'(t).$$

It is clear that the net $\{P_\lambda\}$ is in $\mathrm{appr\,Lat}\,\mathcal{G}$ and that, for each λ in Λ, $\sup\{\|P_\lambda S - SP_\lambda\| : S \in \mathcal{S}\} \geqslant \rho'(t)$. ∎

THEOREM 17. *If $\mathcal{G}$ is a separable commutative C^*-subalgebra of $B(H)$ and $\mathcal{B}$ is an Arzela-Ascoli subalgebra of $B(K)$, then the spatial tensor product $\mathcal{G} \otimes \mathcal{B}$ is an Arzela-Ascoli subalgebra of $B(H \otimes K)$. Moreover, $\Gamma_{\mathcal{G}\otimes\mathcal{B}}(t) \geqslant \min(t/8, \Gamma_{\tilde{\mathcal{B}}}(t/4))$.*

Proof. Suppose $t > 0$ and $\mathcal{S}$ is an infinite t-separated subset of ball $\mathcal{G}$. Let X be the maximal ideal space of $\mathcal{G}$. Then $\mathcal{G} \otimes \mathcal{B}$ is isomorphic to the C^*-algebra $C(X, \mathcal{B})$ of all continuous functions from X to $\mathcal{B}$. Write $\mathcal{S} = \{\varphi_1, \varphi_2, \ldots\}$, and let $\mathcal{W} = \bigcup_{n=1}^{\infty} \operatorname{ran}(\varphi_n)$.

Claim. Either $\mathcal{W}$ contains no finite subset $\mathcal{F}$ such that every point of $\mathcal{W}$ is within $t/4$ of $\mathcal{F}$ or X has no finite Borel partition $\mathcal{P}$ such that, for every set E in $\mathcal{P}$ and every φ in $\mathcal{S}$, $\operatorname{diam}\varphi(E) < t/4$. Assume, via contradiction that the claim is false. Then, for each E in $\mathcal{P}$, we can choose $x(E)$ in E. Then, for every φ in $\mathcal{S}$, $\left\|\varphi - \sum_{E\in\mathcal{P}} \varphi(x(E))\chi_E\right\| < t/4$. And for each E in $\mathcal{P}$ and each φ in $\mathcal{S}$, we can choose $W_{E,\varphi}$ in $\mathcal{F}$ so that $\|W_{E,\varphi} - \varphi(x(E))\| \leqslant t/4$. Hence $\left\|\varphi - \sum_{E\in\mathcal{P}} W_{E,\varphi}\chi_E\right\| < t/2$ for each φ in $\mathcal{S}$. However, since $\mathcal{P}$ and $\mathcal{F}$ are finite, the set of sums of the form $\sum_{E\in\mathcal{P}} A_E\chi_E$ with each A_E in F is a finite set. It follows that there is a finite subset of $\mathcal{S}$ that is within a distance less than $t = 2(t/2)$ to each element of $\mathcal{S}$. This is a contradiction that proves the claim.

Suppose that there is no finite subset of $\mathcal{W}$ that is within $t/4$ of every element of $\mathcal{W}$. Then $\mathcal{W}$ contains an infinite subset $\mathcal{V}$ such that $S, T \in \mathcal{V}$ and $S \neq T$ implies that $\|S - T\| \geqslant t/4$. Since $\mathcal{B}$ is an Arzela-Ascoli algebra, there is a net $\{Q_\lambda\}$ of projections in $\operatorname{appr}\operatorname{Lat}(\mathcal{B})$ and a positive ε such that, for every λ, $\sup\{\|Q_\lambda B - BQ_\lambda\| : B \in \mathcal{W}\} \geqslant \varepsilon$. Moreover , if ρ is a uniform-equicontinuity function for $\mathcal{B}$, we can take $\varepsilon = \rho(t/4)$. For each λ, define $P_\lambda \in C(X, B(K)) \subset B(H \otimes K)$ by $P_\lambda(x) = Q_\lambda$. It follows that, for every λ,

$$\begin{aligned}\sup\{\|P_\lambda\varphi_n - \varphi_n P_\lambda\| : n \geqslant 1\} &= \sup_n \sup_{x\in X} \|Q_\lambda\varphi_n(x) - \varphi_n(x)Q_\lambda\| \\ &= \sup\{\|Q_\lambda B - BQ_\lambda\| : B \in \mathcal{W}\} \geqslant \varepsilon.\end{aligned}$$

Next suppose that X has no finite Borel partition $\mathcal{P}$ such that, for every set E in $\mathcal{P}$ and every $\varphi \in \mathcal{S}$, $\operatorname{diam}\varphi(E) < t/4$. We can imitate the proof of Theorem 5 to find a net $\{P_\alpha\}$ in $\operatorname{appr}\operatorname{Lat}(\mathcal{G} \otimes \mathcal{B})$ so that, for each α, $\sup\|P_\alpha S - SP_\alpha\| \geqslant (t/4)/2 = t/8$.

It follows that $\mathcal{G} \otimes \mathcal{B}$ is indeed an Arzela-Ascoli algebra and that if ρ is a uniform-equicontinuity function for $\mathcal{B}$, then $\rho'(t) = \min(t/8, \rho(t/4))$ is a uniform equicontinuity function for $\mathcal{G} \otimes \mathcal{B}$. ∎

REMARK. We could have used the fact that $\mathcal{G}$ is contained in a commutative AF-algebra $\mathcal{D}$, (e.g., $\mathcal{G}''$ is a direct limit of an increasing net $\{\mathcal{D}_\alpha\}$ of finite-dimensional commutative C^*-algebras). Then $\mathcal{D} \otimes \mathcal{B}$ is the direct limit of the algebras $\mathcal{D}_\alpha \otimes \mathcal{B}$, and each $\mathcal{D}_\alpha \otimes \mathcal{B}$ is isomorphic to a direct sum of copies of $\mathcal{B}$. It follows that each $\mathcal{D}_\alpha \otimes \mathcal{B}$ has ρ as a uniform-equicontinuity function. Therefore, by Theorem 16, $\mathcal{D} \otimes \mathcal{B}$ is a uniformly Arzela-Ascoli algebra. Since $\mathcal{G} \otimes \mathcal{B} \subset \mathcal{D} \otimes \mathcal{B}$, $\mathcal{G} \otimes \mathcal{B}$ is a uniformly Arzela-Ascoli algebra. However, Theorem 16 does not give as good an estimate on $\Gamma_{\mathcal{G}\otimes\mathcal{B}}$. ∎

COROLLARY 18. *Suppose $\{\mathcal{G}_n\}$ is an increasing sequence of C^*-subalgebras whose union is dense in $\mathcal{G}$ and $\mathcal{G}$ and $\mathcal{G} \subset B(H)$. If each $\mathcal{G}_n$ is $*$-isomorphic to the tensor product of a commutative C^*-algebra and a finite-dimensional C^*-algebra, then $\mathcal{G}$ is a uniformly Arzela-Ascoli algebra $\Gamma_{\mathcal{G}}(t) \geqslant t/15360$.*

It is not clear that being an Arzela-Ascoli algebra is preserved under $*$-isomorphisms. For example, if a C^*-subalgebra $\mathcal{G}$ of $B(H)$ can be embedded in an AF algebra, we do not necessarily know if $\mathcal{G}$ is a subset of an AF algebra contained in $B(H)$ (e.g., $C^*(S^* \oplus S)$ and $C^*(S^* \oplus S \oplus S)$ where S is the unilateral shift). The following result shows how such difficulties can be overcome when $\mathcal{G}$ contains no compact operators.

PROPOSITION 19. *If $\mathcal{G} \subset B(H)$, $\mathcal{G} \cap \mathcal{K}(H) = \{0\}$, $\mathcal{G}$ and H are separable, and $\mathcal{G}$ is $*$-isomorphic to an Arzela-Ascoli algebra $\mathcal{B}$ on a separable Hilbert space K, then $\mathcal{G}$ is an Arzela-Ascoli algebra. Moreover $\Gamma_{\mathcal{G}} \geqslant \Gamma_{\mathcal{B}}$.*

Proof. We can assume that $\mathcal{G}$ and H are infinite dimensional. Thus there is a $*$-isomorphism $\sigma : \mathcal{G} \to \mathcal{B}$ such that $\sigma(\mathcal{G}) = \mathcal{B}$ is an Arzela-Ascoli algebra. Let $\tau : \mathcal{G} \to B(H)$ be unitarily equivalent to $\sigma \oplus \sigma \oplus \cdots$; hence $\tau(\mathcal{G}) \cap \mathcal{K}(H) = 0$. It follows from Voiculescu's theorem ([28]) that τ is approximately equivalent to the identity representation on $\mathcal{G}$. It follows from parts (iv) and (v) of Lemma 13 that $\mathcal{G}$ is an Arzela-Ascoli algebra and that $\Gamma_{\mathcal{G}} = \Gamma_{\tau(\mathcal{G})} \geqslant \Gamma_{\sigma(\mathcal{G})} = \Gamma_{\mathcal{B}}$. ∎

COROLLARY 20. *If $\mathcal{G} \subset B(H)$ is $*$-isomorphic to an irrational rotation C^*-algebra, then $\mathcal{G}$ is an Arzela-Ascoli algebra and $\Gamma_{\mathcal{G}}(t) \geqslant t/8$.*

Proof. We know that $\mathcal{G}$ is simple, so $\mathcal{G} \cap \mathcal{K}(H) = \{0\}$. Also, by [23], $\mathcal{G}$ is $*$-isomorphic to a subalgebra of an AF algebra. ∎

We can prove a version of Proposition 19 for cases in which $\mathcal{G} \cap \mathcal{K}(H) \neq 0$.

PROPOSITION 21. *Suppose $\mathcal{G} \subset B(H)$, $\mathcal{B} \subset B(K)$ and $\sigma : \mathcal{G} \to \mathcal{B}$ is a $*$-isomorphism such that, for every a in $\mathcal{G}$,* $\operatorname{rank} \sigma(a) \leqslant \operatorname{rank}(a)$. *If $\mathcal{B}$ is an Arzela-Ascoli algebra, then so is $\mathcal{G}$. Moreover, $\Gamma_{\mathcal{G}} \geqslant \Gamma_{\mathcal{B}}$.*

Proof. It follows from [14] that there is a representation $\tau : \mathcal{G} \to B(H)$ such that τ is approximately equivalent to the identity representation on $\mathcal{G}$ and σ is unitarily equivalent to a direct summand of τ. The rest follows as in the proof of Proposition 19. ∎

REMARKS. (1) This remark is based on the very kind, and greatly appreciated, help from Cornel Pasnicu, who provided me with many references and descriptions of their contents. The result in Corollary 20 raises leads to the question of which algebras are embeddable in AF algebras. The class of AF-embeddable algebras is surprisingly large. For example, M. Pimsner and D. Voiculescu ([23]) proved that any irrational rotation algebra is embeddable in an AF-algebra with the same ordered K_0-group, and G.A. Elliott and D. Evans ([11]) proved that the irrational rotation algebra is an AH algebra (a direct limit of direct sums of commutative algebras with matrix algebras cut down by projections in these algebras). In fact Elliott and Evans ([11]) show that any irrational rotation algebra is an inductive limit of finite direct sums of matrix algebras over $C(\Gamma)$, where Γ is the unit circle. Generalizations of this result to higher-dimensional noncommutative tori have been obtained by Elliot and Lin ([12]), F. Boca ([4]), and Q. Lin ([18]). In [22] M.V. Pimsner proved that a crossed product of a commutative algebra by an action of $\mathbb{Z}$ is AF-embeddable if and only if it is quasidiagonal. Also D. Voiculescu ([29]) obtained results on crossed products of AF-algebras by $\mathbb{Z}$; he proved that a crossed product of any UHF algebra by $\mathbb{Z}$ is AF-embeddable. J. Spielberg ([27]) proved that any residually finite type I C^*-algebra or any separable C^*-algebra with continuous trace is AF-embeddable. Moreover, Spielberg showed that the cone of every type I C^*-algebra is AF-embeddable, and that AF-embeddability is a homotopy invariant among type I C^*-algebras. Other interesting related results are contained in [3], [5], [6], [7], [8], [17] and [24].

(2) A C^*-algebra $\mathcal{G}$ is embeddable in a finite direct sum of algebras $\mathfrak{M}_k(C(X))$ where X is a compact Hausdorff space and $k \leqslant n < \infty$ if and only if each irreducible representation of $\mathcal{G}$ has dimension at most n. An unpublished result of I. Kaplansky states that $\mathcal{G}$ is commutative if and only if $\mathcal{G}$ contains no nonzero nilpotents. We can extend Kaplansky's result by showing that every irreducible representation of $\mathcal{G}$ has dimension at most $n < \infty$ if and only if every nilpotent in $\mathcal{G}$ has order of nilpotence at most n. The "only if" part is trivial. To prove the "if" part, suppose that π is an irreducible representation of $\mathcal{G}$ having dimension greater than n. It follows that there is a positive element A in $\pi(\mathcal{G})$ such that $\sigma(A)$

contains more than n points, and, using the continuous functional calculus, we can choose $n+1$ nonzero, positive, pairwise orthogonal elements $A_1, A_2, \ldots, A_{n+1}$ of $\pi(\mathcal{G})$. Using Kadison's transitivity theorem, we can find elements $C_1, C_2, \ldots, C_n$ in $\pi(\mathcal{G})$ so that $T = A_1C_1A_2 + \cdots + A_nC_nA_{n+1}$ satisfies $T^n \neq 0$. However, $T^{n+1} = 0$. Using a result of C. Olsen and G. Pedersen ([19]), we can find a nilpotent t in $\mathcal{G}$ such that $\pi(t) = T$. Clearly, the order of nilpotence of t exceeds n.

(3) If $\mathcal{G}$ is a separable C^*-subalgebra of $B(H)$ and H is separable, and if every irreducible representation of H is at most n-dimensional, then, because the identity representation on $\mathcal{G}$ is approximately equivalent to a direct sum ρ of irreducible representations ([28]), it follows that $\rho(\mathcal{G})$ is contained in a C^*-subalgebra of $B(H)$ that is a direct sum of algebras of the form $\mathfrak{M}_k(\mathcal{B})$ with $\mathcal{B}$ commutative (diagonalizable) and $k \leqslant n$. Hence $\rho(\mathcal{G})$ is contained in an AF-subalgebra of $B(H)$, which implies that $\Gamma_{\mathcal{G}}(t) \geqslant t/8$.

We can push the ideas in Proposition 19 a little further. This allows us, in some cases, to restrict our attention exclusively to the compact operators in $\mathcal{G}$.

THEOREM 22. *Suppose $\mathcal{G}$ is a separable C^*-algebra on a separable Hilbert space H. Suppose $\mathcal{G}/[\mathcal{G} \cap \mathcal{K}(H)]$ is $*$-isomorphic to a nuclear Arzela-Ascoli algebra $\mathcal{B}$ on a separable Hilbert space. Then:*

(i) *$\mathcal{G}$ is an Arzela-Ascoli algebra if and only if every subset of $G \cap \mathcal{K}(H)$ that is relatively* P-*equicontinuous in $\mathcal{G}$ is totally bounded.*

Moreover, suppose $\mathcal{B}$ is a uniformly Arzela-Ascoli algebra, and there is a function $\rho : (0,2] \to (0,1]$ such that whenever $t > 0$ and $\mathcal{S}$ is an infinite t-separated subset of $\operatorname{ball}(\mathcal{G} \cap \mathcal{K}(H))$, *there is a net $\{P_\lambda\}$ in* $\operatorname{appr}\operatorname{Lat}(\mathcal{G})$ *such that, for each λ,* $\sup\{\|SP_\lambda - P_\lambda S\| : S \in \mathcal{S}\} \geqslant \rho(t)$. *Then*

(ii) *$\mathcal{G}$ is a uniformly Arzela-Ascoli algebra and*

$$\Gamma_{\mathcal{G}}(t) \geqslant \min\left[\Gamma_{\mathcal{B}}\left(\frac{1}{2}\min\left(\frac{t}{2}, \rho(t/2)\right)\right), \frac{1}{2}\rho(t/2)\right].$$

Proof. Suppose $\sigma : \mathcal{G} \to \mathcal{B}$ is a unital surjective $*$-homomorphism and $\ker\sigma = \mathcal{G} \cap \mathcal{K}(H)$. It follows from Voiculescu's theorem ([28]) that $\mathrm{id}_{\mathcal{G}}$ is approximately equivalent to $\mathrm{id}_{\mathcal{G}} \oplus \sigma$. Since $\mathcal{B}$ is nuclear, there is a unital completely positive map $\varphi : \mathcal{B} \to \mathcal{G}$ such that $\sigma \circ \varphi = \mathrm{id}_{\mathcal{B}}$. Suppose $\mathcal{S}$ is a P-equicontinuous subset of $\mathcal{G}$. Then $\{S \oplus \sigma(S) : S \in \mathcal{S}\}$ is a P-equicontinuous subset of $\{A \oplus \sigma(A) : A \in \mathcal{G}\}$. Hence $\sigma(\mathcal{S})$ is a P-equicontinuous subset of $\mathcal{B}$. Since $\mathcal{B}$ is an Arzela-Ascoli algebra, $\sigma(\mathcal{S})$ is totally bounded, which implies $\varphi(\sigma(\mathcal{S}))$ is totally bounded. Define $\gamma : \mathcal{G} \to \mathcal{G} \cap \mathcal{K}(H)$ by $\gamma(A) = A - \varphi(\sigma(A))$. Since $\varphi(\sigma(\mathcal{S}))$ is totally bounded, it follows that $\gamma(\mathcal{S})$ is P-equicontinuous in $\mathcal{G}$ and that $\gamma(\mathcal{S})$ is totally bounded if and only if $\mathcal{S}$ is totally bounded. This proves (i).

Next suppose ρ' is a uniform-equicontinuity function for $\mathcal{B}$, and that the function ρ is defined as in (i) above. Suppose $t > 0$ and $\mathcal{S}$ is an infinite t-separated subset of ball$\mathcal{G}$. Let $s = (1/2)\min(t/2, \rho(t/2)/2)$.

Case 1. $\sigma(\mathcal{S})$ contains an infinite s-separated subset. In this case we can use the fact that $\mathrm{id}_{\mathcal{G}}$ is approximately equivalent to $\mathrm{id}_{\mathcal{G}} \oplus \sigma$ to replace $\mathcal{G}$ with $(\mathrm{id}_{\mathcal{G}} \oplus \sigma)(\mathcal{G})$ and obtain a net $\{Q_\lambda\}$ in $\operatorname{appr}\operatorname{Lat}\sigma(\mathcal{G})$ such that, for every λ, $\sup\{\|Q_\lambda\sigma(S) - \sigma(S)Q_\lambda\| : S \in \mathcal{S}\} \geqslant \rho'(s)$. Let $P_\lambda = 0 \oplus Q_\lambda$ for each λ. Then $\{P_\lambda\}$ is in $\operatorname{appr}\operatorname{Lat}(\mathrm{id}_{\mathcal{G}} \oplus \sigma)(\mathcal{G})$ and $\sup\{\|P_\lambda[S \oplus \sigma(S)] - [S \oplus \sigma(S)]Q_\lambda\| : S \in \mathcal{S}\} \geqslant \rho'(s)$.

Case 2. $\sigma(\mathcal{S})$ contains no infinite s-separated subset. Hence $(\varphi \circ \sigma)(\mathcal{S})$ contains no infinite s-separated subset. Thus there is a finite subset of $(\varphi \circ \sigma)(\mathcal{S})$ that is within s of every point in $(\varphi \circ \sigma)(\mathcal{S})$. Hence there is an S_0 in $\mathcal{S}$ and an infinite subset $\mathcal{S}_0$ of $\mathcal{S}$ such that every point of $(\varphi \circ \sigma)(\mathcal{S}_0)$ has distance less than s from $(\varphi \circ \sigma)(S_0)$. Thus, for every $S, T \in \mathcal{S}_0$ with $S \neq T$, we have

$$\begin{aligned}
\|\gamma(S) - \gamma(T)\| &\geqslant \|S - T\| - \|(\varphi \circ \sigma)(S) - (\varphi \circ \sigma)(T)\| \\
&\geqslant \|S - T\| - \|(\varphi \circ \sigma)(S) - (\varphi \circ \sigma)(S_0)\| \\
&\qquad - \|(\varphi \circ \sigma)(T) - (\varphi \circ \sigma)(T_0)\| \\
&> t - 2s \geqslant t - \frac{t}{2} = \frac{t}{2}.
\end{aligned}$$

Hence, by the definition of ρ, there is a net $\{P_\lambda\}$ in $\operatorname{appr}\operatorname{Lat}\mathcal{G}$ such that, for every λ, $\sup\{\|P_\lambda\gamma(S) - \gamma(S)P_\lambda\| : S \in \mathcal{S}_0\} \geqslant \rho(t/2)$. Suppose $\varepsilon > 0$. Since, for each λ, $\{P_\lambda\}$ is in $\operatorname{appr}\operatorname{Lat}\mathcal{G}$, we can assume that $\|P_\lambda(\varphi \circ \sigma(S_0) - (\varphi \circ \sigma)(S_0)P_\lambda\| < \varepsilon$ for every λ. Hence, for every λ and each S in $\mathcal{S}_0$, we have

$$\begin{aligned}
\|P_\lambda S - SP_\lambda\| &\geqslant \|P_\lambda\gamma(S) - \gamma(S)P_\lambda\| - \|P_\lambda(\varphi \circ \sigma)(S) - (\varphi \circ \sigma)(S)P_\lambda\| \\
&\geqslant \|P_\lambda\gamma(S) - \gamma(S)P_\lambda\| - \|P_\lambda[(\varphi \circ \sigma)(S) - (\varphi \circ \sigma)(S_0)] \\
&\qquad - [(\varphi \circ \sigma)(S) - (\varphi \circ \sigma)(S_0)]P_\lambda\| - \varepsilon \\
&\geqslant \|P_\lambda\gamma(S) - \gamma(S)P_\lambda\| - 2s - \varepsilon.
\end{aligned}$$

Thus $\sup\{\|P_\lambda S - SP_\lambda\| : S \in \mathcal{S}\} \geqslant \rho(t/2) - 2s - \varepsilon \geqslant \rho(t/2) - 2s - \varepsilon \geqslant \rho(t/2) - \rho(t/2)/2 - \varepsilon = \rho(t/2)/2 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary and ρ' was an arbitrary uniform equicontinuity function for $\mathcal{B}$, it follows from Cases 1 and 2 that

$$\Gamma_{\mathcal{G}}(t) \geqslant \min\left[\Gamma_{\mathcal{B}}(s), \frac{1}{2}\rho(t/2)\right] \geqslant \min\Gamma_{\mathcal{B}}\left[\frac{1}{2}\min\left(\frac{t}{2}, \rho(t/2)\right), \frac{1}{2}\rho(t/2)\right]. \quad \blacksquare$$

We apply the preceding theorem to a special case.

THEOREM 23. *Suppose $S \in B(H)$ and S is an isometry. Then $\mathcal{G} = C^*(S)$ is an Arzela-Ascoli algebra with $\Gamma_{\mathcal{G}}(t) \geqslant t/64$.*

Proof. If S is unitary, then $\Gamma_{\mathcal{G}}(t) \geqslant t/4$ (Corollary 15). Otherwise, S is approximately equivalent to a pure isometry ([16]). It follows from part (v) of Lemma 13 that we can assume that S is a unilateral shift operator with multiplicity 1. Note that $C^*(S) \supset \mathcal{K}(H)$, and suppose $\mathcal{S}$ is a subset of ball$\mathcal{G}$, $t > 0$, and, for $S, T \in \mathcal{S}$ with $S \neq T$, we have $\|S - T\| \geqslant t$. Suppose $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H and $Se_n = e_{n+1}$ for each $n \geqslant 1$. For each $n \geqslant 1$, let P_n denote the projection onto $\mathrm{sp}\{e_1, e_2, \ldots, e_n\}$. Since $P_n \mathcal{S} P_n \subset P_n \mathrm{ball}\, B(H) P_n$, which is norm compact, there are elements S, T of $\mathcal{S}$ such that $\|P_n(S-T)P_n\|$ is arbitrarily small, which implies $\|(S - T) - P_n(S - T)P_n\|$ must be arbitrarily close to exceeding t. Hence either $S - P_nSP_n\|$ or $\|T - P_nTP_n\|$ must be close to exceeding $t/2$. Note that, for any operator A, $\|A - P_nAP_n\| \leqslant 2\max\{\|(1 - P_n)AP_n\|, \|(1 - P_n)A^*P_n\|\}$. Hence there is an K in $\mathcal{S}$ such that $\max\{\|(1 - P_n)KP_n\|, \|(1 - P_n)K^*P_n\|\}$ is arbitrary close to exceeding $t/4$. The following claim will prove that if $\rho : (0, 2] \to (0, 1]$ is any function such that $\rho(t) < t/8$, then ρ satisfies the conditions in Theorem 22.

CLAIM. *If K is a compact operator $\varepsilon > 0$, and $n \geqslant 1$, then there is a projection P such that:*

(a) $\|PS - SP\| \leqslant 1/\sqrt{n}$;

(b) $\|PK - KP\| \geqslant \frac{1}{2}\max\{\|(1 - P_n)KP_n\|, \|(1 - P_n)K^*P_n\|) - 2\varepsilon$.

Proof of Claim. Since $\|PK - KP\| = \|PK^* - K^*P\|$, there is no harm is assuming that $\max\{\|(1 - P_n)KP_n\|, \|(1 - P_n)K^*P_n\|\} = \|(1 - P_n)K^*P_n\| = \|P_nK(1 - P_n)\|$. Suppose $\varepsilon > 0$ and choose a unit vector f so that $P_nf = 0$ and $\|P_nKf\| > \|P_nK(1 - P_n)\| - \varepsilon$. We can also assume that there is an $m > 0$ so that $P_{m+n}f = f$ and $\|K(1 - P_{m+n})\| < \varepsilon$. We now use the Berg technique ([2], [10]) to construct P. Let P be the orthogonal projection onto the linear span of the union of the three sets:

$$\left\{\sqrt{\frac{k}{2n}}\, e_{m+k} + \sqrt{1 - \frac{k}{2n}}\, e_{2n+2m+k} : n + 1 \leqslant k \leqslant 2n\right\},$$

$$\left\{\sqrt{\frac{k}{2n}}\, e_k + \sqrt{1 - \frac{k}{2n}}\, e_{2n+m+k} : 1 \leqslant k \leqslant n\right\},$$

$$\left\{\sqrt{\frac{1}{2}}\, e_{n+k} + e_{3n+m+k} : 1 \leqslant k \leqslant m\right\}.$$

Standard computations show that (a) above holds. To show that (b) holds, write $f = a_1e_{n+1} + \cdots + a_me_{n+m}$, and let $g = a_1e_{3n+m+1} + \cdots + a_me_{3n+2m}$. A simple computation shows that

$$Pg = \frac{f+g}{2}, \quad \|Kg\| = \|K(1 - P_{m+n})g\| < \varepsilon, \quad \|PK_g\| < \varepsilon$$

and

$$\|KPg\| \geqslant \left\|\frac{Kf}{2}\right\| - \frac{\varepsilon}{2} \geqslant \frac{\|P_nK(1-P_n)\| - \varepsilon}{2} - \frac{\varepsilon}{2}.$$

Hence

$$\|KP - PK\| \geqslant KPg - PKg\| \geqslant \left(\frac{1}{2}\right)\|P_nK(1-P_n)\| - 2\varepsilon.$$

This proves the claim.

Note that $C^*(S)/[C^*(S) \cap \mathcal{K}(H)]$ is commutative and is therefore nuclear and is isomorphic to an algebra $\mathcal{B}$ with $\Gamma_{\mathcal{B}}(t) \geqslant t/4$. The inequality $\Gamma_{\mathcal{G}}(t) \geqslant t/64$ follows from Theorem 22. ∎

An operator T is quasinormal if T commutes with T^*T. Every quasinormal operator is unitary equivalent to the direct sum of a normal operator and the spatial tensor product $P \otimes S$ where P is a positive operator and S is the unilateral shift operator ([9]).

COROLLARY 24. *Suppose $T \in B(H)$ is a quasinormal operator. Then $C^*(T)$ is an Arzela-Ascoli algebra with $\Gamma_{\mathcal{G}}(t) \geqslant t/256$.*

Proof. It follows from [9] that T is contained in an algebra of the form $\mathcal{D} \oplus (\mathcal{G} \otimes C^*(S))$, where $\mathcal{D}$ and $\mathcal{G}$ are commutative algebras and S is the unilateral shift operator. The inequality follows from Theorems 17 and 23. ∎

Proving that every separable C^*-subalgebra of $B(H)$ is an Arzela-Ascoli algebra is equivalent to showing that $B(H)$ is an Arzela-Ascoli algebra (Theorem 10); this makes an affirmative answer seem less likely. However, we can use a result of Olsen and Zame to reduce the problem to the singly-generated case.

PROPOSITION 25. *Suppose H is an infinite-dimensional separable Hilbert space. The following are true.*

(i) *$B(H)$ is an Arzela-Ascoli algebra if and only if $C^*(T)$ is an Arzela-Ascoli algebra for every T in $B(H)$.*

(ii) *Every separable C^*-subalgebra of $B(H)$ is a uniformly Arzela-Ascoli algebra if and only if $C^*(T)$ is a uniformly Arzela-Ascoli algebra for every T in $B(H)$ if and only if there is a function $\rho : (0, 2] \to (0, 1]$ such that $\Gamma_{\mathcal{G}} \geqslant \rho$ for every separable C^*-subalgebra $\mathcal{G}$ of $B(H)$.*

Proof. Suppose every singly generated C^*-algebra of $B(H)$ is an Arzela-Ascoli algebra, and suppose $\mathcal{G}$ is a separable C^*-subalgebra of $B(H)$. Let $\mathcal{B}$ be the spatial tensor product $\mathcal{G} \otimes \mathcal{K}(H)$. It follows from [20] that $\mathcal{B}$ is singly generated, and is therefore an Arzela-Ascoli algebra. Suppose q is a rank-one projection in $\mathcal{K}(H)$, and let $P = 1 \otimes q$. It follows from Lemma 13 (vi) that $\mathcal{G}$ is an Arzela-Ascoli algebra and that $\Gamma_{\mathcal{G}} \geqslant \Gamma_{\mathcal{B}}$. Hence if $\mathcal{B}$ is uniformly Arzela-Ascoli, then so is $\mathcal{G}$. It also follows from Lemma 13 (vi) that if $\mathcal{G}$ is the C^*-direct sum of a sequence $\{\mathcal{G}_n\}$ of C^*-algebras, then $\Gamma_{\mathcal{G}} \leqslant \inf_{n \geqslant 1} \Gamma_{\mathcal{G}_n}$. This proves that if every separable C^*-algebra is uniformly Arzela-Ascoli, then

$$\rho(t) = \inf\{\Gamma_{\mathcal{G}}(t) : \mathcal{G} \text{ is a separable } C^*\text{-subalgebra of } B(H)\} > 0$$

for every t in $(0, 2]$. ∎

The Arzela-Ascoli theorem yielded an affirmative answer to Question A for commutative algebras of the form $\pi(\mathcal{G})$. For a better result we need an improved version of the Arzela-Ascoli theorem. The key ingredient is a more precise version of the Tietze extension theorem in metric spaces. The idea is based on the proof of the Tietze theorem in [26] combined with the well-known fact that Urysohn functions can be written explicitly for a metric space. I wish to thank my colleague, David Feldman, for suggesting that the algorithmic nature of the proof of Tietze's theorem in [26] might lead to a proof of the generalization below.

Suppose (X, d) is a metric space and $\mathcal{F}$ is a collection of complex-valued functions such that, for each f in $\mathcal{F}$, the domain of f is a closed subset of X. We call $\mathcal{F}$ *equicontinuous* if, for each positive number ε, there is a $\delta > 0$ such that, for every f in $\mathcal{F}$ and every x, y in $\mathrm{dom}(f)$, $[d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon]$.

THEOREM 26. *Suppose (X, d) is a metric space and $\mathcal{F}$ is a uniformly bounded equicontinuous family of functions whose domains are closed subsets of X. Then each f in $\mathcal{F}$ has an extension $\hat{f}$ in $C(X)$ such that the family $\{\hat{f} : f \in \mathcal{F}\}$ is uniformly bounded and equicontinuous.*

Proof. If f and g are functions whose domains are subsets of X, we define $f + g$ to be the sum of f and g restricted to $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. It is clear that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are equicontinuous families, then so are $\mathcal{F}_1 \cup \mathcal{F}_2$ and $\mathcal{F}_1 + \mathcal{F}_2 = \{f + g : f \in \mathcal{F}_1 \text{ and } g \in \mathcal{F}_2\}$. Since $\mathcal{F}$ is uniformly bounded by a positive constant M, there is no harm in assuming that $-1 \leqslant f \leqslant 1$ for every f in $\mathcal{F}$ (otherwise, replace $\mathcal{F}$ with the set of functions of the form $\mathrm{Re}(f/M)$ and $\mathrm{Im}(f/M)$ for $f \in \mathcal{F}$.

Since equicontinuity is unchanged if we replace the metric d with $\frac{d}{1+d}$ (i.e., $\frac{d}{1+d} < r < 1 \Leftrightarrow d < \frac{r}{1-r}$), we can assume that the diameter of X is at most 1.

Let $\mathcal{Y}$ denote the set of all continuous functions whose domain is a closed subset of X and whose range is contained in $[-1, 1]$. Define $\Lambda : \mathcal{Y} \to C(X)$ as follows:

$$(\Lambda f)(x) = 2\frac{d(x, f^{-1}([-1, -\frac{1}{3}]))}{d(x, f^{-1}([-1, -\frac{1}{3}])) + d(x, f^{-1}([\frac{1}{3}, 1]))} - 1,$$

where $d(x, A) = \inf\{d(x, y) : y \in A\}$ for $x \in X$, $\emptyset \neq A \subset X$, and $d(x, \emptyset) = 1$.

CLAIM. *If $\mathfrak{G}$ is an equicontinuous subset of $\mathcal{Y}$, then $\Lambda\mathfrak{G} = \{\Lambda g : g \in \mathfrak{G}\}$ is equicontinuity.*

Proof of the Claim. Choose ρ such that $0 < \rho < 1$, and, for every f in $\mathfrak{G}$ and every $x, y \in \mathrm{dom}(f)$, $[d(x, y) < \rho \Rightarrow |f(x) - f(y)| < 1/3]$. Choose $f \in \mathfrak{G}$ and define $u, v \in C(X)$ by $u(x) = d(x, f^{-1}([-1, -1/3]))$, and $v(x) = d(x, f^{-1}([-1, -1/3])))+ d(x, f^{-1}([1/3, 1]))$. Note that, for $x, y \in X$, $u(x) \leqslant 1$, $|u(x) - u(y)| \leqslant d(x, y)$, $|v(x) - v(y)| \leqslant 2d(x, y)$, $\rho \leqslant v(x) \leqslant 2$.

The claim now follows from

$$\begin{aligned}|(\Lambda f)(x) - (\Lambda f)(y)| &= 2\left|\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)}\right| \\ &= 2\left|\frac{u(x)(v(y) - v(x)) + v(x)(u(x) - u(y))}{v(x)v(y)}\right| \\ &\leqslant 2\frac{(1/2)d(x, y) + 2d(x, y)}{\rho^2} = \frac{8d(x, y)}{\rho^2}. \quad \blacksquare\end{aligned}$$

Next suppose $f \in \mathcal{F}$. We next define, inductively, a sequence $\{\Lambda_n f\}$ in $C(X)$ such that, for each positive integer n, we have

$$|\Lambda_n f| \leqslant \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n-1}, \quad |f - \Lambda_1 f - \cdots - \Lambda_n f| \leqslant \left(\frac{2}{3}\right)^n.$$

Define $\Lambda_1 f = (1/3)\Lambda_f$, and for each positive integer n we define

$$\Lambda_{n+1} f = \left(\frac{2}{3}\right)^n \Lambda_1\left(\left(\frac{3}{2}\right)^n [f - \Lambda_1 f - \cdots - \Lambda_n f]\right).$$

Note that $|\Lambda_1 f| \leqslant 1/3$, and $-2/3 \leqslant f - \Lambda_1 f \leqslant 2/3$.

The required properties follows from induction. It also follows from induction, the equicontinuity of $\mathcal{F}$, and the claim above, that, for each positive integer n, $\Lambda_1\mathcal{F} + \cdots + \Lambda_n\mathcal{F}$ is equicontinuous.

For each f in $\mathcal{F}$ define $\hat{f} = \Lambda_1 f + \Lambda_2 f + \cdots$. It is clear, for each f in $\mathcal{F}$, $\hat{f}|\mathrm{dom}(f) = f$. To show that $\{\hat{f} : f \in \mathcal{F}\}$ is equicontinuous, suppose $\varepsilon > 0$, and choose n so that $(2/3)^n < \varepsilon/3$. We then choose $\delta > 0$ so that, for every x, y in X and every g in $\Lambda_1\mathcal{F} + \cdots + \Lambda_n\mathcal{F}$,

$$[d(x,y) < \delta] \Rightarrow |g(x) - g(y)| < \frac{\varepsilon}{3}.$$

Suppose $f \in \mathcal{F}$, and let $g = \Lambda_1 f + \cdots + \Lambda_n f$. Then $|f - g| < \varepsilon/3$ and $g \in \Lambda_1\mathcal{F} + \cdots + \Lambda_n\mathcal{F}$. Thus, for every x, y in X with $d(x,y) < \delta$, we have

$$|f(x) - f(y)| \leqslant |f(x) - g(x)| + |g(x) - g(y)| + |f(y) - g(y)| < \varepsilon. \quad \blacksquare$$

The following generalization of the Arzela-Ascoli theorem ([26]) follows from the preceding theorem and the Arzela-Ascoli theorem itself.

THEOREM 27. *Suppose (X, d) is a compact metric space and $\mathcal{F}$ is a uniformly bounded equicontinuous family of functions whose domains are closed subsets of X. Then, for every $\varepsilon > 0$, there is a finite subset $\mathfrak{G}$ of $C(X)$ such that, for every f in $\mathcal{F}$, there is a g in $\mathfrak{G}$, such that $|f(x) - g(x)| < \varepsilon$ for every x in* $\operatorname{dom}(f)$.

We can use the generalized Arzela-Ascoli theorem to improve Theorem 5.

THEOREM 28. *Suppose $\mathcal{B}$ is a separable commutative unital C^*-subalgebra of $\ell^\infty(B(H))$. Then* $\operatorname{Alg}\operatorname{Lat}(\eta(\mathcal{B})) = C^*(\eta(\mathcal{B}) \cup \mathcal{Z})$.

Proof. Let X be the maximal ideal space of $\mathcal{B}$. Since $\mathcal{B}$ is separable, X is a compact metric space. Let $\rho : C(X) \to \mathcal{B}$ be the inverse of the Gelfand map. We define $*$-homomorphisms $\rho_n : C(X) \to B(H)$, $n \geqslant 1$, by $\rho(f) = (\rho_1(f), \rho_2(f), \ldots)$. For each $n \geqslant 1$, there is a compact subset X_n of X such that $\ker(\rho_n) = \{f \in C(X) : f|X_n = 0\}$. Hence $C(X)/\ker(\rho_n)$ is $*$-isomorphic to $C(X_n)$. Hence there are $*$-homomorphisms $\tau_n : C(X_n) \to B(H)$ such that $\rho_n(f) = \tau_n(f|X_n)$ for each $n \geqslant 1$ and each f in $C(X)$. As in the proof of Theorem 5, each ρ_n has an extension to the algebra $\operatorname{Bor}(X)$ of bounded complex Borel functions on X, and each τ_n has an extension to $\operatorname{Bor}(X_n)$ so that $\rho_n(f) = \tau_n(f|X_n)$ for each $n \geqslant 1$ and each f in $\operatorname{Bor}(X)$.

Suppose $s \in \operatorname{Alg}\operatorname{Lat}\eta(\mathcal{B})$, and choose $S \in \ell^\infty(B(H))$ so that $\eta(S) = s$. It follows from Theorem 2 that we can assume that there is a sequence $\{f_n\}$ such that, $f_n \in C(X_n)$ for $n \geqslant 1$, and $S = (\tau_1(f_1), \tau_2(f_2), \ldots)$. Following the proof of Theorem 5, we can show that $\{f_1, f_2, \ldots\}$ is equicontinuous. Suppose $\varepsilon > 0$. It follows from the generalized Arzela-Ascoli theorem that there are finitely many functions $g_1, g_2, \ldots, g_m$ in $C(X)$ and a partition $\{E_1, E_2, \ldots, E_m\}$ or $\{1, 2, 3, \ldots\}$ such that, for $1 \leqslant k \leqslant m$ and $n \in E_k$, $\|f_n - g_k|X_n\| < \varepsilon$. For each k, $1 \leqslant k \leqslant m$, let P_k be the characteristic function of E_k. Then $P_k \in \ell^\infty(\mathbb{C})$ and $a = P_1\rho(g_1) + P_2\rho(g_2) + \cdots + P_m\rho(g_m) \in C^*(\mathcal{B} \cup \ell^\infty(\mathbb{C}))$ and $\|S - a\| < \varepsilon$. Hence $s \in C^*(\eta(\mathcal{B}) \cup \mathcal{Z})$. $\blacksquare$

We can use Theorem 23 to prove a more general result.

THEOREM 29. *If s is an isometry in $Q(H)$, then* $\operatorname{Alg}\operatorname{Lat} C^*(s) = C^*(\{s\} \cup \mathcal{Z})$.

Proof. Choose a sequence $A = \{A_n\}$ in $\ell^\infty(B(H))$ so that $\eta(A) = s$. For each $n \geqslant 1$, let $A_n = |A_n| S_n$ be the polar decomposition of A_n. Since $s^* s = 1$, $\|1 - A_n^* A_n\| \to 0$. It follows that $|A_n| \to 1$. Thus $\|A_n - S_n\| \to 0$, and, therefore, $\|1 - S_n^* S_n\| \to 0$. It follows that, for n sufficiently large, S_n is an isometry. Since every non-unitary isometry is approximately equivalent to a pure isometry, we can assume that, for each n, S_n is unitary or S_n is a pure isometry. If S_n is a pure isometry for only finitely many values of n, we can assume that all of the S_n's are unitary and apply Theorem 28. We next suppose that S_n is unitary for only finitely many values of n. In this case we can assume that S_n is a pure isometry for every n. Let S be the unilateral shift of multiplicity 1 acting on H. For each n, there is an isometric unital $*$-homomorphism $\sigma_n : B(H) \to B(H)$ such that $\sigma_n(S) = S_n$. Suppose $T = \{T_n\} \in \ell^\infty(B(H))$ and $\eta(T) \in \operatorname{Alg}\operatorname{Lat} C^*(s)$. It follows from Theorem 2 that we can assume that $T_n \in C^*(S_n)$ for each n. For each n, we can choose $C_n \in C^*(S)$ such that $\sigma_n(C_n) = T_n$.

Assume, via contradiction, that $\{C_1, C_2, \ldots\}$ is not totally bounded. Since $C^*(S)$ is an Arzela-Ascoli algebra, it follows from Theorem 10 that $\eta(\{C_n\}) \notin \operatorname{Alg}\operatorname{Lat} \pi(C^*(S))$. Hence there is a sequence $\{P_n\}$ in $\operatorname{appr}\operatorname{Lat}(C^*(S))$ such that $\limsup \|C_n P_n - P_n C_n\| > 0$. It follows that $\eta(\{\sigma_n(P_n)\}) \in \operatorname{Lat} C^*(s)$, but

$$\begin{aligned} &\|\eta(\{\sigma_n(P_n)\})\eta(\{T_n\}) - \eta(\{T_n\})\eta(\{\sigma_n(P_n)\})\| \\ &\qquad = \limsup \|\sigma_n(P_n)T_n - T_n\sigma_n(P_n)\| \\ &\qquad = \limsup \|\sigma_n(P_n)\sigma_n(C_n) - \sigma_n(C_n)\sigma_n(P_n)\| \\ &\qquad = \limsup \|P_n C_n - C_n P_n\| > 0. \end{aligned}$$

This contradicts the fact that $\eta(\{T_n\}) \in \operatorname{Alg}\operatorname{Lat} C^*(s)$. Hence $\{C_1, C_2, \ldots\}$ is totally bounded. It easily follows, as in the proof of Theorem 2, that $\{T_n\} \in C^*(\{s\} \cup \mathcal{Z})$. This proves the theorem in the case when all but finitely many S_n's are pure isometries.

The remaining case is when infinitely many S_n's are unitary and infinitely many are pure isometries. This follows from the above two cases by identifying $\ell^\infty(B(H))$ with $\ell^\infty(B(H)) \oplus \ell^\infty(B(H))$. ∎

REMARK. With a little more work, we can obtain distance estimates in Theorems 28 and 29 above.

Acknowledgements. The author was partially supported by a grant from the National Science Foundation while this reseach was undertaken. He wishes to thank Cornel

Pasnicu for valuable assistance, David Feldman for helpful insights, and Ileana Ionaşcu for proofreading the manuscript.

REFERENCES

1. W.B. ARVESON, Notes on extensions of C^*-algebras, *Duke Math. J.* **44**(1977), 329–355.
2. I.D. BERG, Index theory for perturbations of direct sums of normal operators and weighted shifts, *Canad. J. Math.* **30**(1978), 1152–1165.
3. B. BLACKADAR, Symmetries of the CAR algebras, *Ann. of Math. (2)* **131**(1990), 589–623.
4. F. BOCA, Metric diophantine approximations and the structure of non-commutative tori, preprint.
5. O. BRATTELI, G.A. ELLIOT, D. EVANS, A. KISHIMOTO, Non commutative spheres II, rational rotations, *J. Operator Theory* **27**(1992), 53–85.
6. O. BRATTELI, D. EVANS, A. KISHIMOTO, The crossed product of irrational rotation algebras by the flip, Operator Algebras, Math. Physics and Low Dimen. Topology, Res. Notes Math., vol. 5, A.K. Peters, Wellesley, MA, 1993, 61–76.
7. O. BRATTELI, A. KISHIMOTO, Non commutative spheres III: irrational rotations, *Comm. Math. Phys.* **147**(1992), 605–624.
8. O. BRATTELI, A. KISHIMOTO, M. RØRDAM, E. STORMER, The crossed product of a UHF algebra by a shift, *Ergodic Theory Dynamical Systems* **13**(1993), 615–626.
9. A. BROWN, On a class of operators, *Proc. Amer. Math. Soc.* **4**(1953), 723–728.
10. K. DAVIDSON, Berg's technique and irrational rotation algebras, *Proc. Roy. Irish Acad. Sect. A* **84**(1984), 117–123.
11. G.A. ELLIOT, D. EVANS, The structure of the irrational rotation C^*-algebra, *Ann. of Math. (2)* **138**(1993), 477-504.
12. G.A. ELLIOT, Q. LIN, Cut-down method in the inductive limit decomposition of non-commutative tori, preprint.
13. D. HADWIN, An asymptotic double commutant theorem for C^*-algebras, *Trans. Amer. Math. Soc.* **244**(1978), 273–297.
14. D. HADWIN, Non-separable approximate equivalence, *Trans. Amer. Math. Soc.* **266** (1981), 203–231.
15. D. HADWIN, Approximately hyperreflexive algebra, *J. Operator Theory* **28**(1992), 51–64.
16. P.R. HALMOS, Limits of shifts, *Acta Sci. Math. (Szeged)* **34**(1973), 131–139.
17. A. KUMJIAN, An involutive automorphism of Bunce-Deddens algebras, *C.R. Math. Rep. Acad. Sci. Canada* **10**(1988), 217–218.
18. Q. LIN, On the structure of 3-dimensional non-commutative tori, preprint.
19. C.L. OLSEN, G. PEDERSEN, Corona C^*-algebras and their applications to lifting problems, *Math. Scand.* **64**(1989), 63–86.
20. C.L. OLSEN, W.R. ZAME, Some C^*-algebras with a single generator, *Trans. Amer. Math. Soc.* **215**(1976), 205–217.
21. G. PEDERSEN, The corona construction, Proc. of the 1988 GPOTS-Wabash Conf., Longman, Harlow, 1990, 49–92.

22. M.V. PIMSNER, Embedding some transformation group C^*-algebras into AF-algebras, *Ergodic Theory Dynamical Systems* **3**(1983), 613–626.
23. M.V. PIMSNER, D. VOICULESCU, Imbedding the irrational rotation algebra into an AF-algebra, *J. Operator Theory* **4**(1980), 201–210.
24. I. PUTNAM, On the topological stable rank of a certain transformation group C^*-algebra, *Ergodic Theory Dynamical Systems* **10**(1990), 197–207.
25. S. ROSENOER, Distance estimates for von Neumann algebras, *Proc. Amer. Math. Soc.* **86**(1982), 248–252.
26. G.F. SIMMONS, *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York 1963.
27. J. SPIELBERG, Embedding C^*-algebra extensions into AF-algebras, *J. Funct. Anal.* **81**(1988), 325–344.
28. D. VOICULESCU, A noncommutative Weyl-von Neumann theorem, *Rev. Roumaine Math. Pures Appl.* **21**(1976), 97–113.
29. D. VOICULESCU, Almost inductive limit automorphisms and embedding into AF-algebras, *Ergodic Theory Dynamical Systems* **6**(1986), 475–484.

DON HADWIN
Mathematics Department
University of New Hampshire
Durham, NH 03824
U.S.A.

Received February 11, 1996.