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EXTREMAL RICHNESS OF MULTIPLIER ALGEBRAS AND CORONA ALGEBRAS OF SIMPLE C^*-ALGEBRAS

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ABSTRACT. A simple unital C^*-algebra A is called extremally rich if the set of one-sided invertible elements is dense in A. We determine some conditions on a separable, simple, infinite dimensional C^*-algebra of real rank zero under which we can decide whether the multiplier algebras $M(A)$, $M(A \otimes K)$ and the corona algebras $Q(A)$, $Q(A \otimes K)$ are extremally rich or not. Our analysis will depend on the existence of a finite trace for A and, when A is an AF algebra, on the number of infinite extremal traces of A and $A \otimes K$.

KEYWORDS: *Simple C^*-algebras, extremal richness, real rank, stable rank.*

AMS SUBJECT CLASSIFICATION: Primary 46L05; Secondary 46L35.

INTRODUCTION

Given a C^*-algebra A, we consider the problem of determining whether the multiplier and the corona algebras of A or the stabilization $A \otimes K$ of A are extremally rich. In the rest of the paper we will assume that K is the set of compact operators on a separable, infinite dimensional Hilbert space. We will also assume all C^*-algebras to be infinite dimensional.

When the answer is affirmative we automatically get bounds on the real rank of the respective algebras. We recall that the real rank of an extremally rich C^*-algebra can be at most one ([20]). So, considering certain C^*-algebras whose multiplier and corona algebras do not have real rank zero, one could hope that this rank would be one, due to extremal richness. The second author has obtained positive results in this direction ([18]) for the multiplier algebra of a σ-unital purely infinite simple C^*-algebra. We will see another proof of this fact in Corollary 3.8.

Let us recall that a unital C^*-algebra A is *extremally rich* if its closed unit ball A_1 equals the convex hull $\mathrm{conv}(\mathfrak{E}(A))$ of the extreme points $\mathfrak{E}(A)$ in A_1. We will see equivalent definitions and properties of extremally rich C^*-algebras in the next section. For simplicity, we shall write in the rest of the paper extreme points of an algebra, but we will mean the extreme points of the closed unit ball of the respective algebra.

We will determine when the multiplier and the corona algebras of a given C^*-algebra are extremally rich by examining extensions. As it turns out (see also the next section), one of the important problems here is to lift extreme points from quotients. Since most of the algebras we will look at are prime or simple, these extreme points will be isometries or co-isometries.

In Section 3 we show that the multiplier algebra of a simple σ-unital C^*-algebra A with a finite trace is not extremally rich (Theorem 3.2). If the algebra A is moreover separable and unital, then $M(A \otimes K)$ is as well not extremally rich. In the rest of this section we consider extensions with a purely infinite simple ideal.

Still, the corona algebra can be extremally rich (Theorem 4.1). However, for a separable simple AF algebra A such that $A \otimes K$ has at least two extremal traces, the corona algebra $Q(A \otimes K)$ is not extremally rich (Theorem 4.9). If A has only finitely many extremal traces of which $n \geqslant 1$ are infinite, then $Q(A)$ is not extremally rich unless $n = 1$ (Proposition 4.13 and Proposition 4.18).

1. PRELIMINARIES

This section contains the basic definitions and results for the property of C^*-algebras called *extremal richness*. This property, which was introduced by L.G. Brown and G.K. Pedersen in [5], is an analog of the topological stable rank one property (asserting that the invertible elements are dense in the algebra ([22])), but with the invertible elements replaced by the so-called *quasi-invertible* elements.

In order to define the quasi-invertible elements, let us first recall that the set $\mathfrak{E}(A)$ of extreme points of the closed unit ball A_1 of a unital C^*-algebra A consists of those partial isometries v in A satisfying $(1 - vv^*)A(1 - v^*v) = 0$. Projections of the form $1 - vv^*, 1 - v^*v$ with $v \in \mathfrak{E}(A)$ will be called defect projections. The set A_{q}^{-1} of quasi-invertible elements of a unital C^*-algebra A is defined as $A^{-1}\mathfrak{E}(A)A^{-1}$, where A^{-1} denotes the set of invertible elements.

Note that in a prime C^*-algebra (or, in particular, a simple C^*-algebra) the set of quasi-invertible elements consists only of the elements of A which are left or right invertible. If the algebra moreover is finite (in the sense that any isometry must be unitary), then the quasi-invertible elements will be just the invertible elements of the algebra.

A unital C^*-algebra A is called extremally rich if the set A_{q}^{-1} of quasi-invertible elements is dense in the algebra. A non-unital C^*-algebra will then be called extremally rich if its unitization $\widetilde{A}$ is extremally rich. Any C^*-algebra with topological stable rank one is extremally rich, since in this case the quasi-invertibles are the same as the invertible elements. But much more is true. Any von Neumann algebra is extremally rich ([20]) and any simple, purely infinite C^*-algebra is extremally rich (see [20], [24] for a proof in the unital case, the non-unital situation being treated in Lemma 3.3).

It turns out that the property of extremal richness is preserved under passing to quotients, hereditary C^*-subalgebras (in particular ideals), and under taking direct sums.

As for the behavior of extremal richness under extensions

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0,$$

it is not enough, as one might hope, to have that the ideal and the quotient are extremally rich and that the extreme points lift from the quotient. More is needed, namely that all spaces of the form pAq are extremally rich for all defect projections p in A and q in J. By Proposition 1.4.8 in [19], the set $\mathfrak{E}(pAq)$ of extreme points in the unit ball of the space pAq consists of partial isometries v in pAq satisfying $(p - vv^*)pAq(q - v^*v) = 0$. Thus, the space pAq will be called extremally rich if $\mathfrak{E}(pAq)$ is non-empty and the set $(pAp)^{-1}\mathfrak{E}(pAq)(qAq)^{-1}$ is dense in pAq. Equivalently, pAq is extremally rich if $(pAq)_1 = \operatorname{conv}(\mathfrak{E}(pAq))$. By convention, the space $\{0\}$ will be extremally rich.

When J has real rank zero the condition for the extension to be extremally rich is:

PROPOSITION 1.1. ([6]) *Let J be a closed ideal of real rank zero in a unital C^*-algebra A, such that pJq is extremally rich for any pair of projections such that $p \in A$ and $q \in J$. Then A is extremally rich if A/J is extremally rich and $\mathfrak{E}(A/J)$ consists only of isometries and co-isometries.*

REMARK 1.2. We will often use the fact that when we have an extension as above and A is extremally rich, then extreme points from the quotient A/J lift to those in the algebra A ([5], Theorem 6.1).

2. LIFTING PROBLEM OF ISOMETRIES

As pointed out in the previous section, if we have a short exact sequence:

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0,$$

we can not in general deduce that A is extremally rich even though this is the case both for J and A/J. L.G. Brown and G.K. Pedersen ([6]) presented an example in which the multiplier algebra of a finite matroid C^*-algebra A is not extremally rich, though it has real rank zero and the corona algebra $M(A)/A = Q(A)$ is a purely infinite simple C^*-algebra, hence extremally rich with real rank zero. However, if A is a σ-unital purely infinite simple C^*-algebra, then $M(A)$ is extremally rich (see [18] and Corollary 3.8 in the next section). The obstruction to having an extremally rich extension can be expressed as a lifting problem of extreme points in A/J to those in A (see Remark 1.2).

In this section we give answers to this problem in some classes of C^*-algebras.

Recall that a simple C^*-algebra is called purely infinite if any non-zero hereditary C^*-subalgebra has an infinite projection. We will need the following fact:

REMARK 2.1. (cf. Proposition 1.5 in [9]) If A is a purely infinite simple C^*-algebra, then any pair of non-zero projections p, q in A will satisfy $p \precsim q$, that is, p is Murray-von Neumann equivalent to a subprojection of q.

The following is a simple modification of an argument of G.A. Elliott ([11]) (see also [30]).

THEOREM 2.2. *Let A be a C^*-algebra and let J be an essential ideal of A. Suppose that J is purely infinite and simple. Then any isometry in $\widetilde{A/J}$ can be lifted to an isometry in $\widetilde{A}$.*

Proof. We may assume that A is unital.

Let v be an isometry in A/J and z be a preimage of v in A. Let π be the canonical quotient map from A to A/J. Then $z^*z - 1$ is contained in J, since $\pi(z^*z-1) = 0$.

Since J has real rank zero by [28], there exists a projection $r \in J$ such that

$$\|(1-r)(z^*z-1)(1-r)\| < 1$$

because J admits an approximate unit consisting of projections by [4]. Thus we get that $x = (1-r)z^*z(1-r)$ is an invertible element in $(1-r)A(1-r)$. Let y be the inverse of x in $(1-r)A(1-r)$.

Put $u = zy^{1/2}$. We have $\pi(x) = \pi(z^*z) = 1$ and $\pi(y) = \pi(y)\pi(x) = \pi(yx) = 1$, hence $\pi(u) = v$. Since $u^*u = y^{1/2}(1-r)z^*z(1-r)y^{1/2} = 1-r$, u is a partial isometry. If $1 - uu^* \neq 0$, then, since J is a purely infinite simple C^*-algebra and an essential ideal in A, it follows from Remark 2.1 that there exists a non-zero projection $t \in (1-uu^*)J(1-uu^*)$ such that $r \precsim t$, that is, there is a partial isometry $w \in J$ such that $w^*w = r$ and $ww^* \leqslant t < 1 - uu^*$. Then $u + w$ is an isometry in A and $\pi(u+w) = v$. If $uu^* = 1$, that is v is a unitary, then with $u' = ((1-r)u)^*$ instead of u the above argument works and gives rise to a partial isometry w' in J such that $u' + w'$ is the required lift. ∎

As pointed out in the beginning of this section, the assumption on J in the previous theorem can not be replaced just by real rank zero, even when J is an AF algebra. Indeed, we get a negative answer as follows. This heavily depends on an idea of N. Christopher Phillips ([21]).

THEOREM 2.3. *Let A be a separable, simple, unital C^*-algebra of real rank zero with a finite trace. In particular A could be an AF algebra. Then there is an isometry in $Q(A \otimes K)$ which can not be lifted to an isometry in $M(A \otimes K)$.*

Proof. We write τ for the semi-finite trace on $A \otimes K$ obtained by tensoring the given finite trace on A by the usual trace on K. Let J_τ be the closure of the set $\{x \in M(A \otimes K) \mid \tau(x^*x) < \infty\}$ and let J be the smallest closed ideal of $M(A \otimes K)$ strictly containing $A \otimes K$ as constructed by H. Lin and S. Zhang ([17]).

Note that $J \subset J_\tau$ comes from the fact that J_τ is a proper ideal in $M(A \otimes K)$ strictly containing $A \otimes K$. Indeed, J_τ is proper since τ is semi-finite and $A \otimes K \neq J_\tau$ by Remark 4.2 in [23].

Taking q to be a projection in $J \setminus A \otimes K$, we get that $\pi(q)$ is an infinite projection in $\pi(J)$ by [17], where π is the canonical quotient map from $M(A \otimes K)$ to $Q(A \otimes K)$. So, there is a partial isometry $v \in Q(A \otimes K)$ such that $v^*v = \pi(q), vv^* < \pi(q)$. Then $w = v + 1 - \pi(q)$ is a proper isometry in $Q(A \otimes K)$. Now, by the same argument as in Theorem 2.2, there exist a partial isometry $u \in M(A \otimes K)$ and a projection $r \in A \otimes K$ such that $\pi(u) = w$ and $u^*u = 1 - r$. Then $\pi(1 - uu^*) = 1 - ww^* = \pi(q) - vv^* \in \pi(J)$, so $1 - uu^* \in J$, hence $\tau(1 - uu^*) < \infty$.

Pick a projection $r_0 \in A \otimes K$ such that $\tau(r_0) > \tau(1 - uu^*)$. Since $1 - r_0 \sim 1 - r$ in $M(A \otimes K)$, there is a partial isometry $m \in M(A \otimes K)$ such that $m^*m = 1 - r_0$ and $mm^* = 1 - r$. Letting $t = um$, it follows that t is a partial isometry and $\pi(t)$ is an isometry. In fact, $t^*t = 1 - r_0$ and $tt^* = uu^*$. Hence, $\tau(1 - t^*t) = \tau(r_0) > \tau(1 - uu^*) = \tau(1 - tt^*)$.

Since $1 - uu^* \in J$, $\varphi(t)$ is a unitary in $M(A \otimes K)/J$, where φ is the canonical quotient map from $M(A \otimes K)$ to $M(A \otimes K)/J$.

Now suppose that there is an isometry $s \in M(A \otimes K)$ which is a lifting of $\pi(t)$. Then $\varphi(s) = \varphi(t)$. From K-theory, Remark 8.1.4 in [25], we know that

$$[1 - s^*s] - [1 - ss^*] = [1 - t^*t] - [1 - tt^*]$$

in $K_0(J)$. Since $\tau_*([1 - t^*t] - [1 - tt^*]) > 0$ we get that $1 - s^*s \neq 0$, which is a contradiction to the fact that s is an isometry. ∎

REMARK 2.4. From the proof of the previous theorem we see that there is an isometry in $\pi(\widetilde{J})$ which can not be lifted to an isometry in $\widetilde{J}$, where $\widetilde{J}$ is the unitization of J with the unit of $M(A \otimes K)$.

3. EXTREMAL RICHNESS OF MULTIPLIER ALGEBRAS

From the previous section we easily get the following result.

THEOREM 3.1. *Let A be a separable unital simple C^*-algebra of real rank zero with a finite trace. Then $M(A \otimes K)$ is not extremally rich.*

Proof. By noticing that $M(A \otimes K)$ is prime we get the statement from Remark 1.2 and Theorem 2.3. ∎

THEOREM 3.2. *Let A be a σ-unital simple C^*-algebra of real rank zero. Assume that A has a finite trace. Then $M(A)$ is not extremally rich.*

Proof. We will follow an idea from [6]. Denote by τ the finite trace on A. Then τ admits an extension $\widetilde{\tau}$ to a faithful finite trace on $M(A)$. It follows from [26] that $M(A)/A$ is purely infinite. Therefore $M(A)/A$ has topological stable rank strictly greater than one since it contains a proper isometry. Thus, if $M(A)$ were extremally rich, then due to the existence of the finite trace, we would get that every extreme point is a unitary, hence the topological stable rank of $M(A)$ must be one by Theorem 5.4 in [20]. This is a contradiction to the fact that the corona algebra has higher topological stable rank ([22]). ∎

Note that in case of a separable, non-elementary, non-unital simple AF algebra, the multiplier algebra $M(A)$ is never extremally rich. Indeed, either A has a finite trace, and then we are done by the previous theorem, or A has no finite traces and therefore is stable by [1]. Hence Theorem 3.1 applies for some full corner pAp of A which is stably isomorphic to A, and hence we get that $M(A)$ is not extremally rich.

In case of a σ-unital purely infinite simple C^*-algebra A, the second author has proved the extremal richness of $M(A)$ directly ([18]). This was done by using an idea of M. Rørdam who in [24] showed that the set

$$\mathrm{ZD}(M(A)) = \{x \in M(A) \mid xy = 0 = zx \text{ for some non-zero } y, z \in M(A)\}$$

of two-sided zero divisors in $M(A)$ is contained in the closure of the set of two-sided invertible elements in $M(A)$. Since the norm closure of $\mathrm{ZD}(M(A))$ consists precisely of all elements in $M(A)$ that are not one-sided invertible ([24]), we deduce that $M(A)$ is extremally rich because it is prime (see Section 1).

Here, we shall give another proof using Proposition 1.1.

LEMMA 3.3. *Let A be a (not necessarily σ-unital) purely infinite simple C^*-algebra. Then A is extremally rich.*

Proof. See [20], [24] for the unital case. In the non-unital case, pick a non-zero projection p in A. Then the hereditary C^*-subalgebra pAp will be full, and hence strongly Morita equivalent to A by [3]. As pAp is a unital purely infinite simple C^*-algebra, it is extremally rich by [20] and [24]. But extremal richness is preserved under strong Morita equivalence ([5], Theorem 5.7) and therefore A is extremally rich. ∎

Another proof of the previous result follows from the following statement.

PROPOSITION 3.4. *Let A be a (not necessarily σ-unital) C^*-algebra with real rank zero. Then A is extremally rich if and only if any proper non-zero hereditary C^*-subalgebra of A is extremally rich.*

Proof. The necessary condition was proved in Theorem 3.5 in [5]. We have only to show the converse.

Let 1 be the identity of the unitization $\widetilde{A}$ of A. Note that it suffices to prove that an element of the form $x+1$ $(x \in A)$ can be approximated by a quasi-invertible element in $\widetilde{A}$.

Since A has real rank zero, for given $\varepsilon > 0$ there is a projection $p \in A$ such that $\|x - pxp\| < \varepsilon/2$. Since pAp is extremally rich, there is a quasi-invertible element $y' \in pAp$ such that $\|p + pxp - y'\| < \varepsilon/2$. From Theorem 1.1 in [5], y' can be assumed of the form uz, where u is an extreme point in the closed unit ball of pAp, and z is an invertible element in pAp. Then $y = (1-p) \oplus y'$ is a quasi-invertible element in $\widetilde{A}$. Indeed, $y = (1 - p \oplus u)(1 - p \oplus z) \in \mathfrak{E}(\widetilde{A})\widetilde{A}^{-1}$.

Moreover,

$$\begin{aligned}\|x+1-y\| &< \|x-pxp\| + \|pxp+1-y\| \\ &< \frac{\varepsilon}{2} + \|pxp+p-y'\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

so we have the approximation. ∎

LEMMA 3.5. *Let A be a C^*-algebra and J be an essential closed ideal of A. Suppose that J is purely infinite and simple. Then pJq is extremally rich for any pair of projections such that $p \in A$ and $q \in J$.*

Proof. We assume that $p \neq 0$ and $q \neq 0$ (otherwise the condition is fulfilled since $\{0\}$ is extremally rich by convention). We may assume that A is unital. Note that $\mathfrak{E}(pJq)$ is not empty. Indeed, since J is essential, pJp is not zero. From Remark 2.1 there is a projection $r \in pJp$ such that $q \precsim r$, that is, there is a partial isometry $u \in J$ such that $u^*u = q$ and $uu^* \leqslant p$. Then $u = puq$, and u is an extreme point in pJq.

The proof of extremal richness of pJq follows from an argument similar to the one in Theorem 10.1 in [20]. Indeed, replacing A in [20] by pJq we get that any element can be approximated by an element in $(pJp)^{-1}\mathfrak{E}(pJq)(qJq)^{-1}$ by Remark 2.1. ∎

THEOREM 3.6. *Let A be a C^*-algebra and J be an essential closed ideal of A. Suppose that J is purely infinite and simple. Then A is extremally rich if and only if A/J is extremally rich and $\mathfrak{E}(A/J)$ consists only of isometries and co-isometries.*

Proof. One direction follows from Proposition 1.1 and Lemma 3.5. The other follows from the fact that extremal richness passes to quotients and that in this case the quotient is a prime algebra. ∎

COROLLARY 3.7. (Osaka, [18]) *Let A be a C^*-algebra and J be a closed two-sided ideal of A. Suppose that J and A/J are purely infinite simple C^*-algebras. Then A is extremally rich.*

Proof. If J is not essential, then A is isomorphic to the direct sum of J and A/J, so A is extremally rich from Section 1 and Lemma 3.3. Therefore, we may assume that J is essential. Hence we get the assertion from Theorem 3.6. ∎

COROLLARY 3.8. *Let A be a σ-unital, non-unital purely infinite simple C^*-algebra. Then $M(A)$ is extremally rich.*

Proof. Since $M(A)/A$ is purely infinite simple by [23], [26], [27], the statement follows from the previous result. ∎

REMARK 3.9. Since there is no known example of a unital simple C^*-algebra which is neither finite with a trace nor purely infinite, we get a reasonable answer concerning the extremal richness of $M(A \otimes K)$, where A is simple with real rank zero and separable and unital.

4. EXTREMAL RICHNESS OF CORONA C^*-ALGEBRAS

Let A be a σ-unital, non-unital purely infinite simple C^*-algebra. Its corona C^*-algebra $Q(A)$ is then purely infinite simple by [23], [26], [27], so it is extremally rich. Generally, corona C^*-algebras of simple, σ-unital C^*-algebras with real rank zero are prime and purely infinite (not necessarily simple) ([26]). But in many cases, these corona algebras will have real rank zero. For example, H. Lin proved in [16] that if A has stable rank one, then $Q(A \otimes K)$ has real rank zero.

We get an affirmative answer for extremal richness in case of some simple C^*-algebras.

THEOREM 4.1. *Let A be a separable, simple C^*-algebra with real rank zero such that $M(A \otimes K)$ has exactly one proper closed two-sided ideal J strictly containing $A \otimes K$. Then $Q(A \otimes K)$ is extremally rich.*

Proof. Let π be the canonical quotient map from $M(A \otimes K)$ to $Q(A \otimes K)$. Then $\pi(J)$ and $M(A \otimes K)/J$ are purely infinite simple C^*-algebras ([31]). Since $M(A \otimes K)/J$ is isomorphic to $Q(A \otimes K)/\pi(J)$, we obtain that $Q(A \otimes K)$ is extremally rich from Corollary 3.7. ∎

REMARK 4.2. Recall from Theorem 3.1 that $M(A \otimes K)$ is not extremally rich when A has a finite trace, even though from Theorem 4.1 we see that the corona algebra can be extremally rich.

A semi-finite trace τ on a C^*-algebra A is said to be order-preserving if for any pair of projections $p, q \in A$, $\tau(p) < \tau(q)$ implies $[p] \leqslant [q]$ in $K_0(A)_+$. Each semi-finite trace τ on A extends to a semi-finite trace on $M(A \otimes K)$, still denoted by τ. If A has real rank zero and an order-preserving trace τ then the closure J_τ of the set $\{x \in M(A \otimes K) \mid \tau(x^*x) < \infty\}$ is the only proper closed two-sided ideal of $M(A \otimes K)$ strictly containing $A \otimes K$ by Proposition 2.9 in [31].

REMARK 4.3. If A is a separable simple unital C^*-algebra of real rank zero with an order-preserving trace, then $Q(A \otimes K)$ is extremally rich by the previous theorem and the above comment. Many C^*-algebras of real rank zero in the class of separable, simple, unital AT-algebras ([13]) (i.e. direct limits of direct sums of matrix algebras over $C(T)$) have such a trace. Examples are finite matroid C^*-algebras ([10]), the irrational rotation algebras A_θ ([2]), and the Bunce-Deddens algebras ([2]).

However, even if A is an AF algebra we can not be certain that its corona algebra is always extremally rich. L.G. Brown has shown ([6]) that the corona algebra of $B \otimes K$, where B is a simple, unital AF algebra with exactly two extremal traces, is not extremally rich. We will here extend this result.

We recall the ideal structure of the multiplier algebra of a simple AF algebra, due to G.A. Elliott ([12]) (see also [14]). Let B be a non-unital, separable AF algebra, and let $D(B)$ be the set of all equivalence classes of projections in B. The element in $D(B)$ represented by the projection r is denoted by $[r]$. Define also the set $\partial D(B) = D(M(B)) \setminus D(B)$. When B is simple, any element in $\partial D(B)$ can be canonically described in terms of the convex cone of lower semicontinuous semi-finite traces on B as follows:

THEOREM 4.4. (Elliott, [12]) *Suppose that B is a simple separable* AF *algebra. Denote by $D(B)'$ the convex cone of non-zero additive maps $\tau : D(B) \to \mathbf{R}^+$, with the topology of pointwise convergence. For each $x \in \partial D(B)$, the function*

$$\widehat{x} : D(B)' \ni \tau \mapsto \tau(x) = \sup \tau([0, x] \cap D(B)) \in \mathbf{R} \cup \{\infty\}$$

is strictly positive, lower semicontinuous, and affine, and furthermore, either $\widehat{x} = \widehat{1}$, or else $\widehat{x} + g = \widehat{1}$ for some strictly positive lower semicontinuous affine function $g : D(B)' \to \mathbf{R} \cup \{\infty\}$. Conversely, any such function on $D(B)'$ is equal to $\widehat{x}$ for a unique x in $\partial D(B)$.

If τ is a semi-finite trace on a separable simple non-elementary AF algebra B, consider its extension to $M(B)$, still denoted by τ, which is obtained as: $\tau(x) = \sup \tau(e_n x e_n)$ for each x in $M(B)_+$, where $(e_n)_n$ is an increasing approximate unit of projections for B. Then the closure J_τ of the set $\{x \in M(B) \mid \tau(x^* x) < \infty\}$ is an ideal of $M(B)$. In particular, if τ is infinite (that is $\tau(1) = \infty$, where 1 is the unit of $M(B)$), then $B \subsetneq J_\tau \subsetneq M(B)$ ([14]).

DEFINITION 4.5. Let $\mathrm{ST}(A)$ be the set of semi-finite traces on A. Then τ in $\mathrm{ST}(A)$ is called *extremal trace* if for $0 \leqslant \sigma \leqslant \tau, \sigma \in \mathrm{ST}(A)$, there is a non-negative number $0 \leqslant \alpha \leqslant 1$ such that $\sigma = \alpha\tau$. Let $\mathrm{EST}(A)$ denote the set of extremal semi-finite traces on A.

From the definition we get the following well-known fact.

LEMMA 4.6. *Let A be a simple unital C^*-algebra and let $\tau \in \mathrm{ST}(A \otimes K)$. For each x in $\mathrm{K}_0(A \otimes K)_+$ set*

$$S_x = \{\sigma \in \mathrm{ST}(A \otimes K) \mid \sigma(x) = \tau(x) < \infty\}.$$

Then $\tau \in \mathrm{EST}(A \otimes K)$ if and only if τ is an extreme point in S_x.

Proof. Assume that $\tau \in \mathrm{EST}(A \otimes K)$ and $\tau = \lambda\sigma_1 + (1-\lambda)\sigma_2$ for some $\sigma_i \in S_x$, $i = 1,2$. Since $\lambda\sigma_1 \leqslant \tau$, there is a non-negative α such that $\lambda\sigma_1 = \alpha\tau$, thus $\alpha = \lambda$ and hence $\tau = \sigma_1$. Similarly we get $\sigma_2 = \tau$. Therefore we obtain that τ is extreme in S_x.

Suppose now that τ is extreme in S_x and assume that $0 \leqslant \sigma \leqslant \tau$ for some σ in $\mathrm{ST}(A \otimes K)$. Then $0 \leqslant \sigma(p) \leqslant \tau(p)$, where p is a representative element for x in $M_n(A \otimes K)$ for some n. We may assume that $n = 1$. If $\sigma(p) = \tau(p)$, then $\sigma, 2(\tau - \frac{1}{2}\sigma) \in S_x$ and

$$\tau = \frac{1}{2}\sigma + \frac{1}{2}\left(2\left(\tau - \frac{1}{2}\sigma\right)\right).$$

Since τ is extreme in S_x we get $\sigma = \tau$. If $\tau(p) \neq \sigma(p)$, then $\lambda\sigma \in S_x$, where $\lambda = \frac{\tau(p)}{\sigma(p)} > 1$. Note that by the simplicity assumption $\tau(p) \neq 0$, and in fact $\tau(p) > 0$. Thus, $\tau(p)(\tau(p) - \sigma(p))^{-1}(\tau - \sigma) \in S_x$, and

$$\tau = \frac{1}{\lambda}\lambda\sigma + \left(1 - \frac{1}{\lambda}\right)\tau(p)(\tau(p) - \sigma(p))^{-1}(\tau - \sigma).$$

Since τ is extreme in S_x, we get $\sigma = \frac{1}{\lambda}\tau$. Therefore, $\tau \in \mathrm{EST}(A \otimes K)$. ∎

Note that the above lemma shows that we can identify an element τ of $\mathrm{EST}(A \otimes K)$, which in fact is an extreme ray, with an extreme point in the convex compact set $S_x = \{\sigma \in \mathrm{ST}(A \otimes K) \mid \sigma(x) = \tau(x) < \infty\}$ where x is a fixed element in $\mathrm{K}_0(A \otimes K)_+$.

With this notation fixed, we will prove the non-extremal richness of a certain type of corona algebras (see Theorem 4.9).

The next result is a converse to Proposition 4.1 (iv) in [23].

LEMMA 4.7. *Let A be a simple, separable and non-elementary* AF *algebra. Let τ be in* $\mathrm{EST}(A \otimes K)$. *Then J_τ is a maximal ideal in $M(A \otimes K)$.*

Proof. Fix an element u in $D(A\otimes K)$ such that $\tau(u) = k > 0$ and consider the set $\{\sigma \in D(A\otimes K)' \mid \sigma(u) = k\}$. We may then regard τ as an extreme point in this set. Let f be the strictly positive function on $D(A \otimes K)'$ defined by $f(\alpha\tau) = k\alpha$ where $\alpha \in (0,\infty)$ and $f(\sigma) = \infty$ if $\sigma \neq \alpha\tau$ for $\alpha \in (0,\infty)$. Then f is a lower semicontinuous affine function on the set $\{\sigma \in D(A\otimes K)' \mid \sigma(u) = k\}$ because τ is an extreme point here. Hence, f can be extended to a lower semicontinuous affine function, still denoted f, on $D(A \otimes K)'$.

Let $J \supsetneq J_\tau$ be an ideal of $M(A \otimes K)$. By [12], there exists a projection, say p, in $J \setminus J_\tau$. Then $[p] \in \partial D(A \otimes K)$. From the previous theorem it follows that $\widehat{[p]}$ is a strictly positive lower semicontinuous affine function on $D(A \otimes K)'$. Moreover, $f + \widehat{[p]} = \hat{1}$. Again from the previous theorem we know that there is a projection $e \in M(A \otimes K)$ corresponding to f, such that $[e] + [p] = [1]$. Choosing now projections $e' \sim e$, $p' \sim p$, $e' \perp p'$ we get $e' + p' \sim 1$. Since $e' \in J_\tau \subset J$ and $p' \in J$ we get that $J = M(A \otimes K)$, thus J_τ is maximal. ∎

The result in the next lemma was obtained by M. Rørdam ([23]) in a more general case. Here we give another proof using the result from Theorem 4.4 due to G.A. Elliott.

LEMMA 4.8. (Rørdam, [23]) *Let A be a separable, simple* AF *algebra and let τ_1, τ_2 be distinct extreme points in* $\mathrm{EST}(A \otimes K)$. *Then $J_{\tau_1} \neq J_{\tau_2}$.*

Proof. Since τ_1, τ_2 are semi-finite, pick a projection p in $A \otimes K$ such that $0 < \tau_i(p) < \infty$, $i = 1, 2$. Set $u = [p]$ and $S_u = \{\sigma \in D(A \otimes K)' \mid \sigma(u) = 1\}$. Let moreover $\tau_i' = \frac{\tau_i}{\tau_i(p)}$, $i = 1, 2$. Note that τ_1', τ_2' are extreme points in S_u. Let now f_i, $i = 1, 2$, be the strictly positive lower semicontinuous affine functions on $D(A \otimes K)'$ defined by $f_i(\alpha\tau_i') = \alpha$ and $f_i(\sigma) = \infty$ if $\sigma \neq \alpha\tau_i'$, $\alpha \in (0,\infty)$ as in the previous lemma. Then $f_1 \neq f_2$ because τ_1 and τ_2 are not proportional. Hence we get as in the previous lemma projections e_1 in $J_{\tau_1'}$ and e_2 in $J_{\tau_2'}$. Since $e_1 \notin J_{\tau_2'} = J_{\tau_2}$ and $e_2 \notin J_{\tau_1'} = J_{\tau_1}$, we get the conclusion. ∎

THEOREM 4.9. *Let A be a separable, simple, unital AF algebra with at least two extreme points in* $\mathrm{EST}(A \otimes K)$. *Then $Q(A \otimes K)$ is not extremally rich.*

Proof. Let τ_1, τ_2 be the extensions to $M(A \otimes K)$ of the two extremal traces on $A\otimes K$. Then, from the previous lemma, $J_{\tau_1} (=: J_1)$, and $J_{\tau_2} (=: J_2)$ are maximal ideals which strictly contain $A \otimes K$. Note that $J_1 + J_2 = M(A \otimes K)$ and

$$M(A \otimes K)/J_1 \cap J_2 \cong J_1/J_1 \cap J_2 \oplus J_2/J_1 \cap J_2.$$

Claim. Each $M(A \otimes K)/J_i$, $i = 1,2$ is a purely infinite simple C^*-algebra. Towards this end we show that any non-zero projection in $M(A \otimes K)/J_i$ is infinite. Indeed, if $i = 1$ (and similarly for $i = 2$) let p be a non-zero projection in $M(A \otimes K)/J_1$. Since $\mathrm{RR}(M(A \otimes K)) = 0$ by [15], there is a projection $q \in M(A \otimes K) \setminus J_1$ which is the preimage of p. From the proof of Lemma 4.7 there are projections e in J_1 and q' in $M(A \otimes K)$ such that $eq' = 0 = q'e$, $e + q' \sim 1$ and $q' \sim q$. Hence, $p \sim 1$. Since $M(A \otimes K)$ has real rank zero, we deduce that $M(A \otimes K)/J_1 \cong J_2/J_1 \cap J_2$ and $M(A \otimes K)/J_2 \cong J_1/J_1 \cap J_2$ also have real rank zero, hence they are both purely infinite simple by Theorem 1.2 in [31].

Hence we have the following exact sequence:

$$0 \to J_1 \cap J_2/A \otimes K \to Q(A \otimes K) \to J_1/J_1 \cap J_2 \oplus J_2/J_1 \cap J_2 \to 0.$$

Since $(J_1/J_1 \cap J_2) \oplus (J_2/J_1 \cap J_2)$ has a non-isometry and non-co-isometry extreme element in its closed unit ball, this can not be lifted to an extreme element in $Q(A \otimes K)$ because $Q(A \otimes K)$ is prime. Hence we get the assertion from Remark 1.2. ∎

Since by Theorem 1.3 in [26] the corona algebra $Q(A \otimes K)$ is purely infinite under the above assumptions, we see that the requirement about simplicity cannot be eliminated from Lemma 3.3.

Notice that from the proof of the previous theorem we can deduce the following:

REMARK 4.10. Assume we are given: $I_1 \subset I_2 \subset B$ where $I_1, I_2/I_1$ and B/I_2 are purely infinite simple C^*-algebras (and hence extremally rich). Then we cannot necessarily conclude that B is extremally rich.

Indeed, if A in the previous theorem has exactly two extremal traces we may take $\pi(J_1 \cap J_2) \subset \pi(J_1) \subset Q(A \otimes K)$, where π is the canonical quotient map from $M(A \otimes K)$ to $Q(A \otimes K)$. Then

$$\pi(J_1)/\pi(J_1 \cap J_2) \cong (\pi(J_1) + \pi(J_2))/\pi(J_2) = Q(A \otimes K)/\pi(J_2).$$

Since $M(A \otimes K)/J_i \cong Q(A \otimes K)/\pi(J_i)$, $i = 1,2$, we get that $\pi(J_1)/\pi(J_1 \cap J_2)$ and $Q(A \otimes K)/\pi(J_1)$ are purely infinite simple. Also, $\pi(J_1 \cap J_2)$ is purely infinite simple by [17] since $J_1 \cap J_2$ is the smallest ideal strictly containing $A \otimes K$ ([14], Theorem 2), but the algebra $Q(A \otimes K)$ is not extremally rich. Recall that we obtained a positive answer in case of only one purely infinite simple ideal such that the corresponding quotient is also purely infinite simple (Corollary 3.7).

From the above results we get as a corollary the following result, which may be well-known. We include it only because it follows easily from what we have just proved.

COROLLARY 4.11. *Let A be a separable, simple, unital* AF *algebra that admits an order-preserving trace. Then A has a unique trace up to multiplication by positive constants.*

Proof. Suppose that A has another trace which is not proportional with the given order-preserving trace. Then $\mathrm{EST}(A \otimes K)$ has at least two points. Hence, from Theorem 4.9, $Q(A \otimes K)$ is not extremally rich, which is a contradiction to Remark 4.3. ∎

QUESTION 4.12. From the previous result we present the following question (which may be well-known): Is it true that if A is a separable, simple, unital C^*-algebra of real rank zero with an order-preserving trace, then A has a unique trace?

We can also get a similar result to Theorem 4.9 in the case of non-stable, non-unital simple AF algebras under some conditions. We use notation and results of H. Lin ([14]).

Recall from [14] that for a separable, simple, non-elementary AF algebra with dimension group G the set $S = S_u(G)$ represents the homomorphisms $\tau : G \to \mathbf{R}$ such that $\tau(G_+) \geqslant 0$ and $\tau(u) = 1$ for some fixed element u in $G_+ \setminus \{0\}$. The set of extreme points of the convex compact set S is denoted by $E(S)$. With $\mathrm{Aff}(S)$ the set of affine, real continuous functions on S one has a positive homomorphism $\theta : G \to \mathrm{Aff}(S)$ which sends a to $\widehat{a}$ defined by $\widehat{a}(\tau) = \tau(a)$. The image of G under θ is a dense additive subgroup in $\mathrm{Aff}(S)$. Let $F = \{\tau \in S \mid \tau(1) = \infty\}$, where 1 is the unit of $M(A)$ and every τ in S is extended to a trace, still denoted by τ, on $M(A)_+$.

PROPOSITION 4.13. *Let A be a separable, simple, non-unital* AF *algebra. Suppose that $E(S)$ has only finitely many points and $F \cap E(S)$ has at least two points. Then $Q(A)$ is not extremally rich.*

We prove first some lemmas.

Note that if $\tau \in F \cap E(S)$, then J_τ is a maximal, proper, closed, two-sided ideal of $M(A)$ by the proof of Theorem 2 in [14].

Recall that a projection p in A is called full if it is not contained in any proper closed two-sided ideal of A.

LEMMA 4.14. *Let A be a separable simple non-unital* AF *algebra for which $E(S)$ has finitely many points. Let r be a full projection in $M(A)$. Suppose that the number of points in $F \cap E(S)$ is n, $n \geqslant 1$, and let J_i, $i = 1, \ldots, n$ be the proper, maximal ideals of $M(A)$ induced by the n extremal traces in $E(S) \cap F$.*

Then there is a projection g in $\bigcap_{i=1}^{n} J_i$ such that $1-g \lesssim r$.

Proof. We will follow the line of the proof of Theorem 2 in [14]. Since $\mathrm{RR}(M(A)) = 0$ by [15], there are orthogonal projections $\{r_k\}$ in A such that $r \sim \sum r_k$ and $r_k \lesssim q_k - q_{k-1} = f_k$ by Theorem 4.1 (ii) in [29], where $\{q_k\}$ is a subsequence of a fixed increasing approximate unit of projections in A.

Since r is full, $\tau(\sum r_k) = \infty$ for each τ in $E(S) \cap F$. Hence there are finite sets $N_1, N_2, \ldots$ of consecutive integers such that for each $m = 1, 2, \ldots$ we have

$$\tau_i(f_m) < \sum_{k \in N_m} \tau_i(r_k), \quad i = 1, 2, \ldots, n,$$

where $\tau_i \in E(S) \cap F$.

For each m let β_m denote the set of τ in $E(S)$ such that

$$\tau(f_m) > \sum_{k \in N_m} \tau(r_k).$$

Then $\beta_m \subset E(S) \setminus F$.

Since $E(S)$ has only finitely many points, for each m we can define a function h_m on $E(S)$ which satisfies:

$$\tau(f_m) > h_m(\tau) > \tau(f_m) - \sum_{k \in N_m} \tau(r_k), \quad \text{when } \tau \in \beta_m,$$

$$0 < h_m(\sigma) < \min\left\{\left(\frac{1}{2}\right)^m, \sigma(f_m)\right\}, \quad \text{when } \sigma \in E(S) \setminus \beta_m.$$

Since S is a simplex we can extend h_m to an affine function on the whole S. Using now the fact that $\theta(G)$ is dense in $\mathrm{Aff}(S)$ we find for each m a projection $g_m < f_m$ such that:

$$0 < \tau(f_m - g_m) < \sum_{k \in N_m} \tau(r_k), \quad \text{when } \tau \in \beta_m,$$

$$0 < \tau(g_m) < \left(\frac{1}{2}\right)^m, \quad \text{when } \tau \in E(S) \setminus \beta_m.$$

Put $g = \sum_{m=1}^{\infty} g_m$. Note that this is well defined because $(g_m)_m$ are orthogonal, since $(f_m)_m$ are so. Then g is in $\bigcap_{i=1}^{n} J_i$, which by Theorem 2 in [14] is equal to the closure of

$$\{a \in M(A) \mid \tau(a^*a) < \infty, \forall \tau \text{ in } E(S)\}.$$

Since for any m in $\mathbb{N}$ and for any τ in S we have

$$\tau(f_m - g_m) < \sum_{k \in N_m} \tau(r_k),$$

it follows from Proposition 4.1 in [1], that $f_m - g_m \precsim \sum_{k \in N_m} r_k$ for any m. Let v_m be a partial isometry in A such that $v_m^* v_m = f_m - g_m$ and $v_m v_m^* \leqslant \sum_{k \in N_m} r_k$. Put $v = \sum_{m=1}^{\infty} v_m$. Then $v \in M(A)$, $v^* v = 1 - g$ and $vv^* \leqslant r$, hence we get the conclusion. ∎

LEMMA 4.15. *Let A be a separable simple non-unital AF-algebra. Suppose that $E(S)$ has only finitely many points and $F \cap E(S)$ has n elements, $n \geqslant 1$. Let $\{J_i\}_{i=1}^n$ be the distinct proper maximal ideals in $M(A)$ corresponding to the elements of $F \cap E(S)$. Let q be a non-zero projection in J_{i_0} for some $i_0 = 1, \ldots, n$. Then*

$$(1-q)(J_{i_0} \cap J_{i_1} \cap \cdots \cap J_{i_k})(1-q) \neq (1-q)(J_{i_1} \cap \cdots \cap J_{i_k})(1-q),$$

for any distinct numbers $\{i_1, \cdots, i_k\} \subset \{1, 2, \ldots, n\} \backslash \{i_0\}$.

Proof. Note that by the proof of Theorem 2 in [14], $J_{i_1} \cap \cdots \cap J_{i_k}$ is not contained in J_{i_0}. Hence, $M(A) = J_{i_1} \cap \cdots \cap J_{i_k} + J_{i_0}$.

Suppose that the equality holds in the conclusion. Then

$$\begin{aligned}
&(1-q)M(A)(1-q)/(1-q)J_{i_0}(1-q) \\
&\quad \cong (1-q)(J_{i_1} \cap \cdots \cap J_{i_k})(1-q)/(1-q)(J_{i_0} \cap J_{i_1} \cap \cdots \cap J_{i_k})(1-q) \\
&\quad \cong 0.
\end{aligned}$$

Thus we would get

$$(1-q)M(A)(1-q) = (1-q)J_{i_0}(1-q),$$

which contradicts the fact that $1 - q \notin J_{i_0}$. ∎

Let φ_i be the canonical quotient map from $M(A)$ to $M(A)/J_i$.

LEMMA 4.16. *Under the same assumptions as in the previous lemma, for $i = 1, \ldots, n$, $n \geqslant 1$, let p in $M(A)/J_i$ be a proper projection. Then there is a full projection q in $M(A)$ such that $\varphi_i(q) = p$.*

Proof. Since $M(A)$ has real rank zero, any non-zero projection in each of the quotients by J_i can be lifted to a non-zero projection in $M(A)$. We may consider the case $i = 1$, since the rest is similar. Let $q \in M(A)$ be a projection such that $\varphi_1(q) = p$. Thus $q \notin J_1$.

In case that $n = 1$ it follows from Theorem 2 in [14] that q is full.

In case that $n \geqslant 2$, if $q \notin J_i$ for any $i = 2, \ldots, n$, then q will be full from the proof of Theorem 2 in [14]. If there is some $i_0 \in \{2, \ldots, n\}$ such that $q \in J_{i_0}$, then from the previous lemma and [12] there exists a projection $e_0 \in (1-q)J_1(1-q)$, but $e_0 \notin J_{i_0}$. Then, $q+e_0 \notin J_{i_0}$, and $\varphi_1(q+e_0) = p$. Repeating the same argument finitely many steps we can find a projection $q' \in M(A)$ such that $q' \notin J_i$ for any $i = 2, \ldots, n$ and $\varphi_1(q') = p$.

Hence q' is full. ∎

PROPOSITION 4.17. *Under the same assumptions as in the above lemma we obtain that $M(A)/J_i$, $i = 1, \ldots, n$, are purely infinite simple C^*-algebras.*

Proof. We already know that $M(A)/J_i$, $1 \leqslant i \leqslant n$ have real rank zero. If we show that any non-zero proper projection in each quotient is infinite, then we are done by Theorem 1.2 in [31].

Indeed, pick a proper projection p in $M(A)/J_1$ (similarly for $i \geqslant 2$). By the previous lemma there is a full projection q in $M(A)$ such that $\varphi_1(q) = p$. Hence, from Lemma 4.14 there is a projection $g \in J_1$ such that $1 - g \precsim q$. Therefore, $\varphi_1(q) = p$ is infinite. ∎

Proof of Proposition 4.13. It follows from Proposition 4.17 and an argument similar to the one in Theorem 4.9. ∎

PROPOSITION 4.18. *If A is a separable simple non-unital* AF *algebra with finitely many extremal traces of which exactly one is infinite, then $Q(A)$ is extremally rich.*

Proof. Note in this case that $M(A)$ has a unique proper closed two-sided ideal, say I, strictly containing A ([14]). For the proof of the proposition we want therefore to use the same method as in Theorem 4.1.

With π the canonical quotient map from $M(A)$ to $Q(A)$ we have that $\pi(I)$ is purely infinite simple by Theorem 1.3 in [26]. Therefore it suffices now to prove that $M(A)/I$ is as well purely infinite simple. But this follows from Proposition 4.17. ∎

REMARK 4.19. In order to complete the discussion of extremal richness of $Q(A)$, for A a separable simple non-unital AF algebra with finitely many extremal traces, note that if A has no infinite extremal traces, then $Q(A)$ is a purely infinite simple C^*-algebra ([14], [26]), hence extremally rich, whereas if A has exactly one infinite extremal trace, then $Q(A)$ is extremally rich by the previous proposition. However, if A has infinitely many extremal traces and is not stable, we do not know that I_τ is a maximal ideal in $M(A)$ for an extremal trace τ on A with $\tau(1) = \infty$. If we did, we would get the same conclusion as in Theorem 4.9.

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