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NORM-CLOSED BIMODULES OF NEST ALGEBRAS

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Abstract. We study the support and essential support functions of a norm-closed bimodule of a nest algebra. An allowable support function pair determines a maximal bimodule. There is also a natural candidate for the minimal bimodule for a given support function pair. We determine precisely when this candidate is the minimal element. In the other cases, this module is still the intersection of all bimodules with a given support function pair, but it is not in this class itself.

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The purpose of this paper is to study norm-closed bimodules of nest algebras. A nest $\mathcal{N}$ is a complete totally ordered sublattice of subspaces of a (separable) Hilbert space $\mathcal{H}$. The nest algebra $\mathcal{T}(\mathcal{N})$ consists of all operators in $\mathcal{B}(\mathcal{H})$ leaving each element of $\mathcal{N}$ invariant. The reader is referred to [2] for the basics of these algebras. For our purposes, a bimodule will be a closed subspace of $\mathcal{B}(\mathcal{H})$ which is closed under left and right multiplication by elements of $\mathcal{T}(\mathcal{N})$.

In [6], Erdos and Power studied weakly closed bimodules of nest algebras. They associated to each bimodule, weakly closed or not, a certain *support function*; and they showed that this function determined the weak operator topology closure of the bimodule. It is also intrinsic in their work that the intersection of this weakly closed bimodule with $\mathfrak{K}$, the compact operators, is the minimal bimodule with given support. (See Proposition 1.2 in Section 1.)

The new ingredient we add to this picture is the notion of an essential support function reflecting where non-compact operators are supported in a bimodule. We characterize the possible *support function pairs* which can occur as the support and essential support of a closed bimodule. It will be shown that there is a natural maximal element with this given support function pair. Under certain necessary conditions on the support function pair, there is also a minimal bimodule in this class. In the remaining cases, the intersection of all bimodules with given support function pair is determined.

The results simplify considerably for nests of *infinite multiplicity.* These are nests which are similar to infinite ampliations of themselves. By the Similarity Theorem ([1]), these are precisely the nests with no finite rank atoms. In this case, the only conditions on a support function pair (Φ, Ψ) are the fact that they are monotone increasing, $\Phi(0) = 0$, $\Psi \leqslant \Phi$ and the support function Φ is left-continuous. There is a minimal bimodule for this pair precisely when the essential support function Ψ is also left-continuous.

The first paper on ideals of nest algebras was Ringrose ([10]) in which he characterizes the radical. A similar radical type condition plays a role in this paper in characterizing the elements of the minimal bimodules associated to an essential support function. There are many other papers on ideals of nest algebras, especially dealing with the question of generators (which will not be much of an issue here). For example, the deep paper of Orr ([8]) characterizes generators of a continuous nest algebra as a two sided ideal. This characterization has been crucial in making the continuous nests the most tractable subclass for many algebraic questions.

If $N \in \mathcal{N}$, let N_- denote the supremum of all strictly smaller elements of the nest. And let N_+ denote the infimum of all strictly greater elements. It is evident that $N\mathcal{B}(\mathcal{H})N_-^\perp$ and $N_+\mathcal{B}(\mathcal{H})N^\perp$ are contained in $\mathcal{T}(\mathcal{N})$. It will be convenient to have the notation $\mathcal{N}_f$ for those elements N of the nest such that $N - N_-$ is a finite dimensional atom. Let $\mathcal{N}_\infty = \mathcal{N} \setminus \mathcal{N}_f$. We will also adopt the notation $\mathcal{N}_\ell$ to consist of those elements $N \in \mathcal{N}$ which are the supremum of a strictly increasing sequence N_k for which $N_k - N_0$ is finite rank for all $k \geqslant 0$. Dually, let $\mathcal{N}_r$ denote the set of elements of $\mathcal{N}$ which are the infimum of a decreasing sequence N_k such that $N_0 - N_k$ is finite rank for all $k \geqslant 0$. These notions will be needed to define the minimal bimodules associated to a support function pair (Φ, Ψ).

1. SUPPORT FUNCTIONS

Let $\mathcal{N}$ be a nest on a separable Hilbert space, and let $\mathcal{T}(\mathcal{N})$ denote the nest algebra of all operators leaving the elements of $\mathcal{N}$ invariant. If $\mathfrak{J}$ is a $\mathcal{T}(\mathcal{N})$-bimodule, its *support function* is defined by

$$\mathrm{Supp}(\mathfrak{J}) : N \mapsto \overline{\mathfrak{J}N} \quad \text{for} \quad N \in \mathcal{N}.$$

It is readily apparent that $\overline{\mathfrak{J}N}$ is invariant for $\mathcal{T}(\mathcal{N})$ and it is a routine exercise to show that it is a subspace; hence it is an element of $\mathcal{N}$. It is easily verified that $\mathrm{Supp}(\mathfrak{J})$ is monotone increasing, left-continuous and $\mathrm{Supp}(\mathfrak{J})(0) = 0$.

Hence we define a *support function* on $\mathcal{N}$ to be a monotone increasing function $\Phi\colon \mathcal{N} \to \mathcal{N}$ such that $\Phi(0) = 0$. If Φ is a support function on $\mathcal{N}$, then define its $\mathcal{T}(\mathcal{N})$-bimodule by

$$\mathrm{Bim}(\Phi) = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq \Phi(N) \text{ for all } N \in \mathcal{N}\}.$$

Our choice of notation is designed to mimic the operations of Lat and Alg. The main result about weak operator topology closed bimodules is:

THEOREM 1.1. ([6]) *If $\mathfrak{J}$ is a $\mathcal{T}(\mathcal{N})$-bimodule and Φ is a support function on $\mathcal{N}$, then* $\mathrm{Bim}(\mathrm{Supp}(\mathfrak{J})) = \overline{\mathfrak{J}}^{\mathrm{WOT}}$ *and* $\mathrm{Supp}(\mathrm{Bim}(\Phi)) = \Phi_-$*, the greatest left-continuous support function on $\mathcal{N}$ such that $\Phi_- \leqslant \Phi$.*

In view of this theorem, call a support function *admissable* if it is left-continuous, i.e. Φ such that $\Phi = \Phi_-$. Left-continuity will also be an important issue for essential support functions.

Clearly $\mathrm{Bim}(\Phi)$ is the maximal $\mathcal{T}(\mathcal{N})$-bimodule with support Φ. It is implicit in [6] that there is a minimal bimodule with support Φ. For the sake of completeness, we give a proof of this, using the fact that every nest algebra contains a norm one approximate identity for the compact operators consisting of finite rank operators in $\mathcal{T}(\mathcal{N})$ ([5]).

PROPOSITION 1.2. *Let Φ be an (admissable) support function on $\mathcal{N}$ and $\mathfrak{J}$ a norm-closed $\mathcal{T}(\mathcal{N})$-bimodule with* $\mathrm{Supp}(\mathfrak{J}) = \Phi$*. Then $\mathfrak{J}$ contains* $\mathrm{Bim}(\Phi) \cap \mathfrak{K}$*. Since* $\mathrm{Supp}\big(\mathrm{Bim}(\Phi) \cap \mathfrak{K}\big) = \Phi$*, this is the minimal closed bimodule contained in $\mathfrak{J}$ with support function Φ.*

Proof. Let $\mathfrak{J}' = \mathfrak{J} \cap \mathfrak{K}$; and let F_n be a norm one approximate identity for the compact operators consisting of finite rank operators in $\mathcal{T}(\mathcal{N})$. For each $J \in \mathfrak{J}$, $F_n J \in \mathfrak{J}'$ for $n \geqslant 1$ and $J = \mathop{\text{SOT--lim}}\limits_{n\to\infty} F_n J$. Thus $\overline{\mathfrak{J}'N}$ contains JN for every $J \in \mathfrak{J}$ and $N \in \mathcal{N}$. Hence $\mathrm{Supp}(\mathfrak{J}') = \mathrm{Supp}(\mathfrak{J})$.

Let $\Phi = \mathrm{Supp}(\mathfrak{J})$. To see that $\mathfrak{J}' = \mathrm{Bim}(\Phi) \cap \mathfrak{K}$, recall that each finite rank element of a nest algebra can be written as the sum of rank one elements of the nest algebra ([5]). Applying this to F_n, it follows that each element $J \in \mathfrak{J}'$ is contained in the closed span of its rank one elements because it is the norm limit of $F_n J$. So let $R = xy^*$ denote a rank one element of $\mathrm{Bim}(\Phi)$. Let N denote the greatest element of $\mathcal{N}$ which is orthogonal to y. Then for every $M > N$ in $\mathcal{N}$, $xy^* M = \mathbb{C}x$. Hence $\Phi(M)$ contains x; whence x belongs to $L = \bigwedge\limits_{M>N} \Phi(M)$.

For any $\varepsilon > 0$, there are vectors z and v in $N^\perp$ such that $\|y - z\| < \varepsilon$ and $\|v\| = 1$ so that vz^* belongs to $\mathcal{T}(\mathcal{N})$. Indeed, if $N_+ > N$, then $z = y$ and any unit vector $v \in N_+ \ominus N$ will suffice. While if $N_+ = N$, one may choose $M > N$ such that $\|My\| < \varepsilon$. So $z = M^\perp y$ and any unit vector v in $M \ominus N$ works.

Now as v is orthogonal to N, it follows that $\overline{\mathcal{T}(\mathcal{N})v}$ is an element of $\mathcal{N}$ strictly larger that N; whence $\overline{\mathfrak{J}v}$ contains L. Choose an element $J \in \mathfrak{J}$ such that $\|Jv - x\| < \varepsilon$. A routine estimate shows that $J(vz^*) = (Jv)z^*$ is close to xy^* for small ε. ∎

Now let us define the *essential support function* of a bimodule $\mathfrak{J}$ as

$$\mathrm{Supp}^{\mathrm{e}}(\mathfrak{J}) : N \mapsto \bigwedge\{L \in \mathcal{N} : L^\perp \mathfrak{J} N \subset \mathfrak{K}\} \quad \text{for} \quad N \in \mathcal{N}$$

and the *support function pair* of $\mathfrak{J}$ to be the pair $(\mathrm{Supp}(\mathfrak{J}), \mathrm{Supp}^{\mathrm{e}}(\mathfrak{J}))$.

Conversely, define an *essential support function* as a support function $\Psi : \mathcal{N} \to \mathcal{N}$ with the following two properties:

(A) $\mathrm{rank}(N_2 - N_1) < \infty$ implies $\Psi(N_2) = \Psi(N_1)$.

(B) $\Psi(N) \in \mathcal{N}_f$ implies $\Psi(N) = \Psi(N)_+$.

If in addition Φ is a left-continuous support function on $\mathcal{N}$ with $\Psi \leqslant \Phi$ and

(B′) $\Psi(N) \in \mathcal{N}_f$ implies $\Psi(N) = \Psi(N)_+ < \Phi(N)$,

then we call (Φ, Ψ) an *admissible pair of support functions.* The next two propositions justify this terminology.

Proposition 1.3. *For a $\mathcal{T}(\mathcal{N})$-bimodule $\mathfrak{J}$,* $(\mathrm{Supp}(\mathfrak{J}), \mathrm{Supp}^{\mathrm{e}}(\mathfrak{J}))$ *is an admissible pair of support functions. In other words,* $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{J})$ *and* $\mathrm{Supp}(\mathfrak{J})$ *satisfy properties* (A), (B) *and* (B′).

Proof. Property (A) follows from the equality

$$L^\perp \mathfrak{J} N_2 = L^\perp \mathfrak{J} N_1 + L^\perp \mathfrak{J}(N_2 - N_1),$$

as the second summand consists of finite rank operators.

Similarly for (B), if $L_0 := \mathrm{Supp}^{\mathrm{e}}(\mathfrak{J})(N)$ belongs to $\mathcal{N}_f$, then

$$L_{0-}^{\perp}\mathfrak{J}N = L_0^{\perp}\mathfrak{J}N + (L_0 - L_{0-})\mathfrak{J}N.$$

So this would consist of compact operators if $L_0^{\perp}\mathfrak{J}N \subset \mathfrak{K}$. Hence it must be the case that $L^{\perp}\mathfrak{J}N$ is contained in $\mathfrak{K}$ for all $L > L_0$, but not for L_0 itself. Therefore

$$L_0 = \mathrm{Supp}^{\mathrm{e}}(\mathfrak{J})(N) = \bigvee_{L > L_0} L = L_{0+}.$$

Since $L_0^{\perp}\mathfrak{J}N$ is non-zero, it follows that $\mathrm{Supp}(\mathfrak{J})(N) > \mathrm{Supp}^{\mathrm{e}}(\mathfrak{J})(N)$, verifying property (B′). ∎

If Ψ is a support function on $\mathcal{N}$, then define

$$\mathrm{Bim}^{\mathrm{e}}(\Psi) := \{X \in \mathcal{B}(\mathcal{H}) : L^{\perp}XN \in \mathfrak{K} \text{ for all } L, N \in \mathcal{N} \text{ such that } L_+ > \Psi(N)\}.$$

This is a $\mathcal{T}(\mathcal{N})$-bimodule since it is clearly a closed subspace and if $A, B \in \mathcal{T}(\mathcal{N})$, then for all $N, L \in \mathcal{N}$ such that $L_+ > \Psi(N)$, one has that

$$L^{\perp}AXBN = L^{\perp}A(L^{\perp}XN)BN$$

is compact. Since $\mathrm{Bim}^{\mathrm{e}}(\Psi)$ contains all compact operators,

$$\mathrm{Supp}(\mathrm{Bim}^{\mathrm{e}}(\Psi))(N) = \begin{cases} 0 & \text{if } N = 0, \\ \mathcal{H} & \text{otherwise.} \end{cases}$$

Moreover, it is evident from the definition that $\mathrm{Bim}^{\mathrm{e}}(\Psi)$ contains all bimodules $\mathfrak{J}$ with $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{J}) \leqslant \Psi$.

PROPOSITION 1.4. *If Ψ is an essential support function, then*

$$\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^{\mathrm{e}}(\Psi)) = \Psi.$$

Proof. Fix $N_0 \in \mathcal{N}$ and set $L_0 = \Psi(N_0)$; and let Φ^{e} be the essential support function of $\mathrm{Bim}^{\mathrm{e}}(\Psi)$. From the definition, $\Phi^{\mathrm{e}} \leqslant \Psi$.

First suppose that $N_0 \in \mathcal{N}_\infty$. When $N_0 = N_{0-}$, choose a sequence $N_k \in \mathcal{N}$ strictly increasing to N_0, and choose unit vectors e_k in $N_{k+1} - N_k$. When $E = N_0 - N_{0-}$ is an infinite rank atom, choose an orthonormal basis e_k for $E\mathcal{H}$.

When $L_0 \in \mathcal{N}_\infty$, choose a basis f_k for $L_0\mathcal{H}$. Then we define a partial isometry by $X = \sum_{k \geqslant 1} f_k e_k^*$. By construction, $X = L_0 X N_0$. Also, XN is finite rank for every $N < N_0$, and $L^{\perp}XN_0$ is not compact for any $L < L_0$. Hence $X \in \mathrm{Bim}^{\mathrm{e}}(\Psi)$ and $\Phi^{\mathrm{e}}(N_0) \geqslant L_0$.

When $L_0 \in \mathcal{N}_f$, property (B) implies that $L_0 = L_{0+}$. Choose a sequence $L_k \in \mathcal{N}$ strictly decreasing to L_0; and choose unit vectors $f_k \in L_k - L_{k+1}$. Then define $X = \sum\limits_{k \geqslant 1} f_k e_k^*$. It is evident that $X = L_0^\perp X N_0$ is not compact but $L^\perp X$ is compact for all $L > L_0$ and XN is compact for all $N < N_0$. Hence $X \in \mathrm{Bim}^{\mathrm{e}}(\Psi)$ and this again demonstrates that $\Phi^{\mathrm{e}}(N_0) \geqslant L_0$.

Finally, suppose that $N_0 \in \mathcal{N}_f$. Let N_1 be the infimum of all $N' < N$ such that $N - N'$ is finite rank. If $N_0 - N_1$ is finite rank, then we have $\Psi(N) = \Psi(N_0)$ and $\Phi^{\mathrm{e}}(N_0) = \Phi^{\mathrm{e}}(N_1)$ by property (A). Thus the result follows from the previous paragraphs applied to N_1. If $N_0 - N_1$ is infinite rank, then $\Psi(N) = \Psi(N_0)$ and $\Phi^{\mathrm{e}}(N) = \Phi^{\mathrm{e}}(N_0)$ for all $N_1 < N \leqslant N_0$. Choose a basis e_k for $N_0 - N_1$ and proceed as above. ∎

We are now able to determine the maximal bimodules associated to any admissible pair of support functions.

Theorem 1.5. *Let $\mathcal{N}$ be a nest, and let (Φ, Ψ) be an admissible pair of support functions on $\mathcal{N}$. Then* $\mathrm{Bim}(\Phi, \Psi) = \mathrm{Bim}(\Phi) \cap \mathrm{Bim}^{\mathrm{e}}(\Psi)$ *is the largest $\mathcal{T}(\mathcal{N})$-bimodule $\mathfrak{J}$ with* $\mathrm{Supp}(\mathfrak{J}) = \Phi$ *and* $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{J}) = \Psi$.

Proof. Since $\mathrm{Bim}(\Phi, \Psi)$ is the intersection of two closed $\mathcal{T}(\mathcal{N})$-bimodules, it is itself a closed bimodule. As $\mathrm{Bim}(\Phi)$ is the largest bimodule with support Φ by Theorem 1.1 and $\mathrm{Bim}^{\mathrm{e}}(\Psi)$ is the largest bimodule with essential support Ψ by Proposition 1.4, it follows that $\mathrm{Bim}(\Phi, \Psi)$ contains all bimodules with the support function pair (Φ, Ψ).

As $\mathrm{Bim}^{\mathrm{e}}(\Psi)$ contains the compact operators, it follows that

$$\mathrm{Bim}(\Phi) \cap \mathfrak{K} \subseteq \mathrm{Bim}(\Phi, \Psi) \subseteq \mathrm{Bim}(\Phi).$$

Hence by Theorem 1.1 and Proposition 1.2, the support function of this bimodule is Φ. Proposition 1.4 shows that $\Phi^{\mathrm{e}} := \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}(\Phi, \Psi))$ satisfies $\Phi^{\mathrm{e}} \leqslant \Psi$. It remains to show that $\Phi^{\mathrm{e}} \geqslant \Psi$. This requires a bit more care than the previous proposition.

Let $N_0 \in \mathcal{N}$, $L_0 = \Psi(N_0)$. First suppose that $L_0 \in \mathcal{N}_\infty$. When $E = N_0 - N_{0-}$ is an infinite rank atom, the operator X constructed above from $E\mathcal{H}$ onto L_0 is the desired operator in $\mathrm{Bim}(\Phi, \Psi)$. This is also the case when there is a decreasing sequence N_k such that $N_0 - N_k$ is finite rank for all $k \geqslant 1$. However, if $N_0 = N_{0-}$, the operator X constructed above need not belong to $\mathrm{Bim}(\Phi)$. But because Φ is left-continuous and $\Phi(N_0) \geqslant \Psi(N_0) = L_0$, given any $L < L_0$, there is an $M < N_0$ such that $\Phi(M) > L$. So if $L_0 - L_{0-}$ is an infinite atom, choose $M < N_0$ so that $\Phi(M) > L_{0-}$; whence $\Phi(M) \geqslant L_0$. Then if we use

$N_1 = M$ in Proposition 1.4, the operator X lies in $\mathrm{Bim}(\Phi, \Psi)$. While if $L_0 = L_{0-}$, choose a sequence L_k strictly increasing to L_0. Then choose a sequence M_k strictly increasing to N_0 such that $\Phi(M_k) > L_k$. Then by choosing unit vectors $e_i \in M_{i+1} \ominus M_i$ and $f_i \in L_i \ominus L_{i-1}$, we obtain $X = \sum_{i \geqslant 1} f_i e_i^*$ in $\mathrm{Bim}(\Phi, \Psi)$ which establishes $\Phi^{\mathrm{e}}(N_0) \geqslant L_0$ as in Proposition 1.4. Finally, when $N_0 - N_1$ is finite rank and N_1 fits the previous case, property (A) implies that $\Psi(N_0) = \Psi(N_1)$ and the operator X already constructed again shows that $\Phi^{\mathrm{e}}(N_0) = \Phi^{\mathrm{e}}(N_1) \geqslant L_0$.

When $L_0 \in \mathcal{N}_f$, property (B) implies that $L_0 = L_{0+}$ and property (B$'$) implies that $\Phi(N_0) = L_1 > L_0$. Let L_k for $k \geqslant 1$ be a strictly decreasing sequence in $\mathcal{N}$ with limit L_0. When $N_0 - N_{0-}$ is an infinite atom, the operator X constructed in Proposition 1.4 lies in $\mathrm{Bim}(\Phi, \Psi)$. When there is a decreasing sequence N_k such that $N_0 - N_k$ is finite rank for all $k \geqslant 1$, properties (A) and (B$'$) show that

$$\Psi(N_0) = \Psi(N_k) < \Phi(N_k) \quad \text{for} \quad k \geqslant 1.$$

It may then be assumed (by dropping the L's to a subsequence) that $L_{k-1} \leqslant \Phi(N_k)$. Let e_k be a unit vector in $N_k - N_{k+1}$ for $k \geqslant 1$. Recall that f_k is a unit vector in $L_k - L_{k+1}$. Set $X = \sum_{k \geqslant 1} f_k e_k^*$ as before. Then it has been arranged that X lies in $\mathrm{Bim}(\Phi)$. The previous proof shows that it also lies in $\mathrm{Bim}^{\mathrm{e}}(\Psi)$ and determines that $\Phi^{\mathrm{e}}(N_0) \geqslant L_0$. Similarly, if $N_0 = N_{0-}$, the left-continuity of Φ shows that there is an $M \in \mathcal{N}$ such that $M < N$ and $\Phi(M) = L_2 > L_0$. Proceeding as before produces the desired result. Finally, the case in which $N_0 - N_1$ is finite rank and N_1 fits one of the previous cases, then property (i) implies that $\Psi(N_0) = \Psi(N_1)$ and the operator X already constructed again shows that $\Phi^{\mathrm{e}}(N_0) = \Phi^{\mathrm{e}}(N_1) \geqslant L_0$. ∎

2. INFINITE MULTIPLICITY

We wish to determine when there is a minimal bimodule for a support function pair (Φ, Ψ); and otherwise determine the intersection of all such bimodules. In this section, we consider an important special case in which the technicalities are reduced. A nest has *infinite multiplicity* if it is similar to the infinite inflation of itself. By the Similarity Theorem ([1]), this happens precisely when every atom is infinite dimensional. For such nests, the properties (A), (B), and (B$'$) are vacuous, so any support function is a valid essential support function.

First we need a simple factorization result.

LEMMA 2.1. *Let $J = L^\perp J N$ be a non-compact element in a $\mathcal{T}(\mathcal{N})$-bimodule $\mathfrak{J}$. Then $\mathfrak{J}$ contains $L_+\mathcal{B}(\mathcal{H})N_-^\perp$.*

Proof. Since $\mathcal{T}(\mathcal{N})$ contains $L_+\mathcal{B}(\mathcal{H})L^\perp$ and $N\mathcal{B}(\mathcal{H})N_-^\perp$, it follows that

$$\mathfrak{J} \supset (L_+\mathcal{B}(\mathcal{H})L^\perp)J(N\mathcal{B}(\mathcal{H})N_-^\perp) = L_+\mathcal{B}(\mathcal{H})N_-^\perp. \quad \blacksquare$$

COROLLARY 2.2. *If $\mathfrak{J}$ is a $\mathcal{T}(\mathcal{N})$-bimodule with essential support function Ψ, then it contains $L\mathcal{B}(\mathcal{H})N_-^\perp$ for every $L, N \in \mathcal{N}$ such that $L_- < \Psi(N)$.*

Proof. From the definition of essential support, there is a non-compact element $J = L_-^\perp J N$ in $\mathfrak{J}$. Thus $\mathfrak{J}$ contains $(L_-)_+\mathcal{B}(\mathcal{H})N_-^\perp$, which contains $L\mathcal{B}(\mathcal{H})N_-^\perp$. $\blacksquare$

In the general situation, there is a more sophisticated factorization, Lemma 3.1, that leads to somewhat larger *rectangles* that live above the support of Ψ. In the infinite multiplicity case, this result is sufficient. In this case, we define a bimodule

$$\mathrm{Bim}^0(\Psi) = \overline{\sum_{\substack{N \in \mathcal{N} \\ L_- < \Psi(N)}} L\mathcal{B}(\mathcal{H})N_-^\perp}.$$

By Corollary 2.2, every bimodule with essential support Ψ contains this one. However, it is easy to verify that $\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi))(N) < \Psi(N)$ whenever Ψ is left-discontinuous at N.

To make the support function of a bimodule equal Φ without changing the essential support, Proposition 1.2 suggests that $\mathrm{Bim}(\Phi) \cap \mathfrak{K}$ be added in. So define

$$\mathrm{Bim}^0(\Phi, \Psi) = \mathrm{Bim}^0(\Psi) + (\mathrm{Bim}(\Phi) \cap \mathfrak{K}).$$

This is a closed bimodule, as is shown in the following lemma.

LEMMA 2.3. *Let Φ be a support function on $\mathcal{N}$ and $\mathfrak{J}$ be a $\mathcal{T}(\mathcal{N})$-bimodule with* $\mathrm{Supp}(\mathfrak{J}) \leqslant \Phi$. *Then*

$$\mathfrak{T} := (\mathrm{Bim}(\Phi) \cap \mathfrak{K}) + \mathfrak{J} = \mathrm{Bim}(\Phi) \cap (\mathfrak{J} + \mathfrak{K})$$

is a closed bimodule with support Φ and essential support $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{J})$.

Proof. The equality $(\mathrm{Bim}(\Phi) \cap \mathfrak{K}) + \mathfrak{J} = \mathrm{Bim}(\Phi) \cap (\mathfrak{J} + \mathfrak{K})$ is easy because $\mathfrak{J}$ is contained in $\mathrm{Bim}(\Phi)$. As $\mathcal{T}(\mathcal{N})$ contains a bounded approximate identity for the compact operators, $\mathfrak{J} \cap \mathfrak{K}$ is weak-$*$ dense in $\mathfrak{J}$. Thus by [3] (see also [2], Theorem 11.6) $\mathfrak{J} \cap \mathfrak{K}$ is an M-ideal in $\mathfrak{J}$ and thus $\mathfrak{J} + \mathfrak{K}$ is norm closed. Hence $\mathfrak{T}$ is the intersection of two closed bimodules, and therefore is itself a closed bimodule.

Because of the inclusions

$$\mathrm{Bim}(\Phi) \cap \mathfrak{K} \subset \mathfrak{T} \subset \mathrm{Bim}(\Phi),$$

it follows from Proposition 1.2 and Theorem 1.1 that the support function of $\mathfrak{T}$ is Φ. Since the addition of compact operators cannot affect the definition of essential support, we obtain that $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{T}) = \mathrm{Supp}^{\mathrm{e}}(\mathfrak{J})$. ∎

Recall that if Ψ is a support function, then Ψ_- denotes the greatest left-continuous support function with $\Psi_- \leqslant \Psi$. Clearly it is defined by

$$\Psi_-(N) = \begin{cases} \Psi(N) & \text{if } N > N_-, \\ \bigvee\limits_{M<N} \Psi(M) & \text{if } N = N_-. \end{cases}$$

The comments above suggest the following result.

THEOREM 2.4. *Let $\mathcal{N}$ be a nest of infinite multiplicity, and let (Φ, Ψ) be an admissible pair of support functions on $\mathcal{N}$. Then $\mathrm{Bim}^0(\Phi, \Psi)$ is a closed $\mathcal{T}(\mathcal{N})$-bimodule with support function pair (Φ, Ψ_-). Moreover, every bimodule with support function pair (Φ, Ψ) contains $\mathrm{Bim}^0(\Phi, \Psi)$.*

Proof. By Lemma 2.3, $\mathrm{Bim}^0(\Phi, \Psi)$ is a closed bimodule. It has support function Φ and essential support function

$$\Phi^{\mathrm{e}} := \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Phi, \Psi)) = \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi)) \leqslant \mathrm{Supp}(\mathrm{Bim}^0(\Psi)) \leqslant \Psi_-.$$

This last inequality follows because

$$\mathrm{Bim}^0(\Psi)N_0 = \overline{\sum_{\substack{N_- < N_0 \\ L_- < \Psi(N)}} L\mathcal{B}(\mathcal{H})(N_0 - N_-)} \subset \Psi_-(N_0)\mathcal{B}(\mathcal{H})N_0.$$

On the other hand, when $N_0 > N_{0-}$, $\mathrm{Bim}^0(\Psi)$ contains $L\mathcal{B}(\mathcal{H})(N_0 - N_{0-})$ for every L with $L_- < \Psi(N_0)$. Therefore if $M < L$, then $M^\perp \mathrm{Bim}^0(\Psi)N_0$ contains $(L - M)\mathcal{B}(\mathcal{H})(N_0 - N_{0-})$, which contains non-compact operators. Thus $\Phi^{\mathrm{e}}(N_0) \geqslant M_+$. Now

$$\bigvee_{\substack{M<L \\ L_- < \Psi(N_0)}} M_+ = \Psi(N_0);$$

whence $\Phi^{\mathrm{e}}(N_0) = \Psi(N_0)$.

The other possibility is that $N_0 = N_{0-}$. In this case, the definition of Ψ_- implies that for each $L < \Psi_-(N_0)$, there is an $N \in \mathcal{N}$ with $N < N_0$ and $\Psi(N) > L$. Thus $\mathrm{Bim}^0(\Psi)$ contains $L\mathcal{B}(\mathcal{H})(N_0 - N)$ and it follows that $\Phi^{\mathrm{e}}(N_0) \geqslant L$. As $L < \Psi_-(N_0)$ was arbitrary, we obtain $\Phi^{\mathrm{e}}(N_0) \geqslant \Psi_-(N_0)$. Therefore $\Phi^{\mathrm{e}} = \Psi_-$.

Suppose that $\mathfrak{J}$ is a $\mathcal{T}(\mathcal{N})$-bimodule with support function pair (Φ, Ψ). By Proposition 1.2, $\mathfrak{J}$ contains $\mathrm{Bim}(\Phi) \cap \mathfrak{K}$. By Corollary 2.2, $\mathfrak{J}$ contains $L\mathcal{B}(\mathcal{H})N_-^\perp$ for all $L, N \in \mathcal{N}$ with $L_- < \Psi(N)$. Consequently, $\mathfrak{J}$ contains $\mathrm{Bim}^0(\Psi)$; whence it contains $\mathrm{Bim}^0(\Phi, \Psi)$. ∎

So we obtain as an immediate consequence that there is a minimal bimodule with support function pair (Φ, Ψ) when Ψ is left-continuous:

Corollary 2.5. *Suppose that* (Φ, Ψ) *is an admissible pair of left-continuous support functions on an infinite multiplicity nest* $\mathcal{N}$. *Then* $\mathrm{Bim}^0(\Phi, \Psi)$ *is the smallest closed* $\mathcal{T}(\mathcal{N})$*-bimodule with support function pair* (Φ, Ψ).

Further, we will show in Theorem 2.8 that $\mathrm{Bim}^0(\Phi, \Psi)$ is the intersection of all closed $\mathcal{T}(\mathcal{N})$-bimodules with support function pair (Φ, Ψ), regardless of whether or not Ψ is left-continuous. We need a characterization of $\mathrm{Bim}^0(\Phi, \Psi)$ which is akin to the Ringrose condition for elements of the radical.

Theorem 2.6. *Let* $\mathcal{N}$ *be a nest of infinite multiplicity, and let* (Φ, Ψ) *be an admissible pair of support functions on* $\mathcal{N}$. *Then the following are equivalent for* $X \in \mathcal{B}(\mathcal{H})$:

(i) X *belongs to* $\mathrm{Bim}^0(\Phi, \Psi)$;

(ii) $X \in \mathrm{Bim}(\Phi)$ *and for each* $P \in \mathcal{N}$ *and* $\varepsilon > 0$, *there are elements* M_0, M_1, L, L_0 *in* $\mathcal{N}$ *such that* $M_{0-} < P < M_1$, $L_{0-} < \Psi(M_0)$, $L_- < \Psi(P)$ *and*

$$\|L_0^\perp X P\|_e < \varepsilon \quad \textit{and} \quad \|L^\perp X M_{1-}\|_e < \varepsilon.$$

When Ψ *is left-continuous, condition* (ii) *may be replaced with the weaker condition:*

(ii′) $X \in \mathrm{Bim}(\Phi)$ *and for each* $P \in \mathcal{N}$ *and* $\varepsilon > 0$, *there are elements* $M > P$ *and* $L_- < \Psi(P)$ *such that* $\|L^\perp X M_-\|_e < \varepsilon$.

Proof. If $X \in \mathrm{Bim}^0(\Phi, \Psi)$, then it can be approximated within ε by a finite sum of the form

$$J = K + \sum_{i=1}^{k} L_i T_i N_{i-}^\perp$$

where $K \in \mathrm{Bim}(\Phi) \cap \mathfrak{K}$, $T_i \in \mathcal{B}(\mathcal{H})$, N_i are distinct elements of $\mathcal{N}$ and $L_i \in \mathcal{N}$ such that $L_{i-} < \Psi(N_i)$. For each $P \in \mathcal{N}$,

(1) let M_0 be the greatest of the N_i's such that $N_{i-} < P$;
(2) let L_0 be the greatest of the L_i's corresponding to these N_i's;
(3) let M_1 be the least of the N_i's strictly greater than P; and
(4) let L be the greatest of the L_i's corresponding to the $N_i \leqslant P$.

Then $L_{0-} < \Psi(M_0)$, $L_- < \Psi(P)$, $L_0^\perp J P = L_0^\perp K P$ and $L^\perp J M_{1-} = L^\perp K M_{1-}$. So

$$\|L_0^\perp X P\|_e \leqslant \|X - J\| + \|K\|_e < \varepsilon$$

and

$$\|L^\perp X M_{1-}\|_e \leqslant \|X - J\| + \|K\|_e < \varepsilon.$$

Notice that two terms are needed only when P is one of the N_i's and $L > \Psi(P)_-$; for otherwise there is an $M' \in \mathcal{N}$ such that $M_0 \leqslant M' < P$ and $\Psi(M') > L_-$.

If (ii$'$) holds and Ψ is left-continuous, then given $P \in \mathcal{N}$ and $\varepsilon > 0$, let $M > P$ and $L_- < \Psi(P)$ be provided so that $\|L^\perp X M_-\|_e < \varepsilon$. By left-continuity, there is an $M_{0-} < P$ such that $L_- < \Psi(M_0)$. Setting $L_0 = L$ yields (ii).

Now suppose that the technical condition (ii) holds. Given $\varepsilon > 0$, each $P \in \mathcal{N}$ is contained in an open interval (M_{0-}, M_1) with the hypotheses of (ii) satisfied. When $P > P_-$, always choose $M_0 = P$ and $L_0 = L$, so that for these terms we have the single norm estimate $\|L_0 X M_{1-}\|_e < \varepsilon$. By compactness, one obtains a finite open subcover. From this, it is easy to extract two sequences of elements of $\mathcal{N}$,

$$0 = M_0 < M_1 < \cdots < M_k = I \quad \text{and} \quad 0 = L_0 < L_1 < \cdots < L_k$$

such that $L_{i-} < \Psi(M_i)$ and $\|L_i^\perp X M_{i+1-}\|_e < \varepsilon$ for $0 \leqslant i < k$ and $\|L_k^\perp X\|_e < \varepsilon$.

These terms are the lower triangular blocks complementary to the upper triangular space $\mathfrak{T} = \sum_{i=1}^{k} L_i \mathcal{B}(\mathcal{H}) M_{i-}^\perp$. There is a formula for the distance to the compact perturbations of a nest algebra ([3]) which is elementary for a finite block form such as this. The formula implies that

$$\operatorname{dist}(X, \mathfrak{T} + \mathfrak{K}) = \max\{\|L_k^\perp X\|_e,\ \|L_i^\perp X M_{i+1-}\|_e,\ 0 \leqslant i < k\} < \varepsilon.$$

Thus there exists $K \in \mathfrak{K}$ and $J \in \mathfrak{T} \subset \operatorname{Bim}^0(\Psi)$ such that $\|X - (J + K)\| < \varepsilon$. As X belongs to $\operatorname{Bim}(\Phi)$,

$$\operatorname{dist}(K, \operatorname{Bim}(\Phi)) = \operatorname{dist}(J + K, \operatorname{Bim}(\Phi)) < \varepsilon.$$

Since K is compact and $\mathcal{T}(\mathcal{N})$ contains an approximate identity for $\mathfrak{K}$, it follows that there is a compact operator $C \in \operatorname{Bim}(\Phi) \cap \mathfrak{K}$ such that $\|K - C\| < \varepsilon$. Therefore $J + C$ is an element of $\operatorname{Bim}^0(\Phi, \Psi)$ within 2ε of X. Since $\varepsilon > 0$ is arbitrary and $\operatorname{Bim}^0(\Phi, \Psi)$ is closed, it follows that $X \in \operatorname{Bim}^0(\Phi, \Psi)$. ∎

To complete the picture in the infinite multiplicity case, we determine the intersection of all bimodules with a given support function pair. This then shows that Corollary 2.5 determines all cases in which there is a minimal element. We need a method for constructing bimodules with left-discontinuous essential supports. For notational convenience, define

$$\mathfrak{R}(L, P) = \overline{\sum_{M < L} M \mathcal{B}(\mathcal{H}) P^\perp} \qquad \text{and} \qquad \Psi_{L,P}(N) = \begin{cases} 0 & N < P, \\ L & N \geqslant P. \end{cases}$$

Let $\mathfrak{B}(\Phi, \Psi)$ denote the collection of all bimodules with support function pair (Φ, Ψ).

LEMMA 2.7. *Suppose that L and P belong to a nest $\mathcal{N}$ of infinite multiplicity, and $P = P_-$. Let Φ be a left-continuous support function on $\mathcal{N}$ such that $\Phi(P) \geqslant L$. Then there is a $\mathcal{T}(\mathcal{N})$-bimodule $\mathfrak{X}$ with support function pair $(\Phi, \Psi_{L,P})$. When $L = L_-$, it can also be arranged that*

$$\lim_{M \uparrow L} \|M^\perp X\|_e = 0 \quad \textit{for all} \quad X \in \mathfrak{X}. \tag{2.1}$$

Furthermore, $\bigcap\{\mathfrak{X} : \mathfrak{X} \in \mathfrak{B}(\Phi, \Psi_{L,P})\} = \mathfrak{R}(L,P) + (\mathrm{Bim}(\Phi) \cap \mathfrak{K})$. Indeed, for $T \in \mathrm{Bim}(\Phi)$,

$$\begin{aligned}\sup_{\mathfrak{X} \in \mathfrak{B}(\Phi, \Psi_{L,P})} \mathrm{dist}(T, \mathfrak{X}) &= \mathrm{dist}\big(T, \mathfrak{R}(L,P) + (\mathrm{Bim}(\Phi) \cap \mathfrak{K})\big) \\ &= \max\big\{\|TP\|_e, \|L^\perp T\|_e, \lim_{M \uparrow L} \|M^\perp T\|_e\big\}.\end{aligned}$$

Proof. Choose a sequence $P_k \in \mathcal{N}$ strictly increasing to P; and choose unit vectors $e_k \in P_{k+1} \ominus P_k$. If $L - L_-$ is an infinite rank atom, choose an orthonormal basis f_k for it. Otherwise, choose a sequence $L_k \in \mathcal{N}$ strictly increasing to L and unit vectors $f_k \in L_{k+1} \ominus L_k$. Since Φ is left-continuous and $\Phi(P) \geqslant L$, we may assume that $\Phi(P_k) > L_k$ for each $k \geqslant 1$. Define $X = \sum\limits_{k \geqslant 1} f_k e_k^*$; and let

$$\mathfrak{X} = \overline{\mathrm{span}\{\mathcal{T}(\mathcal{N}) X \mathcal{T}(\mathcal{N})\}} + (\mathrm{Bim}(\Phi) \cap \mathfrak{K})$$

be the bimodule it generates plus the minimal bimodule for Φ. By arrangement, X belongs to $\mathrm{Bim}(\Phi)$; $X = LXP$; and XN is finite rank for each $N < P$. Thus $\mathfrak{X}N$ is contained in $\mathfrak{K}$ for $N < P$, as is $L^\perp \mathfrak{X}$. But $M^\perp XP$ is not compact if $M < L$. Thus $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{X})(N)$ is 0 for $N < P$ and L for $N \geqslant P$. As $\mathrm{Bim}(\Phi) \cap \mathfrak{K} \subset \mathfrak{X} \subset \mathrm{Bim}(\Phi)$, it follows that $\mathrm{Supp}(\mathfrak{X}) = \Phi$.

When $L = L_-$, let $\mathfrak{X}_k$ be the bimodule constructed as above for the pair L_k and P; and set $\mathfrak{X} = \overline{\sum\limits_{k \geqslant 1} \mathfrak{X}_k}$. Since for any $X \in \mathfrak{X}_k$, $L_k^\perp X$ is compact, it follows that (2.1) holds for X; and hence for any finite linear combination of such terms. Taking limits verifies this condition for every element of $\mathfrak{X}$.

If $\mathfrak{X}$ is any bimodule with $\mathrm{Supp}(\mathfrak{X}) = \Phi$, then it contains $\mathrm{Bim}(\Phi) \cap \mathfrak{K}$ by Proposition 1.2. If in addition, $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{X}) = \Psi_{L,P}$, then Corollary 2.2 shows that it also contains $\mathfrak{R}(L,P)$.

Suppose that $T \in \mathrm{Bim}(\Phi)$. If $L^\perp T$ is not compact, then T does not lie in the bimodule $\mathfrak{X}$ constructed above and $\mathrm{dist}(T, \mathfrak{X}) \geqslant \|L^\perp T\|_e$. Moreover, when $L = L_-$, (2.1) holds for $\mathfrak{X}$ and thus

$$\mathrm{dist}(T, \mathfrak{X}) \geqslant \lim_{M \uparrow L} \|M^\perp T\|_e.$$

Similarly, if TN is not compact for some $N < P$, then T fails to belong to $\mathfrak{X}$ and

$$\operatorname{dist}(T, \mathfrak{X}) \geqslant \sup_{N<P} \|TN\|_e.$$

Finally, suppose that TP is not compact. It is a routine exercise to find orthonormal vectors x_n so that the Tx_n are pairwise orthogonal and $\|Tx_n\| \geqslant \|TP\|_e - 1/n$, and to find projections P_k strictly increasing to P so that $x_n = P_k x_n$ for $n \leqslant k^2$. The bimodule $\mathfrak{X}$ constructed using this sequence P_k contains a dense set of elements of the form

$$Y = \sum_{i=1}^{m} A_i X B_i + K$$

where A_i and B_i belong to $\mathcal{T}(\mathcal{N})$ and K is compact. Given $\varepsilon > 0$, let p be a positive integer such that the $(p+1)$-st singular value of K is less than ε. Consider

$$YP_k = \sum_{i=1}^{m} A_i (XP_k) B_i P_k + KP_k.$$

As X is constructed using the sequence P_k, we have $\operatorname{rank}(XP_k) = k-1$. Hence the right hand side is the sum of an operator of rank at most km and KP_k, which is a rank p operator plus one of norm less than ε. So $(T-Y)|\operatorname{span}\{x_n : k \leqslant n \leqslant k^2\}$ is bounded below by $\|TP\|_e - 1/k - \varepsilon$ on a subspace of dimension at least $k(k-m-1)-p$. Therefore $\operatorname{dist}(T, \mathfrak{X}) \geqslant \|TP\|_e$.

Evidently, $T \in \operatorname{Bim}(\Phi)$ belongs to $\mathfrak{R}(L,P) + (\operatorname{Bim}(\Phi) \cap \mathfrak{K})$ if and only if TP and $L^\perp T$ are compact and (2.1) holds. Therefore the intersection of all bimodules in $\mathfrak{B}(\Phi, \Psi)$ is indeed $\mathfrak{R}(L,P) + (\operatorname{Bim}(\Phi) \cap \mathfrak{K})$. It remains to evaluate the distance to this bimodule. Since the given estimate is clearly a lower bound, suppose that $T \in \operatorname{Bim}(\Phi)$ and

$$\max\{\|TP\|_e, \|L^\perp T\|_e, \lim_{M \uparrow L} \|M^\perp T\|_e\} = r.$$

As in the case of nest algebras themselves ([2], Theorem 12.1), given $T \in \operatorname{Bim}(\Phi)$, there is a compact operator in $\operatorname{Bim}(\Phi) \cap \mathfrak{K}$ such that $\|T-K\| = \|T\|_e$. In particular, applying this to $T - LTP^\perp$, we perturb T by such an element K so that

$$\max\big\{\|(T-K)P\|, \|L^\perp(T-K)\|\big\} \leqslant r.$$

Now the Parrott–Davis–Kahan–Weinberger Lemma ([9], [4]) (see also [2], Lemma 9.1) shows that there is an element $X \in L\mathcal{B}(\mathcal{H})P^\perp$ such that $\|T-K-X\| \leqslant r$. When $L = L_-$, choose $M < L$ sufficiently large that $\|M^\perp T\|_e < r + \varepsilon$. Apply the same procedure to M in the place of L to obtain a compact element $K \in \operatorname{Bim}(\Phi) \cap \mathfrak{K}$ and an element $X \in M\mathcal{B}(\mathcal{H})P^\perp \subset \mathfrak{R}(L,P)$ such that

$$\|T-K-X\| \leqslant \max\big\{\|TP\|_e, \|M^\perp T\|_e\big\} < r + \varepsilon.$$

Then it follows that $\operatorname{dist}(T, \mathfrak{R}(L,P) + (\operatorname{Bim}(\Phi) \cap \mathfrak{K})) \leqslant r$. ∎

Now we can prove the main result of this section concerning minimal bimodules in the infinite multiplicity setting.

Theorem 2.8. *Let $\mathcal{N}$ be a nest of infinite multiplicity, and let (Φ, Ψ) be an admissible pair of support functions on $\mathcal{N}$. Then the intersection of all closed $\mathcal{T}(\mathcal{N})$-bimodules with support function pair (Φ, Ψ) is $\mathrm{Bim}^0(\Phi, \Psi)$. In particular, this set of modules has a minimal element only when Ψ is left-continuous.*

Proof. Theorem 2.4 shows that $\mathrm{Bim}^0(\Phi, \Psi)$ is contained in every $\mathcal{T}(\mathcal{N})$-bimodule with support function pair (Φ, Ψ). However, by the same result, the essential support function of this bimodule will be Ψ_-. Theorem 1.5 shows that the intersection is contained in $\mathrm{Bim}(\Phi, \Psi) = \mathrm{Bim}(\Phi) \cap \mathrm{Bim}^e(\Psi)$.

Natural candidates for bimodules with support function pair (Φ, Ψ) are readily obtained as follows. Enumerate the points of left-discontinuity of Ψ as $\{P_k\}$. For each k, let $\mathfrak{X}_k$ be the bimodule constructed in Lemma 2.7 for the pair of projections $(\Psi(P_k), P_k)$. Then let

$$\mathfrak{T} = \overline{\mathrm{Bim}^0(\Phi, \Psi) + \sum_k \mathfrak{X}_k}.$$

Each $T \in \mathfrak{T}$ can be approximated by a finite sum $A + \sum_{k=1}^{m} X_k$ consisting of an element $A \in \mathrm{Bim}^0(\Phi, \Psi)$ and terms $X_k \in \mathfrak{X}_k$.

Suppose that $T \in \mathcal{B}(\mathcal{H})$ lies in the intersection of all bimodules with support function pair (Φ, Ψ). If T does not belong to $\mathrm{Bim}^0(\Phi, \Psi)$, then Theorem 2.6 implies that there is an element $P \in \mathcal{N}$ and an $\varepsilon > 0$ so that either

(a) for every $M_- < P$ and $L_- < \Psi(M)$ in $\mathcal{N}$, one has $\|L^\perp T P\|_e > \varepsilon$, or

(b) for every $M > P$ and $L_- < \Psi(P)$, one has $\|L^\perp T M_-\|_e > \varepsilon$.

In particular, T belongs to the bimodule $\mathfrak{T}$ constructed above. So after subtracting off a term $A \in \mathrm{Bim}^0(\Phi, \Psi)$, T can be approximated within $\varepsilon > 0$ by a finite sum $\sum_{k=1}^{m} X_k$, where $X_k \in \mathfrak{X}_k$.

Consider case (a) first. Notice that if $P_k > P$, $\mathfrak{X}_k P$ is contained in $\mathfrak{K}$; and if $P_k < P$, $\Psi(P_k)^\perp \mathfrak{X}$ is contained in $\mathfrak{K}$ and by (1) of Lemma 2.7, when $\Psi(P_k) = \Psi(P_k)_-$, there is an $L < \Psi(P_k) \leqslant \Psi(P)$ such that $\|L^\perp X_k\|_e$ is arbitrarily small. Thus the only term that can have any effect on (a) is the term X_{k_0} when $P = P_{k_0}$. Then by Lemma 2.7, condition (a) implies that there is another bimodule $\mathfrak{X}'_{k_0}$ corresponding to the pair of projections $(\Psi(P), P)$ such that $\mathrm{dist}(X_{k_0}, \mathfrak{X}'_{k_0}) > \varepsilon$. Let $\mathfrak{T}'$ be the bimodule in $\mathfrak{B}(\Phi, \Psi)$ obtained by using $\mathfrak{X}'_{k_0}$ instead of $\mathfrak{X}_{k_0}$. The same analysis then shows that

$$\mathrm{dist}(T, \mathfrak{T}') \geqslant \mathrm{dist}(X_{k_0}, \mathfrak{X}'_{k_0}) > \varepsilon$$

which is a contradiction.

Case (b) is similar. If $P_k \leqslant P$, there is an L with $L_- < \Psi(P_k) \leqslant \Psi(P)$ such that $\|L^\perp X_k\|_e$ is arbitrarily small. And if $P_k > P_+$, taking $P < M < P_k$ yields that $X_k M$ is compact. Finally, when $P < P_+ = P_{k_0}$, proceed as in case (a) to reach a contradiction.

Thus T lies in $\mathrm{Bim}^0(\Phi, \Psi)$ as claimed. ∎

3. MINIMAL BIMODULES

In this section, we determine the minimal bimodule with a given support function pair (Φ, Ψ) for an arbitrary nest, when it exists. We need several more sophisticated factorization results along the lines of Lemma 2.1.

LEMMA 3.1. *Let $\mathfrak{J}$ be a $\mathcal{T}(\mathcal{N})$-bimodule. Suppose that $J = L_0^\perp J N$ is a non-compact element of $\mathfrak{J}$ and L_∞ is the supremum of a strictly increasing sequence L_k such that $L_k - L_0$ are finite rank for all $k \geqslant 0$. Then $\mathfrak{J}$ contains $L_\infty \mathcal{B}(\mathcal{H}) N_-^\perp$.*

Dually, if $J = L^\perp J N_0$ is a non-compact element in $\mathfrak{J}$ and N_∞ is the infimum of a strictly decreasing sequence N_k such that $N_0 - N_k$ are finite rank for all $k \geqslant 0$, then $\mathfrak{J}$ contains $L_+ \mathcal{B}(\mathcal{H}) N_\infty^\perp$.

Finally, suppose that $J = L_0^\perp J N_0$ is a non-compact element in $\mathfrak{J}$, L_∞ is the supremum of a strictly increasing sequence L_k and N_∞ is the infimum of a strictly decreasing sequence N_k such that $L_k - L_0$ and $N_0 - N_k$ are finite rank for all $k \geqslant 0$. Then $\mathfrak{J}$ contains $L_\infty \mathcal{B}(\mathcal{H}) N_\infty^\perp$.

Proof. Suppose that we are in the setting of the first paragraph. We may assume that $\|J\|_e > 1$. It follows that $L_k^\perp J \neq 0$ for all $k \geqslant 1$. Hence $\mathrm{Supp}(\mathfrak{J})(N) \geqslant \bigvee_{k \geqslant 0} L_k = L_\infty$. By Proposition 1.2, $\mathfrak{J}$ contains $L_\infty \mathfrak{K} N_-^\perp$.

For any finite k, there is a unit vector $x \in N$ such that $\|Jx\| > 1$ and Jx is orthogonal to the finite dimensional space $L_k - L_0$ (and thus is orthogonal to L_k). Moreover, this vector x may be chosen to be orthogonal to any given finite dimensional subspace of N. Recursively, construct an orthonormal sequence x_n in N and an increasing sequence of positive integers k_n so that

(i) $Jx_n = y_n$ is orthogonal to L_{k_n},

(ii) $\|L_{k_{n+1}}^\perp y_n\| < 2^{-n}$, and

(iii) $z_n = L_{k_{n+1}} y_n$ has $\|z_n\| > 1$.

Note that the z_n's are pairwise orthogonal and bounded by $\|J\|$.

Let $A \in L_\infty \mathcal{B}(\mathcal{H}) N_-^\perp$. Choose an orthonormal basis f_n for $N_-^\perp$. Let $u_n = Af_n$; choose integers j_n so that $\|L_{j_n}^\perp u_n\| < 2^{-n}$; and set $v_n = L_{j_n} u_n$. Then by dropping to a subsequence of the sequence k_n, it may be supposed that $k_n \geqslant j_n$ for every $n \geqslant 1$. Then

$$T = \sum_{n \geqslant 1} x_n f_n^* \quad \text{and} \quad S = \sum_{n \geqslant 1} v_n z_n^* \|z_n\|^{-2}$$

are bounded elements of $\mathcal{T}(\mathcal{N})$. Moreover,

$$JT = \sum_{n \geqslant 1} y_n f_n^* = \sum_{n \geqslant 1} z_n f_n^* + K$$

where $K = \sum\limits_{n \geqslant 1} (L_{k_{n+1}}^\perp y_n) f_n^*$ is compact. Thus

$$A - SJT = \sum_{n \geqslant 1} u_n f_n^* - \sum_{n \geqslant 1} v_n f_n^* - SK = \sum_{n \geqslant 1} (L_{j_n}^\perp u_n) f_n^* - SK$$

which is compact in $L_\infty \mathfrak{K} N_-^\perp$. Consequently, A belongs to $\mathfrak{J}$.

The second case of the lemma is analogous to the first. So consider the combined case in which $J = L_0^\perp J N_0$ is a non-compact element in $\mathfrak{J}$, L_∞ is the supremum of a strictly increasing sequence L_k and N_∞ is the infimum of a strictly decreasing sequence N_k such that $L_k - L_0$ and $N_0 - N_k$ are finite rank for all $k \geqslant 0$. Proceeding as above, we can find unit vectors x_n in N_0, vectors z_n in $L_0^\perp$ and two increasing sequences j_n and k_n with $x_n \in N_{j_n} - N_{j_{n+1}}$ and $z_n \in L_{k_{n+1}} - L_{k_n}$ such that $\sum\limits_{k \geqslant 1} \|Jx_n - z_n\| < \infty$. Then as before, any operator A in $L_\infty \mathcal{B}(\mathcal{H}) N_\infty^\perp$ will factor through the operator J modulo a compact operator; and thus lies in $\mathfrak{J}$. ∎

Recall that $\mathcal{N}_\ell$ ($\mathcal{N}_r$) denotes the set of elements of $\mathcal{N}$ which are the limit of a strictly increasing (decreasing) sequence L_k such that $L_k - L_0$ are all finite rank. In this case, property (A) implies that $\Psi(L_k) = \Psi(L_0)$ for all $k \geqslant 1$. Define

$$\Psi_+(N) = \inf_{M_+ > N} \Psi(M).$$

Corollary 3.2. *Let $\mathfrak{J}$ be a $\mathcal{T}(\mathcal{N})$-bimodule with essential support Ψ. Then for every $L, N \in \mathcal{N}$, $\mathfrak{J}$ contains*

(i) *$L\mathcal{B}(\mathcal{H})N_-^\perp$ when $L_- < \Psi(N)$;*
(ii) *$\Psi(N)\mathcal{B}(\mathcal{H})N_-^\perp$ when $\Psi(N) \in \mathcal{N}_\ell$;*
(iii) *$L\mathcal{B}(\mathcal{H})N^\perp$ when $L_- < \Psi_+(N)$ and $N \in \mathcal{N}_r$;*
(iv) *$\Psi_+(N)\mathcal{B}(\mathcal{H})N^\perp$ when $\Psi_+(N) \in \mathcal{N}_\ell$ and $N \in \mathcal{N}_r$.*

Proof. Item (i) is just Corollary 2.2.

(ii) If $\Psi(N) \in \mathcal{N}_\ell$, then there is a strictly increasing sequence L_k such that $L_k - L_0$ is finite rank for all $k \geqslant 1$ and $\bigvee_{k\geqslant 0} L_k = \Psi(N)$. As $L_0 < \Psi(N)$, there is an element $J = L_0^\perp J N$ in $\mathfrak{J}$ which is non-compact. Thus by Lemma 3.1, $\mathfrak{J}$ contains $\Psi(N)\mathcal{B}(\mathcal{H})N_-^\perp$.

(iii) Suppose that $N \in \mathcal{N}_r$; say N is the infimum of a strictly decreasing sequence N_k such that $N_0 - N_k$ is finite rank for all $k \geqslant 1$. When $L < \Psi_+(N) = \Psi(N_0)$, there is a non-compact element $J = L^\perp J N_0$ in $\mathfrak{J}$. Hence by Lemma 3.1, $\mathfrak{J}$ contains $L\mathcal{B}(\mathcal{H})N^\perp$.

(iv) Likewise, if $N \in \mathcal{N}_r$ and $\Psi_+(N) \in \mathcal{N}_\ell$, then the arguments of case (ii) and (iii) combine to yield a non-compact element $J = L_0^\perp J N_0$ in $\mathfrak{J}$. Hence $\Psi_+(N)\mathcal{B}(\mathcal{H})N^\perp$ belongs to $\mathfrak{J}$ by Lemma 3.1. ∎

This corollary allows us to define a candidate for a lower bound for the bimodules with essential support Ψ. Define the bimodule $\mathrm{Bim}^0(\Psi)$ to be the norm closure of the sum

$$\sum_{\substack{N\in\mathcal{N}\\ L_-<\Psi(N)}} L\mathcal{B}(\mathcal{H})N_-^\perp + \sum_{\Psi(N)\in\mathcal{N}_\ell} \Psi(N)\mathcal{B}(\mathcal{H})N_-^\perp + \sum_{\substack{N\in\mathcal{N}_r\\ L_-<\Psi_+(N)}} L\mathcal{B}(\mathcal{H})N^\perp + \sum_{\substack{N\in\mathcal{N}_r\\ \Psi_+(N)\in\mathcal{N}_\ell}} \Psi_+(N)\mathcal{B}(\mathcal{H})N^\perp.$$

However, it is easy to verify that $\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi))(N) < \Psi(N)$ when Ψ is left-discontinuous at N. This inequality also occurs when $\Psi(N) \in \mathcal{N}_f$.

Combining the observations so far, we have:

Lemma 3.3. *Let $\mathcal{N}$ be a nest and let (Φ, Ψ) be an admissible pair of support functions. Then the bimodule*

$$\mathrm{Bim}^0(\Phi,\Psi) := (\mathrm{Bim}(\Phi)\cap\mathfrak{K}) + \mathrm{Bim}^0(\Psi) = \mathrm{Bim}(\Phi)\cap(\mathrm{Bim}^0(\Psi)+\mathfrak{K})$$

is a closed bimodule with support Φ and essential support at most Ψ. Moreover, every bimodule with support function pair (Φ,Ψ) contains $\mathrm{Bim}^0(\Phi,\Psi)$.

Proof. By Lemma 2.3, $\mathrm{Bim}^0(\Phi,\Psi)$ is a closed bimodule. It has support function Φ and essential support function

$$\Phi^{\mathrm{e}} := \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Phi,\Psi)) = \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi)) \leqslant \mathrm{Supp}(\mathrm{Bim}^0(\Psi)) \leqslant \Psi.$$

If $\mathfrak{J}$ is a $\mathcal{T}(\mathcal{N})$-bimodule with support function pair (Φ,Ψ), then $\mathfrak{J}$ contains $\mathrm{Bim}^0(\Psi)$ by Corollary 3.2. ∎

Now consider the question of recovering the essential support function Ψ from $\operatorname{Bim}^0(\Psi)$. We begin with the support of a simple *rectangular* bimodule. Define

$$\mathfrak{R}(L_0, N_0) = \begin{cases} \overline{\sum_{L_- < L_0} L\mathcal{B}(\mathcal{H})N_0^\perp} & \text{if } L_0 \notin \mathcal{N}_\ell, \\ L_0\mathcal{B}(\mathcal{H})N_0^\perp & \text{if } L_0 \in \mathcal{N}_\ell. \end{cases}$$

LEMMA 3.4. *Let $L_0, N_0 \in \mathcal{N}$ and let L_1 denote the infimum of all elements $L \in \mathcal{N}$ such that $L_0 - L$ is finite rank. Then the essential support function of $\mathfrak{R}(L_0, N_0)$ is*

$$\Phi^{\mathrm{e}}(N) = \begin{cases} 0 & \textit{if } N \leqslant N_0 \textit{ or } N - N_0 \textit{ is finite rank,} \\ L_1 & \textit{otherwise.} \end{cases}.$$

Proof. If $N \leqslant N_0$ or $N - N_0$ is finite rank, then $\mathfrak{R}(L_0, N_0)N$ is zero or consists of finite rank operators, and thus $\Phi^{\mathrm{e}}(N) = 0$. If $N > N_0$ and $N - N_0$ is infinite rank, then for $M \leqslant L_0$, we have

$$M^\perp \mathfrak{R}(L_0, N_0)N = (L_0 - M)\mathcal{B}(\mathcal{H})(N - N_0) \quad \text{if} \quad L \in \mathcal{N}_\ell$$

and

$$M^\perp \mathfrak{R}(L_0, N_0)N = \sum_{L < L_0} (L - M)\mathcal{B}(\mathcal{H})(N - N_0) \quad \text{if} \quad L \in \mathcal{N} \setminus \mathcal{N}_\ell.$$

This is contained in the compact operators precisely when $L - M$ is finite rank for $L = L_0$ or all $L < L_0$ in the two cases. When $L_0 = L_{0-}$, the only M which qualifies is L_0 $(= L_1)$. When $L_0 > L_{0-}$, the infimum of such M's is L_1. Thus $\Phi^{\mathrm{e}}(N) = L_1$. ∎

This lemma points out a difficulty in achieving a given essential support function using the rectangles $\mathfrak{R}(L, N)$ that live *below* the function Ψ. If $\Psi(N_0) = L_0 = L_{0+}$ belongs to $\mathcal{N}_f$, then the essential support of $\operatorname{Bim}^0(\Psi)$ at N_0 will be strictly less than L_0.

EXAMPLE 3.5. Let $\{N_t : 0 \leqslant t \leqslant 1\}$ be the Volterra nest on $L^2(0,1)$. Consider the nest on $\mathcal{H} = L^2(0,1) \oplus \mathbb{C} \oplus L^2(0,1)$ given by $L_t = N_t \oplus 0 \oplus 0$ and $M_t = L^2(0,1) \oplus \mathbb{C} \oplus N_t$ for $0 \leqslant t \leqslant 1$. Define a function

$$\Psi(N) = \begin{cases} 0 & N = 0, \\ M_0 & 0 < N \leqslant M_0, \\ M_1 & N > M_0. \end{cases}$$

Then Ψ is a left-continuous function satisfying the properties (A) and (B) and

$$\mathrm{Bim}^0(\Psi) = \overline{\sum_{t>0} M_0\mathcal{B}(\mathcal{H})L_t^\perp + \sum_{t<1} M_t\mathcal{B}(\mathcal{H})M_0^\perp}.$$

A simple computation shows that the essential support function is given by

$$\Phi^{\mathrm{e}}(N) = \begin{cases} 0 & N = 0, \\ L_1 & 0 < N \leqslant M_0, \\ M_1 & N > M_0. \end{cases}$$

EXAMPLE 3.6. Another difficulty is that the essential support of $\mathrm{Bim}^0(\Psi)$ also need not be left-continuous. Consider a nest $\mathcal{N} = \{0, L_k, M_k, N_k, \mathcal{H} : k \geqslant 0\}$ where L_0 is infinite dimensional, $L_k - L_0$, $M_k - M_0$ and $N_k - N_0$ are k-dimensional for $k \geqslant 0$, $M_0 = \bigvee_{k\geqslant 0} L_k$, $N_0 = \bigvee_{k\geqslant 0} M_k$ and $\mathcal{H} = \bigvee_{k\geqslant 0} N_k$. Define Ψ_j for $1 \leqslant j \leqslant 4$ by

N	$\Psi_1(N)$	$\Psi_2(N)$	$\Psi_3(N)$	$\Psi_4(N)$
$N = 0$	0	0	0	0
$L_0 \leqslant N < M_0$	L_0	L_0	L_0	L_0
$M_0 \leqslant N < N_0$	M_0	L_0	L_0	L_0
$N_0 \leqslant N < \mathcal{H}$	N_0	M_0	L_0	L_0
$N = \mathcal{H}$	$\mathcal{H}$	N_0	M_0	L_0

Then a calculation shows that $\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi_j)) = \Psi_{\min\{j+1,4\}}$. The function Ψ_4 is the largest left-continuous function less than Ψ_1 satisfying the conclusions of the following lemma.

LEMMA 3.7. *Let Ψ be a function on a nest $\mathcal{N}$ such that* $\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi)) = \Psi$. *Then Ψ is a left-continuous function such that property* (A) *holds and* $\Psi(N) \in \mathcal{N}_\infty$ *for each $N \in \mathcal{N}$.*

Proof. By Proposition 1.3, Ψ must satisfy properties (A) and (B).

If Ψ were left-discontinuous at $N_0 = N_{0-}$, then let $L_0 = \bigvee_{N<N_0} \Psi(N)$. It is evident from the definition of $\mathrm{Bim}^0(\Psi)$ that $\mathrm{Bim}^0(\Psi)N_0 = L_0\mathrm{Bim}^0(\Psi)N_0$. Hence

$$\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi))(N_0) \leqslant \mathrm{Supp}(\mathrm{Bim}^0(\Psi))(N_0) \leqslant L_0 < \Psi(N_0).$$

So Ψ must be left-continuous.

Also, if $A = \Psi(N_0) - \Psi(N_0)_-$ is a non-zero finite rank atom, then

$$\Psi(N_0)_-^{\perp}\mathrm{Bim}^0(\Psi)N_0 \subset A\mathcal{B}(\mathcal{H})N_0 \subset \mathfrak{K}.$$

Hence $\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi))(N_0) \leqslant \Psi(N_0)_- < \Psi(N_0)$. Thus $\Psi(N) \in \mathcal{N}_\infty$. In particular, property (B) is vacuous for these functions. ∎

The main result of this section characterizes the minimal bimodule with support function pair (Φ, Ψ) when it exists.

Theorem 3.8. *Suppose that Ψ is a left-continuous support function on $\mathcal{N}$ satisfying property* (A) *and $\Psi(N) \in \mathcal{N}_\infty$ for each $N \in \mathcal{N}$. Then* $\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi)) = \Psi$. *Thus for functions of this type, the bimodule* $\mathrm{Bim}^0(\Phi, \Psi)$ *is the minimal bimodule with support function pair* (Φ, Ψ).

Proof. Let Φ^{e} denote the essential support function of $\mathrm{Bim}^0(\Psi)$. Fix $N_0 \in \mathcal{N}$ and set $L_0 = \Psi(N_0)$.

Suppose that $N_0 = N_{0-}$ is not in $\mathcal{N}_\ell$. Then $N_0 - N$ is infinite rank for each $N < N_0$. Now $\mathrm{Bim}^0(\Psi)$ contains $\mathfrak{R}(\Psi(N), N)$. So by Lemma 3.4, $\Phi^{\mathrm{e}}(N_0) \geqslant M$ where M is the infimum of elements in $\mathcal{N}$ differing from $\Psi(N)$ by a finite rank. By hypothesis however, $M = \Psi(N)$. Therefore, by left-continuity,

$$\Phi^{\mathrm{e}}(N_0) \geqslant \bigvee_{N<N_0} \Psi(N) = \Psi(N_0).$$

Next suppose that $N_0 \in \mathcal{N}_\ell$ and let N_k be a sequence increasing to N_0 such that $N_k - N_1$ is finite rank for all $k \geqslant 1$. By property (A), $\Psi(N_k) = \Psi(N_1)$ for all $k \geqslant 1$. Whence by left-continuity, $L_0 = \Psi(N_0) = \Psi(N_k)$ for all $k \geqslant 1$. Therefore $\mathrm{Bim}^0(\Psi)$ contains $\mathfrak{R}(L_0, N_1)$. So by Lemma 3.4, $\Phi^{\mathrm{e}}(N_0) \geqslant L_1$ where L_1 is the infimum of elements differing from L_0 by a finite rank. By hypothesis, $L_1 = L_0$; so $\Phi^{\mathrm{e}}(N_0) = L_0$.

If $N_0 - N_{0-}$ is infinite rank, $\mathrm{Bim}^0(\Psi)$ contains $\mathfrak{R}(L_0, N_{0-})$. Again by Lemma 3.4 and the hypothesis on L_0, it follows that $\Phi^{\mathrm{e}}(N_0) = \Psi(N_0)$.

Finally, when $N_0 - N_{0-}$ is finite rank, let N_k denote the maximal decreasing sequence of elements in $\mathcal{N}$ with $N_0 - N_k$ finite rank. Either this list contains a least element N_{k_0} or it has an infimum N_∞. In the first case, N_{k_0} is one of the earlier cases. So by property (A),

$$\Phi^{\mathrm{e}}(N_0) = \Phi^{\mathrm{e}}(N_{k_0}) = \Psi(N_{k_0}) = \Psi(N_0).$$

In the latter case, N_∞ belongs to $\mathcal{N}_r$. Therefore $\mathrm{Bim}^0(\Psi)$ contains $\mathfrak{R}(L_0, N_\infty)$. So again, $\Phi^{\mathrm{e}}(N_0) = \Psi(N_0)$.

Lemma 3.3 shows that every bimodule with support function pair (Φ, Ψ) contains $\mathrm{Bim}^0(\Phi, \Psi)$. Since

$$\mathrm{Bim}^0(\Psi) \subset \mathrm{Bim}^0(\Phi, \Psi) \subset \mathrm{Bim}^0(\Phi, \Psi) + \mathfrak{K},$$

it follows that

$$\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Phi, \Psi)) = \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi)) = \Psi.$$

So this is the minimal bimodule in the class $\mathfrak{B}(\Phi, \Psi)$. ∎

A simple recursive argument shows what happens to an arbitrary function Ψ through repeated use of the operations Bim^0 and $\mathrm{Supp}^{\mathrm{e}}$.

Corollary 3.9. *Given a function Ψ from $\mathcal{N}$ into itself, define a decreasing net indexed on the ordinals by $\Psi_1 = \Psi$, $\Psi_{\alpha+1} = \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi_\alpha))$ and $\Psi_\alpha = \bigwedge_{\beta<\alpha} \Psi_\beta$ for limit ordinals. This net converges to the function Ψ_-, the greatest left-continuous function less than Ψ satisfying the hypotheses of Theorem* 3.8.

Proof. Lemma 3.7 shows that this net will continue to strictly decrease if it does not satisfy the hypotheses of Theorem 3.8. On the other hand, at every stage, $\Psi_\alpha \geqslant \Psi_-$ and hence

$$\mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi_\alpha)) \geqslant \mathrm{Supp}^{\mathrm{e}}(\mathrm{Bim}^0(\Psi_-)) = \Psi_-.$$

Thus, by a cardinality argument, it follows that the net is eventually constant; and therefore must equal Ψ_-. ∎

This suggests a need to understand the auxillary function Ψ_- associated to Ψ which is the greatest left-continuous function smaller than Ψ which satisfies property (A) and such that $\Psi(N) \in \mathcal{N}_\infty$ for each $N \in \mathcal{N}$. Say that an interval (M, N) of $\mathcal{N}$ is of *ordinal type* if it has the order structure of an ordinal and all its atoms are finite dimensional. The elements of this interval may be parametrized as N_α for all ordinals $\alpha < \alpha_0$. By property (A), $\Psi_-(N_{\alpha+1}) = \Psi_-(N_\alpha)$ for each α. And by left-continuity, each limit ordinal β must have

$$\Psi_-(N_\beta) = \bigvee_{\alpha<\beta} \Psi_-(N_\alpha).$$

Hence by transfinite induction, it follows that Ψ_- is constant on each ordinal interval. Look at Example 3.6 again to see this phenomenon exhibited.

The union of an increasing union of intervals of ordinal type need not be ordinal because there may be a left limit point. Nevertheless, each ordinal interval is contained in a maximal interval which is the increasing union of ordinal intervals. Call this a super-ordinal interval. Distinguish the two cases as being of ordinal type or not. The ordinal case has an initial element $N \in \mathcal{N}_\infty$ such that no interval $[M, N]$ has ordinal type for $M < N$. An element $N \in \mathcal{N}$ such that $N = N_-$ and such that no interval $[M, N]$ has ordinal type for $M < N$ will be called a *left limit of non-ordinal type*. We define a map Γ which takes each element in a maximal super-ordinal interval to its left endpoint; and takes other points to themselves.

Because Ψ_- cannot take values L such that $L \in \mathcal{N}_f$, we also need to consider super-ordinal* intervals to be the complements of the super-ordinal intervals of the complementary nest. Let Ω denote the map which sends each element of a super-ordinal* interval to the infimum of that interval; and other elements to themselves. It must be the case that $\Omega(N) \in \mathcal{N}_\infty$; and this is the greatest element less than or equal to N with this property. Notice that the range of Ω is closed under left limits. The following lemma is easy, and is left to the interested reader.

Lemma 3.10. *Given an increasing function Ψ of $\mathcal{N}$ into itself, the greatest left-continuous function less than Ψ satisfying the hypotheses of Theorem* 3.8 *is given by*

$$\begin{aligned}
\Psi_-(N) &= \begin{cases} 0 & \text{if } N = 0, \\ \Omega\Psi(N) & \text{if } N - N_- \text{ is infinite rank}, \\ \bigvee\limits_{M<N} \Omega\Psi\Gamma(M) & \text{if } N = N_- \text{ is a left limit of non-ordinal type}, \\ \bigwedge\limits_{L>\Gamma(N)} \Omega\Psi(L) & \text{if } N \text{ belongs to a non-ordinal type super-ordinal} \\ & \qquad \text{interval}, \\ \Psi_-\Gamma(N) & \text{if } N \text{ belongs to an ordinal type super-ordinal} \\ & \qquad \text{interval}. \end{cases} \\
&= \bigwedge_{L_+>\Gamma(N)} \bigvee_{M_-<L} \Omega\Psi(M).
\end{aligned}$$

4. THE GENERAL CASE

In this section, we determine the intersection of all bimodules associated to a support function pair (Φ, Ψ) in the general case.

Example 4.1. Let $\mathcal{H}$ be the Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ and $\mathcal{N}$ be the nest whose non-trivial elements are $N_k = \operatorname{span}\{e_n : n \leqslant k\}$ for $k \in \mathbb{Z}$.

There are only three possible essential support functions, i.e., increasing functions from $\mathcal{N}$ to itself satisfying properties (A) and (B); namely the function $\mathbf{0}$ that sends all $N \in \mathcal{N}$ to 0, the function Ψ that sends $\mathcal{H}$ to itself and all other $N \in \mathcal{N}$ to 0, and the function $\mathbf{1}$ that sends all $N \in \mathcal{N} \setminus \{0\}$ to $\mathcal{H}$.

The first and third functions are left-continuous, and give bimodules whose supports are easy to understand. $\operatorname{Bim}(\Phi, \mathbf{0})$ consists of all operators T in $\operatorname{Bim}(\Phi)$ such that $N_k^{\perp} T$ is compact for all k, or equivalently for $k = 0$. Therefore

$$\operatorname{Bim}(\Phi, \mathbf{0}) = \operatorname{Bim}(\Phi) \cap (N_0 \mathcal{B}(\mathcal{H}) + \mathfrak{K}).$$

The ideal $\operatorname{Bim}(\Phi) \cap \mathfrak{K}$ is the minimal ideal with this support pair by Proposition 1.2.

If the essential support is $\mathbf{1}$, then so is the support. Clearly, $\operatorname{Bim}(\mathbf{1}, \mathbf{1}) = \mathcal{B}(\mathcal{H})$. In fact, this is the only bimodule with essential support $\mathbf{1}$. Indeed, $0 \in \mathcal{N}_r$ and $\mathbf{1}_+(0) = \mathcal{H}$ belongs to $\mathcal{N}_\ell$. Hence by Corollary 3.2 (iv), any bimodule with essential support $\mathbf{1}$ contains all of $\mathcal{B}(\mathcal{H})$.

We turn to the second function, Ψ. It is easy to verify that $\operatorname{Bim}^0(\Psi) = 0$; and thus $\operatorname{Bim}^0(\Phi, \Psi) = \operatorname{Bim}(\Phi) \cap \mathfrak{K}$. For convenience, let $\Phi = \operatorname{Id}$ be the identity function on $\mathcal{N}$. We will show that the intersection of all bimodules with support function pair $(\operatorname{Id}, \Psi)$ is $\mathfrak{K} \cap \mathcal{T}(\mathcal{N})$. Hence the essential support function of this intersection is $\mathbf{0}$, the greatest left-continuous function less than Ψ.

Let P be the diagonal projection onto the span of $\{e_{2^n} : n \geqslant 0\}$. Let $\mathfrak{J}$ be the norm-closed ideal generated by P and $\mathcal{T}(\mathcal{N}) \cap \mathfrak{K}$. Then $\operatorname{Supp}(\mathfrak{J}) = \Phi$. By property (A), $\operatorname{Supp}^{\mathrm{e}}(\mathfrak{J})(N_k) = 0$ for all k. As $N_k^{\perp} P$ is non-compact for each k, $\operatorname{Supp}^{\mathrm{e}}(\mathfrak{J})(\mathcal{H}) = \mathcal{H}$. Thus the essential support function is Ψ.

Let $U = \sum\limits_{n \geqslant 1} e_{-n} e_n^*$ be the partial isometry that sends e_n to e_{-n} for $n > 0$. We claim that $U \notin \mathfrak{J}$. If it were, we could find elements $A_n, B_n \in \mathcal{T}(\mathcal{N})$ and $K \in \mathcal{T}(\mathcal{N}) \cap \mathfrak{K}$ so that

$$\Big\| U - \Big(K + \sum_{n=1}^{m} A_n P B_n \Big) \Big\| < \frac{1}{3}.$$

Choose k so large that the k-th singular value of K is less than 1/3. Choose an integer p and let E be the projection onto the the span of $\{e_n : |n| \leqslant 2^p\}$. Then EPE has rank p. Since E is semi-invariant for $\mathcal{T}(\mathcal{N})$, we have

$$X := E\Big(K + \sum_{n=1}^{m} A_n P B_n\Big)E = EKE + \sum_{n=1}^{m} EA_n EPEB_n E.$$

Each term in the sum has rank at most p and hence this sum has rank at most mp. And EKE has at most k singular values greater than 1/3. Hence at most $mp + k$ singular values of X are greater than 1/3. Since $\|EUE - X\| < 1/3$, the $(mp + k + 1)$-st singular value of EUE is less than 2/3. On the other hand, EUE is a partial isometry of rank 2^p, so the first 2^p singular values of EUE are all 1, a contradiction for large p.

Clearly this argument can be extended. Indeed, given any non-compact operator in $\mathcal{T}(\mathcal{N})$, the ideal generated by this operator contains a partial isometry U of the form $\sum\limits_{n\geqslant 1} e_{-k_n} e_{k_n}^*$ for some increasing sequence k_n. Adapting the above argument, we can find a diagonal projection P so that the ideal generated by P and $\mathcal{T}(\mathcal{N}) \cap \mathfrak{K}$ does not contain U. As before, this ideal has support $\Phi = \mathrm{Id}$ and essential support Ψ. Thus, the intersection of all ideals with support function pair (Id, Ψ) is $\mathfrak{K} \cap \mathcal{T}(\mathcal{N})$.

The following Ringrose style characterization of $\mathrm{Bim}^0(\Phi, \Psi)$ can be proved by the same method as Theorem 2.6. To avoid many cases, define a set-valued function ψ by defining $\psi(N)$ according to the following table:

	$\Psi(N) = \Psi(N)_- \notin \mathcal{N}_\ell$	$\Psi(N) \in \mathcal{N}_\ell$ or $\Psi(N) > \Psi(N)_-$
$N \notin \mathcal{N}_r$	$[0, \Psi(N))$	$[0, \Psi(N)]$
$N \in \mathcal{N}_r$	$[0, \Psi_+(N))$	$[0, \Psi_+(N)]$

Theorem 4.2. *Let (Φ, Ψ) be an admissible pair of support functions on a nest $\mathcal{N}$. Then the following are equivalent for $X \in \mathcal{B}(\mathcal{H})$:*

(i) *X belongs to* $\mathrm{Bim}^0(\Phi, \Psi)$;

(ii) $X \in \mathrm{Bim}(\Phi)$ *and for each $P \in \mathcal{N}$ and $\varepsilon > 0$, there are elements M_0, M_1, L, L_0 in $\mathcal{N}$ with $M_{0-} < P < M_1$, $L_0 \in \psi(M_0)$ and $L \in \psi(P)$ such that*

$$\|L_0^\perp X P\|_e < \varepsilon \quad \textit{and} \quad \|L^\perp X M_{1-}\|_e < \varepsilon.$$

A few more constructions of bimodules are needed. The first is a variant on Lemma 2.7 to deal with left-discontinuities of Ψ. The second is the analogue for right-discontinuities of Ψ. Recall that

$$\Psi_{L,P} = \begin{cases} 0 & \text{if } N < P, \\ L & \text{if } N \geqslant P. \end{cases} \quad \text{and} \quad \mathfrak{R}(L,P) = \begin{cases} \overline{\sum_{N_- < L} N\mathcal{B}(\mathcal{H})P^\perp} & \text{if } L \notin \mathcal{N}_\ell, \\ L\mathcal{B}(\mathcal{H})P^\perp & \text{if } L \in \mathcal{N}_\ell. \end{cases}$$

LEMMA 4.3. *Suppose that L and P belong to a nest $\mathcal{N}$ and Φ is a left-continuous support function on $\mathcal{N}$. Assume that either*

(i) *$P = P_-$, $L \in \mathcal{N}_\infty$, and $\Phi(P) \geqslant L$, or*

(ii) *$P \in \mathcal{N}_\infty$, $L = L_+ \in \mathcal{N}_f$ and $\Phi(P) > L$.*

Then there is a $\mathcal{T}(\mathcal{N})$-bimodule $\mathfrak{X}$ with $\operatorname{Supp}(\mathfrak{X}) = \Phi$ *and* $\operatorname{Supp}^{\mathrm{e}}(\mathfrak{X}) = \Psi_{L,P}$. *When $L = L_- \notin \mathcal{N}_\ell$, it can also be arranged that*

$$\lim_{M \uparrow L} \|M^\perp X\|_e = 0 \quad \textit{for all} \quad X \in \mathfrak{X}.$$

Furthermore, $\bigcap\{\mathfrak{X} : \mathfrak{X} \in \mathfrak{B}(\Phi, \Psi_{L,P})\} = \mathfrak{R}(L, P_-) + (\operatorname{Bim}(\Phi) \cap \mathfrak{K})$. Indeed, for $T \in \operatorname{Bim}(\Phi)$,

$$\begin{aligned} \sup_{\mathfrak{X} \in \mathfrak{B}(\Phi, \Psi_{L,P})} \operatorname{dist}(T, \mathfrak{X}) &= \operatorname{dist}\big(T, \mathfrak{R}(L, P_-) + (\operatorname{Bim}(\Phi) \cap \mathfrak{K})\big) \\ &= \max\big\{\|TP\|_e, \|L^\perp T\|_e, \lim_{M \uparrow L} \|M^\perp T\|_e\big\} \end{aligned}$$

where the limit term occurs only when $L = L_- \notin \mathcal{N}_\ell$.

Proof. When conditions (i) hold for L, P, and Φ, the proof is almost word for word the same as that of Lemma 2.7, with a minor observation about points in $\mathcal{N}_\ell$. So we omit the details.

Suppose that the conditions (ii) hold. If $P = P_-$, choose a sequence $P_k \in \mathcal{N}$ strictly increasing to P; and choose unit vectors $e_k \in P_{k+1} \ominus P_k$. If instead, $A = P \ominus P_-$ is infinite dimensional, choose an orthonormal basis e_k for the atom A. If $P = P_-$, then we may suppose that $\Phi(P_1) = L_1 > L$ since Φ is left-continuous and $\Phi(P) > L$. If $P > P_-$, then set $P_1 = P_-$ and $L_1 = \Phi(P)$.

Choose a strictly decreasing sequence L_k with infimum L and choose unit vectors $f_k \in L_k \ominus L_{k+1}$. Set $X = \sum_{k \geqslant 1} f_k e_k^*$ and

$$\mathfrak{X} = \overline{\operatorname{span}(\mathcal{T}(\mathcal{N}) X \mathcal{T}(\mathcal{N}))} + (\operatorname{Bim}(\Phi) \cap \mathfrak{K}).$$

Notice that $X = (L_1 - L)X(P - P_1)$ is not compact, and that XN and $M^\perp X$ are finite rank if $N < P$ and $M > L$. It follows that $\mathfrak{X}N$ and $M^\perp \mathfrak{X}$ are contained in the compact operators when $N < P$ and $M > L$. Thus $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{X})(N)$ equals 0 when $N < P$ and is always at most L. Since $X \in \mathfrak{X}$, it follows that $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{X})(P) \geqslant L$. Hence $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{X}) = \Psi_{L,P}$. Since $L_1\mathcal{B}(\mathcal{H})P_1^{\perp}$ is contained in $\mathrm{Bim}(\Phi)$, the bimodule $\mathfrak{X}$ is contained in $\mathrm{Bim}(\Phi)$. Hence $\mathfrak{X}$ is wedged between $\mathrm{Bim}(\Phi) \cap \mathfrak{K}$ and $\mathrm{Bim}(\Phi)$ and therefore $\mathrm{Supp}(\mathfrak{X}) = \Phi$.

Note that under the conditions (ii), $\mathfrak{R}(L, P_-) = L\mathcal{B}(\mathcal{H})P_-^{\perp}$. Any bimodule $\mathfrak{X}$ with $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{X})(P) = L$ contains $L\mathcal{B}(\mathcal{H})P_-^{\perp}$ by Corollary 2.2. Since $\mathrm{Supp}(\mathfrak{X}) = \Phi$, it also contains $\mathrm{Bim}(\Phi) \cap \mathfrak{K}$ by Proposition 1.2.

On the other hand, suppose that $T \in \mathrm{Bim}(\Phi)$ is not in $L\mathcal{B}(\mathcal{H})P_-^{\perp} + (\mathrm{Bim}(\Phi) \cap \mathfrak{K})$. This means that either TP_- or $L^{\perp}T$ is non-compact.

If TP_- is not compact, then using the bimodule $\mathfrak{X}$ constructed above, we have $\mathfrak{X}N$ contained in $\mathfrak{K}$ for every $N_- < P$. So $P = P_-$, and as in Lemma 2.7, there will be a sequence P_k strictly increasing to P such that the first k^2 singular values of $T(P_{k+1} - P_k)$ are all close to $\|TP_-^{\perp}\|_e$. Following the argument of Lemma 2.7, we deduce that $\mathrm{dist}(T, \mathfrak{X}) \geqslant \|TP_-^{\perp}\|_e$.

If $L^{\perp}T$ is not compact, then again using the bimodule $\mathfrak{X}$ constructed above, we have $M^{\perp}\mathfrak{X}$ contained in $\mathfrak{K}$ for every $M > L$. Thus there is a sequence L_k strictly decreasing to L such that the first k^2 singular values of $(L_k - L_{k+1})T$ are all close to $\|L^{\perp}T\|_e$. If $P > P_-$, then we can use the bimodule $\mathfrak{X}$ constructed above. If $P = P_-$, then choose the sequence P_k and, if necessary, discard the first few terms of the sequence L_k so that $\Phi(P_1) \geqslant L_1$. Then again the rank argument shows that $\mathrm{dist}(T, \mathfrak{X}) \geqslant \|L^{\perp}T\|_e$.

That the distance to $L\mathcal{B}(\mathcal{H})P_-^{\perp} + (\mathrm{Bim}(\Phi) \cap \mathfrak{K})$ equals the maximum of these two terms follows from an application of the Parrott–Davis–Kahan–Weinberger Lemma as in Lemma 2.7. ∎

For the next lemma, we need to define:

$$\Psi'_{L,P} = \begin{cases} 0 & N \leqslant P, \\ L & N > P. \end{cases} \quad \text{and} \quad \mathfrak{R}'(L, P) = \begin{cases} \overline{\sum_{N > P} L\mathcal{B}(\mathcal{H})N^{\perp}} & \text{if } P \notin \mathcal{N}_r, \\ L\mathcal{B}(\mathcal{H})P^{\perp} & \text{if } P \in \mathcal{N}_r. \end{cases}$$

Lemma 4.4. *Suppose that L and P belong to $\mathcal{N}$ such that $P = P_+$, $L \in \mathcal{N}_f$ and $L = L_+$. Let Φ be a left-continuous support function on $\mathcal{N}$ such that $\Phi(N) > L$ for every $N > P$.*

Then there is a $\mathcal{T}(\mathcal{N})$-bimodule $\mathfrak{X}$ with $\mathrm{Supp}(\mathfrak{X}) = \Phi$ and $\mathrm{Supp}^{\mathrm{e}}(\mathfrak{X}) = \Psi'_{L,P}$. The intersection of all bimodules in $\mathfrak{B}(\Phi, \Psi'_{L,P})$ is $\mathfrak{R}'(L, P) + (\mathrm{Bim}(\Phi) \cap \mathfrak{K})$. Indeed, for $T \in \mathrm{Bim}(\Phi)$,

$$\begin{aligned} \sup_{\mathfrak{X} \in \mathfrak{B}(\Phi, \Psi'_{L,P})} \mathrm{dist}(T, \mathfrak{X}) &= \mathrm{dist}\big(T, \mathfrak{R}'(L, P) + (\mathrm{Bim}(\Phi) \cap \mathfrak{K})\big) \\ &= \max\big\{\|TP\|_e, \|L^{\perp}T\|_e, \lim_{N \downarrow P} \|TN^{\perp}\|_e\big\}. \end{aligned}$$

where the limit term is used only if $P \notin \mathcal{N}_r$.

Proof. While this lemma can be proved directly using the methods of the previous proof, we instead reduce it to the previous lemma using duality. If $\mathfrak{A}$ is a $\mathcal{T}(\mathcal{N})$-bimodule, then $\mathfrak{A}^*$ is a $\mathcal{T}(\mathcal{N}^\perp)$-bimodule. Also, if $\Phi = \operatorname{Supp}(\mathfrak{A})$, then $\Phi^* = \operatorname{Supp}(\mathfrak{A}^*)$ where

$$\Phi^*(L^\perp) = \bigwedge\{N^\perp : N \in \mathcal{N} \text{ and } \Phi(N) \leqslant L\}.$$

To see this, observe that $\operatorname{Supp}(\mathfrak{A}^*)(L^\perp) = \bigwedge\{N^\perp : N \in \mathcal{N} \text{ and } N^\perp\mathfrak{A}^*L^\perp = \mathfrak{A}^*L^\perp\}$ and that $N^\perp\mathfrak{A}^*L^\perp = \mathfrak{A}^*L^\perp$ holds if and only if $L \geqslant \Phi(N)$. It follows that $(\Phi^*)^* = \Phi$ for any support function Φ. Similarly, if $\Psi = \operatorname{Supp}^{\mathrm{e}}(\mathfrak{A})$, then $\Psi^* = \operatorname{Supp}^{\mathrm{e}}(\mathfrak{A}^*)$. For P and L as above, then

$$\Psi'_{L,P} = (\Psi_{P^\perp,L^\perp})^* \quad \text{and} \quad \mathfrak{R}'(L,P) = \mathfrak{R}(P^\perp, L^\perp)^*.$$

We can apply case (i) of Lemma 4.3 with P replaced by $L^\perp$, L by $P^\perp$ and Φ by Φ^*. This gives a $\mathcal{T}(\mathcal{N}^\perp)$-bimodule, $\mathfrak{Y}$, with $\operatorname{Supp}(\mathfrak{Y}) = \Phi^*$ and $\operatorname{Supp}^{\mathrm{e}}(\mathfrak{Y}) = \Psi_{P^\perp,L^\perp}$. Letting $\mathfrak{X} = \mathfrak{Y}^*$, it is easy to check that $\mathfrak{X}$ has the required properties. ∎

THEOREM 4.5. *Let (Φ,Ψ) be an admissible pair of support functions on a nest $\mathcal{N}$. Then the intersection of all bimodules with support function pair (Φ,Ψ) is* $\operatorname{Bim}^0(\Phi,\Psi)$. *Therefore, there is a minimal element for this class of bimodules precisely when the hypotheses of Theorem* 3.8 *are satisfied.*

Proof. Lemma 3.3 shows that $\operatorname{Bim}^0(\Phi,\Psi)$ is contained in every $\mathcal{T}(\mathcal{N})$-bimodule with support function pair (Φ,Ψ). However, by Lemma 3.7, the essential support function of this bimodule will be strictly less than Ψ if Ψ does not satisfy the hypotheses of Theorem 3.8. While if these hypotheses are satisfied, Theorem 3.8 implies that $\operatorname{Bim}^0(\Phi,\Psi)$ is the minimal bimodule with support function pair (Φ,Ψ).

Natural candidates for bimodules with support function pair (Φ,Ψ) are readily obtained as follows. Enumerate the points $\{Q_k\}$ which are in $\mathcal{N}_f$ and also in the range of Ψ. Let M_k denote the infimum of $N \in \mathcal{N}$ such that $\Psi(N) = Q_k$. Distinguish the two cases depending on whether $\Psi(M_k) = Q_k$ or not. For each k, let $\mathfrak{Y}_k$ be the bimodule constructed for the pair $L = Q_k$ and $P = M_k$ using either case (ii) of Lemma 4.3 or Lemma 4.4 depending on whether $\Psi(M_k) = Q_k$ or not. Also let $\{N_j\}$ enumerate the points of left-discontinuity of Ψ such that $\Psi(N_j) \in \mathcal{N}_\infty$. For each j, let $\mathfrak{X}_j$ be the bimodule constructed in case (i) of Lemma 4.3 for the pair $P = N_j$ and $L = \Psi(N_j)$. Then let

$$\mathfrak{T} = \overline{\operatorname{Bim}^0(\Phi,\Psi) + \sum_j \mathfrak{X}_j + \sum_k \mathfrak{Y}_k}.$$

Each $T \in \mathfrak{T}$ can be approximated by a finite sum

$$T = A + \sum_{j=1}^{m} X_j + \sum_{k=1}^{n} Y_k \tag{4.1}$$

where $A \in \mathrm{Bim}^0(\Phi, \Psi)$, $X_j \in \mathfrak{X}_j$ and $Y_k \in \mathfrak{Y}_k$.

Suppose that T is an operator of the form of equation (4.1) that is not in $\mathrm{Bim}^0(\Phi, \Psi)$. By Theorem 4.2, there is an element $P \in \mathcal{N}$ and an $\varepsilon > 0$ so that either

(a) for every $M_- < P$ and $L \in \psi(M)$ in $\mathcal{N}$, one has $\|L^\perp TP\|_e \geqslant \varepsilon$, or

(b) for every $M > P$ and $L \in \psi(P)$, one has $\|L^\perp TM_-\|_e \geqslant \varepsilon$.

We can now proceed as in the infinite multiplicity case. Using the pair $(P, \Psi(P))$ and the appropriate case of Lemma 4.3 or 4.4, construct a closed bimodule with support function pair (Φ, Ψ) which does not contain T.

Since an operator in $\mathfrak{T}$ can be approximated by one of the form of equation (4.1), the intersection of $\mathfrak{T}$ with the family of bimodules constructed for $(P, \Phi(P))$ with $P \in \mathcal{N}$ is $\mathrm{Bim}^0(\Phi, \Psi)$. Thus the intersection of all bimodules in $\mathfrak{B}(\Phi, \Psi)$ is $\mathrm{Bim}^0(\Phi, \Psi)$. ∎

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