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THE DAUGAVET EQUATION FOR OPERATORS NOT FIXING A COPY OF $C[0,1]$

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ABSTRACT. We prove the norm identity $\|\mathrm{Id}+T\| = 1+\|T\|$, which is known as the Daugavet equation, for operators T on $C(S)$ not fixing a copy of $C[0,1]$, where S is a compact Hausdorff space without isolated points.

KEYWORDS: *Daugavet equation, operators not fixing* $C[0,1]$.

AMS SUBJECT CLASSIFICATION: Primary 47B38; Secondary 47B07.

1. INTRODUCTION

An operator $T : X \to X$ on a Banach space is said to satisfy the *Daugavet equation* if

$$\|\mathrm{Id} + T\| = 1 + \|T\|; \tag{1.1}$$

this terminology is derived from Daugavet's theorem that a compact operator on $C[0,1]$ satisfies (1.1). Many authors have established the Daugavet equation for various classes of operators, e.g., the weakly compact ones, on various spaces; we refer to [1], [2], [4], [8], [11], [12], [14] and the references in these papers for more information.

The most far-reaching result for operators on $L_1[0,1]$ is due to Plichko and Popov ([8], Theorem 9.2 and Theorem 9.8) who proved that an operator on $L_1[0,1]$ which does not fix a copy of $L_1[0,1]$ satisfies the Daugavet equation. (As usual, $T : X \to X$ fixes a copy of a Banach space E if there is a subspace $F \subset X$ isomorphic to E such that $T|F$ is an (into-) isomorphism.) In this paper we shall establish the corresponding result for operators on spaces of continuous functions; that is, we shall prove for an operator $T: C(S) \to C(S)$ not fixing a copy of $C[0,1]$ that T satisfies the Daugavet equation provided the compact space S fails to have isolated points.

In the next section we shall give a proof of this result which relies on a deep result due to Rosenthal ([9]), and in Section 3 we develop another approach which is rather self-contained and measure theoretic in spirit. Since this proof appears to be more revealing than the one in Section 2, we found it worthwhile to present both arguments. Very recently we learnt that Kadets and Popov ([5]) found still another proof of this theorem.

Our results are valid for real and complex spaces; we denote the scalar field by $\mathbb{K}$.

2. OPERATORS NOT FIXING A COPY OF $C[0,1]$

Either of our approaches to the main result depends on the analysis of the representing kernel of an operator $T : C(S) \to C(S)$; this is the family of Borel measures on S defined by $\mu_s = T^*\delta_s$, $s \in S$. In [11] (see also [12]) a necessary and sufficient condition on the kernel of T was established in order that T satisfy the Daugavet equation. Here we record a special case which suffices for our needs.

LEMMA 2.1. *If the representing kernel $(\mu_s)_{s\in S}$ of a bounded linear operator $T: C(S) \to C(S)$ satisfies*

$$\inf_{s\in U} |\mu_s(\{s\})| = 0 \tag{2.1}$$

for all nonvoid open subsets U of S, then T satisfies the Daugavet equation.

Proof. For the convenience of the readers we shall present the (easy) proof of this lemma. Let $\varepsilon > 0$ and consider the open set $U = \{s : \|\mu_s\| > \|T\| - \varepsilon\}$. By (2.1), there is some $s \in U$ such that

$$|1 + \mu_s(\{s\})| \geqslant 1 + |\mu_s(\{s\})| - \varepsilon.$$

Consequently,

$$\begin{aligned} \|\mathrm{Id} + T\| &\geqslant \|(\mathrm{Id} + T)^*(\delta_s)\| = \|\delta_s + \mu_s\| = |1 + \mu_s(\{s\})| + |\mu_s|(S\backslash\{s\}) \\ &\geqslant 1 + |\mu_s(\{s\})| + |\mu_s|(S\backslash\{s\}) - \varepsilon = 1 + \|\mu_s\| - \varepsilon \geqslant 1 + \|T\| - 2\varepsilon, \end{aligned}$$

since $s \in U$, and the Daugavet equation follows. ∎

Proposition 2.2. *Suppose S is a compact Hausdorff space without isolated points. If $T : C(S) \to C(S)$ is a bounded linear operator such that* $\operatorname{ran}(T^*)$ *is separable, then T satisfies the Daugavet equation.*

Proof. Let $(\mu_s)_{s\in S}$ be the representing kernel of T, and let $\{\mu_{s_n} : n \in \mathbb{N}\}$ be dense in $\{\mu_s : s \in S\}$. We define $\mu = \sum 2^{-n}|\mu_{s_n}|$; then all the μ_s are absolutely continuous with respect to μ. Hence $\mu_s(\{s\}) \neq 0$ only if $|\mu|(\{s\}) \neq 0$. Therefore, $\{s \in S : \mu_s(\{s\}) \neq 0\}$ is countable, and, since S is perfect, its complement is dense by Baire's theorem. This implies that the condition of Lemma 2.1 is satisfied, and the proposition is proved. ∎

Proposition 2.2 obviously covers the case of compact operators, operators factoring through a space with a separable dual, and, in the case of metric S, weakly compact operators; for the latter observe that $T^{**}\big(C(S)^{**}\big) \subset C(S)$ so that T^{**} and, consequently, T^* have separable ranges. But even for nonmetrizable perfect S the Daugavet equation for weakly compact operators on $C(S)$ can be derived from the argument of Proposition 2.2; one only has to recall that by a theorem of Bartle, Dunford and Schwartz ([3], p. 306) also in this case all the μ_s are absolutely continuous with respect to some finite measure μ. This seems to yield the easiest proof of the Daugavet equation for these classes of operators.

The most important application of Proposition 2.2, however, is to prove the following theorem which is the main result of this paper.

Theorem 2.3. *Let S be a compact Hausdorff space without isolated points. If $T : C(S) \to C(S)$ does not fix a copy of $C[0,1]$, then T satisfies the Daugavet equation.*

Proof. If S is metrizable, we can invoke a result due to Rosenthal ([9]) (see also [10]) who has shown in the metrizable setting that an operator $T : C(S) \to C(S)$ fixes a copy of $C[0,1]$ if and only if $\operatorname{ran}(T^*)$ is nonseparable. So, in this case Theorem 2.3 follows from Proposition 2.2.

The general case will be reduced to the metrizable one by means of a lemma that was kindly pointed out to us by H.P. Rosenthal.

Lemma 2.4. *Let S be a compact Hausdorff space without isolated points, let X be a separable subspace of $C(S)$, and let $T : C(S) \to C(S)$ be an operator. Then there exists a closed subspace A of $C(S)$ containing X which is algebraically and isometrically isomorphic to some $C(K)$, where K is a compact metric space without isolated points, such that $T(A) \subset A$.*

Taking this lemma for granted we may complete the proof of Theorem 2.3 as follows. Assume $\|\mathrm{Id} + T\| < 1 + \|T\|$. Pick a sequence of functions $f_1, f_2, \ldots$ in the unit ball of $C(S)$ with $\|T\| = \sup_n \|Tf_n\|$. Let X be the closed linear span of the f_n and choose A according to Lemma 2.4. Clearly we have

$$\|(\mathrm{Id} + T)|A\| \leqslant \|\mathrm{Id} + T\| < 1 + \|T\| = 1 + \|T|A\|;$$

hence, by the first part of the proof $T|A$ and thus T fixes a copy of $C[0,1]$. (By the results of [9] we can even deduce that T actually fixes an isometric copy of $C[0,1]$.) ∎

Proof of Lemma 2.4. Let us agree to call a closed self-adjoint subalgebra of a complex $C(S)$-space containing the constants or a closed subalgebra of a real $C(S)$-space containing the constants simply a C^*-subalgebra. For a C^*-subalgebra A of $C(S)$, an atom is a non-zero $\{0,1\}$-valued function $f \in A$ such that whenever $g \in A$, $0 \leqslant g \leqslant f$, we have $g = 0$ or $g = f$. Note that $C(S)$ has atoms if and only if S has isolated points.

Let $B_1 \subset B_2 \subset \cdots$ be C^*-subalgebras of $C(S)$ and put $B = \overline{\mathrm{lin}} \bigcup_n B_n$. We claim for an atom $f \in B$ that f belongs to some B_n and hence is an atom of B_n. In fact, there is some n and some $g \in B_n$ such that $\|f - g\| < 1/2$. If $S_0 = \{s : f(s) = 0\}$ and $S_1 = \{s : f(s) = 1\}$, then $|g(s)| < 1/2$ on S_0 and $|1 - g(s)| < 1/2$ on S_1. Hence there is a continuous function φ on $\mathbb{K}$ satisfying $\varphi \circ g = \chi_{S_1} = f$; consequently, $f \in B_n$.

Turning to the lemma, we denote by B_1 the C^*-subalgebra generated by X, which is separable. Therefore, B_1 contains at most countably many atoms. For each atom a of B_1 there exists a function $f_a \in C(S)$, different from both 0 and a, such that $0 \leqslant f_a \leqslant a$, by the perfectness of S. Let B_2 denote the (separable) C^*-subalgebra generated by B_1 and those f_a; note that by construction a is no longer an atom of B_2. We continue defining separable C^*-subalgebras $B_1 \subset B_2 \subset \cdots$ of $C(S)$ inductively in such a way that B_{n+1} is generated by B_n and functions f_a as above, a an atom of B_n. If $B = \overline{\mathrm{lin}} \bigcup_n B_n$, then B is a separable C^*-subalgebra of $C(S)$ without any atoms, by the previous paragraph.

Let $A_1 = B$ and $X_1 = \overline{\mathrm{lin}} \bigcup_n T^n(A)$; we clearly have $T(X_1) \subset X_1$. We repeat the above construction to obtain an atomless separable C^*-subalgebra $A_2 \supset X_1$. Next we let $X_2 = \overline{\mathrm{lin}} \bigcup_n T^n(A_2)$, find $A_3 \supset X_2$ and keep going, thus obtaining atomless C^*-subalgebras $A_1 \subset A_2 \subset \cdots$. Now $A = \overline{\mathrm{lin}} \bigcup_n A_n$ satisfies the conclusion of Lemma 2.4; being a separable unital C^*-algebra, A is algebraically and isometrically isomorphic to a $C(K)$-space where K is metrizable, and since A does not have atoms by the argument above, K does not have isolated points. ∎

We remark that the converse of Theorem 2.3 is clearly false as is most easily demonstrated by the operator $T = -\mathrm{Id}$ on $C[0,1]$. We also note that in our setting T does not fix a copy of $C[0,1]$ if and only if T does not fix a copy of ℓ^1 if and only if T is weakly sequentially precompact, i.e., if (f_n) is a bounded sequence in $C(S)$, then (Tf_n) admits a weak Cauchy subsequence.

We finally extend Proposition 2.2 to the class of nicely embedded Banach spaces introduced in [12] so that we obtain a unified approach to several results of that paper.

Let S be a Hausdorff topological space, and let $C^{\mathrm{b}}(S)$ be the sup-normed Banach space of all bounded continuous scalar-valued functions. The functional $f \mapsto f(s)$ on $C^{\mathrm{b}}(S)$ is denoted by δ_s. We say that a linear map $J : X \to C^{\mathrm{b}}(S)$ on a Banach space X is a *nice embedding* and that X is *nicely embedded* into $C^{\mathrm{b}}(S)$ if J is an isometry such that for all $s \in S$ the following properties hold:

(N1) For $p_s := J^*(\delta_s) \in X^*$ we have $\|p_s\| = 1$.

(N2) $\mathrm{lin}\{p_s\}$ is an L-summand in X^*.

The latter condition means that there are projections Π_s from X^* onto $\mathrm{lin}\{p_s\}$ such that

$$\|x^*\| = \|\Pi_s(x^*)\| + \|x^* - \Pi_s(x^*)\| \quad \forall x^* \in X^*.$$

We will also need the equivalence relation

$$s \sim t \ \text{ if and only if } \ \Pi_s = \Pi_t$$

on S. Then s and t are equivalent if and only if p_s and p_t are linearly dependent, which implies by (N1) that $p_t = \lambda p_s$ for some scalar of modulus 1. The equivalence classes of this relation are obviously closed.

We will consider the following nondiscreteness condition:

(N3) None of the equivalence classes $Q_s = \{t \in S : s \sim t\}$ contains an interior point.

If the set $\{p_s : s \in S\}$ is linearly independent, this simply means:

(N3′) S does not contain an isolated point.

By (N2), the p_s are linearly independent as soon as they are pairwise linearly independent.

PROPOSITION 2.5. *Let S be a Baire topological Hausdorff space and suppose that X is nicely embedded into $C^{\mathrm{b}}(S)$ so that additionally* (N3) *holds. If $T : X \to X$ is a bounded linear operator with* $\operatorname{ran}(T^*)$ *separable, then T satisfies the Daugavet equation.*

Proof. We stick to the above notation and put $q_s = T^*(p_s)$. By Proposition 2.1 in [12] it is sufficient to prove that

$$S' = \{t \in S : \ \Pi_t(q_s) = 0 \ \forall\, s\}$$

is dense in S. If $\{q_{s_n} : n \in \mathbb{N}\}$ denotes a dense subset of $\{q_s : s \in S\}$, then $t \notin S'$ if and only if $\Pi_t(q_{s_n}) \neq 0$ for some n. This implies that $S \setminus S'$ consists of at most countably many equivalence classes for the equivalence relation $\sim$ (cf. [12], Lemma 2.3), and by (N3) and Baire's theorem S' must be dense. ∎

It is proved in [12] that the following classes of Banach spaces satisfy the assumptions of Proposition 2.5:

(i) X is a function algebra whose Choquet boundary does not contain isolated points;

(ii) X is a real L^1-predual space so that the set $\operatorname{ex} B_{X^*}$ of extreme functionals does not contain isolated points (for the weak* topology);

(iii) X is a complex L^1-predual space for which the quotient space $\operatorname{ex} B_{X^*}/\!\sim$ does not contain isolated points, where $\sim$ means linear dependence;

(iv) X is a space of type C_Λ for certain subsets Λ of abelian discrete groups.

3. A DIFFERENT APPROACH

In this section we present a direct measure theoretic approach to Theorem 2.3. As above we only have to take care of metrizable spaces S since the general case reduces to this one (see Section 2).

So we suppose that (S, d) is a compact metric space without isolated points, and we fix a diffuse probability measure λ on S whose support is S. To show the existence of such a measure, consider a countable basis of open sets $O_n \subset S$. Since each O_n is perfect, we know from a theorem of Bessaga and Pełczyński (see [7], p. 52) that there are diffuse probability measures λ_n with $\operatorname{supp} \lambda_n \subset O_n$; it remains to define $\lambda = \sum 2^{-n}\lambda_n$.

Now let $T : C(S) \to C(S)$ be a bounded linear operator and $(\mu_s)_{s\in S}$ its representing kernel. It is enough to show that T fixes a copy of $C[0,1]$ if (2.1) of

Lemma 2.1 fails for some open set U. Thus we assume, for some open set $U \subset S$ and $\alpha > 0$, that

$$|\mu_s(\{s\})| \geqslant \alpha \quad \forall\, s \in U. \tag{3.1}$$

We shall also assume without loss of generality that $\|T\| = 1$.

We shall decompose the μ_s into their atomic and diffuse parts. It is instrumental for our argument that this can be accomplished in a measurable fashion, as was proved by Kalton ([6], Theorem 2.10). More precisely we can write

$$\mu_s = \sum_{n=1}^{\infty} a_n(s)\delta_{\sigma_n(s)} + \nu_s$$

where

(i) each $a_n : S \to \mathbb{K}$ is measurable for the completion Σ_λ of the Borel sets of S with respect to λ;

(ii) each $\sigma_n : S \to S$ is Σ_λ-Borel-measurable;

(iii) each ν_s is diffuse, and $s \mapsto |\nu_s|$ is Σ_λ-Borel-measurable (we are considering the weak* Borel sets of the unit ball of $C(S)^*$);

(iv) $|a_1(s)| \geqslant |a_2(s)| \geqslant \cdots$ for all $s \in S$;

(v) $\sigma_n(s) \neq \sigma_m(s)$ for all $s \in S$ whenever $n \neq m$;

(vi) $\sum_{n=1}^{\infty} |a_n(s)| \leq 1$ for all $s \in S$.

Put $\beta = \lambda(U) > 0$. Applying the Egorov and Lusin theorems, we may find a compact subset $S_1 \subset S$ with $\lambda(S_1) \geq 1 - \beta/2$ such that

$$\begin{aligned} &\text{(i)} && \text{each } a_n|S_1 \text{ is continuous on } S_1;\\ &\text{(ii)} && \text{each } \sigma_n|S_1 \text{ is continuous on } S_1;\\ &\text{(iii)} && \textstyle\sum |a_n(s)| \text{ converges uniformly on } S_1;\\ &\text{(iv)} && \lim_{n\to\infty} \sup_{s\in S_1} |\nu_s|\big(B(t,\tfrac{1}{n})\big) \to 0 \text{ for all } t \in S; \end{aligned} \tag{3.2}$$

here $B(t, r)$ denotes the closed ball with centre t and radius r. To prove (iv) fix a countable dense subset Q of S. Let $\mathcal{U}_n$ be the countable set of all finite covers of S by means of open balls of radius $1/n$ with centres in Q. Then

$$\varphi_n(s) = \inf_{\mathcal{C}\in\mathcal{U}_n} \sup_{B\in\mathcal{C}} |\nu_s|(B)$$

defines a sequence of Σ_λ-measurable functions, and we have $\varphi_n \to 0$ pointwise, since the ν_s are diffuse and S is compact. By Egorov's theorem (φ_n) tends to 0 uniformly on a large subset of S. This easily implies our claim.

After discarding relatively isolated points of S_1, if necessary, we come up with a perfect subset S_1 with the above properties; note that $U \cap S_1 \neq \emptyset$ since this intersection has positive λ-measure. Next we pick $N \in \mathbb{N}$ such that

$$\sum_{n>N} |a_n(s)| \leqslant \frac{\alpha}{3}, \quad \forall\, s \in S_1. \tag{3.3}$$

Let $s \in U \cap S_1$. Then

$$\alpha \leqslant |\mu_s(\{s\})| = \Big| \sum_{n=1}^{\infty} a_n(s) \delta_{\sigma_n(s)}(\{s\}) \Big|.$$

Consequently $\sigma_k(s) = s$ for some k; in addition, we must have

$$|a_k(s)| \geqslant \alpha \tag{3.4}$$

for this k so that $k \leqslant N$. This shows that

$$U \cap S_1 = \bigcup_{l=1}^{N} \{s \in U \cap S_1 : \sigma_l(s) = s\}.$$

Each of these N sets is relatively closed in $U \cap S_1$, so one of them, with index k say, contains a relatively interior point s_0. Therefore, there exists some $\delta > 0$ with the following properties:

For $B_l = \sigma_l\big(S_1 \cap B(s_0, \delta)\big)$, $l = 1, \ldots, N$, we have

(a) $B_k = S_1 \cap B(s_0, \delta) \subset S_1 \cap U$, and $\sigma_k(s) = s$ for all $s \in B_k$;

(b) $B_l \cap B(s_0, \delta) = \emptyset$ for $l \neq k$ (since $s_0 = \sigma_k(s_0) \neq \sigma_l(s_0)$ and these functions are continuous on S_1);

(c) $|\nu_s|\big(B(s_0, \delta)\big) \leqslant \alpha/3$ for all $s \in S_1$ (see (iv) of (3.2)).

Now let $V = S_1 \cap B(s_0, \delta/2)$. This is a compact perfect subset of S and thus $C(V)$ is isomorphic to $C[0, 1]$ by Milutin's theorem ([13], Theorem III.D.19). Also let $W = \{s \in S : d(s, s_0) \geqslant \delta\}$. Again, this is a compact subset of S, $V \cap W = \emptyset$, and we have from (a) and (b) above

(a$'$) $\sigma_k(s) = s$ for all $s \in V$;

(b$'$) $B_l \subset W$ for all $l = 1, \ldots, N$, $l \neq k$.

By the Borsuk-Dugundji theorem (see e.g. [13], Corollary III.D.17) there exists an isometric linear extension operator $L : C(V \cup W) \to C(S)$. Let $E = \{f \in C(V \cup W) : f|W = 0\}$ and $F = L(E)$; thus $F \cong E \cong C(V)$, and F is isomorphic to $C[0, 1]$.

We finally prove that $T|F$ is an isomorphism. In fact, let us show that

$$\|Tf\| \geqslant \frac{\alpha}{3}\,\|f\|, \quad \forall\, f \in F.$$

Suppose $f \in F$ with $\|f\| = 1$. Then $\|f|V\| = 1$ and $f|W = 0$. Pick $s \in V$ with $|f(s)| = 1$. Then we have (recall that $V \subset S_1 \cap U$)

$$\begin{aligned}
\|Tf\| \geqslant |Tf(s)| &= \Big|\sum_{n=1}^{\infty} a_n(s) f\big(\sigma_n(s)\big) + \int_S f\,\mathrm{d}\nu_s\Big| \\
&\geqslant |a_k(s)| - \sum_{n>N} |a_n(s)| - \int_S |f|\,\mathrm{d}|\nu_s| \quad \text{(by (a$'$) and (b$'$))} \\
&\geqslant \alpha - \frac{\alpha}{3} - |\nu_s|\big(B(s_0,\delta)\big) \quad \text{(by (3.4), (3.3) and since } f|W = 0) \\
&\geqslant \frac{\alpha}{3} \quad \text{(by (c))}.
\end{aligned}$$

This completes the second proof of our theorem. ∎

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