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COCYCLE CONJUGACY CLASSES OF SHIFTS ON THE HYPERFINITE II_1 FACTOR. II

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Dedicated to Richard V. Kadison on the occasion of his 70th birthday

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Abstract. R.T. Powers has constructed a family of unital endomorphisms of the hyperfinite II_1 factor R, each of which has range a subfactor of index 2, and each of which has no non-trivial invariant subalgebras. A cocycle conjugacy invariant for a Powers shift σ is the commutant index, viz., the first index k for which the range of σ has non-trivial relative commutant. We show that all of the Powers shifts of commutant index 2 are cocycle conjugate.

Keywords: *Endomorphism, commutant index, cocycle conjugacy, hyperfinite* II_1 *factor.*

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1. INTRODUCTION

An endomorphism σ on a unital C^*-algebra is called a shift if σ commutes with the adjoint operation and if the scalar multiples of the identity form the only proper C^*-subalgebra globally invariant under σ. Recently a number of papers has appeared which investigate the structure of the cocycle conjugacy classes of shifts. Cocycle conjugacy for shifts is an equivalence relation which is closely related to the notion of outer conjugacy for automorphisms of von Neumann algebras. In the situation when σ is a shift on $B(H)$, $\sigma(B(H))$ is a type I subfactor. The dimension of its commutant acts as a deficiency index and is easily shown to be a cocycle conjugacy invariant. It is perhaps surprising that the structure of the conjugacy classes of shifts on $B(H)$ has proven to be quite complicated [1], [2], [12], [13], [14].

For other factors N, the Jones index $[N : \sigma(N)]$ plays an analogous role as a cocycle conjugacy invariant for σ. In this paper we study a family of shifts on the hyperfinite II_1 factor R which are called binary shifts. A binary shift σ on R has minimal deficiency index $[R : \sigma(R)] = 2$. As we describe below, there is a one to one correspondence between conjugacy classes of binary shifts and sequences, or bitstreams, $a_0\, a_1\, a_2\, \ldots$ of 0's and 1's which lack a certain symmetry property. The construction of these shifts is related to, and was originally motivated by, the construction of sequences of projections used by V. Jones to produce subfactors of index $4\cos^2(\pi/n)$, ([10], see the remark at the end of the next section).

Recently H. Narnhofer, E. Størmer, and W. Thirring have used examples of automorphisms related to binary shifts to demonstrate that the tensor product formula for the Connes-Størmer entropy may fail, i.e., there is an automorphism α on R for which $h(\alpha \otimes \alpha) > h(\alpha) + h(\alpha)$ ([15], Corollary 4.3). For related results on entropy and endomorphisms of R, see [5], [6] and [21].

In the case of binary shifts there is another important numerical invariant for cocycle conjugacy. This is the first k for which $\sigma^k(R)$ has a non-trivial relative commutant in R. We shall call this k the *commutant index* for the binary shift σ. From [10], Corollary 2.2.4 it follows that $k \geqslant 2$, and, at the other extreme, there are known examples of binary shifts for which $\sigma^k(R)^{\mathrm{c}} \cap R$ is trivial for all positive integers k. These are the binary shifts whose corresponding bitstreams satisfy a certain aperiodicity condition. It is also known that for $k = 3, 4, \ldots$, the commutant index is not a complete cocycle conjugacy invariant, i.e., for each such k there are shifts which share this index but which are not cocycle conjugate [3], [7], [8], (see also [4]). The situation when k is infinite is not well understood: indeed, it is not known whether there is only one or perhaps even uncountably many equivalence classes of shifts with infinite commutant index.

In this paper we focus on those binary shifts which have commutant index 2. We show below in Corollary 4.10 that any two binary shifts sharing this index are cocycle conjugate. The proof of this result relies upon the use of the notion of congruence for Toeplitz matrices over finite fields [16], Chapter IV. A pair A, B, of n by n matrices with coefficients in a field F are said to be congruent if there is an invertible matrix U such that $U^{\mathrm{t}} A U = B$ (U^{t} means transpose of U). It is easily seen that congruence is an equivalence relation. For each binary shift σ of commutant index 2 we define a sequence of Toeplitz matrices associated with σ. We shall use a congruence result relating sequences of a pair of binary shifts to produce an operator Y in the unitary group $\mathcal{U}(R)$ which implements the cocycle conjugacy between the shifts.

This paper is organized as follows. In Section 2 we define binary shifts and introduce some of the known results which are helpful in proving our main result. In Section 3 we define, for every binary shift of commutant index 2, a sequence of Toeplitz matrices associated with the shift. We review some of the known results about the equivalence relation of congruence among square matrices, and prove some results about congruence between sequences of Toeplitz matrices corresponding to a pair of binary shifts of commutant index 2. In Section 4 we prove our main result, Corollary 4.10. The reader may prefer to skim the technical results in Section 3 and then read Section 4 prior to a careful reading of Section 3.

2. BINARY SHIFTS ON THE HYPERFINITE II_1 FACTOR

In this section we review some known results about the structure of binary shifts which we shall require to prove our main result in Section 4. The main references for these results are [17], [20], [19]. The reader is referred to the end of this section for a specific example of a binary shift.

Let F be the field consisting of two elements $\{0, 1\}$. Select a sequence, or bitstream, of elements $(a) = \{a_n : n = 0, 1, 2, \ldots\}$ in F with the property that $a_0 = 0$. Accordingly, let $V = \{v_0, v_1, \ldots\}$ be a sequence of hermitian unitary elements which satisfy the commutation relations $v_j v_k = (-1)^{a_{k-j}} v_k v_j$ for $j \leqslant k$. The set V generates the algebra consisting of linear combinations, over the field of complex numbers $\mathbb{C}$, of ordered words in the v_j's, where by an ordered word we refer to any element of the form $v = v_0^{k_0} v_1^{k_1} \cdots v_n^{k_n}$, for some n, and for $k_0, k_1, \ldots, k_n \in F$. Note that by using the commutation relations, the product of any pair of ordered words can be rewritten as a scalar multiple of an ordered word. In fact, if $v' = v_0^{j_0} v_1^{j_1} \cdots v_n^{j_n}$ is another ordered word, $vv' = \pm v_0^{k_0+j_0} v_1^{k_1+j_1} \cdots v_n^{k_n+j_n}$. Notice also that either $v^* = v$ or $v^* = -v$.

There is a well-defined linear functional tr which is defined on the algebra $\mathcal{A}$ of linear combinations of ordered words by $\mathrm{tr}(I) = 1$, $\mathrm{tr}(v) = 0$ for any non-trivial ordered word in the generators v_j. It is easy to verify that tr is tracial, i.e., $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $A, B \in \mathcal{A}$.

Theorem 2.1. ([17], Lemma 3.3; [20], Theorem 2.3) *The functional* tr *is a positive linear tracial functional on* $\mathcal{A}$. $\mathcal{A}$ *is simple if and only if the sequence* $\ldots, a_2, a_1, a_0, a_1, a_2, \ldots$ *is not periodic. Hence in the aperiodic case, the* GNS *representation induced on* $\mathcal{A}$ *by* tr *is a faithful representation of* $\mathcal{A}$ *whose weak closure is isomorphic to the hyperfinite* II_1 *factor* R.

From now on we shall consider only those bitstreams which are aperiodic in the sense of the theorem. We shall also suppress the notation of the GNS representation induced by tr and identify the elements of $\mathcal{A}$ with their image in R under the representation.

It is straightforward to show that there is a unital endomorphism σ defined on ordered words by $\sigma(v) = v_1^{k_0} v_2^{k_1} \cdots v_{n+1}^{k_n}$ and extended to all of $\mathcal{A}$ by linearity. This endomorphism has a unique extension to R which we also denote by σ. We refer to endomorphisms on R constructed in this way as binary shifts.

THEOREM 2.2. ([17]) *The mapping defined by $\sigma(v_j) = v_{j+1}$, $j = 0, 1, 2, \ldots$ has a unique extension to a shift on R. The image $\sigma(R)$ of R is a subfactor of R of Jones subfactor index 2, i.e., $[R : \sigma(R)] = 2$.*

Let $\mathcal{A}_n$, $n = 0, 1, 2, \ldots$ be the subalgebra of R generated by $v_0, v_1, \ldots, v_n$. It is easy to see that $\mathcal{A}_n$ is an algebra of dimension 2^{n+1} (the number of ordered words in the generators v_0 through v_n). Let d_n be the dimension of the center of $\mathcal{A}_n$. Then $d_n = 2^{c_n}$ for some $c_n \in \{0, 1, 2, \ldots\}$ ([19], Theorem 5.4). In fact, the algebra $\mathcal{A}_n$ decomposes as the direct sum of d_n copies of 2^{m_n} by 2^{m_n} matrices, where $m_n = \frac{1}{2}(n + 1 - c_n)$.

DEFINITION 2.3. We refer to the sequence $\{c_0, c_1, \ldots\}$ as the *center sequence* for the shift σ corresponding to the bitstream $a_0\, a_1 \cdots$.

THEOREM 2.4. (Unimodality condition) ([19], Theorem 5.4; cf., [9], Theorem 15.6) *The center sequence consists of a disjoint union of infinitely many finite strings of the form $1\,2\,3 \cdots m-1\, m\, m-1 \cdots 2\,1\,0$. The value of m may vary in the sequence.*

Since the theorem implies that $c_n = 0$ for infinitely many n, so that $\mathcal{A}_n$ is a matrix algebra for infinitely many n, the following result is immediate.

COROLLARY 2.5. ([19], Corollary 5.5) *The uniform closure of the algebra $\bigcup \mathcal{A}_n$ is a* UHF *algebra of type 2^∞.*

THEOREM 2.6. ([19], Lemma 6.2) *Let n be an index for which $c_n = 0$ and such that, for some k, $c_{n+j} = j$ for $0 \leqslant j \leqslant k$. Then there is a word $z = v_0^{k_0} v_1^{k_1} \cdots v_{n+1}^{k_{n+1}}$ which generates the center of $\mathcal{A}_{n+1}$. The word z has the property that $k_0 = k_{n+1} = 1$. Moreover, z is "flip-symmetric" in the sense that $z = v_0^{k_{n+1}} v_1^{k_n} \cdots v_{n+1}^{k_0}$. For $1 \leqslant j \leqslant k$, the center of $\mathcal{A}_{n+j}$ is the algebra of dimension 2^j generated by the words z, $\sigma(z), \ldots, \sigma^{j-1}(z)$.*

Now suppose we have a pair β, σ of binary shifts on the hyperfinite II_1 factor. We say that β and σ are *conjugate* if there is an automorphism γ of R such that $\sigma = \gamma \circ \beta \circ \gamma^{-1}$. We say that β and σ are *cocycle conjugate* if there exists a unitary element Y in $\mathcal{U}(R)$ such that $\mathrm{Ad}(Y) \circ \sigma$ and β are conjugate.

THEOREM 2.7. ([17], Theorem 3.6) *A pair of binary shifts are conjugate if and only if their corresponding sequences of hermitian unitary generators satisfy the same commutation relations, i.e., β and σ are conjugate if and only if they correspond to the same bitstream sequence in F.*

As in the introduction we define the *commutant index* for a binary shift to be the first integer k for which the subfactor $\sigma^k(R)$ has a non-trivial relative commutant index in R, or to be ∞ if no such k exists. (Note that $\sigma^k(R) = \{v_k, v_{k+1}, \ldots\}''$.) It is straightforward to verify that the commutant index is a cocycle conjugacy invariant ([17], Theorem 3.10).

THEOREM 2.8. ([20], Corollary 2.4) *A binary shift has finite commutant index if and only if the bitstream $a_0\, a_1\, \cdots$ has the property that for some p the subsequence $a_p\, a_{p+1}\, \cdots$ of the bitstream is periodic.*

We shall require the following result about binary shifts with finite commutant index.

THEOREM 2.9. ([20], Corollary 2.4) *Suppose σ is a binary shift with finite commutant index k. Then the von Neumann algebra $\sigma^k(R)^{\mathrm{c}} \cap R$ is a two-dimensional algebra generated by a word w in the hermitian unitary generators $\{v_0, v_1, \ldots\}$ of σ. In fact, w has the form $v_0^{r_0} v_1^{r_1} \cdots v_m^{r_m}$, for some m, where $r_0 = 1$ (and where we assume that m has been chosen so $r_m = 1$).*

We finish this section by deriving a useful result about the center sequence of a binary shift of commutant index 2.

THEOREM 2.10. *Let m be as above, and let $M > m$ be such that $C_M = 0$. Then for $n \geqslant M$, $c_n = 0$ if n is odd and $c_n = 1$ if n is even.*

Proof. Suppose there is a first integer $n > M$ such that $c_n = 2$. Then by Theorem 2.6 there are words z and $\sigma(z)$ in the center $\mathfrak{Z}(\mathcal{A}_n)$ of $\mathcal{A}_n$ such that $z = v_0^{k_0} v_1^{k_1} \cdots v_{n-1}^{k_{n-1}}$ and $k_0 = 1$. Hence $\sigma(z)$ is an ordered word which begins with v_1 and commutes with w. Since w commutes with $\sigma(z)$ and also with $\sigma^2(R)$, it follows that w also commutes with v_1. But then w commutes with $v_1, v_2, \ldots$, so that w commutes with all of $\sigma(R)$, a contradiction since $\sigma(R)^{\mathrm{c}} \cap R = \mathbb{C}I$, by [10]. Hence by contradiction, $c_n \leqslant 1$. By Theorem 2.4, the center sequence must therefore be of the form $1\,0\,1\,0 \cdots$ or $0\,1\,0\,1 \cdots$ for $n \geqslant M$. Since $\dim(\mathcal{A}_n) = 2^{n+1}$, $\mathcal{A}_n$ can be isomorphic to a matrix algebra only if n is odd, whence $c_n = \log_2(\mathfrak{Z}(\mathcal{A}_n))$ can be 0 only if n is odd. ∎

Remark 2.11. In the sequel we shall have occasion to refer to the following fixed binary shift. We shall also use the same notation introduced here. Consider the bitstream $0\,1\,0\,0\,\cdots$ and let τ be the corresponding shift on R. Then τ has hermitian unitary generators which we shall write as $u_0, u_1, \ldots$. The bitstream dictates that a pair of generators u_j, u_k will anticommute if $|j-k| = 1$ and will commute otherwise. It is easy to see that τ has commutant index 2, $\tau^2(R)^{\mathrm{c}} \cap R = \{u_0\}''$. (It is also easy to see that this is the only binary shift with commutant index 2, up to conjugacy, where $m = 0$ as in Theorem 2.9.) We shall refer to τ as the Jones shift. This name stems from the observation that the sequence of projections $e_j = \frac{1}{2}(I + u_j)$ satisfies the identities $e_j e_k = e_k e_j$ for $|j-k| \neq 1$, and $e_j e_{j\pm1} e_j = \frac{1}{2} e_j$ as in [10], Theorem 4.1.1.

3. TOEPLITZ MATRICES AND CONGRUENCE

In this section and throughout the remainder of the paper we shall assume that τ is the Jones shift on the hyperfinite II_1 factor R, with corresponding bitstream $0\,1\,0\,0\,0\,0\,\cdots$, and with hermitian unitary operators $\{u_0, u_1, \ldots\}$ which generate R and on which τ satisfies $\tau(u_j) = u_{j+1}$. As discussed in the remark at the end of the previous section, τ is a binary shift with commutant index 2.

From now on we use σ to denote a fixed binary shift on R with commutant index 2, with bitstream $a_0\,a_1\,a_2\,\cdots$ distinct from $0\,1\,0\,0\,0\,0\,\cdots$, with hermitian unitary generators, $\{v_0, v_1, \ldots\}$ satisfying $\{v_0, v_1, \ldots\}'' = R$, and with word $w = v_0^{r_0} v_1^{r_1} \cdots v_m^{r_m}$ (where we assume $r_m = 1$) generating the relative commutant of $\sigma^2(R)$ in R. As remarked at the end of the previous section, since $a_0\,a_1\,a_2\,\cdots$ is distinct from $0\,1\,0\,0\,0\,0\,\cdots$, $m > 0$.

R.T. Powers and the author have shown (unpublished) that there are countably many non-conjugate binary shifts of commutant index 2 on R. In fact we have shown that for any integer $m > 2$, there are precisely 2^{m-2} distinct vectors of the form $(r_0, r_1, \ldots, r_m)$ over F, with $r_0 = r_m = 1$, such that

(i) for some bitstream $a_0\,a_1\,a_2\,\cdots$ with corresponding shift σ and generators $\{v_0, v_1, \ldots\}$, the word $v_0^{r_0} \cdots v_m^{r_m}$ generates $\sigma^2(R)^{\mathrm{c}} \cap R$, and

(ii) the shifts of commutant index 2 induced by distinct vectors are non-conjugate.

In this section we also introduce some notation and terminology in order to establish some connections between a number of the results cited in the previous section and of some results which pertain to sequences of Toeplitz matrices with coefficients in the field F. We shall also briefly review some of the well-known results regarding the equivalence relation of congruence between a pair of square matrices over F. In the following section these results will be used to establish Theorem 4.9 showing that σ and τ are cocycle conjugate

For any non-negative integer n, let $\mathcal{A}$ be any $n+1$ by $n+1$ matrix over F. It will be convenient for us always to label the columns of A so that the initial column of A is the 0th column and the last is the nth column. Similarly for the rows of A. Let F^n be the vector space of dimension $n+1$ consisting of column vectors of the form $x = (x_0, x_1, \ldots, x_n)^{\mathrm{t}}$ (where t denotes transpose). We view A as acting as a linear transformation on the vector space F^n via the mapping by left multiplication by A, i.e., $x \to Ax$.

For $j = 0, 1, \ldots, n$, let e_j be the column vector which has a 1 in the jth position and 0's elsewhere. If $n < s$ it will often be convenient to view F^n as a subspace of the vector space F^s, and therefore e_j may be viewed simultaneously as an element of both F^n and F^s.

We use F^∞ to denote the infinite-dimensional vector space consisting of infinitely long column vectors $(x_0, x_1, \ldots)^{\mathrm{t}}$ over F which are finitely non-zero. In the obvious way we may view F^n as a subspace of F^∞ for any $n \in \mathbb{N}$. It will occasionally be useful to speak of linear transformations on F^∞. In particular, we use S to denote the shift which satisfies $Se_j = e_{j+1}$ for all non-negative j, and which extends by linearity to all of F^∞.

For $n \in \mathbb{N}$, let F_0^n be n-dimensional subspace of F^n spanned by $\{e_1, e_2, \ldots, e_n\}$, and let F_0^∞ be the span in F^∞ of $\{e_1, e_2, \ldots\}$.

Next we introduce notation to be used for elementary row and column operations. Let i, j be distinct non-negative integers. For any $n \in \mathbb{N}$ satisfying $n \geqslant \max\{i, j\}$ the notation E_{ij} will be used to denote the elementary transformation, or elementary operator, which has 1's along the main diagonal, a 1 in the (i, j) position, and 0's elsewhere. It is straightforward to verify that AE_{ij} is the matrix obtained from A by adding the ith column of A to the jth column of A. Similarly $E_{ij}A$ is the matrix obtained by adding the jth row of A to the ith row of A. Note that $E_{ij}^{-1} = E_{ij}$ and that $E_{ij}^{\mathrm{t}} = E_{ij}$.

DEFINITION 3.1. (cf. [16], Chapter IV) Let n be a non-negative integer. We say that a pair of $n+1$ by $n+1$ matrices A and B over F are *congruent* if there exists an invertible $n+1$ by $n+1$ matrix U with coefficients in F such that $U^{\mathrm{t}}AU = B$.

Note that if A is symmetric, i.e., $A^t = A$, then so is any matrix congruent to A. By direct calculation we obtain the following useful lemma.

LEMMA 3.2. *Suppose A is a symmetric matrix over F with 0 main diagonal. If E is any elementary transformation, $E^{\mathrm{t}}AE$ is a symmetric matrix with 0 main diagonal.*

In general it is an open question to determine whether a pair of matrices A and B are congruent. For certain families of matrices, however, the answer is very straightforward. The following theorem is adequate for our purposes. First we need a definition.

DEFINITION 3.3. For $n \in \mathbb{N}$ let J_n be the $n+1$ by $n+1$ triple diagonal matrix with 1's along the secondary diagonals and 0's on the main diagonal. Let J be the 2 by 2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = J_1$

LEMMA 3.4. *J_n has rank n if n is even and rank $n+1$ if n is odd.*

Proof. Clear. ∎

THEOREM 3.5. *Let A be a symmetric matrix over F with 0's along the main diagonal. Then A has even rank. If $\operatorname{rank}(A) = 2q$ then A is congruent, via a product of elementary transformations, to the matrix consisting of q copies of J along the main diagonal and 0's elsewhere. If B is also symmetric with 0 diagonal, A and B are congruent if and only if they have the same rank.*

Proof. From the proof of [16], Theorem IV.6, A is congruent via a product of elementary matrices to a matrix with the desired form. It is obvious that rank is preserved under congruence, so $\operatorname{rank}(A) = 2q$. The remaining claim is a restatement of [16], Theorem IV.11. ∎

For n a non-negative integer let A_n be the $n+1$ by $n+1$ Toeplitz matrix associated with the bitstream $a_0\, a_1\, \cdots$ and defined by

$$A_n = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_n \\ a_1 & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{n-1} \\ a_2 & a_1 & a_0 & a_1 & \cdot & \cdot & \cdot & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n-1} & a_{n-2} & \cdot & \cdot & \cdot & \cdot & a_0 \end{bmatrix}.$$

Clearly A_n is symmetric and since $a_0 = 0$, A_n has 0 main diagonal. Note that J_n is the Toeplitz matrix corresponding to the bitstream $0\,1\,0\,0\,0\cdots$. Using the commutation relations on the generators v_j, $j = 0, 1, 2, \ldots$, the following lemma is easily verified.

Lemma 3.6. *An ordered word* $z = v_0^{k_0} v_1^{k_1} \cdots v_n^{k_n}$*, is in the center of the finite-dimensional von Neumann algebra* $\mathcal{A}_n$ *if and only if the vector* $(k_0, \ldots, k_n)^{\mathrm{t}}$ *is in the kernel of the matrix* A_n*.*

As a consequence of the lemma we see that the entries c_n of the center sequence for the shift σ coincide with the nullity of the matrix A_n.

Lemma 3.7. *Let* $n > m$ *and let* $w = v_0^{r_0} v_1^{r_1} \cdots v_m^{r_m}$*, be the ordered word generating* $\sigma^2(R)^{\mathrm{c}} \cap R$*. Then if* $\mathbf{r} = (r_0, r_1, \ldots, r_m, 0, \ldots, 0)^{\mathrm{t}}$*,* $A_n \mathbf{r} = (*, 1, 0, \ldots, 0)^{\mathrm{t}}$*, where* $*$ *may be either* 0 *or* 1*.*

Proof. Since w commutes with $v_2, v_3, \ldots$, one sees as in the previous lemma that the dot products (modulo 2) of all but the first 2 rows of A_n with $\mathbf{r}$ is 0. On the other hand, w anticommutes with v_1: for if w were to commute with v_1, w would commute with $\sigma(R) = \{v_1, v_2, \ldots\}''$, a contradiction, since $\sigma(R)^{\mathrm{c}} \cap R = \mathbb{C}I$ ([10]). Since w anticommutes with v_1 the corresponding dot product (modulo 2) of the second row of A_n with $\mathbf{r}$ must be 1. ∎

Proposition 3.8. *For* $n > M$ *(where* M *is as in Theorem* 2.10*) the matrices* A_n *and* J_n *are congruent.*

Proof. Both matrices are symmetric with 0 diagonal. By Theorem 2.10, c_n is 0 for n odd and 1 for n even. Hence A_n has rank $n+1$ for n odd and rank n for n even. By Lemma 3.4, so does J_n, so by the preceding theorem A_n and J_n are congruent. ∎

We need to be more explicit about the type of elementary tranformations used to realize the congruence just established. This is the goal of the next two results.

Theorem 3.9. *Let* n *be (an odd) positive integer for which* A_n *has full rank. Then* A_n *is congruent to* J_n *via a product* $\mathcal{E}$ *of elementary operations* E_{ij} *satisfying* $i \neq 0$ *and* $j \neq 0$*.*

Proof. By Proposition 3.8, A_n and J_n are congruent. Since A_n has full rank, one of the entries a_i, $1 \leqslant i \leqslant n$, must be non-zero. If $a_1 = 0$ then replace A_n with the congruent matrix $E_{1i} A_n E_{i1} = E_{i1}^{\mathrm{t}} A_n E_{i1}$, to obtain a congruent matrix with 1's in the (0,1) and (1,0) positions. In order to eliminate any non-zero entries in the $(0, j)$ and $(j, 0)$ positions $(j > 1)$ multiply this matrix on the right by elementary matrices of the form E_{1j} and on the left by their transposes. We obtain a matrix A_{n1} congruent to A_n with initial (0th) row and column all 0's except in the (1,0) and (0,1) positions. If A_{n1} has no 1's in row 1 other than the (1,0) entry then relabel A_{n1} as A_{n2} and move on to row 2 and column 2. Otherwise use elementary transformations of the form E_{j2} or E_{2j} with $3 \leqslant j \leqslant n$ to obtain a matrix A_{n2} congruent to A_n and of the form

$$\begin{bmatrix} 0 & 1 & 0 & 0 & & 0 \\ 1 & 0 & 1 & 0 & & 0 \\ 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & & & * & \\ 0 & 0 & & & & \end{bmatrix}.$$

Hence A_{n2} either has the form above or

$$\begin{bmatrix} 0 & 1 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & 0 & & & \\ 0 & 0 & & & \\ & & & * & \\ 0 & 0 & & & \end{bmatrix}.$$

Continuing in this fashion, and using the fact that A_n has full rank, it is clear that we can use products of elementary matrices of the required form to obtain a matrix A_{nn} congruent to A_n and having the form

$$\begin{bmatrix} J_{k_1} & & & 0 \\ & J_{k_2} & & \\ & & \ddots & \\ 0 & & & J_{k_m} \end{bmatrix}.$$

Since $n+1 = \operatorname{rank}(A_n) = \sum_{i=1}^{m} \operatorname{rank}(J_{k_i})$, each of the matrices J_{k_i} must have maximal rank. In particular, J_{k_i} has an even number of rows and columns. Since such a J_{k_i} is congruent to a matrix with J's along the main diagonal and 0's elsewhere, we may alter A_{nn} using elementary column transformations of the prescribed form to obtain a matrix B_n congruent to A_{nn} (and hence, to A_n) and of the form

$$\begin{bmatrix} J_{k_1} & & & & & 0 \\ & J & & & & \\ & & J & & & \\ & & & J & & \\ & & & & \ddots & \\ 0 & & & & & J \end{bmatrix}.$$

Let $\mathbf{R} = E_{k_1+2,k_1} E_{k_1+4,k_1+2} \cdots E_{n,n-2}$. It is easy to check that $\mathbf{R}^{\mathrm{t}} B_n \mathbf{R} = J_n$. Hence we have shown that A_n is congruent to J_n using products of elementary column transformations of the prescribed form. ∎

Theorem 3.10. *Let $p \in \mathbb{N}$ be even, $p > 2$, and such that A_{p-1} has full rank p. Then there is a product W_p of elementary transformations E_{ij}, $i, j \in \{1, 2, \ldots, p\}$, $i \neq p$, such that $W_p^{\mathrm{t}} A_p W_p = J_p$.*

Proof. Since $\mathrm{rank}(A_p) = \mathrm{rank}(A_{p-1}) = p$, there is a single vector $\mathbf{z} = (z_0, \ldots, z_p)^{\mathrm{t}}$ spanning the kernel of A_p. By Theorem 2.6, $z_0 = 1 = z_p$. Hence if $E_1 = E_{1p}^{z_1} E_{2p}^{z_2} \cdots E_{p-1,p}^{z_{p-1}}$,

$$E_1^{\mathrm{t}} A_p E_1 = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdot & \cdot & \cdot & a_{p-1} & a_0 \\ a_1 & a_0 & a_1 & a_2 & \cdot & \cdot & \cdot & a_{p-2} & a_1 \\ a_2 & a_1 & a_0 & a_1 & \cdot & \cdot & \cdot & a_{p-3} & a_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{p-1} & a_{p-2} & a_{p-3} & \cdot & \cdot & \cdot & \cdot & a_0 & a_{p-1} \\ a_0 & a_1 & a_2 & & & & & a_{p-1} & 0 \end{bmatrix}.$$

By the previous result there is a product $\mathcal{E}$ of elementary tranformations E_{ij} (with $1 \leqslant i, j \leqslant p-1$) such that $\mathcal{E}^{\mathrm{t}} A_{p-1} \mathcal{E} = J_{p-1}$. Then

$$\mathcal{E}^{\mathrm{t}} E_1^{\mathrm{t}} A_p E_1 \mathcal{E} = \left[\begin{array}{ccccccccc} & & & & & & & & 0 \\ & & & & & & & & 1 \\ & & & & J_{p-1} & & & & 0 \\ & & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & \cdot \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{array}\right].$$

Now letting $E_2 = E_{2p} E_{4p} \cdots E_{p-2,p}$ it is straightforward to show that

$$E_2^{\mathrm{t}} \mathcal{E}^{\mathrm{t}} E_1^{\mathrm{t}} A_p E_1 \mathcal{E} E_2 = J_p.$$

Set $W_p = E_1 \mathcal{E} E_2$. ∎

Remark 3.11. For the remainder of this section and in the next section, we choose a fixed even integer $p > M$, where M is as in Theorem 2.10. By Theorem 2.10 and Lemma 3.6, for $n > p$, A_n has full rank $n+1$ if n is odd and rank n if n is even.

Theorem 3.12. *Let M and p be as above. Then for each $n > p$, there is a product W_n of elementary transformations such that:*

(i) $W_n^{\mathrm{t}} A_n W_n = J_n$;

(ii) *for any* $j \leqslant n$, $W_{n+2}^{-1} e_j = W_n^{-1} e_j$ *and* $W_{n+2} e_j = W_n e_j$;

(iii) $W_n^{-1} e_0 = e_0 = W_n e_0$;

(iv) *if* $1 \leqslant j \leqslant n$, $W_n^{-1}e_j$ *and* $W_n e_j$ *lie in the finite linear span of the vectors* $\{e_1, e_2, \ldots\}$*;*

(v) *if* $1 \leqslant j \leqslant p$, $W_n^{-1}e_j$ *and* $W_n e_j$ *lie in the linear span of the vectors* $\{e_1, \ldots, e_p\}$*;*

(vi) *if* $j > p$*, then both* $W_n^{-1}e_j$ *and* $W_n e_j$ *lie in the linear span of the vectors* $\{e_1, \ldots, e_j\}$.

Proof. As above, let $w = v_0^{r_0} v_1^{r_1} \cdots v_m^{r_m}$ be the ordered word generating $\sigma^2(R)^{\mathrm{c}} \cap R$. By Lemma 3.7, $A_n(r_0, r_1, \ldots, r_m, 0, \ldots, 0)^{\mathrm{t}} = (c, 1, 0, \ldots, 0)^{\mathrm{t}}$, ($c$ is either 0 or 1). Equivalently, the dot product in F of the row $(a_1 a_0 a_1 a_2 \cdots a_{n-1})$ of A_n with $(r_0, r_1, \ldots, r_m, 0, \ldots, 0)^{\mathrm{t}}$ is 1, and the dot products of the subsequent rows with this vector are 0. Enumerating the columns of A_n as $c_0, c_1, \ldots, c_n$, it is clear from the dot product relations above that the linear combination $r_m c_{n-m} + r_{m-1} c_{n-m+1} + \cdots + r_0 c_n$ is the column vector $(0, 0, \ldots, 0, 1, c)^{\mathrm{t}}$. Let

$$\mathcal{E}_n = E_{n-m,n}^{r_m} E_{n-m+1,n}^{r_{m-1}} \cdots E_{n-1,n}^{r_1}.$$

Then

$$A_n \mathcal{E}_n = \left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & A_{n-1} & & \vdots \\ & & & 0 \\ \hline & & & 1 \\ * & \cdots\cdots\cdots & * & c \end{array}\right]$$

and therefore

$$\mathcal{E}_n^{\mathrm{t}} A_n \mathcal{E}_n = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & 0 \\ & & A_{n-1} & & & \vdots \\ & & & & & 0 \\ \hline & & & & & 1 \\ 0 & 0 & \cdots\cdots\cdots & 0 & 1 & 0 \end{array}\right].$$

We continue in this fashion, constructing products $\mathcal{E}_{n-1}, \ldots, \mathcal{E}_{p+1}$ of elementary column matrices such that

$$\mathcal{E}_{p+1}^{\mathrm{t}}\mathcal{E}_{p+2}^{\mathrm{t}}\cdots\mathcal{E}_{n}^{\mathrm{t}}A_n\mathcal{E}_n\cdots\mathcal{E}_{p+2}\mathcal{E}_{p+1} =$$

$$\left[\begin{array}{ccccccccccccc}
 & & & \vdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
 & & & \vdots & \vdots & \vdots & & & & & & & \\
 & A_p & & \vdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
 & & & \vdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
- & - & - & - & & & & & & & & & \\
0 & \cdots & 0 & 1 & 0 & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
 & & & & & & \cdot & \cdot & \cdot & & & & \\
 & & & & & & & \cdot & \cdot & \cdot & & & \\
 & & & & & & & & & & & & \\
 & & & & & & & & & \cdot & \cdot & \cdot & \\
0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & & & & & & 0 & 0 & 1 & 0
\end{array}\right].$$

Call this matrix B_n. Next we consider the full rank matrix A_{p-1} embedded in A_p in the corner of the matrix above. Since $\operatorname{rank}(A_{p-1}) = \operatorname{rank}(A_p) = p$, we may apply Theorem 3.10 to find W_p such that $W_p^{\mathrm{t}}A_pW_p = J_p$. Also by the theorem, we may assume that W_p is a product of elementary transformations E_{ij} with $1 \leqslant i$, $j \leqslant p$, and $i \neq p$. Since $i \neq p$, it follows that $W_p^{\mathrm{t}}B_nW_p = J_p$. This proves (i).

Note that $E_{st}e_j = e_j + \delta_{tj}e_s$. By construction W_n is a product of elementary transformations of the form E_{st}, where neither s nor t is 0: properties (iii) and (iv) follow from this observation.

Now suppose $1 \leqslant j \leqslant p$. Clearly $W_p^{-1}e_j$ and W_pe_j are in the span of $\{e_1, \ldots, e_p\}$. But for $n > p$, $W_n = \mathcal{E}_n\mathcal{E}_{n-1}\cdots\mathcal{E}_{p+1}W_p$, where $\mathcal{E}_k$ is a product of elementary transformations of the form E_{ik}, for some $i \in \{1, 2, \ldots, k-1\}$. Since $\mathcal{E}_ke_r = e_r$ whenever $k > r$, it is clear that $W_ne_j = W_pe_j$. Similarly, $W_n^{-1}e_j = W_p^{-1}\mathcal{E}_{p+1}^{-1}\cdots\mathcal{E}_n^{-1}e_j = W_p^{-1}e_p$, giving (v).

If $j > p$, $W_ne_j = \mathcal{E}_n\mathcal{E}_{n-1}\cdots\mathcal{E}_{j+1}W_je_j$. Arguing as above it is clear that W_je_j lies in the span of $\{e_1, \ldots, e_j\}$ and that such a vector is fixed by $\mathcal{E}_{j+1}, \ldots, \mathcal{E}_n$. Hence $W_ne_j = W_je_j$. Similarly, $W_n^{-1}e_j = W_j^{-1}\mathcal{E}_{j+1}^{-1}\cdots\mathcal{E}_n^{-1}e_j = W_j^{-1}e_j$. Hence we have established condition (vi) and completed the verification of condition (ii). ∎

Notation 3.13. As an application of the preceding result it makes sense to speak of a transformation W on the linear space spanned by $\{e_0, e_1, \ldots\}$ which is defined by $We_j = \lim W_ne_j$.

Using the proof of the preceding theorem we give some more detailed information about the vectors We_j for $j > p$. For $n \geqslant j$,

$$W_n e_j = \mathcal{E}_n \mathcal{E}_{n-1} \cdots \mathcal{E}_{p+1} W_p e_j = \mathcal{E}_n \mathcal{E}_{n-1} \cdots \mathcal{E}_{p+1} e_j = \mathcal{E}_n \cdots \mathcal{E}_{j+1} \mathcal{E}_j e_j = \mathcal{E}_j e_j.$$

Recalling from above that $\mathcal{E}_j = E_{j-m,j}^{r_m} E_{j-m+1,j}^{r_{m-1}} \cdots E_{j-1,j}^{r_1}$, we obtain $We_j = \mathcal{E}_j e_j = (0, \ldots, 0, r_m, r_{m-1}, \ldots, r_1, 1, 0, \ldots, 0)^{\mathrm{t}} = r_m e_{j-m} + r_{m-1} e_{j-m+1} + \cdots + r_1 e_{j-1} + e_j$. This establishes the following result.

COROLLARY 3.14. *Let p be as above. Let S be the shift defined on F^∞ by $Se_k = e_{k+1}$ for all $k = 0, 1, 2, \ldots$. Then for any $j > p$, $We_{j+1} = SWe_j$, i.e., $WS = SW$ on the subspace of F^∞ spanned by $\{e_{p+1}, e_{p+2}, \ldots\}$.*

4. QUADRATIC FORMS AND COCYCLE CONJUGACY

In this section we show how to use the results of the previous section to establish the result that any binary shift with relative commutant index 2 is cocycle conjugate to the Jones shift. As above, let τ be the Jones shift and let σ be any other binary shift having relative commutant index 2.

Recall that any symmetric matrix A over a field gives rise to a quadratic form on the vector space of the corresponding dimension via the inner product $(x, y) = x^{\mathrm{t}} Ay$. In particular, for $n \in \mathbb{N}$ we have a quadratic form corresponding to the matrix A_n. It is easy to see that

$$e_j^{\mathrm{t}} A_n e_k = a_{k-j}, \tag{4.1}$$

for $j \leqslant k \leqslant n$. This equation suggests that it is useful to construct a correspondence between the generators v_j and the standard basis vectors e_j. Indeed let $\mathcal{W}_\sigma$ be the subgroup of the unitary group $\mathcal{U}(R)$ generated by the words in the generators v_j, let $\widetilde{\mathcal{W}_\sigma}$ be the quotient group $\mathcal{W}_\sigma/\{\pm 1\}$, and define a map $\lambda : \widetilde{\mathcal{W}_\sigma} \to F^\infty$ by $\lambda(v_0^{k_0} v_1^{k_1} \cdots v_n^{k_n}) = \sum_{i=0}^{n} k_i e_i = (k_0, \ldots, k_n, 0, 0, \ldots,)^{\mathrm{t}}$. It is easy to check that λ is an isomorphism. The result below follows immediately from Equation (4.1) and the bilinearity of quadratic forms.

LEMMA 4.1. *A pair of words $w = v_0^{k_0} v_1^{k_1} \cdots v_n^{k_n}$, $w' = v_0^{j_0} v_1^{j_1} \cdots v_n^{j_n}$ commute (respectively, anticommute) if and only if $\lambda(w)^{\mathrm{t}} A_n \lambda(w')$ has the value* 0 *(respectively,* 1*).*

Next we show how to construct a sequence $\{w_0, w_1, \ldots\}$ of words in the generators $u_0, u_1, \ldots$ which have the same commutation relations as the generators of the binary shift σ. Suppose $p \in \mathbb{N}$ is as in Theorem 3.12. Suppose $j \leqslant k$ are non-negative integers. From (4.1) and Theorem 3.12, for any $n \geqslant k$,

$$\begin{aligned} a_{j-k} &= \lambda(v_j)^{\mathrm{t}} A_n \lambda(v_k) = e_j^{\mathrm{t}} A_n e_k = e_j^{\mathrm{t}} (W_n^{\mathrm{t}})^{-1} J_n W_n^{-1} e_k \\ &= (W_n^{-1} e_j)^{\mathrm{t}} J_n W_n^{-1} e_k = (W^{-1} e_j)^{\mathrm{t}} J_n (W^{-1} e_k). \end{aligned}$$

Define a mapping χ from vectors in F^∞ to words in the generators of the Jones shift, by setting $\chi\Big(\sum\limits_{i=0}^{n} k_i e_i\Big) = w = u_0^{k_0} u_1^{k_1} \cdots u_n^{k_n}$. Define $w_j = \chi(W^{-1} e_j)$, $j = 0, 1, 2, \ldots$. From the preceding lemma and calculation, the words w_j satisfy the same pairwise commutation relations as do the words v_j. Also since W_n is invertible for any $n > p$, it follows from parts (iii) and (vi) of Theorem 3.12 that $\{W^{-1}e_0, W^{-1}e_1, \ldots, W^{-1}e_n\}$ is a basis for the vector space spanned by $\{e_0, e_1, e_2, \ldots, e_n\}$. Hence the von Neumann algebra $\{w_0, \ldots, w_n\}''$ coincides with $\{u_0, \ldots, u_n\}''$. Since $W^{-1}e_0 = e_0$ from Theorem 3.12 (iii), $w_0 = u_0$. Also note from the same corollary that for $j \geqslant 1$, w_j is a word in the generators $u_1, u_2, \ldots$. Hence we have shown the following.

LEMMA 4.2. *The words $w_j = \chi(W^{-1}e_j)$, $j = 0, 1, 2, \ldots$, satisfy the same pairwise commutation relations as do the words v_j. Moreover the sequence $\{w_0, w_1, \ldots\}$ generates R as a von Neumann algebra, i.e., $\{w_0, w_1, \ldots\}'' = R$. Also $w_0 = u_0$ and for $j \geqslant 1$, $w_j \in \{u_1, \ldots\}'' = \tau(R)$.*

As an important step in showing that the shifts σ and τ are cocycle conjugate, we will show first that there exists a binary shift on R which takes w_j, $j = 0, 1, 2, \ldots$ to a scalar multiple of w_{j+1}, and that this shift is cocycle conjugate to τ. The linear transformation defined below plays a crucial role in this step.

DEFINITION 4.3. Let φ be the linear transformation defined on the vector space F_0^∞ by $\varphi(e_j) = W^{-1}SWS^{-1}e_j$.

LEMMA 4.4. *φ is an invertible linear transformation on F_0^∞.*

Proof. First it must be checked that φ actually maps F_0^∞ into itself. $\varphi(e_1) = W^{-1}SWS^{-1}(e_1) = W^{-1}SWe_0 =$ (by Theorem 3.12 (iii)) $W^{-1}Se_0 = W^{-1}e_1$ which by Theorem 3.12 lies in F_0^p. For $j \geqslant 1$, $\varphi(e_{j+1}) = W^{-1}SWe_j \in W^{-1}(F_0^\infty) \subset F_0^\infty$, by Theorem 3.12 (iv). Hence $\varphi(F_0^\infty) \subset F_0^\infty$. It is clear that φ is injective. Surjectivity follows from parts (v) and (vi) of Theorem 3.12. ∎

The following result is just a restatement of Corollary 3.14.

LEMMA 4.5. *$\varphi(e_j) = e_j$ for $j > p$.*

LEMMA 4.6. *For any $j, k \geqslant 1$ and for n exceeding j, k, p, $\varphi(e_j)^{\mathrm{t}} J_n \varphi(e_k)$ is 1 if $|j-k| = 1$, and is 0 otherwise.*

Proof. By Theorem 3.12, for $n > \max\{j, k, p\}$, $\varphi(e_j)$ and $\varphi(e_k)$ lie in the linear space spanned by $\{e_1, \ldots, e_{n-1}\}$. Then

$$\begin{aligned}\varphi(e_j)^{\mathrm{t}} J_n \varphi(e_k) &= (W^{-1}SWS^{-1}e_j)^{\mathrm{t}} J_n (W^{-1}SWS^{-1}e_k)\\ &= (W_n^{-1}SW_nS^{-1}e_j)^{\mathrm{t}} J_n (W_n^{-1}SW_nS^{-1}e_k)\\ &= (SW_ne_{j-1})^{\mathrm{t}} (W_n^{-1})^{\mathrm{t}} J_n (W_n^{-1})(SW_ne_{k-1})\\ &= (SW_ne_{j-1})^{\mathrm{t}} A_n (SW_ne_{k-1}).\end{aligned}$$

It is easy to check, using the symmetry of A_n, $(Sx)^{\mathrm{t}} A_n Sy = x^{\mathrm{t}} A_n y$ for any vectors in the span of $\{e_0, \ldots, e_{n-1}\}$, so

$$\begin{aligned}\varphi(e_j)^{\mathrm{t}} J_n \varphi(e_k) &= (W_ne_{j-1})^{\mathrm{t}} A_n (W_ne_{k-1}) = e_{j-1}^{\mathrm{t}} (W_n^{\mathrm{t}} A_n W_n) e_{k-1}\\ &= e_{j-1}^{\mathrm{t}} J_n e_{k-1} = e_j^{\mathrm{t}} J_n e_k.\end{aligned}$$

It is clear from direct calculation that $e_j^{\mathrm{t}} J_n e_k$ is 1 for $|j-k|$ and 0 otherwise, and we are done. ∎

The mapping φ permits us to define an isomorphism on the subgroup $\tau(\mathcal{W}_\tau)$ of $\mathcal{U}(R)$ of the words generated by $\{u_1, u_2, \ldots\}$. For $j = 1, 2, \ldots$, set $y_j = \chi(\varphi(e_j))$. Multiplying y_j by $\sqrt{-1}$ if necessary, we may assume that the y_j are hermitian unitary elements of $\tau(\mathcal{W}_\tau)$. Define a mapping π from ordered words $u = u_1^{k_1} \cdots u_n^{k_n}$ in $\tau(\mathcal{W}_\tau)$ to $\tau(\mathcal{W}_\tau)$ by $\pi(u) = y_1^{k_1} \cdots y_n^{k_n}$. By the preceding lemma the y_j's satisfy the same commutation relations as do the u_j's, and so it is not difficult to show that π extends to a $*$-homomorphism on all of $\tau(\mathcal{W}_\tau)$. Moreover, since φ is invertible on F_0^∞, π is actually a $*$-isomorphism of this group.

THEOREM 4.7. (cf. [19], Theorem 3.7) *There is a unitary element $Y \in \tau(\mathcal{U}(R))$ such that, for all words z in $\tau(\mathcal{W}_\tau)$, $\mathrm{Ad}(Y)(z) = \pm\pi(z)$.*

Proof. From the preceding paragraph π is a $*$-isomorphism on the group $\tau(\mathcal{W}_\tau)$ mapping words in the u_j's to scalar multiples of words. Fix $n > p$ and even. By Lemma 4.5, π fixes u_j for $j > p$, so π gives rise to an automorphism of the finite-dimensional von Neumann subalgebra $\mathcal{B}_n = \{u_1, \ldots, u_n\}''$ on R. By Theorem 2.10, $\mathcal{A}_q = \{u_0, \ldots, u_q\}''$ is a type I factor for q sufficiently large and odd, so $\mathcal{B}_n = \tau(\mathcal{A}_{n-1})$ is a type I factor. Thus there exists a unitary element $Y_n \in \mathcal{B}_n$ such that $\pi|\mathcal{B}_n = \mathrm{Ad}(Y_n)$. Similarly, there exists a unitary element $Y_{n+2} \in \mathcal{B}_{n+2}$ such that $\pi|\mathcal{B}_{n+2} = \mathrm{Ad}(Y_{n+2})$. Then $Y_n^* Y_{n+2} \in \mathcal{B}_n^{\mathrm{c}} \cap \mathcal{B}_{n+2}$. It is not difficult to show that this algebra is generated by the words $u_1u_3\cdots u_{n-1}u_{n+1}$ and u_{n+2}. Note, however, that Y_{n+2} and Y_n both commute with u_{n+2} (since $\mathrm{Ad}(Y_{n+2})(u_{n+2}) = u_{n+2}$ and since $Y_n \in \mathcal{B}_n$). Hence $Y_n^*Y_{n+2}$ must be of the form $aI + bu_{n+2}$, for some $a, b \in \mathbb{C}$. Since $Y_n^*Y_{n+2}$ is unitary, $|a|^2 + |b|^2 = 1$ and $\mathrm{Re}(a\bar{b}) = 0$. On the other hand,

$$\begin{aligned} u_{n+1} &= \mathrm{Ad}(Y_{n+2})(u_{n+1}) = Y_n(a + bu_{n+2})u_{n+1}(\bar{a} + \bar{b}u_{n+2})Y_n^* \\ &= Y_nu_{n+1}(a - bu_{n+2})(\bar{a} + \bar{b}u_{n+2})Y_n^* \\ &= (|a|^2 - |b|^2)Y_nu_{n+1}Y_n^* + 2\,\mathrm{Im}(a\bar{b})Y_nu_{n+1}u_{n+2}Y_n^*, \end{aligned}$$

hence $0 = \mathrm{Im}(a\bar{b}) = \mathrm{Re}(a\bar{b})$. If $b = 0$ then Y_n is a scalar multiple of Y_{n+2}. Then $\mathrm{Ad}(Y_n)|\mathcal{B}_{n+2} = \mathrm{Ad}(Y_{n+2})|\mathcal{B}_{n+2} = \pi|\mathcal{B}_{n+2}$. Moreover, since Y_n commutes with $u_{n+3}, u_{n+4}, \ldots$, we have in fact $\mathrm{Ad}(Y_n) = \pi$ on all of R, so we may take $Y = Y_n$. If $Y_n = bY_{n+2}u_{n+2}$, then $\mathrm{Ad}(Y_n)(u_{n+1}) = -\mathrm{Ad}(Y_{n+2})(u_{n+1}) = -u_{n+1} = -\pi(u_{n+1})$, and $\mathrm{Ad}(Y_n)(u_{n+j}) = u_{n+j} = \pi(u_{n+j})$ for $j \geqslant 2$. Hence $\mathrm{Ad}(Y_n)(z) = \pi(z)$ for all words $z = U(Q)$ with $n + 1 \notin Q$, and $\mathrm{Ad}(Y_n)(z) = -\pi(z)$ if $n + 1 \in Q$, and once again we may take $Y = Y_n$. ∎

REMARK 4.8. See [19], Theorem 3.7 for a more explicit characterization of the operator Y.

THEOREM 4.9. *Let Y be as in the previous theorem. The endomorphism $\mathrm{Ad}(Y) \circ \tau$ is a binary shift on R which is conjugate to σ. Hence σ and τ are cocycle conjugate.*

Proof. By Theorem 3.12, $w_0 = u_0 = \chi(W^{-1}e_0)$, and for $j \in \mathbb{N}$, w_j is a scalar multiple of the word $\chi(W^{-1}e_j)$ in the generators u_i, $i \in \mathbb{N}$. The w_j's satisfy the same commutation relations $w_jw_k = (-1)^{a_{k-j}}w_kw_j$, as do the v_j's.

We prove by induction that $\mathrm{Ad}(Y) \circ \tau(w_j) = \pm w_{j+1}$. First,

$$\begin{aligned} \mathrm{Ad}(Y) \circ \tau(w_0) &= Y^*u_1Y = \pm\pi(u_1) = \pm\chi(\varphi(e_1)) = \pm\chi(W^{-1}SWS^{-1}e_1) \\ &= \pm\chi(W^{-1}SWe_0) = \pm\chi(W^{-1}Se_0) = \pm\chi(W^{-1}e_1) = \pm w_1. \end{aligned}$$

Suppose $\mathrm{Ad}(Y) \circ \tau((w_j) = \pm w_{j+1}$ for $0 \leqslant j \leqslant k - 1$. Since $\tau \circ \chi = \chi \circ S$,

$$\begin{aligned} \mathrm{Ad}(Y) \circ \tau(w_k) &= \pm\mathrm{Ad}(Y)(\tau(\chi(W^{-1}e_k))) = \pm\mathrm{Ad}(Y)(\chi(SW^{-1}e_k)) \\ &= \pm\pi(\chi(SW^{-1}e_k)). \end{aligned}$$

Since $\pi \circ \chi|F_0^\infty = \pm\chi \circ \varphi|F_0^\infty$,

$$\begin{aligned} \mathrm{Ad}(Y) \circ \tau(w_k) &= \pm\chi(\varphi(SW^{-1}e_k) = \pm\chi(W^{-1}SWS^{-1}(SW^{-1}e_k)) \\ &= \pm\chi(W^{-1}e_{k+1}) = \pm w_{k+1}. \end{aligned}$$

Hence the induction holds. By Theorem 3.12 and the paragraph preceding Theorem 4.7, $R = \{w_0, w_1, \dots\}''$, so $\mathrm{Ad}(Y) \circ \tau$ is a binary shift. Since the generators $\pm w_j$ of $\mathrm{Ad}(Y) \circ \tau$ satisfy the same commutation relations as the generators v_j do for σ, $\mathrm{Ad}(Y) \circ \tau$ is conjugate to σ by [17], Theorem 3.6. ∎

Since cocycle conjugacy is an equivalence relation we have the following.

COROLLARY 4.10. *If σ, ρ are binary shifts on R, each of which has commutant index* 2*, then σ and ρ are cocycle conjugate.*

REMARK 4.11. As mentioned in the introduction, the commutant index of a binary shift is a cocycle conjugacy invariant. In [19], R.T. Powers and the author conjectured that a pair of binary shifts with finite commutant index are cocycle conjugate if and only if

(i) they have the same commutant index, and

(ii) their center sequences coincide for all but finitely many entries (see Definition 2.3).

The second condition is automatically satisfied in commutant index 2 case (Theorem 2.10).

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REFERENCES

1. O. BRATTELI, P.E.T. JORGENSEN, Endomorphisms of $B(H)$. II: finitely correlated states on $\mathcal{O}_n$, University of Iowa, preprint, 1995.
2. O. BRATTELI, P.E.T. JORGENSEN, G.L. PRICE, Endomorphisms of $B(H)$, *Proc. Sympos. Pure Math.* **59**(1996), 93–138.
3. D. BURES, H. YIN, Outer conjugacy of shifts on the hyperfinite II_1 factor, *Pacific J. Math.* **142**(1990), 245–257.
4. M. CHODA, Shifts on the hyperfinite II_1 factor, *J. Operator Theory* **17**(1987), 223–235.
5. M. CHODA, Entropy for canonical shifts. II, *Publ. Res. Inst. Math. Sci.* **27**(1991), 461–489.
6. M. CHODA, Entropy for canonical shifts, *Trans. Amer. Math. Soc.* **334**(1992), 827–849.
7. M. ENOMOTO, M. NAGISA, Y. WATATANI, H. YOSHIDA, Relative commutant algebras of Powers' binary shifts on the hyperfinite II_1 factor, *Math. Scand.* **68**(1991), 115–130.

8. M. ENOMOTO, Y. WATATANI, Powers' binary shifts on the hyperfinite factor of type II_1, *Proc. Amer. Math. Soc.* **105**(1989), 371–374.
9. I.S. IOHVIDOV, *Hankel and Toeplitz matrices and forms*, Birkhäuser, Boston 1982.
10. V.F.R. JONES, Index for subfactors, *Invent. Math.* **72**(1983), 1–25.
11. V.F.R. JONES, On a family of almost commuting endomorphisms, *J. Funct. Anal.* **122**(1994), 84–90.
12. P.E.T. JORGENSEN, A duality for endomorphisms of von Neumann algebras, *J. Math. Phys.* **37**(1996), 1521–1538.
13. M. LACA, Endomorphisms of $B(H)$ and Cuntz algebras, *J. Operator Theory* **30**(1993), 85–108.
14. M. LACA, Gauge invariant states on $\mathcal{O}_\infty$, *J. Operator Theory* **30**(1993), 381–396.
15. H. NARNHOFER, E. STØRMER, W. THIRRING, C^*-dynamical systems for which the tensor product formula for entropy fails, *Ergodic Theory Dynamical Systems* **15**(1995), 961–968.
16. M. NEWMAN, *Integral Matrices*, Academic Press, New York 1972.
17. R.T. POWERS, An index theory for semigroups of $*$-endomorphisms of $B(H)$ and type II_1 factors, *Canad. J. Math.* **40**(1988), 86–114.
18. R.T. POWERS, G.L. PRICE, Binary shifts on the hyperfinite II_1 factor, *Contemp. Math.* , Amer. Math. Soc., vol. 145, 1993, pp. 453–464.
19. R.T. POWERS, G.L. PRICE, Cocycle conjugacy classes of shifts on the hyperfinite II_1 factor, *J. Funct. Anal.* **121**(1994), 275–295.
20. G.L. PRICE, Shifts on type II_1 factors, *Canad. J. Math.* **39**(1987), 492–511.
21. H. YIN, Entropy of certain noncommutative shifts, *Rocky Mountain J. Math.* **20** (1990), 651–656.

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