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A DECOMPOSITION THEOREM FOR OPERATORS ON L^1

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ABSTRACT. The operator space $\mathcal{L}(L^1)$, as a Banach lattice, can be decomposed into four bands: the Radon-Nikodym band, the Dunford-Pettis band, the Rosenthal band, and the Enflo band. Thus, each operator in $\mathcal{L}(L^1)$ can be decomposed uniquely into the sum of four operators, so that each member of the decomposition has a characterization in terms of natural videly discussed operator-theoretic invariants in Banach space theory.

KEYWORDS: *operator, decomposition, Banach lattice, band.*

AMS SUBJECT CLASSIFICATION: 46A32, 46B28, 46B42, 46E27, 47B38.

This paper is based on work by Kalton ([12]), Bourgain ([3], [4]), Rosenthal ([14]), and Enflo-Starbird ([8]). Section 1 mainly states the required notation and known results, and Section 2 gives the results of this paper.

The main result of this paper is that each operator $T \in \mathcal{L}(L^1)$ can be uniquely written as a sum of operators which are respectively Radon-Nikodym, non-representable Dunford-Pettis (we will describe this in more detail later), Rosenthal and Enflo. In fact, there exists a unique decomposition of $\mathcal{L}(L^1)$

$$\mathcal{L}(L^1) = \mathcal{L}_{\mathrm{RN}} \oplus \mathcal{L}_{\mathrm{SDP}} \oplus \mathcal{L}_{\mathrm{R}} \oplus \mathcal{L}_{\mathrm{E}}$$

with the following properties:

(a) each subspace on the right is a band;
(b) $T \in \mathcal{L}_{\mathrm{RN}}$ iff T is Radon-Nikodym;

(c) $T \in \mathcal{L}_{\rm SDP}$ iff $T = 0$ or T is Dunford-Pettis and for any $0 \neq S \in \mathcal{L}(L^1)$ with $|S| \leqslant |T|$, S is not Radon-Nikodym;

(d) $T \in \mathcal{L}_{\rm R}$ iff $T = 0$ or T is Rosenthal, and for any $0 \neq S \in \mathcal{L}(L^1)$ with $|S| \leqslant |T|$, S is Rosenthal;

(e) $T \in \mathcal{L}_{\rm E}$ iff $T = 0$ or T is Enflo and for any $0 \neq S \in \mathcal{L}(L^1)$ with $|S| \leqslant |T|$, S is Enflo.

In particular, we are going to show that the Enflo part of an operator in $\mathcal{L}(L^1)$ has a similar form to that given in Corollary 1.6.

1. PRELIMINARIES

Throughout this paper, Σ_0 denotes the Lebesgue σ-algebra on $[0,1]$ and $\mathcal{B}$ denotes the Borel σ-algebra on $[0,1]$. For an arbitrary topological space K, $\mathcal{B}(K)$ is the Borel σ-algebra on K, and $\mathcal{U}(K)$ is the σ-algebra of universally measurable sets.

In what follows, λ is the Lebesgue measure on $[0,1]$ and $L^1 = L^1([0,1],\lambda)$ is the space of all equivalence classes of Lebesgue integrable functions on $[0,1]$. Each measure space (X,Σ,ν) in this paper will be taken to be a measure space with ν purely non-atomic unless specially indicated. A purely non-atomic σ-subalgebra $\mathcal{A}$ of Σ is a σ-subalgebra with $\nu|\mathcal{A}$ purely non-atomic.

As usual, $L^\infty = L^\infty(\lambda) = (L^1)^*$ is the space of all bounded Lebesgue measurable functions with the uniform convergence norm.

For a compact metric space K, $C(K)$ is the space of bounded continuous functions, and $M(K) = C(K)^*$, the space of regular Borel measures on K. $\mathcal{B}^*$ is the weak* Borel σ-algebra on $M[0,1]$, and $\mathcal{U}^* = \mathcal{U}(M(K))$. The subspace of atomic measures in $M[0,1]$ is denoted by $M_{\rm a}$, and the subspace of continuous measures is denoted by $M_{\rm c}$. Recall that

$$M[0,1] = M_{\rm a} \oplus M_{\rm c}.$$

As usual, $\mathcal{L}(X,Y)$ is the space of all bounded operators from a Banach space X to a Banach space Y. If $X = Y$, then we denote $\mathcal{L}(X,Y)$ by $\mathcal{L}(X)$.

For $E \subset [0,1]$, χ_E is the characteristic function of E. If T is an operator on some space of functions on $[0,1]$, we denote $T\chi_E$ by TE.

The operator $T|E$ on a space X of functions on $[0,1]$ is defined by

$$(T|E)(f) = T(f|E), \quad \text{for all } f \in X.$$

If T is an operator on a space X of measurable functions with respect to some σ-algebra Σ on [0,1], and $\mathcal{A}$ is a σ-subalgebra of Σ, then $T|\mathcal{A}$ is the restriction of T to the space of all $\mathcal{A}$-measurable functions that lie in X.

A *bush* is a sequence of finite measurable partitions $(E_{n,i})$, $i = 1, \dots, m_n$, $n = 1, 2, \dots$, of a measurable subset $E_{0,1}$ of [0,1], such that:

(a) $m_0 = 1$, $\lambda(E_{0,1}) > 0$;

(b) for each n, $\bigcup_{i=1}^{m_n} E_{n,i} = E_{0,1}$;

(c) for each n, $E_{n,i} \cap E_{n,j} = \emptyset$ if $i \neq j$;

(d) for each n and each j, $1 \leqslant j \leqslant m_{n+1}$, there is an i, $1 \leqslant i \leqslant m_n$ with $E_{n+1,j} \subset E_{n,i}$;

(e) $\max_{1 \leqslant i \leqslant m_n} 1(E_{n,i}) \to 0$ as $n \to \infty$.

A *tree* is a bush $(E_{n,i})$, $1 \leqslant i \leqslant m_n$, $n = 1, 2, \dots$, in which $m_n = 2^n$ and $E_{n,i} = E_{n+1,2(i-1)} \cup E_{n+1,2i}$. A *binary tree* is a tree $(E_{n,i})$ with $\lambda(E_{n,i}) = (1/2^n)\lambda(E_{0,1})$ for all n and i. $(\Delta_{n,i})$ denotes the usual binary tree on $[0, 1)$, i.e. $\Delta_{n,i} = \big[(i-1)/2^n, i/2^n\big)$.

Let $T \in \mathcal{L}(L^1, X)$. Then T is *Radon-Nikodym* if there is a $g \in L^\infty(X)$ such that for each $f \in L^1$

$$Tf = \int fg \, \mathrm{d}\lambda.$$

T is *Dunford-Pettis* if T maps weakly compact subsets to relatively compact subsets.

If $T \in \mathcal{L}(L^1)$, then T is *Enflo* if it fixes a subspace which is isomorphic to L^1; T is *Rosenthal* if T is neither Enflo nor Dunford-Pettis. We will call an operator $T \in \mathcal{L}(L^1)$ which satisfies the condition (d) (resp. (e)) in the beginning of the paper a *pure Rosenthal* (resp. *Enflo*) *operator*.

It is well known that the spaces $\mathcal{L}(L^1)$, $\mathcal{L}(L^\infty)$, $\mathcal{L}(M(K))$ and $\mathcal{L}(C(K)^{**})$ are order complete Banach lattices (see, e.g. [16] or [1]). We will use these facts without mentioning them. Throughout this paper, T^+ and T^- denote the positive part and the negative part of an operator T respectively, and $|T|$ denotes the total variation of T.

The following are some known results that we need. First, we state a representation theorem by Kalton ([12]).

THEOREM 1.1. *Let K be a compact metric space, μ a probability measure on K, and (X, Σ, ν) a measure space. Then*

$$T : L^1(K, \mathcal{B}(K), \mu) \to L^1(X, \Sigma, \nu)$$

is a bounded linear operator iff it has the form

$$Tf(x) = \int_K f(t) \, \mathrm{d}\mu_x(t) \quad \nu\text{-a.e. } x \in X \tag{1.1}$$

where $x \to \mu_x$ is a Σ–$\mathcal{B}^*$ measurable map of X into $M(K)$, satisfying

$$\sup_{\mu(B)>0} \frac{1}{\mu(B)} \int\limits_K |\mu_x|(B)\,\mathrm{d}\nu(x) = M < \infty. \tag{1.2}$$

In this case, $\|T\| = M$. The map $x \to \mu_x$ is unique up to sets of μ-measure zero.

REMARK 1.2. Given a map $x \to \mu_x$ satisfying (1.2), the operator in (1.1) is well-defined, i.e. its definition is independent of the member chosen from an equivalent class in $L^1(\mu)$. In fact, for a μ-null set E and any $n \in \mathbb{N}$, one can choose an open set U with $E \subset U$ and $\mu(U) < 1/n$, and then

$$\int\limits_K |\mu_x|(E)\,\mathrm{d}\nu(x) < \int\limits_K |\mu_x|(U)\,\mathrm{d}\nu(x) < M/n \to 0,$$

and so $\int\limits_K |\mu_x|(E)\,\mathrm{d}\nu(x) = 0$. Thus, if $\lambda(E \triangle F) = 0$, one has

$$\mu_x(E) = \mu_x(F), \quad \lambda\text{-a.e.}$$

Since the measurable simple functions are dense in $L^1(K, \mathcal{B}(K), \mu)$, it easily follows that if $f = g$, λ-a.e., then

$$\int f\,\mathrm{d}\mu_x = \int g\,\mathrm{d}\mu_x \quad \lambda\text{-a.e.}$$

This is very important for later discussions.

THEOREM 1.3. ([12]) *Let K be a compact metric space. Then there exist $\mathcal{U}^*$ measurable mappings $b_n : M(K) \to \mathbb{R}$, ($n \in \mathbb{N}$), $\mathcal{U}^*$–$\mathcal{B}(K)$ measurable mappings $h_n : M(K) \to K$, ($n \in \mathbb{N}$), and a $\mathcal{U}^*$–$\mathcal{B}^*$ measurable mapping $\varphi : M(K) \to M_{\mathrm{c}}(K)$ such that*

$$|b_n(\mu)| \geqslant |b_{n+1}(\mu)|, \quad n \in \mathbb{N},\ \mu \in M(K);$$

$$h_n(\mu) \neq h_m(\mu), \quad n \neq m,\ \mu \in M(K);$$

and

$$\mu = \sum_{n=1}^{\infty} b_n(\mu)\delta_{h_n}(\mu) + \varphi(\mu), \quad \mu \in M(K).$$

THEOREM 1.4. ([12]) *Let K be an infinite Polish space, μ a probability measure on K, and (X, Σ, ν) a measure space. Then*

$$T : L^1(K, \mathcal{B}(K), \mu) \to L^1(X, \Sigma, \nu)$$

is a bounded linear operator iff it has the form

$$Tf(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n(x)) + \int_K f(s) \mathrm{d}\rho_x(s) \tag{1.3}$$

where

(i) $a_n : X \to \mathbb{R}$ *is Borel measurable with* $|a_n(x)| \geqslant |a_{n+1}(x)|$ ν*-a.e.;*
(ii) $\sigma_n : X \to K$ *is* Σ–$\mathcal{B}$ *measurable with* $\sigma_n(x) \neq \sigma_m(x)$, $m \neq n$, $x \in X$*;*
(iii) $x \to \rho_x$ *is* Σ–$\mathcal{B}^*$ *measurable from* X *into* $M(K)$ *with* $\rho_x \in M_{\mathrm{c}}(K)$ ν*-a.e.;*
(iv) $\sup\limits_{\mu(B)>0} \frac{1}{\mu(B)} \Big[\sum\limits_{n=1}^{\infty} \int\limits_{\sigma_n^{-1}(B)} |a_n(x)| \,\mathrm{d}\nu(x) + \int\limits_X |\rho_x|(B) \,\mathrm{d}\nu(x) \Big] = M < \infty.$

In this case, $\|T\| = M$.

REMARK 1.5. By Kalton ([12]), one also has

$$T^{\pm} f(x) = \sum_{n=1}^{\infty} a_n^{\pm}(x) f(\sigma_n(x)) + \int_K f(s) \,\mathrm{d}\rho_x^{\pm}(s)$$

and

$$|T| f(x) = \sum_{n=1}^{\infty} |a_n(x)| f(\sigma_n(x)) + \int_K f(s) \,\mathrm{d}|\rho_x|(s).$$

On the right side of (1.3), we denote the first part by T_{a} and the second part by T_{c}, and call them the *purely atomic part* and the *purely continuous part* of T, respectively. If T has only purely atomic part (resp. purely continuous part), then we say that T is *purely atomic* (resp. *purely continuous*).

By Theorem 1.4, if $x \to \nu_x$ represents T_{a}, then for almost all $x \in X$, ν_x has the form

$$\nu_x = \sum_{n=1}^{\infty} a_n(x) \delta_{\sigma_n(x)}. \tag{1.4}$$

Conversely, if $x \to \nu_x$ represents an operator $T \in \mathcal{L}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$ such that ν_x has the form (1.4) for almost all $x \in X$, then by the proof of Theorem 1.4 in [12], T is purely atomic.

We thus have the following corollary (cf. [15]).

COROLLARY 1.6. *With the notation of Theorem* 1.4*, every bounded linear operator*

$$T : L^1(K, \mathcal{B}(K), \mu) \to L^1(X, \Sigma, \nu)$$

can be uniquely written as

$$T = T_{\mathrm{a}} + T_{\mathrm{c}}.$$

Moreover, one has

$$|T| = |T_{\mathrm{a}}| + |T_{\mathrm{c}}| = |T|_{\mathrm{a}} + |T|_{\mathrm{c}}$$

where T_{a} has the form

$$T_{\mathrm{a}} = \sum_n A_n$$

with

$$A_n f(x) = a_n(x) f(\sigma_n(x)).$$

In Corollary 1.6, $\sum\limits_n A_n$ is the *strong ℓ^1-sum*. This means that there exists a $K < \infty$, such that for all $f \in L^1(\mu)$, $\sum \|A_n f\| \leqslant K\|f\|$ and $T_{\mathrm{a}} f = \sum A_n f$. It is known that

$$K \leqslant \|T\|,$$

and each A_n maps disjoint functions to disjoint functions, i.e. if $|f| \wedge |g| = 0$, a.e., then $|A_n f| \wedge |A_n g| = 0$, a.e. ([15]). We call such operators *atoms*.

We can also write T_{a} as

$$T_{\mathrm{a}} = {\sum_n}' A_n,$$

where $\sum'$ denotes the *pointwise sum* of operators, which is defined as follows.

For $1 \leqslant p \leqslant \infty$, $f_n \in L^p$, $n = 1, 2, \ldots$, we say that ${\sum\limits_n}' f_n$ exists if the pointwise sum $\sum |f_n(x)|$ belongs to L^p; then ${\sum\limits_n}' f_n$ denotes the pointwise (a.e.) sum of f_n's, which of course belongs to L^p. For a sequence of operators $(T_n) \subset \mathcal{L}(L^p)$, we say that ${\sum\limits_n}' T_n$ exists if $\sum' T_n f$ exists for all $f \in L^p$. In this case, it follows easily from the closed graph theorem that

$$f \to {\sum_n}' T_n f$$

defines a bounded linear operator on L^p.

We note that:

(i) if $T_n \in \mathcal{L}(L^p)$, $n = 1, 2, \ldots$, are such that ${\sum\limits_n}' T_n$ exists, then also ${\sum\limits_n}' T_n^*$ exists, and $\Big({\sum\limits_n}' T_n\Big)^* = {\sum\limits_n}' T_n^*$;

(ii) given $T_n \in \mathcal{L}(L^1)$, $n = 1, 2, \ldots$, then if $\sum_n T_n$ is a strong ℓ^1-sum, then ${\sum\limits_n}' T_n$ exists, and if the T_n's are all positive, the converse is true.

Let $\mathcal{L}_{\rm a} = \mathcal{L}_{\rm a}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$ denote the set of all purely atomic operators, and $\mathcal{L}_{\rm c} = \mathcal{L}_{\rm c}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$ the set of all purely continuous operators. Then by Corollary 1.6, both $\mathcal{L}_{\rm a}$ and $\mathcal{L}_{\rm c}$ are 1-complemented closed ideals (and so are projection bands) of $\mathcal{L}(L^1)$, and

$$\mathcal{L}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu)) = \mathcal{L}_{\rm a} \oplus \mathcal{L}_{\rm c} \tag{1.5}$$

(see [9]).

DEFINITION 1.7. Let $T : L^1 \to L^1$ be a bounded linear operator, and $\mathbf{B} = (E_{n,i})$ be a bush of subsets of [0,1]. Define

$$\lambda_T(\mathbf{B}) = \lim_n \max_i |TE_{n,i}|.$$

In particular, if $\mathbf{B} = (\Delta_{n,i})$ we denote $\lambda_T(\mathbf{B})$ by λ_T.

If $\mathcal{A}$ is the σ-algebra generated by $\mathbf{B} = (E_{n,i})$ we also denote $\lambda_T(\mathbf{B})$ by $\lambda_T(\mathcal{A})$.

The function $\lambda_T(\mathbf{B})$ is called the *Enflo-Starbird maximal function* with respect to the bush $\mathbf{B}$, and λ_T is simply called the *Enflo-Starbird maximal function.* The existence of the limit in the definition was proved in [8]. Later, we will show that λ_T is exactly the function $|a_1|$ with a_1 as in Theorem 1.4, and thus $\lambda_T(\mathcal{A})$ is independent of the choice of the generating bush.

T is said to be *sign-preserving* if there exists a set S of positive measure and a $\delta > 0$, so that

$$\|Tf\| \geqslant \delta$$

for all f with

$$\int f \, \mathrm{d}\mu = 0 \quad \text{and} \quad |f| = \chi_S \text{ a.e.}$$

This definition was initially given by Rosenthal ([15]).

THEOREM 1.8. ([15]) *Let $T \in \mathcal{L}(L^1)$. Then the following are equivalent:*

(i) *T has an atomic part;*

(ii) *λ_T is a non-zero element in L^1;*

(iii) *T is sign preserving;*

(iv) *there is a set S with $\lambda(S) > 0$ so that $T|S$ is an isomorphism.*

THEOREM 1.9. ([8]) *$T \in \mathcal{L}(L^1)$ is Enflo iff $\lambda_T(\mathbf{B}) \neq 0$ for some bush $\mathbf{B}$.*

THEOREM 1.10. ([14]) *There exists an operator in $\mathcal{L}(L^1)$ which is neither Dunford-Pettis nor Enflo. In other words, the Rosenthal operators exist.*

Bourgain ([4]) shows that the space of all Dunford-Pettis operators in $\mathcal{L}(L^1)$ is a solid sublattice. To prove this, he gives the following:

PROPOSITION 1.11. *For $T : L^1 \to L^1$, the following are equivalent:*

(i) *T is a Dunford-Pettis operator;*

(ii) *$T1_p$ is compact for $1 < p \leqslant \infty$;*

(iii) *$T1_\infty$ is compact.*

($1_p : L^p \to L^1$, $1 < p \leqslant \infty$, *are the canonical maps.*)

To show that all Dunford-Pettis operators form a band, we need some results from the theory of Banach lattices.

Let X, Y be vector spaces and $\mathcal{B}(X, Y)$ be the vector space of all bilinear forms on $X \times Y$. Define the bilinear form $x \otimes y$ on $\mathcal{B}(X, Y)$ by

$$x \otimes y(f) = f(x, y).$$

Then, the canonical map

$$\chi : (x, y) \to x \otimes y$$

is a bilinear map from $X \times Y$ into $\mathcal{B}(X, Y)^*$. The linear span of $\chi(X \times Y)$ in $\mathcal{B}(X, Y)^*$ is called the *tensor product* of X and Y, and is denoted by $X \otimes Y$.

If X and Y are Banach spaces, one can define different norms on $X \otimes Y$. What we need here is the so-called *ε-norm*, which is defined as

$$\|u\|_\varepsilon = \sup\{|(x^* \otimes y^*)(u)| : (x^*, y^*) \in B_{\mathrm{a}}(X^*) \times B_{\mathrm{a}}(Y^*)\}.$$

In this paper, a bounded linear operator $T : L^\infty \to L^1$ is called *integral* if

$$b_T \in (L^\infty \otimes_\varepsilon L^\infty)^*,$$

where b_T is defined by

$$b_T(f, g) = \langle Tf, g\rangle, \quad f, g \in L^\infty.$$

The *integral norm* $\|T\|_{\mathrm{i}}$ of T is defined by

$$\|T\|_{\mathrm{i}} = \|b_T\|.$$

Recall that an operator $T : L^\infty \to L^1$ is *regular* if T has the form $T = T^+ - T^-$ with both T^+ and T^- positive. The space of all such operators is denoted by $\mathcal{L}^{\mathrm{r}}(L^\infty, L^1)$.

THEOREM 1.12. ([16])
(i) *$\mathcal{L}^{\mathrm{r}}(L^\infty, L^1)$ is an* AL*-space with the integral norm.*
(ii) *If T is positive, then $\|T\|_{\mathrm{i}} = \|T\|$.*

Since $\mathcal{L}(L^1)$ is a Banach lattice, if we define $i_\infty : L^\infty \to L^1$ to be the canonical map, then for each $T \in \mathcal{L}(L^1)$,

$$Ti_\infty = T^+ i_\infty - T^- i_\infty,$$

i.e., any operator of the form Ti_∞ is a member of $\mathcal{L}^{\mathrm{r}}(L^\infty, L^1)$. In order to apply the lattice property of Theorem 1.12, we need the following facts about AL-spaces.

THEOREM 1.13. *Let X be an* AL*-space. Then:*
(i) *X is order complete, that is, for each non-empty majorized set $M \subset X$, $\sup M$ exists in X.*
(ii) *Each upward directed ($\leqslant$) norm bounded family in X converges in norm.*

2. THE RESULTS OF THIS PAPER

LEMMA 2.1. *Let K be a compact metrix space, $\mu \in M(K)$ and $B(\mu)$ be the band generated by μ. Then*

$$B(\mu) = L^1(K, \mu) = \{\nu \in M(K) : \nu \ll \mu\},$$

and both $B(\mu)$ and $B(\mu)^\perp$ are weak-Borel subsets of $M(K)$.*

Proof. Let $\mathcal{O}$ be a countable base of the topology on K consisting of balls. For each m, k and disjoint $U_1, \dots, U_n \in \mathcal{O}$ with

$$|\mu|\Big(\bigcup_{1 \leqslant j \leqslant n} U_j\Big) < \frac{1}{m+k},$$

write

$$V(U_1, \dots, U_n, m, k) = \Big\{\nu \in M(K) : |\nu|\Big(\bigcup_{1 \leqslant j \leqslant n} U_j\Big) < \frac{1}{m}\Big\}$$

Then $V(U_1, \dots, U_n, m, k)$ is a weak*-Borel set, and so is the set

$$A = \bigcap_m \bigcup_k \bigcap_{(U_1, \dots, U_n)} V(U_1, \dots, U_n, m, k).$$

We claim that

$$A = B(\mu).$$

In fact, if $\nu \in B(\mu)$, then $\nu \leqslant \mu$. Thus for each m, there exists a k such that for any open sets $U_1, \dots, U_n$, with

$$|\mu|\Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big) < \frac{1}{m+k},$$

one has

$$|\nu|\Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big) < \frac{1}{m}.$$

This implies $\nu \in A$.

Conversely, if $\nu \in A$, let $\nu = \nu_1 + \nu_2$, where $|\nu_1| \ll |\mu|$ and $|\nu_2| \perp |\mu|$. Assume $|\nu_2| \neq 0$; then there exists a Borel set E such that

$$|\nu_2|(E) > 0, \quad |\mu|(E) = 0.$$

Choose m such that

$$\frac{1}{m} < |\nu_2|(E).$$

Then, by the definition of A, there exists a k such that

$$\nu \in \bigcap_{(U_1,\dots,U_n)} V(U_1, \dots, U_n, m, k).$$

Since $\nu_2 \perp \mu$, we can choose $U_1, \dots, U_n \in \mathcal{O}$, such that

$$|\mu|\Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big) < \frac{1}{m+k}$$

and

$$|\nu_2|\Big(E \triangle \Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big)\Big) < |\nu_2|(E) - \frac{1}{m}.$$

Then, by the definition of $V(U_1, \dots, U_n, m, k)$, one has

$$|\nu_2|\Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big) \leqslant |\nu|\Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big) \leqslant \frac{1}{m},$$

and so

$$|\nu_2|(E) < |\nu_2|\Big(E \triangle \Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big)\Big) + |\nu_2|\Big(\bigcup_{1\leqslant j\leqslant n} U_j\Big) < |\nu_2|(E),$$

which is a contradiction. Hence $\nu_2 = 0$ and $\nu = \nu_1 \in B(\nu)$. This proves that $B(\mu)$ is a weak*-Borel set.

For the second part, let $O_1, \dots, O_n \in \mathcal{O}$ with

$$|\mu|\Big(\bigcup_{1\leqslant j\leqslant n} O_j\Big) > |\mu| - \frac{1}{k}.$$

Define

$$V(O_1, \dots, O_n, m, k) = \Big\{\nu : |\nu|\Big(\bigcup_{1\leqslant j\leqslant n} O_j\Big) < \frac{1}{m}\Big\}$$

and

$$B = \bigcap_m \bigcap_k \bigcup_{(O_1,\dots,O_n)} V(O_1, \dots, O_n, m, k).$$

Then, since for each $\nu \in B(\mu)^\perp$ there is a Borel set E such that

$$|\mu|E = \|\mu\| \quad \text{and} \quad |\nu|E = 0,$$

one has that for fixed k, m there exist open sets $O_1, \dots, O_n$, such that

$$|\mu|\Big(\bigcup_{1\leqslant j\leqslant n} O_j\Big) > \|\mu\| - \frac{1}{k}$$

$$|\nu|\Big(\bigcup_{1\leqslant j\leqslant n} O_j\Big) < \frac{1}{m}.$$

Thus

$$\nu \in \bigcup_{(O_1,\dots,O_n)} V(O_1, \dots, O_n, m, k).$$

Fix an m. Then, since for each k there exists a $V(O_1, \dots, O_n, m, k)$ which contains ν, we have

$$\nu \in \bigcap_k \bigcup_{(O_1,\dots,O_n)} V(O_1, \dots, O_n, m, k),$$

and since this holds for each m, we have $\nu \in B$. So $B(\mu)^\perp \subset B$.

On the other hand, if $\nu \notin B(\mu)^\perp$, then $|\nu| \wedge |\mu| \neq 0$, and so there exists a Borel set F with $|\mu|(F) \neq 0$ such that

$$|\mu|(E) \neq 0 \Leftrightarrow (|\nu| \wedge |\mu|)(E) \neq 0$$

for all Borel subset $E \subset F$. Choose m such that

$$(|\nu| \wedge |\mu|)(F) > \frac{1}{m}.$$

Since $|\nu| \wedge |\mu| \leqslant |\mu|$, there exists a k such that for all Borel sets G

$$|\mu|(G) < \frac{1}{k} \Rightarrow |\nu \wedge \mu|(G) < |\nu \wedge \mu|(E) - \frac{1}{m}.$$

It follows that

$$|\mu|\Big(\bigcup_{1\leqslant j\leqslant n} O_j\Big) > |\mu| - \frac{1}{k} \Rightarrow |\nu|\Big(\bigcup_{1\leqslant j\leqslant n} O_j\Big) > \frac{1}{m}$$

for $O_1, \dots, O_n \in \mathcal{O}$. Thus $\nu \notin \bigcup_{(O_1,\dots,O_n)} V(O_1, \dots, O_n, m, k)$ for all k, and so $\nu \notin B$. This shows that $B \subset B(\mu)^{\perp}$. ∎

LEMMA 2.2. *Let K be a compact metric space, and let φ be as in Theorem 1.3. Then for any Borel measure μ on K, there exist $\mathcal{U}^*$–$\mathcal{B}^*$ measurable mappings $\varphi_1 : M(K) \to B(\mu)$ and $\varphi_2 : M(K) \to B(\mu)^{\perp}$ such that $\varphi = \varphi_1 + \varphi_2$.*

Proof. Let $M(K) \times L^1(\mu)$ take the product weak*-topology on $M(K) \times M(K)$. Consider the map $\tau : M(K) \times L^1(\mu) \to M(K)$ defined by

$$\tau(\nu, f) \to \nu - f\,\mathrm{d}\mu.$$

It is w*-continuous and so is a Borel map. Let

$$A = (2\mathrm{Ba}\,(M(K)) \times 2\mathrm{Ba}\,(L^1(\mu)) \cap \tau^{-1}(\mathrm{Ba}\,(M(K))$$

and let $\tau' = \tau|A$. Then since $\mathrm{Ba}\,(M(K)) \times \{0\} \subset A$,

$$\tau' : A \to \mathrm{Ba}\,(M(K))$$

is a surjection. By Lemma 2.1, $B = \tau'^{-1}(\mathrm{Ba}\,(B(\mu)^{\perp}))$ is a Borel subset of A, and so is a Suslin space.

Define $\sigma : B \to \mathrm{Ba}\,(M(K))$ by

$$\sigma(\nu, f) = \nu;$$

then σ is a weak*-continuous surjection. Thus, by Theorem 2.2 in [12] (a consequence of Kuratowski-Ryll-Nardzewski selection theorem), there is a $\mathcal{U}^*$–$\mathcal{B}^*$ measurable map

$$h : \mathrm{Ba}\,(M(K)) \to B,$$

such that

$$\sigma(h(\nu)) = \nu.$$

Let $h(\nu) = (\nu, f)$, and

$$\psi(\nu) = \tau'(h(\nu)) = \nu - f\,\mathrm{d}\mu;$$

thus

$$(\nu - f\,\mathrm{d}\mu) \perp \mu.$$

Then $\psi : \mathrm{Ba}\,(M(K)) \to B(\mu)^{\perp}$ is $\mathcal{U}^*$–$\mathcal{B}^*$ measurable. Since

$$M(K) = B(\mu) \otimes B(\mu)^{\perp},$$

if Q is the band projection of $M(K)$ onto $B(\mu)^{\perp}$, then ψ is in fact the restriction of Q to $\mathrm{Ba}\,(M(K))$. Thus Q is $\mathcal{U}^*$–$\mathcal{B}^*$ measurable. Let $\varphi_1 = (I - Q)\varphi$, $\varphi_2 = Q\varphi$, and the conclusion follows. ∎

The next result is the first step of the proof of the main theorems in this paper.

PROPOSITION 2.3. *With the same hypothesis as in Theorem* 1.4, T *is a bounded linear operator iff it has the form*

$$Tf(x) = \sum_{n=1}^{\infty} a_n(x) f(\sigma_n(x)) + \int\limits_K f(s)\,\mathrm{d}\rho_x(s) + \int\limits_K f(s) g_x(s)\,\mathrm{d}\mu(s), \ \ a.e.$$

where for all n, a_n *and* σ_n *are as in Theorem* 1.4, *and*

(i) $x \to g_x\,\mathrm{d}\mu$ *is* Σ–$\mathcal{B}^*$ *measurable from* $[0,1]$ *to* $M(K)$ *with* $g_x \in L^{\infty}(\mu)$, *a.e.;*

(ii) $x \to \rho_x$ *is* Σ–$\mathcal{B}^*$ *measurable from* $[0,1]$ *to* $M_{\mathrm{c}}(K)$ *with* $\rho_x \perp \mu$, *a.e.*

Proof. Consider $L^1(K, \mathcal{B}(K), \mu)$ as a band in $M(K)$. Then the conclusion follows by Theorem 1.4 and Lemma 2.2. ∎

REMARK 2.4. Let

$$\mathcal{L}_{\mathrm{RN}} = \mathcal{L}_{\mathrm{RN}}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$$

be the space of all representable operators, and

$$\mathcal{L}_{\mathrm{s}} = \mathcal{L}_{\mathrm{s}}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$$

be the space of all operators T satisfying that, if $X \to \mu_x$ is the Kalton representation for T as in Theorem 1.1, then $\mu_x \perp \lambda$ for almost all $x \in [0,1]$. By [9], both $\mathcal{L}_{\mathrm{s}}$ and $\mathcal{L}_{\mathrm{RN}}$ are 1-complemented closed ideals (and so are projection bands) in $\mathcal{L}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$, and

$$\mathcal{L}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu)) = \mathcal{L}_{\mathrm{RN}} \oplus \mathcal{L}_{\mathrm{s}}. \tag{2.1}$$

Thus, each $T \in \mathcal{L}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$ can be uniquely written as

$$T = T_{\mathrm{RN}} + T_{\mathrm{s}}$$

with $T_{\mathrm{RN}} \in \mathcal{L}_{\mathrm{RN}}$ and $T_{\mathrm{s}} \in \mathcal{L}_{\mathrm{s}}$. Moreover,

$$|T| = |T_{\mathrm{RN}}| + |T_{\mathrm{s}}| = |T|_{\mathrm{RN}} + |T|_{\mathrm{s}}. \tag{2.2}$$

By Proposition 2.3, the related band projections are weak*-Borel maps.

We call T_{RN} the *Radon-Nikodym part* (or *representable part*) of T and T_{s} the *singular part* of T. It is easy to see that T_{s} is exactly the sum of the first two parts in the equations of Proposition 2.3, and T_{RN} is the third part. $\mathcal{L}_{\mathrm{RN}}$ and $\mathcal{L}_{\mathrm{s}}$ are called the *Radon-Nikodym band* and the *singular band* of $\mathcal{L}(L^1)$, respectively.

The next several lemmas are used to split off a non-Enflo part from the singular part of the operators in $\mathcal{L}(L^1)$, which we do in Proposition 2.13.

LEMMA 2.5. *Suppose $T \in \mathcal{L}(L^1)$. Let $\mathbf{B} = (E_{n,i})$ be a bush on $[0,1]$. For fixed n, i, let*

$$\mathbf{B}_{n,i} = \mathbf{B} \cap E_{n,i} = \{E_{n,i} \cap E_{m,j} : m, j \text{ are arbitrary}\}.$$

Then for every m,

$$\lambda_T(\mathbf{B}) = \max_j \lambda_T(\mathbf{B}_{m,j})$$

where $\lambda_T(\cdot)$ is the Enflo-Starbird maximal function.

Proof. Since for $n \geqslant m$,

$$g_n(\mathbf{B}) = \max_i |TE_{n,i}| \geqslant \max_i \{|TE_{n,i}| : E_{n,i} \subset E_{m,i}\} = g_n(\mathbf{B}_{m,i}),$$

by the definition of $\lambda_T(\cdot)$, we have

$$\lambda_T(E) \geqslant \max_j \lambda_T(\mathbf{B}_{m,j}) \quad \text{a.e.}$$

On the other hand, for almost all t and every $\delta > 0$, if n is large enough, then

$$\lambda_T(\mathbf{B})(t) < g_n(\mathbf{B})(t) + \frac{\delta}{2}$$

and

$$g_n(\mathbf{B}_{m,j}) < \lambda_T(\mathbf{B}_{m,j})(t) + \frac{\delta}{2}.$$

Thus

$$\begin{aligned} \lambda_T(\mathbf{B})(t) < \max_i |TE_{n,i}(t)| + \frac{\delta}{2} &= \max_j \max_i \{|TE_{n,i}(t)| : E_{n,i} \subset E_{m,j}\} \\ &\leqslant \max_j \lambda_T(\mathbf{B}_{m,j})(t) + \delta. \end{aligned}$$

Since δ is arbitrary,

$$\lambda_T(E)(t) \leqslant \max_j \lambda_T(E_{m,j})(t).$$

The lemma is proved. ∎

LEMMA 2.6. *Suppose $T \in \mathcal{L}(L^1)$. Then*

$$\lambda_T = |a_1|, \quad a.e.,$$

where a_1 is as in Theorem 1.4 (i)*, and*

$$\lambda_T = \lambda_{|T|} = \max(\lambda_{T^+}, \lambda_{T^-}).$$

Proof. Let a_n, σ_n, $n = 1, 2, \ldots$, be as in Theorem 1.4, and $T = T_{\mathrm{a}} + T_{\mathrm{c}}$ be as in Corollary 1.6. Then

$$\max_i |T\Delta_{n,i}| \leqslant \max_i |T_{\mathrm{a}}\Delta_{n,i}| + \max_i |T_{\mathrm{c}}\Delta_{n,i}|.$$

By the definition of T_{c} and Theorem 1.4 (iii), we obtain

$$\max_i |T_{\mathrm{c}}\Delta_{n,i}| \to 0, \quad \text{a.e.}$$

Thus $\lambda_T \leqslant \lambda_{T_{\mathrm{a}}}$, a.e.

Now by Corollary 1.6, there exists a $K < \infty$, such that

$$\sum \|A_n f\| \leqslant K\|f\|, \quad \text{for all } f \in L^1.$$

So by the definition of A_n's and by the bounded convergence theorem,

$$\sum_n |a_n(x)| < \infty$$

for almost all $x \in [0,1]$. Fix such an $x \in [0,1]$ with

$$\lambda_T(x) \leqslant \lambda_{T_{\mathrm{a}}}(x),$$

and let $F_{n,i} = \{k : \sigma_k(x) \in \Delta_{n,i}\}$. Then since by Theorem 1.4 (ii),

$$\sigma_k(x) \neq \sigma_j(x), \quad \text{if } k \neq j,$$

one has

$$\max_i |T_{\mathrm{a}}\Delta_{n,i}|(x) = \max_i \left| \sum_{k \in F_{n,i}} a_k(x) \right| \leqslant \max_i \sum_{k \in F_{n,i}} |a_k(x)|.$$

But since for $k \neq j$, there exists an n, such that

$$\sigma_k(x) \in \Delta_{n,i_k},\ \sigma_j(x) \in \Delta_{n,i_j} \text{ and } \Delta_{n,i_k} \cap \Delta_{n,i_j} = \emptyset,$$

one has

$$\max_i \sum_{k\in F_{n,i}} |a_n(x)| \to \max_n |a_n(x)| = |a_1(x)|,$$

and so

$$\lambda_T(x) \leqslant |a_1(x)|.$$

On the other hand, let $\sigma_1(x) \in \Delta_{n,i_n}$, $i = 1, 2, \ldots$. Then

$$\begin{aligned}\lambda_T(x) &= \lim_n \max_i |T\Delta_{n,i}(x)| \geqslant \overline{\lim_n} |T\Delta_{n,i_n}(x)| \\ &\geqslant |a_1(x)| - \overline{\lim_n} \Big| \sum_{i\neq k\in F_{n,i_n}} a_k(x) \Big| - \lim |T_{\mathrm{c}}\Delta_{n,i_n}(x)| = |a_1(x)|.\end{aligned}$$

The first part of the lemma follows.

The second part of the lemma follows by the remark after Theorem 1.4. ∎

LEMMA 2.7. *Let $T \in \mathcal{L}(L^1)$. Suppose that $\mathbf{B}_1$ and $\mathbf{B}_2$ are two bushes on $[0,1]$ that generate the same atomless σ-algebra $\mathcal{A}$ on a subset A of $[0,1]$. Then*

$$\lambda_T(\mathbf{B}_1) = \lambda_T(\mathbf{B}_2) \quad \text{a.e.}$$

Hence $\lambda_T(\mathcal{A})$ is well defined.

Proof. Through a linear isometry between L^1-spaces, we can suppose $\mathcal{A} = \mathcal{B}$. Thus, we need only show that, for any bush $\mathbf{B} = (E_{n,i})$ on $[0,1]$ which generates $\mathcal{B}$, the following holds:

$$\lambda_T(\mathbf{B}) = \lambda_T = |a_1| \quad \text{a.e.}$$

But again, through a linear isometry on L^1 into L^1, we can suppose that all $E_{n,i}$ are intervals. Then, replacing $(\Delta_{n,i})$ by $(E_{n,i})$, we can repeat word for word the argument in the proof of last lemma. ∎

We have already defined the purely atomic part T_{a} and the purely continuous part T_{c} for $T \in \mathcal{L}(L^1(K, \mathcal{B}(K), \mu), L^1(X, \Sigma, \nu))$ with K a compact metric space. Generally, if (X_1, Σ_1, ν_1) is a separable non-atomic probability space, it is well known that $L^1(X_1, \Sigma_1, \nu_1)$ is linearly isometric to L^1. In particular, the isometry can be realized through an isomorphism between measure algebras (see, e.g., [11]), which we call a *regular isometry.* Such an isometry maps characteristic functions to characteristic functions, and so is also a lattice isomorphism.

Now let $S : L^1(X_1, \Sigma_1, \nu_1) \to L^1$ be a regular isometry. For $T \in \mathcal{L}(L^1(X_1, \Sigma_1, \nu_1), L^1(X, \Sigma, \nu))$, define

$$T_{\mathrm{a}} = (TS^{-1})_{\mathrm{a}} S \quad \text{and} \quad T_{\mathrm{c}} = (TS^{-1})_{\mathrm{c}} S.$$

Then

$$T = T_{\mathrm{a}} + T_{\mathrm{c}}.$$

Moreover, T_{a} and T_{c} are unique. In fact, if $S : L^1(X_1, \Sigma_1, \nu_1) \to L^1$ is another linear isometry, let

$$T'_{\mathrm{a}} = (TS'^{-1})_{\mathrm{a}} S' \quad \text{and} \quad T'_{\mathrm{c}} = (TS'^{-1})_{\mathrm{c}} S'.$$

Suppose $T_{\mathrm{a}} - T'_{\mathrm{a}} \neq 0$; then $(T_{\mathrm{a}} - T'_{\mathrm{a}})S^{-1} \neq 0$ is purely atomic, and $(T_{\mathrm{c}} - T'_{\mathrm{c}})S^{-1}$ is purely continuous (if it is not zero). Thus

$$0 \neq (T_{\mathrm{a}} - T'_{\mathrm{a}})S^{-1} + (T_{\mathrm{c}} - T'_{\mathrm{c}})S^{-1} = (T_{\mathrm{a}} + T_{\mathrm{c}} - T'_{\mathrm{a}} - T'_{\mathrm{c}})S^{-1} = (T - T)S^{-1} = 0.$$

which is a contradiction. So $T_{\mathrm{a}} = T'_{\mathrm{a}}$. Similarly, $T_{\mathrm{c}} = T'_{\mathrm{c}}$.

Thus we can call T_{a} and T_{c} the *purely atomic part* and the *purely continuous part* of T, respectively. By the definition of T_{a} and Corollary 1.6, T_{a} has the form

$$T_{\mathrm{a}} = {\sum}' A_n,$$

where A_n is an atom for each n and the sum is a strong ℓ^1-sum.

LEMMA 2.8. *Let $L^1(X_i, \Sigma_i, \mu_i)$, $i = 1, 2, 3$, be separable and atomless, and*

$$T_i : L^1(X_i, \Sigma_i, \mu_i) \to L^1(X_{i+1}, \Sigma_{i+1}, \mu_{i+1}), \quad i = 1, 2$$

be purely atomic. Then $T = T_2 T_1$ is purely atomic.

Proof. First suppose $L^1(X_i, \Sigma_i, \mu_i) = L^1$, $i = 1, 2$. Let the maps $t \to \mu_t$, $t \in [0, 1]$, and $x \to \nu_x$, $x \in X_3$, represent T_1 and T_2 respectively. Then by (1.4)

$$\mu_t = \sum_n a_n(t) \delta_{\sigma_n}(t), \quad \lambda\text{-a.e.}$$

$$\nu_x = \sum_m b_m(x) \delta_{\tau_m}(x), \quad \mu_3\text{-a.e.}$$

(for the definitions of $a_n, b_m, \sigma_n, \tau_m$, $n, m = 1, 2, \ldots$, see Theorem 1.4). Then it is easy to check that

$$x \to \sum_{m,n} a_n(\tau_m(x)) b_m(x) \delta_{\sigma_n \circ \tau_m}(x)$$

represents $T_2 T_1$, and so by the note below (1.4), $T = T_2 T_1$ is purely atomic.

Now suppose $L^1(X_2, \Sigma_2, \mu_2) = L^1$ and $S = T_2$ is a regular isometry. Let

$$S_1 : L^1(X_1, \Sigma_1, \nu_1) \to L^1$$

be a regular isometry; then, by definition, $T^1 S_1^{-1}$ is purely atomic. Let

$$T = ST_1 S_1^{-1} = T_{\mathrm{a}} + T_{\mathrm{c}};$$

then, since S is a lattice isomorphism as well as a linear isometry,

$$S^{-1} T_{\mathrm{c}} = 0 \Rightarrow T_{\mathrm{c}} = 0.$$

Hence T is purely atomic which implies that ST_1 is purely atomic.

Generally, let $S_1 : L^1(X_i, \Sigma_i, \mu_i) \to L^1$, $i = 1, 2$, be regular isometries. Then $T_i S_i^{-1}$, $i = 1, 2$, are purely atomic, and so by the previous argument, the maps

$$T_2 S_2^{-1} : L^1 \to L^1(X_3, \Sigma_3, \mu_3)$$

and

$$S_2 T_1 : L^1(X_1, \Sigma_1, \mu_1) \to L^1$$

are purely atomic. Hence,

$$(T_2 S_2^{-1})(S_2 T_1 S_1^{-1}) = (T_2 T_1) S_1^{-1} : L^1 \to L^1(X_3, \Sigma_3, \mu_3)$$

is purely atomic, and so $T_2 T_1$ is purely atomic (by definition). ∎

LEMMA 2.9. *Let $\mathcal{A}$ be an atomless σ-subalgebra of Σ_0 on $[0,1]$, and $E \in \Sigma_0$ with $1 > \lambda(E) > 0$. Let $\sigma(\mathcal{A}, E)$ be the σ-algebra generated by $\mathcal{A} \cup \{E\}$. Then*

$$(T|\mathcal{A})_{\mathrm{a}} = (T|\sigma(\mathcal{A}, E))_{\mathrm{a}}|\mathcal{A}.$$

Proof. Let $\mathcal{A}(E) = \mathcal{A} \cap E = \{F \cap E : F \in \mathcal{A}\}$, and define the operator $S_E : L^1(\mathcal{A}) \to L^1(\mathcal{A}(E))$ by

$$S_E f = f|E, \quad f \in L^1(\mathcal{A}).$$

Then, to prove the lemma, we need only to show that

$$(T|\mathcal{A})_{\mathrm{a}} = (T|\mathcal{A}(E))_{\mathrm{a}} S_E + (T|\mathcal{A}(E^{\mathrm{c}}))_{\mathrm{a}} S_{E^{\mathrm{c}}}. \tag{2.3}$$

First we note that each of the operators $T|\mathcal{A}$, $(T|\mathcal{A}(E))S_E$, $(T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}}$ is an operator from $L^1(\mathcal{A})$ into L^1, and

$$(T|\mathcal{A}) = (T|\mathcal{A}(E))S_E + (T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}}.$$

Let $S : L^1(\mathcal{A}) \to L^1$ be a regular isometry; then

$$(T|\mathcal{A})S^{-1} = (T|\mathcal{A}(E))S_E S^{-1} + (T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}} S^{-1}$$

where $(T|\mathcal{A})S^{-1}, (T|\mathcal{A}(E))S_E S^{-1}, (T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}} S^{-1} \in \mathcal{L}(L^1)$. By (1.5), one has

$$[(T|\mathcal{A})S^{-1}]_{\mathrm{a}} = [(T|\mathcal{A}(E))S_E S^{-1}]_{\mathrm{a}} + [(T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}} S^{-1}]_{\mathrm{a}},$$

and

$$[(T|\mathcal{A})S^{-1}]_{\mathrm{a}} S = [(T|\mathcal{A}(E))S_E S^{-1}]_{\mathrm{a}} S + [(T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}} S^{-1}]_{\mathrm{a}} S.$$

By definition, the last equality is the same as

$$(T|\mathcal{A})_{\mathrm{a}} = [(T|\mathcal{A}(E))S_E]_{\mathrm{a}} + [(T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}}]_{\mathrm{a}}.$$

Thus, to show that (2.3) holds, we need only to show

$$[(T|\mathcal{A}(E))S_E]_{\mathrm{a}} = (T|\mathcal{A}(E))_{\mathrm{a}} S_E$$

$$[(T|\mathcal{A}(E^{\mathrm{c}}))S_{E^{\mathrm{c}}}]_{\mathrm{a}} = (T|\mathcal{A}(E^{\mathrm{c}}))_{\mathrm{a}} S_{E^{\mathrm{c}}}.$$

We are going to show the first equality, and then the second one can be similarly obtained.

Consider the following equality

$$(T|\mathcal{A}(E))S_E = (T|\mathcal{A}(E))_{\mathrm{a}} S_E + (T|\mathcal{A}(E))_{\mathrm{c}} S_E.$$

Since S_E is purely atomic, by Lemma 2.8, so is $(T|\mathcal{A}(E))_{\mathrm{a}} S_E$. On the other hand, let $\mathbf{B} = (E_{n,i})$ be an arbitrary bush which generates $\mathcal{A}$; then $\mathbf{B}_0 = (E_{n,i} \cap E)$ is a bush which generates $\mathcal{A}(E)$. Since

$$\max_i |[(T|\mathcal{A}(E))_{\mathrm{c}} S_E](E_{n,i})| = \max_i |(T|\mathcal{A}(E))_{\mathrm{c}}(E \cap E_{n,i})|, \quad i = 1, 2, \ldots,$$

by the definition of an Enflo-Starbird function, one has

$$\lambda_{(T|\mathcal{A}(E))_{\mathrm{c}} S_E}(\mathbf{B}) = \lambda_{(T|\mathcal{A}(E))_{\mathrm{c}}}(\mathbf{B}_0).$$

But, by Theorem 1.8, $\lambda_{(T|\mathcal{A}(E))_{\mathrm{c}}}(\mathbf{B}_0) = 0$, λ-a.e., so

$$\lambda_{(T|\mathcal{A}(E))_{\mathrm{c}} S_E}(\mathbf{B}) = 0, \quad \lambda\text{-a.e.}$$

Again by Theorem 1.8, $(T|\mathcal{A}(E))_{\mathrm{c}} S_E$ is purely continuous. By the uniqueness of the representation in Theorem 1.4, the conclusion follows. ∎

REMARK 2.10. The equation (2.3) is also true for arbitrary finite partitions of $[0, 1]$.

For a fixed atomless σ-subalgebra $\mathcal{A}$, we now present a certain sub-band $\mathcal{L}_{\mathcal{A}}$ contained in $\mathcal{L}_{\mathrm{E}}$, consisting of T with $|T|_{\mathcal{A}}$ purely atomic. We first define the operation $T \to T_{\mathcal{A}}$ corresponding to the band projection, and later verify the band properties.

LEMMA 2.11. *Let $T \in \mathcal{L}(L^1)$, and let $\mathcal{A}$ be an atomless σ-subalgebra of Σ_0. Then there exists a $T_{\mathcal{A}} \in \mathcal{L}(L^1)$ such that:*

(i) *$T_{\mathcal{A}}|\mathcal{A}$ is purely atomic;*

(ii) *$(T - T_{\mathcal{A}})|\mathcal{A}$ is purely continuous;*

(iii) *${T_{\mathcal{A}}}^{\pm} = (T^{\pm})_{\mathcal{A}}$.*

Proof. If $\lambda_T(\mathcal{A}) = 0$, this is trivial. So suppose $\lambda_T(\mathcal{A}) \neq 0$.

First suppose that T is positive, and let $\mathcal{A}(E)$ be as in Lemma 2.9. Define

$$T_{n,i} = (T|\mathcal{A}(\Delta_{n,i}))_{\mathrm{a}} S_{\Delta_{n,i}}.$$

Then $T_{n,i} \in \mathcal{L}(L^1(\mathcal{A}), L^1)$, and

$$T_{n,i} = \sum \{T_{m,j} : \Delta_{m,j} \subset \Delta_{n,i}\}$$

for all $1 \leqslant i \leqslant 2^n$, $n = 1, 2, \dots$. Define

$$T_{\mathcal{A}} \Delta_{n,i} = T_{n,i} \Delta_{n,i}.$$

Then $T_{\mathcal{A}}$ induces an operator in $\mathcal{L}(L^1)$. In fact, for any function f of the form $f = \sum\limits_{1 \leqslant k \leqslant m} \alpha_k \chi_{\Delta_{n_k,i_k}}$ with $\alpha_k \in \mathbb{R}$, define

$$T_{\mathcal{A}} f = \sum_{1 \leqslant k \leqslant m} \alpha_k T_{\mathcal{A}} \Delta_{n_k,i_k}.$$

Then, by (2.3), it is easy to see that $T_{\mathcal{A}}$ is a linear operator on X, the linear span of $\{\Delta_{n,i}\}$, with $\|T_{\mathcal{A}}\| \leqslant \|T\|$. Since X is dense in L^1, $T_{\mathcal{A}}$ can be extended to an operator in $\mathcal{L}(L^1)$.

Now since

$$T_{\mathcal{A}} \Delta_{n,i} \leqslant T \Delta_{n,i}.$$

one has

$$T_{\mathcal{A}} \leqslant T.$$

By the definition of $T_{\mathcal{A}}$, for $f \in L^1(\mathcal{A})$,

$$T_{\mathcal{A}}(f) = T_{0,1}(f) = (T|\mathcal{A})_{\mathrm{a}}(f),$$

so $T_{\mathcal{A}}|\mathcal{A}$ is purely atomic, i.e., $T_{\mathcal{A}}$ satisfies (i). Since $(T - T_{\mathcal{A}})|\mathcal{A} = T|\mathcal{A} - T_{\mathcal{A}}|\mathcal{A}$, $(T - T_{\mathcal{A}})|\mathcal{A}$ is purely continuous, and (ii) is proved. Since T is positive, (iii) is trivial.

For the general case, consider T^+ and T^- respectively. Let

$$T_{\mathcal{A}} = (T^+)_{\mathcal{A}} - (T^-)_{\mathcal{A}}.$$

Then, since $0 \leqslant (T^{\pm})_{\mathcal{A}} \leqslant T^{\pm}$ and $T^+ \wedge T^- = 0$, one has

$$(T^+)_{\mathcal{A}} \wedge (T^-)_{\mathcal{A}} = 0.$$

Since $T_{\mathcal{A}}$ can be uniquely written as

$$T_{\mathcal{A}} = T_{\mathcal{A}}{}^+ - T_{\mathcal{A}}{}^-$$

with $T_{\mathcal{A}}{}^+ \wedge T_{\mathcal{A}}{}^- = 0$, one must have that $T_{\mathcal{A}}{}^{\pm} = (T^{\pm})_{\mathcal{A}}$. This proves (iii).

Let $\mathbf{B} = (E_{n,i})$ be any bush which generates $\mathcal{A}$. Then

$$|(T - T_{\mathcal{A}})E_{n,i}| \leqslant |T^+ - (T^+_{)}\mathcal{A}|E_{n,i} + |T^- - (T^-)_{\mathcal{A}}|E_{n,i} = |T - T_{\mathcal{A}}|E_{n,i}.$$

Thus, by definition of the Enflo-Starbird function,

$$0 \leqslant \lambda_{(T-T_{\mathcal{A}})}(\mathbf{B}) \leqslant \lambda_{T^+-(T^+)_{\mathcal{A}}}(\mathbf{B}) + \lambda_{T^--(T^-)_{\mathcal{A}}}(\mathbf{B}) = 0.$$

This proves (ii).

It remain to show (i). Let $S : L^1(\mathcal{A}) \to L^1$ be a regular isometry; then

$$(T_{\mathcal{A}}|_{\mathcal{A}})S^{-1} = (T_{\mathcal{A}}{}^+|_{\mathcal{A}})S^{-1} - (T_{\mathcal{A}}{}^-|_{\mathcal{A}})S^{-1}.$$

Let $x \to \nu_x^{\pm}$ be the map from [0,1] into $M[0,1]$ which represents $(T_{\mathcal{A}}{}^{\pm}|_{\mathcal{A}})S^{-1}$ respectively; then by (1.4), for almost all $x \in [0,1]$, $\nu_x^{\pm}$ has the forms

$$\nu_x^{\pm} = \sum_{n=1}^{\infty} a_n^{\pm}(x)\delta_{\sigma_n^{\pm}}(x),$$

where for each n, $a_n^{\pm}$ and $\sigma_n^{\pm}$ satisfy the conditions in Theorem 1.4. Thus, the map

$$x \to \sum_{n=1}^{\infty} a_n^+(x)\delta_{\sigma_n^+}(x) - \sum_{n=1}^{\infty} a_n^-(x)\delta_{\sigma_n^-}(x)$$

represents $(T_{\mathcal{A}}|_{\mathcal{A}})S^{-1}$, and so by the note below (1.4), $(T_{\mathcal{A}}|_{\mathcal{A}})S^{-1}$ is purely atomic. By the definition of $(T_{\mathcal{A}}|_{\mathcal{A}})_{\mathrm{a}}$, $T_{\mathcal{A}}|_{\mathcal{A}}$ is purely atomic.

REMARK 2.12. (i) For a given atomless σ-subalgebra $\mathcal{A}$ of Σ_0, let $T = E_{\mathcal{A}}$ be the conditional expectation operator relative to $\mathcal{A}$, and $\mathcal{A}_n$ be the algebra generated by $\mathcal{A} \cup \left\{\Delta_{n,i} : 1 \leqslant i \leqslant 1/2^n\right\}$. Then one has

$$T = T_{\mathcal{A}} = T_{\mathcal{A}_n}, \quad n = 1, 2, \ldots .$$

In fact, in this case, let $T_{n,i}$ be as in the proof of the lemma; then

$$T_{n,i} = T_{\mathcal{A}_n} | \Delta_{n,i}.$$

(ii) For a given arbitrary operator $T \in \mathcal{L}(L^1)$,

$$T_{\mathcal{A}} = \lim_n (T|\mathcal{A}_n)_{\mathrm{a}} E_{\mathcal{A}_n}$$

where the limit exists in the strong operator topology.

(iii) It is possible that $T_{\mathcal{A}} \neq 0$ but $T_{\mathcal{A}}|\mathcal{A} = 0$. For example, let $S : L^1[0,1/2] \to L^1$ be an isometry, $R : L^1[1/2,1] \to L^1[0,1/2]$ be the translation operator defined by

$$Rf(x) = f\left(x - \frac{1}{2}\right), \quad f \in L^1\left[\frac{1}{2}, 1\right],$$

and define $T : L^1 \to L^1$ by

$$Tf = S\left(f \,\Big|\, \left[0, \frac{1}{2}\right]\right) - SR\left(f \,\Big|\, \left[\frac{1}{2}, 1\right]\right).$$

Take $\mathcal{A} = \{E \cup RE : E \in \Sigma_0 \cap [1/2, 1]\}$. Then $T = T_{\mathcal{A}} \neq 0$, but $T_{\mathcal{A}}|\mathcal{A} = 0$.

LEMMA 2.13. *Let $\mathcal{A}$ be a given atomless σ-subalgebra of Σ_0, and $\mathcal{L}_{\mathcal{A}}$ be the set of all operators of the form $T = T_{\mathcal{A}}$ with $T_{\mathcal{A}}$ satisfying the conditions in Lemma* 2.11. *Then:*

(i) *$\mathcal{L}(\mathcal{A})$ is a band in $\mathcal{L}(L^1)$;*
(ii) *$T \in \mathcal{L}_{\mathcal{A}}$ iff $|T|$ is purely atomic on $L^1(\mathcal{A})$;*
(iii) *for each $T \in (\mathcal{L}_{\mathcal{A}})^{\perp}$, $T|\mathcal{A}$ is purely continuous.*

Proof. (i) If $T, S \in \mathcal{L}_{\mathcal{A}}$, then by Lemma 2.11, $T^{\pm} + S^{\pm}$ is purely atomic on $L^1(\mathcal{A})$. Since $0 \leqslant (T+S)^{\pm} \leqslant T^{\pm} + S^{\pm}$, by (1.5), $(T+S)^{\pm}$ is purely atomic on $L^1(\mathcal{A})$. This shows that $\mathcal{L}_{\mathcal{A}}$ is a linear subspace of $\mathcal{L}(L^1)$. A similar argument shows that $\mathcal{L}_{\mathcal{A}}$ is an ideal of $\mathcal{L}(L^1)$. Finally, let $\{T_\alpha : \alpha \in D\}$ be a majorized upward directed family of positive operators in $\mathcal{L}_{\mathcal{A}}$, and suppose that

$$T = \sup_\alpha T_\alpha$$

exists in $\mathcal{L}(L^1)$. If T is not purely atomic on $L^1(\mathcal{A})$, i.e., $(T|\mathcal{A})_{\mathrm{c}} \neq 0$, then there exists an $\alpha \in D$ such that $(T|\mathcal{A})_{\mathrm{c}} \wedge (T_\alpha|\mathcal{A}) \neq 0$. By (1.5), $(T|\mathcal{A})_{\mathrm{c}} \wedge (T_\alpha|\mathcal{A})$ is purely atomic. But

$$\lambda_{(T|\mathcal{A})_{\mathrm{c}} \wedge (T_\alpha|\mathcal{A})}(\mathcal{A}) \leqslant \lambda_{(T|\mathcal{A})_{\mathrm{c}}} = 0,$$

a contradiction to Theorem 1.9.

(ii) If $T \in \mathcal{L}_{\mathcal{A}}$, then by Lemma 2.11, $|T|$ is purely atomic on $L^1(\mathcal{A})$. Conversely, if $|T|$ is purely atomic on $L^1(\mathcal{A})$, then by (1.5) (through a regular isometry), $0 \leqslant T^{\pm} \leqslant |T|$ is purely atomic on $L^1(\mathcal{A})$. Thus, by Lemma 2.11, $T_{\mathcal{A}} = T$, i.e. $T \in \mathcal{L}_{\mathcal{A}}$.

(iii) Let $T \in (\mathcal{L}_{\mathcal{A}})^{\perp}$. Assume $(T|\mathcal{A})_{\mathrm{a}} \neq 0$. Let $S : L^1(\mathcal{A}) \to L^1$ be a regular isometry; then, by the definition of $(T|\mathcal{A})_{\mathrm{a}}$, $(TS^{-1})_{\mathrm{a}} \neq 0$. By Theorem 1.8,

$$\lambda TS^{-1} \neq 0,$$

which is equivalent to saying that $\lambda_{T|\mathcal{A}} \neq 0$ (since S is an isometry). Let $\mathbf{B} = (E_{n,i})$ be any bush which generates $\mathcal{A}$; then by definition,

$$\lambda_{T|\mathcal{A}} = \lim_n \max_i \big|(T|_{\mathcal{A}})(E_{n,i})\big| = \lim_n \max_i |TE_{n,i}| \leqslant \lim_n \max_i |T|E_{n,i} = \lambda_{|T|}(\mathcal{A}).$$

Thus $\lambda_{|T|}(\mathcal{A}) \neq 0$, which implies that $\lambda_{T^+}(\mathcal{A}) \neq 0$ or $\lambda_{T^-} \neq 0$. By Lemma 2.11 (iii), $T_{\mathcal{A}} \neq 0$. But $T_{\mathcal{A}} \in \mathcal{L}_{\mathcal{A}}$ and $|T_{\mathcal{A}}| \leqslant |T|$, a contradiction to the assumption that $T \in (\mathcal{L}_{\mathcal{A}})^{\perp}$. ∎

Let $\mathcal{L}_{\mathrm{NE}}$ be the set of all non-Enflo operators. We shall show next that $\mathcal{L}_{\mathrm{NE}}$ is a band. It then follows from Proposition 2.15 below that $\mathcal{L}_{\mathrm{E}} = (\mathcal{L}_{\mathrm{NE}})^{\perp}$.

LEMMA 2.14. *$\mathcal{L}_{\mathrm{NE}}$ is a band of $\mathcal{L}(L^1)$.*

Proof. Let $T_1, T_2 \in \mathcal{L}_{\mathrm{NE}}$; then for any bush $\mathbf{B} = (E_{n,i})$,

$$\max_i |(T_1 + T_2)E_{n,i}| \leqslant \max_i |T_1 E_{n,i}| + \max_i |T_2 E_{n,i}|,$$

and so by the definition of the Enflo-Starbird maximal function, one has

$$\lambda_{T_1+T_2}(\mathbf{B}) \leqslant \lambda_{T_1}(\mathbf{B}) + \lambda_{T_2}(\mathbf{B}).$$

By Theorem 1.9,

$$\lambda_{T_1}(\mathbf{B}) = \lambda_{T_2}(\mathbf{B}) = 0,$$

so $\lambda_{T_1+T_2}(\mathbf{B}) = 0$, which implies that $T_1 + T_2 \in \mathcal{L}_{\mathrm{NE}}$. This shows that $\mathcal{L}_{\mathrm{NE}}$ is a linear subspace of $\mathcal{L}(L^1)$.

Suppose $T \in \mathcal{L}_{\mathrm{NE}}$, $S \in \mathcal{L}(L^1)$ and $|S| \leqslant |T|$. By a result in [8], $|T|$ is a non-Enflo operator, so for any given bush $\mathbf{B}$,

$$\lambda_{|S|}(\mathbf{B}) \leqslant \lambda_{|T|}(\mathbf{B}) = 0$$

and so $S \in \mathcal{L}_{\mathrm{NE}}$. This shows that $\mathcal{L}_{\mathrm{NE}}$ is an ideal of $\mathcal{L}(L^1)$.

Finally, let $\{T_\alpha : \alpha \in D\}$ be a majorized upward directed family of positive operators in $\mathcal{L}_{\mathrm{NE}}$, and assume that $T = \sup_\alpha T_\alpha$ exists in $\mathcal{L}(L^1)$. If T is an Enflo operator, then by Theorem 1.9, there is a bush $\mathbf{B}$ such that

$$\lambda_T(\mathbf{B}) \neq 0.$$

Let $\mathcal{A}$ be the (atomless) σ-subalgebra of Σ_0 generated by $\mathbf{B}$; then by Lemma 2.11, $T_{\mathcal{A}} \neq 0$ and $0 \leqslant T_{\mathcal{A}} \leqslant T$. But then there exists at least one $\alpha \in D$ such that $T_{\mathcal{A}} \wedge T_\alpha \neq 0$. Since

$$0 \leqslant T_{\mathcal{A}} \wedge T_\alpha \leqslant T_\alpha$$

$T_{\mathcal{A}} \wedge T_\alpha$ is non-Enflo, and so is purely continuous on $L^1(\mathcal{A})$. But by Lemma 2.13, $T_{\mathcal{A}} \wedge T_\alpha \in \mathcal{L}_{\mathcal{A}}$, a contradiction. ∎

Let $\mathcal{L}_{\mathrm{u}} = \mathcal{L}_{\mathrm{s}} \cap \mathcal{L}_{\mathrm{NE}}$; then by (2.1) and Lemma 2.14, it is easy to see that

$$\mathcal{L}(L^1) = \mathcal{L}_{\mathrm{RN}} \oplus \mathcal{L}_{\mathrm{u}} \oplus \mathcal{L}_{\mathrm{E}}. \tag{2.4}$$

The bands $\mathcal{L}_{\mathrm{E}}, \mathcal{L}_{\mathrm{NE}}$, and $\mathcal{L}_{\mathrm{u}}$ are called the *Enflo band*, the *non-Enflo band* and the *singular continuous band* respectively.

PROPOSITION 2.15. *Let $T \in \mathcal{L}(L^1)$. Then T can be uniquely written as,*

$$T = T_{\mathrm{E}} + T_{\mathrm{u}} + T_{\mathrm{RN}}, \tag{2.5}$$

where

(i) $T_{\mathrm{E}} = \sum_{n=1}^{\infty} T_n$, *with $\sum_n T_n$ ℓ^1-strongly convergent, such that for each n there exists a non-atomic subalgebra $\mathcal{A}_n$ of Σ_0, such that $T_n^{\pm}|\mathcal{A}_n$ is purely atomic, and such that for all $m < n$ $T_n^{\pm}|\mathcal{A}_m$ is purely continuous and $T_n \perp T_m$.*

(ii) T_{u} *has the form $T_{\mathrm{u}} f(x) = \int_K f(s)\mathrm{d}\rho_x(s)$ such that for almost all x, $\rho_x \in M_{\mathrm{c}}(K)$, $\rho_X \perp \lambda$ and for each atomless σ-subalgebra $\mathcal{A}$ of Σ_0, $T_{\mathrm{u}}|\mathcal{A}$ is purely continuous.*

(iii) T_{RN} *is Radon-Nikodym, as defined below Proposition* 2.3.

Furthermore, one has

$$|T| = \sum_n |T_n| + |T_{\mathrm{RN}}| + |T_{\mathrm{u}}| = \sum_n |T|_n + |T|_{\mathrm{RN}} + |T|_{\mathrm{u}}. \tag{2.6}$$

Proof. Let $P_{\mathrm{RN}} : \mathcal{L}(L^1) \to \mathcal{L}_{\mathrm{RN}}$, $P_{\mathrm{u}} : \mathcal{L}(L^1) \to \mathcal{L}_{\mathrm{u}}$ and $P_{\mathrm{E}} : \mathcal{L}(L^1) \to \mathcal{L}_{\mathrm{E}}$ be band projections. Let

$$T_{\mathrm{RN}} = P_{\mathrm{RN}}(T), \quad T_{\mathrm{u}} = P_{\mathrm{u}}(T) \quad \text{and} \quad T_{\mathrm{E}} = P_{\mathrm{E}}(T).$$

Then it is easy to see that T_{u} satisfies (ii) and T_{RN} satisfies (iii). By (2.4), we need only show that T_{E} has the form in (i).

By Lemma 2.13, for a given atomless σ-subalgebra $\mathcal{A}$, there exists a band projection $P_{\mathcal{A}} : \mathcal{L}(L^1) \to \mathcal{L}_{\mathcal{A}}$, and so for each $S \in \mathcal{L}(L^1)$, one has $P_{\mathcal{A}}|S| = |P_{\mathcal{A}}S|$.

Let $\mathfrak{A} = \{\mathcal{A} : \mathcal{A}$ is an atomless σ-subalgebra of $\Sigma_0\}$, and let

$$\alpha = \sup\Big\{ \int |P_{\mathcal{A}}(T)|1 \, \mathrm{d}\lambda : \mathcal{A} \in \mathfrak{A} \Big\}.$$

CLAIM. There exist operators $T_1, T_2, \dots$ in $\mathcal{L}(L^1)$, atomless σ-subalgebra $\mathcal{A}_1, \mathcal{A}_2, \dots$ and integers $n_1 < n_2 < \cdots$ with the following properties:

(1) $\sum\limits_{k=1}^{n} T_k^{\pm} \leqslant T^{\pm}$ and $T_j \perp T_k$, for all n, and all $j < k \leqslant n$;

(2) $T_k|\mathcal{A}_k$ is purely atomic and $T_k|\mathcal{A}_{k-1}$ is purely continuous for all $k > 1$ and $T_j \perp T_k = 0$, all j, k, $j \neq k$;

(3) If $S_n = T - \sum\limits_{k<n} T_k$, then for all m and any atomless σ-subalgebra $\mathcal{A}$,

$$\int |(S_k)_{\mathcal{A}}|1 \, \mathrm{d}\lambda \leqslant \frac{1}{2^m}\alpha, \quad \text{for all } k > n_m.$$

In fact, let $S_0 = T$; then we can define $T_n = P_{\mathcal{A}_n}(S_{n-1})$, $n \geqslant 1$, in the following way: let n_1 be the largest $n \in \mathbb{N}$ such that for each $k < n$ there exists $\mathcal{A}_{k+1} \in \mathfrak{A}$ with

$$\int |P_{\mathcal{A}_{k-1}}(S_k)|1 \, \mathrm{d}\lambda > \frac{1}{2}\alpha.$$

Since, if this equality is satisfied for all $k < n$, one has

$$\frac{n-1}{2}\alpha \leqslant \int \sum_{k<n} |P_{\mathcal{A}_{k+1}}(S_k)|1 \, \mathrm{d}\lambda \leqslant \int |T|1 \, \mathrm{d}\lambda,$$

such an n_1 exists.

In general, assume that n_m is defined. If there exists no σ-subalgebra $\mathcal{A}$, such that

$$\int |P_{\mathcal{A}}(S_{n_m})|1 \, \mathrm{d}\lambda > \frac{1}{2^{m+1}}\alpha,$$

we define $n_{m+1} = n_m$; otherwise, take a subalgebra $\mathcal{A} = \mathcal{A}_{n_m+1}$ such that the above inequality is satisfied, and define $T_{n_m+1} = P_{\mathcal{A}_{n_m+1}}(S_{n_m})$, and repeat an argument similar to that in the first step to find n_{m+1}.

Thus, we get sequences $(T_n) \subset \mathcal{L}(L^1)$ and $(n_m) \subset \mathbb{N}$ such that (3) is satisfied. Since $P_{\mathcal{A}_n}$ is a band projection for each n, by the definition of T_n's, (1) is satisfied. Condition (2) follows by Lemma 2.13 (iii) and the fact that $S_n \in (\mathcal{L}_{\mathcal{A}_1} + \dots + \mathcal{L}_{\mathcal{A}_n})^{\perp}$ (note that we need not have $\mathcal{L}_{\mathcal{A}_j} \perp \mathcal{L}_{\mathcal{A}_k}$ for $j \neq k$).

Now (1) implies that

$$T' = \sum_{n=0}^{\infty} T_n$$

is a strong ℓ^1-sum. Since for any i, there is no atomless σ-subalgebra $\mathcal{A}$ such that

$$\int |P_{\mathcal{A}}(T - T')|1 \, \mathrm{d}\lambda > \frac{1}{2^i},$$

we have $P_{\mathcal{A}}(T - T') = 0$ for all $\mathcal{A} \in \mathfrak{A}$. This implies $T - T' \in \mathcal{L}_{\mathrm{NE}}$. So, to finish the proof, we need only show $T' \in \mathcal{L}_{\mathrm{E}}$.

Assume $T' \notin \mathcal{L}_{\mathrm{E}}$. Without loss of generality, we can suppose that T is positive. Since $\mathcal{L}_{\mathrm{NE}}$ is a band, T' can be uniquely written as

$$T' = S_1 + S_2,$$

with $S_1 \in \mathcal{L}_{\mathrm{NE}}$, $S_2 \in \mathcal{L}_{\mathrm{E}}$, $S_1, S_2 \geqslant 0$ and $S_1 \neq 0$. Since

$$0 \leqslant S_1 \leqslant T' = \sum_n T_n = \sup_k \sum_{n \leqslant k} T_n,$$

one has

$$S_1 = \sup_n \Big(S_1 \wedge \sum_{n \leqslant k} T_n \Big).$$

Thus, there exists a $k < \infty$, such that $S_1 \wedge \sum\limits_{n \leqslant k} T_n \neq 0$, and so there exists an $n \leqslant k$ such that $S_1 \wedge T_n \neq 0$. But then, $T_n | \mathcal{A}_n$ has a purely continuous part, contradicting Condition (i).

Finally, (2.6) follows by the fact that all operators P_{RN}, P_{u} and $P_{\mathcal{A}_n}$, $n = 1, 2, \dots$ are band projections. ∎

REMARK 2.16. In general, the T_n's are not unique.

For $T \in \mathcal{L}(L^1)$, let T_n, $n = 1, 2, \dots,$ T_{u} and T_{RN} be as in Proposition 2.15. We will call each T_n, $n = 1, 2, \dots$ a *conditional atomic part*, and call T_{RN}, T_{u} the *Radon-Nikodym part* and the *singular continuous part*, respectively. T_{E} is, as we defined in Section 1, the *pure Enflo part*. We also denote the part $T_{\mathrm{RN}} + T_{\mathrm{u}}$ by T_{NE}, and call it the *non-Enflo part*.

Next, we wish to give a further decomposition of T_{u} by using Theorems 1.11, 1.12 and 1.13.

LEMMA 2.17. *$\mathcal{L}_{\mathrm{DP}}(L^1)$, the space of all Dunford-Pettis operators, is a band of $\mathcal{L}(L^1)$.*

Proof. The fact that $\mathcal{L}_{\mathrm{DP}}(L^1)$ is a solid sublattice, i.e. an ideal, was shown in [4]. To show that $\mathcal{L}_{\mathrm{DP}}$ is a band, we need only show that the supremum of a majorized upward directed family $\{S_\alpha : \alpha \in D\}$ of positive Dunford-Pettis operators is still a Dunford-Pettis operator.

Suppose

$$S = \sup_\alpha S_\alpha \in \mathcal{L}(L^1).$$

Then $(S_\alpha i_\infty)$ is a subset of the regular operators in $\mathcal{L}(L^\infty, L^1)^+$ dominated by Si_∞. Since $\mathcal{L}^{\mathrm{r}}(L^\infty, L^1)$ with the integral norm is an AL-space, and since every AL-space is order complete,

$$S_0 = \sup_\alpha S_\alpha i_\infty$$

exists in $\mathcal{L}(L^\infty, L^1)^+$; since the continuity and the order continuity are the same in AL-spaces, and $S_0 - S_\alpha \geqslant 0$, by Proposition 1.11 (i),

$$S_\alpha i_\infty \to S_0$$

in the integral norm, and hence in the operator norm. Thus, since $S_\alpha i_\infty$ is compact, so is S_0. But S_0 is dominated by Si_∞, and $S \in \mathcal{L}(L^1)$, so for each $f \in (L^\infty)^+$, one has

$$S_0 f \leqslant Si_\infty f,$$

and so

$$\|S_0 f\|_1 \leqslant \|Si_\infty f\|_1 = \|Sf\|_1.$$

Hence S_0 induces an operator $S_0' : L^1 \to L^1$ with

$$\|S_0'\|_1 \leqslant \|S\|_1$$

and

$$S_\alpha \leqslant S_0' \leqslant S$$

in $\mathcal{L}(L^1)$. By the definition of S,

$$S_0' = S,$$

and so Si_∞ is compact. Thus, by Theorem 1.13, S is Dunford-Pettis. ∎

PROPOSITION 2.18. *Let $T \in \mathcal{L}(L^1)$. Then T can be uniquely written as*

$$T = T_{\mathrm{DP}} + T_{\mathrm{NDP}}$$

such that

(i) T_{DP} *is Dunford-Pettis;*

(ii) *for any* $0 \neq S \in \mathcal{L}(L^1)$, $|S| \leqslant |T_{\mathrm{NDP}}|$, S *is not Dunford-Pettis;*

(iii) $|T| = |T_{\mathrm{DP}}| + |T_{\mathrm{NDP}}|$.

Proof. By Lemma 2.17, there exists a band projection P_{DP} from $\mathcal{L}(L^1)$ onto $\mathcal{L}_{\mathrm{DP}}(L^1)$. Let $T_{\mathrm{DP}} = P_{\mathrm{DP}}(T)$, and $T_{\mathrm{NDP}} = (I - P_{\mathrm{DP}})(T)$; then T_{DP} and T_{NDP} satisfy (iii). By the definition of $\mathcal{L}_{\mathrm{DP}}$, T_{DP} is Dunford-Pettis. Since $T_{\mathrm{NDP}} \in \mathcal{L}_{\mathrm{NDP}} = \mathcal{L}_{\mathrm{DP}}^{\perp}$, there exists no $S \in \mathcal{L}_{\mathrm{DP}}$ with $|S| \leqslant |T_{\mathrm{NDP}}|$. So T_{NDP} satisfies (ii). ∎

T_{DP} and T_{NDP} will be called the *Dunford-Pettis part* and the *non-Dunford-Pettis part* of T, respectively. $\mathcal{L}_{\mathrm{DP}}$ will be called the *Dunford-Pettis band*, and $\mathcal{L}_{\mathrm{NDP}}$ the *non-Dunford-Pettis band*. We also denote

$$\mathcal{L}_{\mathrm{SDP}} = \mathcal{L}_{\mathrm{u}} \cap \mathcal{L}_{\mathrm{DP}}, \quad \mathcal{L}_{\mathrm{R}} = \mathcal{L}_{\mathrm{u}} \cap \mathcal{L}_{\mathrm{NDP}}$$

and call them the *singular Dunford-Pettis band* and the *Rosenthal band* respectively.

Now, by these definitions, and combining (2.1), (2.4) and Proposition 2.18, we can give the decomposition on $\mathcal{L}(L^1)$ which we have claimed at the beginning of this section

$$\mathcal{L}(L^1) = \mathcal{L}_{\mathrm{RN}} \oplus \mathcal{L}_{\mathrm{SDP}} \oplus \mathcal{L}_{\mathrm{R}} \oplus \mathcal{L}_{\mathrm{E}}. \tag{2.7}$$

The following is the complete version of the main result of this paper.

THEOREM 2.19. *Each $T \in \mathcal{L}(L^1)$ can be uniquely written as*

$$T = T_{\mathrm{RN}} + T_{\mathrm{SDP}} + T_{\mathrm{R}} + T_{\mathrm{E}},$$

where T_{RN} is Radon-Nikodym, T_{SDP} is singular Dunford-Pettis, T_{R} is pure Rosenthal and T_{E} is pure Enflo. All operators have norm $\leqslant \|T\|$, or more precisely,

$$T^{\pm} = T_{\mathrm{RN}}^{\pm} + T_{\mathrm{SDP}}^{\pm} + T_{\mathrm{R}}^{\pm} + T_{\mathrm{E}}^{\pm},$$

and so

$$|T| = |T_{\mathrm{RN}}| + |T_{\mathrm{SDP}}| + |T_{\mathrm{R}}| + |T_{\mathrm{E}}| = |T|_{\mathrm{RN}} + |T|_{\mathrm{SDP}} + |T|_{\mathrm{R}} + |T|_{\mathrm{E}}.$$

Proof. This follows by Proposition 2.15 and 2.18. ∎

T_{R} is the *pure Rosenthal part* of T as defined in Section 1, and we call T_{SDP} the *singular Dunford-Pettis part.*

By what we have shown up to now, it is easy to get the following.

PROPOSITION 2.20. (i) $[\mathcal{L}_{\mathrm{RN}} \circ \mathcal{L}(L^1)] \cup [\mathcal{L}(L^1) \circ \mathcal{L}_{\mathrm{RN}}] \subset \mathcal{L}_{\mathrm{RN}}$*;*

(ii) $[\mathcal{L}_{\mathrm{SDP}} \circ \mathcal{L}_{\mathrm{s}}] \cup [\mathcal{L}_{\mathrm{s}} \circ \mathcal{L}_{\mathrm{SDP}}] \subset \mathcal{L}_{\mathrm{RN}} \oplus \mathcal{L}_{\mathrm{SDP}}$*;*

(iii) $[\mathcal{L}_{\mathrm{R}} \circ (\mathcal{L}_{\mathrm{R}} \oplus \mathcal{L}_{\mathrm{E}})] \cup [(\mathcal{L}_{\mathrm{R}} \oplus \mathcal{L}_{\mathrm{E}}) \circ \mathcal{L}_{\mathrm{R}}] \subset \mathcal{L}_{\mathrm{RN}} \oplus \mathcal{L}_{\mathrm{SDP}} \oplus \mathcal{L}_{\mathrm{R}}$ (*where* $A \circ B = \{ST : S \in A,\, T \in B\}$).

REMARK 2.21. By a result in [13] (or in [2]), for $T \in \mathcal{L}_{\mathrm{SDP}}$ there is always a non-zero $S \in \mathcal{L}_{\mathrm{SDP}}$, such that $T \circ S \in \mathcal{L}_{\mathrm{SDP}}$. By [14], there is a non-zero $T \in \mathcal{L}_{\mathrm{R}}$ such that $T^2 \in \mathcal{L}_{\mathrm{R}}$. In both cases, a convolution operator was considered. The properties of such operators are closely related to the properties of the measures involved. Using Proposition 2.20, we will give a decomposition of $M[0,1]$ into some related bands.

Recall that a *convolution operator* in $\mathcal{L}(L^1(G))$, where G is a compact abelian group, is an operator $T\mu$ defined by $T_\mu f = f * \mu$, or more precisely,

$$T_\mu f(x) = \int f(xt)\mathrm{d}\mu(t) = \int f(t)\mathrm{d}\mu(x^{-1}t), \quad f \in L^1(G),$$

where $\mu \in M(G)$.

Identifying the points 0 and 1 in [0,1], one can regard [0,1] as the quotient group of $\mathbb{R}$ modulo 1, which is isomorphic to the circle group $\mathbb{T}$, and so they have the same dual group $\mathbb{Z}$. In this case, $C[0,1]$ is regarded as the space of all continuous functions with $f(0) = f(1)$, and the point measures δ_1 and δ_0 are regarded as the same.

In [6], it was shown that $\widehat{\mu} \in c_0(\mathbb{Z})$ iff T_μ is Dunford-Pettis. By Theorem 2.19, we obtain the following:

THEOREM 2.22. *Denote by* M_{RN} *(resp.* M_{DPS}*,* M_{R} *or* M_{E}*) the set of all* $\mu \in M[0,1]$ *such that* T_μ *is a Radon-Nikodym operator (resp. a singular Dunford-Pettis, a pure Rosenthal, or a pure Enflo operator). Then*

$$M[0,1] = M_{\mathrm{RN}} \oplus M_{\mathrm{SDP}} \oplus M_{\mathrm{R}} \oplus M_{\mathrm{E}}.$$

In fact, M_{RN}*,* M_{SDP}*,* M_{R} *and* M_{E} *form an orthogonal band decomposition of* $M[0,1]$.

Proof. We have that $M[0,1]$ is an AL-space and, defining $\varphi : M[0,1] \to \mathcal{L}(L^1)$ by $\varphi(\mu) = T_\mu$ for all $\mu \in M[0,1]$, that φ is a lattice homomorphism. Now the conclusion follows immediately from Theorem 2.19 and the claim: if B is a band in $\mathcal{L}(L^1)$, then $\varphi^{-1}(B)$ is a band in $M[0,1]$.

In turn, since φ is a lattice homomorphism, it follows that fixing a band B in $\mathcal{L}(L^1)$, then $\varphi^{-1}(B)$ is a lattice ideal in $M[0,1]$. Thus, let A be a non-empty subset of $\varphi^{-1}(B)$ so that $\mu = \sup A$ exists in $M[0,1]$; we need only to show that $\mu \in \varphi^{-1}(B)$. But since $M[0,1]$ is an AL-space, there exists a sequence in $\varphi^{-1}(B)$ so that $\mu_n \to \mu$ is norm. Hence, since B is closed, and φ is continuous, $\varphi^{-1}(B)$ is closed, so $\mu \in \varphi^{-1}(B)$. ∎

REMARK 2.23 Note that the Fourier-Stieltjes transformation maps an AL-space into an AM-space, and this map is not a lattice homomorphism. Also note that by our main theorem and [14], M_{R} is non-zero.

A measure in M_{SDP} is sometimes called a *Rajchman measure*, and is discussed in [10]. M_{E} includes all singular idempotent measures. It looks like not very much is known about the measures in M_{R}.

For the last part of this paper, let us consider some useful facts on the Cantor group $2^{\mathbb{N}}$.

On $2^{\mathbb{N}}$, the dual group consists of the set of Walsh functions $\{w_1, w_2, \ldots\}$. Regard $\{w_1, w_2, \ldots\}$ as the unit basis in ℓ^1 (i.e. $\|\sum \alpha_i w_i\|_{\ell^1} = \sum |\alpha_i|$), and let $R : \ell^1(w_n) \to C(2^{\mathbb{N}})$ be the natural map. The Fourier-Stieltjes transformation can be regarded as the adjoint map $R^* : M(K) \to \ell^\infty(w_n)$. Since the closed linear span of (w_n) equals $L^1(2^{\mathbb{N}})$, R maps $\ell^1(w_n)$ to a dense subspace of $C(2^{\mathbb{N}})$, and so R^* is an injection. Now, for $f \in L^1(2^{\mathbb{N}}, m)$, where m is the Haar measure on $2^{\mathbb{N}}$, $R^* f(w_n)$ is nothing but the n-th coefficient of the expansion of f with respect to the biorthogonal system (w_n) in $L^1(2^{\mathbb{N}})$.

Using this, we wish to reveal some relations between an operator $T \in \mathcal{L}(L^1)$ and its representation measures, which we will do in a subsequent paper. Here, we show only that $M_{\mathrm{DP}}(2^{\mathbb{N}})$ is a weak* Borel subset of $M(2^{\mathbb{N}})$, and since $C[0,1]$ can be embedded into $C(2^{\mathbb{N}})$ in a natural way, the same is true for $M[0,1]$.

We need the following lemma.

LEMMA 2.24. $\mathrm{Ba}\, c_0$ *is a weak* Borel subset of* $\mathrm{Ba}\, L^\infty$.

Proof. Let

$$A_{n,k} = \Big\{ x \in \mathrm{Ba}\, \ell^\infty : |x(n)| \leqslant \frac{1}{k} \Big\}.$$

Then $A_{n,k}$ is a weak* close set of $\operatorname{Ba} L^\infty$, so $B_{m,k} = \bigcap_{n \geqslant m} A_{n,k}$ is a weak* closed set, and $C_k = \bigcup_{m\in\mathbb{N}} B_{m,k}$ is a weak* Baire-1 set. Let

$$D = \bigcap_{k\in\mathbb{N}} C_k;$$

then D is a Baire-2 set. We show that $\operatorname{Ba} c_0 = D$.

First, if $x \in c_0$, then for m_k large enough,

$$|x(m)| \leqslant \frac{1}{k}, \qquad \text{for all } m \geqslant m_k,$$

i.e. $x \in B_{m_k,m} \subset C_k$ for each k, hence $x \in D$. Conversely, if $x \in D$, then $x \in C_k$ for each k, so there is m_k such that $x \in B_{m_k,m}$, and so, for $m \geqslant m_k$, $|x(m)| \leqslant 1/k$. This holds for each k, hence $x(m) \to 0$ as $m \to \infty$, i.e. $x \in \operatorname{Ba} c_0$. ∎

PROPOSITION 2.25. *$M_{\mathrm{DPS}}(2^{\mathbb{N}})$ is a weak* Borel subset of $M(2^{\mathbb{N}})$.*

Proof. Let R be as defined before Lemma 2.24; then $R^*\operatorname{Ba} M(2^{\mathbb{N}})$ is weak* compact. Since R maps $\ell^1(w_n)$ to a dense subspace of $C(2^{\mathbb{N}})$, R^* is an injection. Let $D_{n,i}$ be the usual tree on $2^{\mathbb{N}}$; then the characteristic function of $D_{n,i}$ is continuous and is contained in $R\ell^1(w_n)$, so the linear span of all such functions in dense in $C(2^{\mathbb{N}})$. By Lemma 2.24, $c_0(w_n) \cap R^*\operatorname{Ba} M(2^{\mathbb{N}})$ is a weak* Borel set. So, since R^* is weak* continuous, the set

$$[(R^*)^{-1}(c_0 \cap R^*\operatorname{Ba} M(2^{\mathbb{N}}))] \cap \operatorname{Ba} M(2^{\mathbb{N}})$$

is weak* Borel, and is exactly the set

$$\operatorname{Ba}(M_{\mathrm{DPS}}(2^{\mathbb{N}}) \oplus M_{\mathrm{RN}}(2^{\mathbb{N}})).$$

Thus, by Lemma 2.1,

$$\operatorname{Ba} M_{\mathrm{DPS}}(2^{\mathbb{N}}) = \operatorname{Ba}(M_{\mathrm{DPS}}(2^{\mathbb{N}}) \oplus M_{\mathrm{RN}}(2^{\mathbb{N}})) \cap M_{\mathrm{RN}}(2^{\mathbb{N}})^{\perp}$$

is a weak* Borel set. The conclusion now trivially follows. ∎

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