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ISOMETRIES AND JORDAN-ISOMORPHISMS ONTO C^*-ALGEBRAS

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ABSTRACT. Let A be a C^*-algebra, and B a complex normed non-associative algebra. We prove that, if B has an approximate unit bounded by one, then, for every linear isometry F from B onto A, there exists a Jordan-isomorphism $G : B \to A$ and a unitary element u in the multiplier algebra of A such that $F(x) = uG(x)$ for all x in B. We also prove that, if G is an isometric Jordan-isomorphism from B onto A, then there exists a self-adjoint element φ in the centre of the multiplier algebra of the closed ideal of A generated by the commutators satisfying $\|\varphi\| \leqslant 1$ and

$$G(xy) = \frac{1}{2}(G(x)G(y) + G(y)G(x) + \varphi(G(x)G(y) - G(y)G(x)))$$

for all x, y in B.

KEYWORDS: *C^*-algebras, isometries, Jordan-isomorphisms.*

AMS SUBJECT CLASSIFICATION: Primary 46L05; Secondary 46L70.

0. INTRODUCTION

The aim of this paper is to prove the following two theorems.

THEOREM A. *Let A be a non-zero C^*-algebra, B a non-associative normed complex algebra with an approximate unit bounded by one, and F a linear isometry from B onto A. Then F can be written in the form $F = L \circ G$, where G is an isometric Jordan-homomorphism from B onto A, and L is the operator of left multiplication on A by a suitable unitary element in the multiplier algebra of A.*

THEOREM B. *Let A be a C^*-algebra, B a complex normed non-associative algebra, and $F : B \to A$ a surjective isometric Jordan-homomorphism. Then there exists a self-adjoint element φ in the centre of the multiplier algebra of the closed ideal of A generated by the commutators satisfying $\|\varphi\| \leqslant 1$ and*

$$F(uv) = \frac{1}{2}(F(u)F(v) + F(v)F(u) + \varphi(F(u)F(v) - F(v)F(u)))$$

for all u, v in B.

Theorems A and B above are non-associative variants of Theorems 7 and 10, respectively, in the paper of R.V. Kadison ([17]). The reader is also referred to [6], [13], [14], [15], [19], [20], [21], [31], and [34] for related results. We note that Theorems A and B provide previously unknown results in the associative setting (see Corollaries 1.3, 2.8, and 2.9, and Remark 2.10).

1. ISOMETRIES

Given a vector space X, each bilinear mapping from $X \times X$ into X will be called a *product* on X. For u in X, those products f on X satisfying $f(u, x) = f(x, u) = x$ for all x in X are called *u-admissible*. A product f on (the vector space of) an algebra A is said to be *Jordan-admissible* if, for every x, y in A, the equality $f(x, y) + f(y, x) = xy + yx$ holds. Unless explicitly stated otherwise, all products on a normed space X will be assumed to be continuous, so that each product f on X has a natural norm $\|f\|$ given by

$$\|f\| := \sup\{\|f(x, y)\| : x, y \in X, \|x\| \leqslant 1, \|y\| \leqslant 1\}.$$

It is easy to see that, if X is a normed space, and if u is a norm-one element in X, then the set of all norm-one u-admissible products on X is a face of the closed unit ball of the normed space of all products on X (see for instance [22], Lemma 1.5). It follows that, if A is a norm-unital normed algebra, and if $\mathbf{1}$ denotes the unit of A, then every norm-one Jordan-admissible product on A is $\mathbf{1}$-admissible. Note also that, if A is a non-zero C^*-algebra, and if f is a Jordan-admissible product on A, then $\|f\| \geqslant 1$, and therefore the set of all norm-one Jordan-admissible products on A is a convex subset of the Banach space of all products on A.

Let X be a normed space, and u be a norm-one element in X. The set of states of X relative to $u, D(X, u)$, is defined by

$$D(X, u) := \{\phi \in X^* : \|\phi\| = \phi(u) = 1\}.$$

For complex X, an element x in X is said to be *hermitian* (relative to u) if $\phi(x)$ belongs to $\mathbb{R}$ whenever ϕ is in $D(X,u)$, and the set of all hermitian elements in X is denoted by $H(X,u)$. A *Vidav algebra* is a norm-unital complete normed complex algebra A satisfying $A = H(A,\mathbf{1}) \oplus \mathrm{i}H(A,\mathbf{1})$, where $\mathbf{1}$ stands for the unit of A. For such an algebra, the mapping $x + \mathrm{i}y \to x - \mathrm{i}y$ $(x, y \in H(A,\mathbf{1}))$ is called the *natural involution* of A.

Now let A be a non-zero C^*-algebra with a unit $\mathbf{1}$, and f be a norm-one $\mathbf{1}$-admissible product on A. Denote by A_1 and A_2 the normed algebras consisting of the Banach space of the associative Vidav algebra A and the product

$$f_1 : (x,y) \to 2^{-1}(xy + yx) \quad \text{and} \quad f_2 : (x,y) \to 2^{-1}(f(x,y) + f(y,x)),$$

respectively. The definition of a Vidav algebra only involves the underlying Banach space and one distinguished element, the unit. Therefore, A_1 and A_2 are commutative Vidav algebras as well. Since they have the same unit (equal to the unit $\mathbf{1}$ of A), they have also the same natural involution (equal to the C^*-involution of A), which is multiplicative on A_1 in an obvious manner and also on A_2 thanks to [25], Theorem 1. Now, since the mapping $F : x \to x$ from A_1 to A_2 is a surjective linear isometry preserving the units, the argument in the proof of the implication (i) $\Rightarrow$ (ii) of [15], Lemma 6, shows that F is an algebra isomorphism, so $f_1 = f_2$, and so the product f on A is Jordan-admissible. The results we have just shown are collected in the next lemma.

Lemma 1.1. *Let A be a C^*-algebra with unit $\mathbf{1}$, and f a product on A with $\|f\| \leqslant 1$. Then f is Jordan-admissible if and only if it is $\mathbf{1}$-admissible.*

Recall that a *Jordan-homomorphism* between two algebras A and B is a linear mapping $F : A \to B$ such that $F(x^2) = (F(x))^2$ for all x in A (equivalently, $F(xy + yx) = F(x)F(y) + F(y)F(x)$ for all x, y in A). Let X be a normed space, and u be a norm-one element in X. We say that u is a *vertex* of the closed unit ball of X if the conditions $x \in X$ and $\phi(x) = 0$ for all ϕ in $D(X,u)$ imply $x = 0$. It is well-known and easy to see that the vertex property for u implies that u is an extreme point of the closed unit ball of X.

Corollary 1.2. *Let A be a C^*-algebra, and B a norm-unital complex normed non-associative algebra. If there exists a surjective linear isometry from B to A, then A has a unit, and the surjective linear isometries from B to A are the mappings of the form $b \to uF(b)$, where u is a unitary element in A, and F is an isometric Jordan-homomorphism from B onto A.*

Proof. Let G be a linear isometry from B onto A. By the non-associative version of the Bohnenblust-Karlin theorem ([8], see for instance [29], Theorem 1.5), the unit (say v) of B is a vertex of the closed unit ball of B, hence $u := G(v)$ is a vertex of the closed unit ball of A, and therefore A has a unit $\mathbf{1}$ ([30], Proposition 1.6.1) and u is a unitary element of A ([8], Example 4.1). Now, consider the linear isometry $F : b \to u^*G(b)$ from B onto A. Then the mapping

$$f : (x, y) \to F(F^{-1}(x)F^{-1}(y))$$

from $A \times A$ to A is a $\mathbf{1}$-admissible product on A with $\|f\| \leqslant 1$. Finally, apply Lemma 1.1. ∎

Proof of Theorem A. Since B is linearly isometric to a C^*-algebra and every product on a C^*-algebra is Arens regular ([26], [33]), it follows that the product of B^{**} (equal to the third Arens transpose of that of B) is w^*-continuous in each of its variables. Then, since B has an approximate unit bounded by one, a straightforward verification shows that B^{**} has a unit and that such a unit has norm equal to one (take a w^*-limit point in B^{**} of the approximate unit of B). By Corollary 1.2, there exists a unitary element u in A^{**} and an isometric Jordan-homomorphism Φ from B^{**} onto A^{**} such that $F^{**}(\beta) = u\Phi(\beta)$ for all β in B^{**}. For an arbitrary element x in A, we can write $x = F(b)$ for a suitable b in B, so that

$$xu^*x = F(b)u^*F(b) = u\Phi(b)u^*u\Phi(b) = u\Phi(b)\Phi(b) = u\Phi(b^2) = F(b^2) \in A.$$

By [1], Proposition 4.4, u belongs to the multiplier algebra of A. Therefore Φ maps B onto A, and the mapping $G : b \to \Phi(b)$ from B onto A is a Jordan-isomorphism. ∎

Theorem A, together with [28], Theorem 2, leads directly to the following result.

COROLLARY 1.3. *An associative normed complex algebra B is a C^*-algebra (for its own product and norm, and a suitable involution) if (and only if) B is linearly isometric to a C^*-algebra and has an approximate unit bounded by one.*

2. JORDAN-ISOMORPHISMS

This section is devoted to prove Theorem B and derive its main corollaries.

Lemma 2.1. *Let A be a non-zero C^*-algebra, f a norm-one Jordan-admissible product on A, and P a primitive ideal of A. Then there exists a real number ρ with $0 \leqslant \rho \leqslant 1$ and such that $f(x,y) - \rho xy - (1-\rho)yx$ belongs to P for all x, y in A.*

Proof. For every product h on a normed space X, denote by h^{r} the product on X defined by $h^{\mathrm{r}}(x,y) := h(y,x)$ for all x, y in X, and let $h^{***} : X^{**} \times X^{**} \to X^{**}$ stand for the third Arens transpose of h, ([7]). Then, denoting by g the C^*-product of A and keeping in mind that C^*-algebras are Arens regular, we have

$$f^{***} + f^{\mathrm{r}***} = g^{***} + g^{\mathrm{r}***} = g^{***} + g^{***\mathrm{r}}.$$

Regard the bidual A^{**} of A as a W^*-algebra under the product g^{***}, which in the following will be denoted by juxtaposition. It follows that, if $\mathbf{1}$ denotes the unit of A^{**}, then the product $2^{-1}(f^{***} + f^{\mathrm{r}***})$ on A^{**} is $\mathbf{1}$ -admissible. By the face property of the set of norm-one $\mathbf{1}$ -admissible products, also f^{***} is $\mathbf{1}$-admissible. Now take an irreducible representation π of A on a Hilbert space H in such a way that the kernel of π is the given primitive ideal P. It is known that there exists a central projection e in A^{**} such that the W^*-algebra $BL(H)$ (of all bounded linear operators on H) can be identified with the W^*-algebra eA^{**} in such a way that, up to that identification, π becomes the mapping $x \to ex$ from A to eA^{**} (see for instance [24], Theorem 3.8.2). Since f^{***} is a norm-one $\mathbf{1}$-admissible product on A^{**} and e is a central projection in A^{**}, it follows from Lemma 1.1 and [29], Lemma 5.15, that, for all α, β in A^{**}, the equality

$$f^{***}(e\alpha, e\beta) = ef^{***}(\alpha, \beta)$$

holds. As a consequence, we have $f^{***}(eA^{**}, eA^{**}) \subseteq eA^{**}$ and the mapping $(m,n) \to f^{***}(m,n)$ from $eA^{**} \times eA^{**}$ to eA^{**} is a norm-one e-admissible product on eA^{**}. Then, since eA^{**} and $BL(H)$ are isomorphic as W^*-algebras and e is the unit of eA^{**}, we may apply the determination of norm-one Id_H-admissible products on $BL(H)$ ([29], Theorem 5.17) to get the existence of a real number ρ with $0 \leqslant \rho \leqslant 1$ and

$$f^{***}(m,n) = \rho mn + (1-\rho)nm$$

for all m, n in eA^{**}. Finally, for x, y in A, we have

$$e(f(x,y) - \rho xy - (1-\rho)yx) = f^{***}(ex, ey) - \rho(ex)(ey) - (1-\rho)(ey)(ex) = 0,$$

hence $f(x,y) - \rho xy - (1-\rho)yx$ belongs to $\mathrm{Ker}(\pi) = P$, as required. ∎

Remark 2.2. Let A be a non-zero C^*-algebra, f be a norm-one product on A, and let $\mathbf{1}$ denote the unit of A^{**}. We have seen in the above proof that, if f is Jordan-admissible on A, then f^{***} is $\mathbf{1}$-admissible on A^{**}. The converse is also true: if f^{***} is $\mathbf{1}$-admissible on A^{**}, then, by Lemma 1.1, f^{***} is Jordan-admissible on A^{**}, hence f is Jordan-admissible on A. It follows from the face property of the set of norm-one $\mathbf{1}$ -admissible products on A^{**} that the set of all norm-one Jordan-admissible products on A is a face of the closed unit ball of the Banach space of all products on A.

From now on, for a C^*-algebra $A, \mathrm{Prim}(A)$ will denote the set of all primitive ideals of A endowed with the hull-kernel topology.

Lemma 2.3. *Let A be a non-zero C^*-algebra, and f a norm-one Jordan-admissible product on A. For each Q in* $\mathrm{Prim}(A)$*, set*

$$h(Q) := \{\rho \in [0,1] : f(x,y) - \rho xy - (1-\rho)yx \in Q \textit{ for all } x, y \textit{ in } A\}.$$

Let $\{P_\lambda\}_{\lambda\in\Lambda}$ be a net in $\mathrm{Prim}(A)$ *converging to some $P \in \mathrm{Prim}(A)$, $\{\rho_\lambda\}_{\lambda\in\Lambda}$ a net of real numbers with $\rho_\lambda \in h(P_\lambda)$ for all λ in Λ, and ρ be a limit point of the net $\{\rho_\lambda\}_{\lambda\in\Lambda}$. Then ρ belongs to $h(P)$.*

Proof. Let ε be an arbitrary positive number. For λ in Λ, choose $\mu(\lambda)$ in Λ such that $\mu(\lambda) \geqslant \lambda$ and $|\rho_{\mu(\lambda)} - \rho| \leqslant \varepsilon$. Then, for x, y in A and λ in Λ, we have

$$f(x,y) - \rho xy - (1-\rho)yx + P_{\mu(\lambda)} = (\rho_{\mu(\lambda)} - \rho)[x,y] + P_{\mu(\lambda)},$$

hence

$$\|f(x,y) - \rho xy - (1-\rho)yx + P_{\mu(\lambda)}\| \leqslant \varepsilon \|[x,y]\|,$$

where, as usual, we have denoted by $[x,y]$ the commutator $xy - yx$. Since, for x, y in A, the set

$$\{Q \in \mathrm{Prim}(A) : \|f(x,y) - \rho xy - (1-\rho)yx + Q\| \leqslant \varepsilon \|[x,y]\|\}$$

is closed in $\mathrm{Prim}(A)$ ([11], Proposition 3.3.2), and P belongs to the closure of the set $\{P_{\mu(\lambda)} : \lambda \in \Lambda\}$, it follows

$$\|f(x,y) - \rho xy - (1-\rho)yx + P\| \leqslant \varepsilon \|[x,y]\|$$

for all x, y in A. Now the proof is concluded by letting $\varepsilon \to 0$. ∎

The *centroid* $\Gamma(A)$ of an algebra A is the set of all linear operators φ on A satisfying $\varphi(xy) = x\varphi(y) = \varphi(x)y$ for all x, y in A. If the algebra A has a unit, then $\Gamma(A)$ naturally identifies with the centre of A. If A is a C^*-algebra, then $\Gamma(A)$ is a closed commutative subalgebra of the Banach algebra of all bounded linear operators on A and, endowed with the operator norm and the involution $*$ defined for φ in $\Gamma(A)$ by $\varphi^*(x) := (\varphi(x^*))^*$ for all x in A, becomes a C^*-algebra. Moreover, $\Gamma(A)$ identifies with the centre of the multiplier algebra of A so that, according to the Dauns-Hofmann theorem [24], Corollary 4.4.8, for φ in $\Gamma(A)$ and P in $\mathrm{Prim}(A)$, there is a unique complex number $\widehat{\varphi}(P)$ such that $\varphi x - \widehat{\varphi}(P)x$ belongs to P for all x in A, and $\varphi \to \widehat{\varphi}$ becomes a $*$-isomorphism from $\Gamma(A)$ onto the C^*-algebra of all bounded continuous complex-valued functions on $\mathrm{Prim}(A)$.

From now on, for a C^*-algebra A, A_0 will denote the closed (two-sided) ideal of A generated by the commutators in A.

Lemma 2.4. *Let A be a non-zero C^*-algebra. Then, for φ in $\Gamma(A_0)$, we have*

$$2\|\varphi\| = \sup\{\|\varphi[x,y]\| : x, y \in A, \|x\| \leqslant 1, \|y\| \leqslant 1\}.$$

Proof. Let φ be in $\Gamma(A_0)$, and ε be an arbitrary positive number. Then there exist Q in $\mathrm{Prim}(A_0)$ and ρ in $\mathbb{C}$ satisfying $\varphi b - \rho b \in Q$ for all b in A_0 and $|\rho| \geqslant \|\varphi\| - \varepsilon$. Writing $Q = A_0 \cap P$ for a suitable P in $\mathrm{Prim}(A)$ not containing A_0, the C^*-algebra A/P is not commutative so, by Kaplansky's characterization of commutativity ([18], Appendix III, Theorem B), there exists a norm-one element α in A/P with $\alpha^2 = 0$. Putting $\beta := \alpha + \alpha^*$ and $\gamma := \alpha - \alpha^*$, we easily verify that the equalities $\|\beta\| = \|\gamma\| = 1$ and $\|[\beta,\gamma]\| = 2$ are true. Now, choose z, t in A with $z \in \beta, t \in \gamma$, and $\max\{\|z\|, \|t\|\} \leqslant 1 + \varepsilon$, and set $x := \|z\|^{-1}z$ and $y := \|t\|^{-1}t$. Then x and y are norm-one elements in A, and we have

$$\begin{aligned}(1+\varepsilon)^2\|\varphi[x,y]\| &\geqslant \|z\|\,\|t\|\,\|\varphi[x,y]\| = \|\varphi[z,t]\| \\ &\geqslant \|\varphi[z,t] + P\| = \|\rho[z,t] + P\| \\ &= \|\rho[\beta,\gamma]\| = 2|\rho| \geqslant 2(\|\varphi\| - \varepsilon). \quad \blacksquare\end{aligned}$$

For every C^*-algebra A, we denote by A_{sa} the self-adjoint part of A.

Proposition 2.5. *Let A be a non-zero C^*-algebra. For φ in $\Gamma(A_0)$, let f_φ stand for the product on A defined by $f_\varphi(x,y) := \frac{1}{2}(xy + yx + \varphi[x,y])$. Then $\varphi \to f_\varphi$ is an isometric affine bijection from the closed unit ball of $\Gamma(A_0)_{\mathrm{sa}}$ onto the set of all norm-one Jordan-admissible products on A.*

Proof. Let φ be in the closed unit ball of $\Gamma(A_0)_{\mathrm{sa}}$. Then clearly f_φ is a Jordan-admissible product on A. For P in $\mathrm{Prim}(A)$ not containing $A_0, P \cap A_0$

is a primitive ideal of A_0, so there exists ρ in $\mathbb{R}$ with $|\rho| \leqslant 1$ and satisfying $\varphi b - \rho b \in P \cap A_0$ for all b in A_0. The same conclusion is obviously true (with $\rho = 0$, for example) if the primitive ideal P of A contains A_0. Therefore, for all P in $\mathrm{Prim}(A)$ and all x, y in A, we have

$$\|f_\varphi(x,y)+P\| = \frac{1}{2}\|xy+yx+\rho[x,y]+P\| = \frac{1}{2}\|(1+\rho)xy+(1-\rho)yx+P\| \leqslant \|x\|\,\|y\|,$$

hence

$$\|f_\varphi(x,y)\| = \sup\{\|f_\varphi(x,y)+P\| : P \in \mathrm{Prim}(A)\} \leqslant \|x\|\,\|y\|.$$

It follows that f_φ is a norm-one Jordan-admissible product on A.

Now, clearly, the mapping $\varphi \to f_\varphi$ from the closed unit ball of $\Gamma(A_0)_{\mathrm{sa}}$ into the set of all norm-one Jordan-admissible products on A is affine, and, by Lemma 2.4, it is isometric.

Let f be an arbitrary norm-one Jordan-admissible product on A. By Lemma 2.1, for P in $\mathrm{Prim}(A)$ there is $\rho(P)$ in $\mathbb{R}$ with $0 \leqslant \rho(P) \leqslant 1$ and such that $f(x,y) - \rho(P)xy - (1-\rho(P))yx \in P$ for all x, y in A, and it is clear that such a number $\rho(P)$ is uniquely determined whenever P does not contain A_0. Now, denoting by Ω the open subset of $\mathrm{Prim}(A)$ consisting of all primitive ideals of A which do not contain A_0, Lemma 2.3 gives us that the mapping $P \to \rho(P)$ from Ω to $[0,1]$ is continuous. Since the mapping $P \to A_0 \cap P$ from Ω to $\mathrm{Prim}(A_0)$ is a homeomorphism ([11], Proposition 3.2.1), it follows from the Dauns-Hofmann theorem that there exists some ψ in $\Gamma(A_0)$ with $0 \leqslant \psi \leqslant 1$ and such that $\psi b - \rho(P)b \in P$ for all P in Ω and all b in A_0. Putting $\varphi := 2\psi - 1, \varphi$ belongs to the closed unit ball of $\Gamma(A_0)_{\mathrm{sa}}$ and, for all x, y in A and all P in Ω, we have $f_\varphi(x,y) = yx + \psi[x,y]$, hence

$$f_\varphi(x,y) + P = yx + \rho(P)[x,y] + P = \rho(P)xy + (1-\rho(P))yx + P = f(x,y) + P.$$

The equality $f_\varphi(x,y) + P = f(x,y) + P$ we have just obtained for all x, y in A and P in Ω, remains also true for P in $\mathrm{Prim}(A)\backslash\Omega$, since then we have $xy + P = yx + P$ for all x, y in A, and Lemma 2.1 applies. It follows that $f = f_\varphi$. ∎

Proof of Theorem B. If the C^*-algebra A is non-zero, then the mapping $(x,y) \to F(F^{-1}(x)F^{-1}(y))$ from $A \times A$ to A is a norm-one Jordan-admissible product on A. Now apply Proposition 2.5. ∎

Let A be a C^*-algebra. If ψ is in $\Gamma(A)$ with $0 \leqslant \psi \leqslant 1$, then

$$(x, y) \to \psi xy + (1-\psi)yx$$

is a norm-one Jordan-admissible product on A. Indeed, for x, y in A, we have $\psi xy + (1-\psi)yx = \frac{1}{2}(xy + yx + (2\psi - 1)[x, y])$, $\|2\psi - 1\| \leqslant 1$, and the restriction of $2\psi - 1$ to A_0 can be seen as an element of $\Gamma(A_0)_{\mathrm{sa}}$ (use that $A_0 = A_0^2$), hence Proposition 2.5 applies. In general, not every norm-one Jordan-admissible product on A is of the above form, even if A has a unit and the given product is associative (see for instance [31], Example 2.3).

Examples like the one quoted above are rather artificial. In fact, the next proposition is devoted to provide an intrinsic characterization of those C^*-algebras allowing the pathology in such examples, and shows in particular that "most" C^*-algebras do not admit such a pathology. A C^*-algebra A is said to be *boundedly centrally closed* if, for every closed essential ideal I of A, the restriction mapping $\Gamma(A) \to \Gamma(I)$ is surjective. This is equivalent to require the same condition for every closed ideal I of A. For, if I is a closed ideal of a C^*-algebra A, then $J := I \oplus \mathrm{Ann}_A(I)$ is a closed essential ideal of A and the restriction mapping $\Gamma(J) \to \Gamma(I)$ is clearly surjective. All prime C^*-algebras are boundedly centrally closed, because if A is a prime C^*-algebra, and if I is a non-zero closed ideal of A, then I is a prime C^*-algebra, so $\Gamma(I)$ is a prime commutative C^*-algebra, and so $\Gamma(I)$ reduces to the complex multiples of the identity operator on I. On the other hand, the proof of [2], Theorem 2 shows that a C^*-algebra A is boundedly centrally closed if and only if the annihilator of every ideal of A is of the form Ae for some central projection e in A, and therefore all AW^*-algebras are boundedly centrally closed. Also, for every C^*-algebra A, the local multiplier algebra $M_{\mathrm{loc}}(A)$ of A, introduced by G.A. Elliot ([12]) and G.K. Pedersen ([23]), and the bounded central closure ${}^{\mathrm{c}}A$ of A (see [4] p. 165 for the definition) are boundedly centrally closed C^*-algebras (see [2], Theorem 2 and [4], Proposition 3.10, respectively). Note that, if A is an AW^*-algebra, then $M_{\mathrm{loc}}(A) = A$ ([3], Proposition 3.3). Note also that boundedly centrally closed C^*-algebras can be characterized among all C^*-algebras A by the property that $\mathrm{Prim}(A)$ is extremely disconnected (i.e., the closure in $\mathrm{Prim}(A)$ of every open subset is open ([4], Proposition 2.9)).

COROLLARY 2.6. *For a C^*-algebra A, consider the following assertions:*

(i) *A is either prime, an AW^*-algebra, or of the form $M_{\mathrm{loc}}(C)$ or ${}^{\mathrm{c}}C$ for some C^*-algebra C.*

(ii) *A is boundedly centrally closed.*

(iii) *The restriction mapping $\Gamma(A) \to \Gamma(A_0)$ is surjective.*

(iv) *Every norm-one Jordan-admissible product on* A *is of the form*

$$(x, y) \to \psi xy + (1 - \psi)yx$$

for some ψ *in* $\Gamma(A)$.

(v) *Every norm-one Jordan-admissible product on* A *is of the form*

$$(x, y) \to \psi xy + (1 - \psi)yx$$

for some ψ *in* $\Gamma(A)$ *with* $0 \leqslant \psi \leqslant 1$.

(vi) *For every complex normed non-associative algebra* B*, and every surjective isometric Jordan-homomorphism* $F : B \to A$*, there exists an element* ψ *in* $\Gamma(A)$ *with* $0 \leqslant \psi \leqslant 1$ *satisfying* $F(uv) = \psi F(u)F(v) + (1 - \psi)F(v)F(u)$ *for all* u, v *in* B.

Then (i) $\Rightarrow$ (ii)*,* (ii)$\Rightarrow$ (iii)*, and* (iii)*,* (iv)*,* (v)*, and* (vi) *are equivalent.*

Proof. In view of the above comments, only the equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) require proofs. The implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) follow from Proposition 2.5 and [27], Theorem 1, respectively. (v) $\Rightarrow$ (vi) because, if B and F are as in (vi), then the mapping $(x, y) \to F(F^{-1}(x)F^{-1}(y))$ from $A \times A$ to A is a norm-one Jordan-admissible product on A. Finally, let us prove that (vi) implies (iii). To verify (iii), it is enough to show that every element φ in the closed unit ball of $\Gamma(A_0)_{\mathrm{sa}}$ is the restriction to A_0 of some element in $\Gamma(A)$. But Proposition 2.5 gives us that, for such a φ, the normed space of A endowed with the product $(x, y) \to \frac{1}{2}(xy + yx + \varphi[x, y])$ is a complex normed non-associative algebra (say B) and, clearly, the mapping $x \to x$ from B to A is a surjective isometric Jordan-homomorphism. By the assumption (vi), there exists ψ in $\Gamma(A)$ satisfying

$$\frac{1}{2}(xy + yx + \varphi[x, y]) = \psi xy + (1 - \psi)yx = \frac{1}{2}(xy + yx + (2\psi - 1)[x, y])$$

for all x, y in A. This implies $\varphi = (2\psi - 1)|A_0$. ∎

For a normed space X, we denote by $\Pi(X)$ the normed space of all products on X.

COROLLARY 2.7. *Let* A *be a non-zero* C^**-algebra, and* f *a Jordan-admissible product on* A*. Then the following assertions are equivalent:*

(i) f *is associative (continuity of* f *is not required here).*

(ii) f *is an extreme point of the closed unit ball of* $\Pi(A)$.

(iii) f *is an extreme point of the set of all norm-one Jordan-admissible products on* A.

(iv) *There exists a self-adjoint unitary element* φ *in* $\Gamma(A_0)$ *such that*

$$f(x,y) = \frac{1}{2}(xy + yx + \varphi[x,y])$$

for all x, y *in* A.

Proof. (i) ⇒ (ii) By assumption (i) and [28], Theorem 2, the Banach space of A, endowed with the product f and the C^*-involution of A, is a C^*-algebra. Now apply Proposition 1.6.6 of [30], Proposition 2.5, and Remark 2.2.

(ii) ⇒ (iii) This implication is clear.

(iii) ⇒ (iv) Since the extreme points of the closed unit ball of $\Gamma(A_0)_{\mathrm{sa}}$ are the self-adjoint unitary elements in $\Gamma(A_0)$ ([30], Proposition 1.6.3), the existence of φ in $\Gamma(A_0)$ as required in (iv) follows from assumption (iii) and Proposition 2.5.

(iv) ⇒ (i) Let φ be the element in $\Gamma(A_0)$ given by assumption (iv). We have $\varphi^2 = 1$ and $\varphi(ab) = a\varphi(b), \varphi(ba) = \varphi(b)a$ whenever a is in A and b is in A_0. Hence, for all x, y, z in A, we find

$$\begin{aligned}
4f(f(x,y),z) &= 2f(xy + yx + \varphi[x,y], z) \\
&= (xy + yx + \varphi[x,y])z + z(xy + yx + \varphi[x,y]) \\
&\qquad + \varphi[xy + yx + \varphi[x,y], z] \\
&= 2(xyz + zyx) + 2\varphi(xyz - zyx) \\
&= x(yz + zy + \varphi[y,z]) + (yz + zy + \varphi[y,z])x \\
&\qquad + \varphi[x, yz + zy + \varphi[y,z]] \\
&= 2f(x, yz + zy + \varphi[y,z]) = 4f(x, f(y,z)). \quad ∎
\end{aligned}$$

Let A be a C^*-algebra. It follows from the above corollary that the C^*-product of A is an extreme point of the closed unit ball of $\Pi(A)$. Actually a better result holds, namely the C^*-product of A is a vertex of the closed unit ball of $\Pi(A)$ ([16]).

COROLLARY 2.8. *Let* A *be a* C^**-algebra,* B *a complex associative algebra, and* $F : B \to A$ *a surjective Jordan-isomorphism. Then there exists a self-adjoint unitary element* φ *in* $\Gamma(A_0)$ *such that*

$$F(uv) = \frac{1}{2}(F(u)F(v) + F(v)F(u) + \varphi[F(u), F(v)])$$

for all u, v *in* B.

Proof. The mapping $(x,y) \to F(F^{-1}(x)F^{-1}(y))$ from $A \times A$ to A is an associative Jordan-admissible product on A. Now apply Corollary 2.7. ∎

Let A and B be algebras over the same field, and $F : B \to A$ be a bijective linear mapping such that there exist ideals P, Q of B satisfying that $B = P \oplus Q$, $F|P$ is a homomorphism, and $F|Q$ is an anti-homomorphism. Then we say that F is the *sum of an isomorphism and an anti-isomorphism.* Mappings F as above are Jordan-isomorphisms. However, it is not true in general that surjective Jordan-$*$-isomorphisms between C^*-algebras behave in such a manner (cf. the already quoted example in [31]).

Corollary 2.9. *Let A be a boundedly centrally closed C^*-algebra, B a complex associative algebra, and $F : B \to A$ a surjective Jordan-isomorphism. Then F is the sum of an isomorphism and an anti-isomorphism.*

Proof. By Corollary 2.8, there exists a self-adjoint unitary element φ in $\Gamma(A_0)$ such that

$$F(uv) = \frac{1}{2}(F(u)F(v) + F(v)F(u) + \varphi[F(u), F(v)])$$

for all u, v in B. Let I be the closed essential ideal of A defined by $I := A_0 \oplus \mathrm{Ann}_A(A_0)$, and ϕ be the unique element in $\Gamma(I)$ extending $\frac{1}{2}(1 + \varphi)$ and vanishing on $\mathrm{Ann}_A(A_0)$. Since A is boundedly centrally closed, there exists ψ in $\Gamma(A)$ such that $\psi|I = \phi$. Since ϕ is a projection in $\Gamma(I)$ and the restriction mapping $\Gamma(A) \to \Gamma(I)$ is an injective $*$-homomorphism, ψ is a projection in $\Gamma(A)$ and we have

$$F(uv) = \psi F(u)F(v) + (1 - \psi)F(v)F(u)$$

for all u, v in B. Then it is routine to verify that $P := F^{-1}(\psi A)$ and $Q := F^{-1}((1-\psi)A)$ are ideals of B satisfying that $B = P \oplus Q$, $F|P$ is a homomorphism, and $F|Q$ is an anti-homomorphism. ∎

An alternative proof of Corollary 2.9 is provided in [5], Chapter 6, by applying the purely algebraic result in [9], Theorem 2.3.

Remark 2.10. Since surjective Jordan-homomorphisms from C^*-algebras to C^*-algebras are continuous (see for instance the introduction of [32]), and closed Jordan-ideals of C^*-algebras are ideals ([10], Theorem 5.3), the following variants of Corollaries 2.8 and 2.9 hold. If F is a Jordan-homomorphism from the C^*-algebra B onto the C^*-algebra A, then there exists a self-adjoint unitary element φ in $\Gamma(A_0)$ such that

$$F(uv) = \frac{1}{2}(F(u)F(v) + F(v)F(u) + \varphi[F(u), F(v)])$$

for all u, v in B. If in addition A is boundedly centrally closed, then F is "the sum of an epimorphism and an anti-epimorphism" in the sense of [31].

3. AN APPLICATION

We conclude this paper with a geometric characterization of C^*-algebras among non-associative normed complex algebras. The proof involves Theorem A and the arguments in the proof of Theorem B.

THEOREM 3.1. *Let B be a non-associative normed complex algebra. Then the following assertions are equivalent:*

(i) *B is a C^*-algebra with respect to the given product and norm.*

(ii) *B is linearly isometric to a C^*-algebra, has an approximate unit bounded by one, and the product of B is an extreme point of the closed unit ball of* $\Pi(B)$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from Corollary 2.7. Let us assume that (ii) holds. Then there exists a linear isometry from B onto some C^*-algebra A. By Theorem A, there also exists an isometric Jordan homomorphism F from B onto A. Since the product of B is an extreme point of the closed unit ball of $\Pi(B)$, and the mapping

$$f \to [(x,y) \to F\{f(F^{-1}(x), F^{-1}(x))\}]$$

from $\Pi(B)$ to $\Pi(A)$ is a surjective linear isometry, it follows that $h:(x,y) \to F(F^{-1}(x)F^{-1}(x))$ is a Jordan-admissible product on A and an extreme point of the closed unit ball of $\Pi(A)$. By Corollary 2.7 and [28], Theorem 2, the Banach space of A, endowed with the product h and the C^*-involution of A, is a C^*-algebra (say C). Finally, F becomes an isometric algebra isomorphism from B onto the C^*-algebra C. ∎

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