J. OPERATOR THEORY
40(1998), 133–146

ALEKSANDROV MEASURES USED IN ESSENTIAL NORM INEQUALITIES FOR COMPOSITION OPERATORS

JONATHAN E. SHAPIRO

Communicated by William B. Arveson

ABSTRACT. Recently, Aleksandrov measures have been used to find new formulas for the essential norms of composition operators. Here it is shown how these measures also provide information about the essential norm of the difference of two composition operators. The utility of the Aleksandrov measure approach to the study of composition operators is illustrated in the final section of this paper, where these measures are used to supply new proofs of several known results.

KEYWORDS: *Composition operator, essential norm, Hardy space, Aleksandrov measure.*

AMS SUBJECT CLASSIFICATION: Primary 47B38; Secondary 30H05.

1. INTRODUCTION

Several theorems regarding composition operators have related properties of the composition operator to the existence and size of the angular derivative of the inducing function. An early example of this is the theorem which tells us that if a holomorphic self-map of the unit disk has an angular derivative at any boundary point, then its associated composition operator is not compact (see [17]). The converse does not hold, and later Joel Shapiro discovered necessary and sufficient conditions for compactness of such composition operators which involve properties the Nevanlinna counting function (see [13]). Other conditions for compactness of the composition operator, by D. Sarason, Shapiro, and C. Sundberg (see [10], [15]) involving the Aleksandrov measures corresponding to the inducing function, more directly generalize the angular derivative condition. In this paper we pursue further the relationship between the properties of these measures, particularly the parts of these measures singular with respect to Lebesgue measure, and the properties of the related composition operators. It will be shown that the Aleksandrov measures prove useful in providing new proofs for or even improving existing results of Berkson, Shapiro, Sundberg, MacCluer, and Cowen, relating to essential norm estimates for composition operators, and that the Aleksandrov measure approach might be of further use in outstanding isolation problems, and other problems involving composition operators.

Some general facts about composition operators are provided in Section 2, including Shapiro's formula for the essential norm of a composition operator. In Section 4 we discuss the formula found by J. Cima and A. Matheson ([2]), $\|C_\varphi\|_{\mathrm{e}}^2 = \sup_{\lambda \in \partial\mathbb{D}} \|\mu_\lambda^{\mathrm{s}}\|$, and prove some generalizations to inequalities in the case when we have combinations of more than one composition operator, after a description of properties of the family of measures $\{\mu_\lambda\}$, that we provide in Section 3. The main theorems are used to discuss the question of compactness of the difference of two composition operators in Section 5. Other related consequences of the various Aleksandrov measure formulas are discussed in Section 6. New proofs are given for several known results, and it is shown that this approach can provide sufficient conditions for equality in an inequality found by Cowen ([3], Theorem 2.4).

Some notes on notation: throughout this paper we will use φ and ψ to denote holomorphic self-maps of the unit disk $\mathbb{D}$. As such, φ has boundary values at almost every point on the unit circle $\partial\mathbb{D}$, which can be defined by taking radial (or nontangential) limits, and we will use φ (with argument on the unit circle) to denote those, too. All integrals, unless otherwise specified, are on the unit circle. Normalized Lebesgue measure on the unit circle will be denoted m.

2. COMPOSITION OPERATORS ON H^2

A composition operator C_φ, induced by a holomorphic self-map of the unit disk φ, can be defined on many different Banach spaces of analytic functions by the formula $C_\varphi f = f \circ \varphi$. The study of composition operators thus can be seen to involve significantly, and provide links between operator theory and analytic function theory. Our analysis here will be confined to the behavior of composition operators on the Hardy space H^2, which can be defined as the space of all analytic functions with square-summable Taylor coefficients. For our purposes, it will be more useful to think of H^2 as the space of analytic functions f on the unit disk

$\mathbb{D}$ which have the property that $\lim\limits_{r\to 1^-} \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$ exists, and when it does, this limit will be the square of the norm of the function f. As an operator on H^2, C_φ is bounded. This is not at all obvious, but it is a very important fact which is most easily seen as a consequence of Littlewood's subordination principle (see [5], Theorem 1.7 or [7]). The composition operator C_φ is also bounded on the other H^p spaces, as well as the Bergman spaces and others, but we will concentrate our attention here to its action on H^2, which has been a setting for many interesting problems concerning composition operators. Additionally, the compactness of C_φ in H^2 (or any H^p, $0 < p < \infty$) is equivalent to the compactness of C_φ in all other H^p; see [17], Theorem 6.1.

By the essential norm of a composition operator, we mean its distance, in the operator norm, from the space of compact operators on H^2. That is, the essential norm $\|C_\varphi\|_e = \inf\left\{\|C_\varphi - T\| :\ T \text{ compact on } H^2\right\}$.

Shapiro, in [13], gives a formula for the essential norm of a composition operator in purely function-theoretic terms.

THEOREM 2.1. [Shapiro] *For φ a holomorphic self-map of the unit disk,*

$$\|C_\varphi\|_e^2 = \overline{\lim_{|a|\to 1}}\ \frac{N_\varphi(a)}{-\log|a|}.$$

Here $N_\varphi(a)$ is the Nevanlinna counting function for φ, which can be defined as $\sum\limits_{\varphi(w)=a} \log\frac{1}{|w|}$, and $\overline{\lim}$ is used to denote the upper limit. This has the immediate corollary which gives necessary and sufficient conditions for compactness of a composition operator.

COROLLARY 2.2. *A composition operator C_φ is compact on H^2 if and only if*

$$\lim_{|a|\to 1}\ \frac{N_\varphi(a)}{-\log|a|} = 0.$$

3. THE MEASURES μ_λ

For any holomorphic self-map of the unit disk φ and $\lambda \in \partial\mathbb{D}$, the real part of the function $\frac{\lambda+\varphi(z)}{\lambda-\varphi(z)}$ is a positive harmonic function which can be written as $\mathrm{Re}\left(\frac{\lambda+\varphi(z)}{\lambda-\varphi(z)}\right) = \int P(\zeta,z)\mathrm{d}\mu_\lambda(\zeta)$ for some positive Borel measure μ_λ, where $P(\zeta,z) = \frac{1-|z|^2}{|\zeta-z|^2}$ is the Poisson kernel for $z \in \mathbb{D}$. In this way we associate a family of measures $\{\mu_\lambda\}_{\lambda\in\partial\mathbb{D}}$ with the function φ. These measures have played an interesting role in the study of the de Branges-Rovnyak space $\mathcal{H}(\varphi)$ (see [11], Chapter III). They also provided the main basis for definition and study of "relative angular derivatives" by the author in [12]. The measures μ_λ have the following properties:

(i) We have the Herglotz integral representation,

$$\frac{\lambda+\varphi(z)}{\lambda-\varphi(z)} = \int \frac{\zeta+z}{\zeta-z}\mathrm{d}\mu_\lambda(\zeta) + \mathrm{i}\,\mathrm{Im}\left(\frac{\lambda+\varphi(0)}{\lambda-\varphi(0)}\right).$$

(ii) All positive Borel measures on $\partial\mathbb{D}$ are associated with holomorphic functions in this way.

(iii) The absolutely continuous part of μ_λ, denoted $\mu_\lambda^{\mathrm{a.c.}}$, is given by $\mathrm{d}\mu_\lambda^{\mathrm{a.c.}} = \frac{1-|\varphi|^2}{|\lambda-\varphi|^2}\mathrm{d}m$, where m is normalized Lebesgue measure on the unit circle.

(iv) For any λ, the measure μ_λ is singular if and only if φ is an inner function, i.e., $|\varphi| = 1$ almost everywhere on $\partial\mathbb{D}$.

(v) The norm of μ_λ is given by

$$\|\mu_\lambda\| = \int \mathrm{d}\mu_\lambda = \int P(\zeta,0)\mathrm{d}\mu_\lambda(\zeta) = \mathrm{Re}\left(\frac{\lambda+\varphi(0)}{\lambda-\varphi(0)}\right) = \frac{1-|\varphi(0)|^2}{|\lambda-\varphi(0)|^2}.$$

(vi) The measure μ_λ has an atom at a point z_0 on the unit circle if and only if φ has an angular derivative at the point z_0, and in this case $\mu_\lambda(\{z_0\}) = \frac{1}{|\varphi'(z_0)|}$. See [11], Theorem VI-7 or [12], Special Case 7.1, 7.3.

(vii) For μ_λ^{s}-a.e. $\zeta \in \partial\mathbb{D}$, the function $\varphi(z) \to \lambda$ as $z \to \zeta$ nontangentially, where μ_λ^{s} denotes the singular part of μ_λ.

4. MAIN THEOREMS

In this section we will state and prove the main theorems of the paper. We will begin by mentioning a result which was proved by Cima and Matheson in [2], which provides the Aleksandrov measure formula for the essential norm of a composition operator.

Theorem 4.1. ([2]) *Let φ be a holomorphic self-map of the unit disk with corresponding measures μ_λ, $\lambda \in \partial\mathbb{D}$. Then*

$$\|C_\varphi\|_{\mathrm{e}}^2 = \sup_{\lambda\in\partial\mathbb{D}} \|\mu_\lambda^{\mathrm{s}}\|.$$

This theorem has the immediate corollary.

Corollary 4.2. *A composition operator C_φ is compact on H^2 if and only if for all $\lambda \in \partial\mathbb{D}$ the measure μ_λ is absolutely continuous.*

Indeed, this last corollary was originally proved in a totally different way by Sarason, Shapiro, and Sundberg (see [10], [15]). It first demonstrated the use of Aleksandrov measures to understand properties of composition operators. Cima and Matheson, with their result above, provided the more direct proof, asked for by Sarason in [11], p. 84, of the equivalence of the compactness conditions in Corollaries 4.2 and 2.2. This result, then, generalized the angular-derivative criterion by providing necessary and sufficient conditions for compactness of a composition operator (in a way different from the generalization provided by Shapiro in [13]). Using the notion of relative angular derivatives (see [12]), one can interpret Corollary 4.2 as saying that C_φ is compact on H^2 if and only if φ has no angular derivative "relative" to any inner function u.

In the rest of this section, we will follow along the lines of Cima and Matheson in [2], but consider the more general case of the essential norm of the difference of two composition operators, and eventually, the essential norm of any linear combination of composition operators. These theorems are motivated by the theorems in the work of B. MacCluer ([8], Theorems 2.2 and 3.1), which have the similar goal of finding lower bounds for the essential norms of composition operator differences or linear combinations, but which involve the angular derivative instead of Aleksandrov measures. In Section 5 this relationship will be explored. The theorems are also motivated by their relationship to the questions of isolation of composition operators. In particular, these theorems are motivated by their use to provide alternate proofs for some theorems of Shapiro and Sundberg ([16], Theorem 2.3, Corollary 2.4) which give information about the isolation of composition operators. These appear here in Section 6.

For questions concerning the essential norms of differences or linear combinations of composition operators, we will not be able to get exact answers, as we have above for the essential norm of a single composition operator, but we will be able to get better lower bounds than were available before, and from this derive better information about the structure of the space of composition operators on H^2.

THEOREM 4.3. *Let φ and ψ be holomorphic self-maps of the unit disk with corresponding measures μ_λ and ν_λ, $\lambda \in \partial\mathbb{D}$. If for some λ, μ_λ^{s} and ν_λ^{s} are mutually singular, then*

$$\|C_\varphi - C_\psi\|_{\mathrm{e}}^2 \geqslant \|\mu_\lambda^{\mathrm{s}}\| + \|\nu_\lambda^{\mathrm{s}}\|.$$

Proof. We begin the proof by noting that, as in [13], $\|T\|_{\mathrm{e}}^2 \geqslant \|Tf_a\|_2^2$ for any operator T on H^2, where f_a is the normalized reproducing kernel for $a \in \mathbb{D}$, so

$$\|C_\varphi - C_\psi\|_{\mathrm{e}}^2 \geqslant \|(C_\varphi - C_\psi)f_a\|_2^2.$$

We can now write

$$\begin{aligned} \|(C_\varphi - C_\psi)f_a\|_2^2 &= \langle (C_\varphi - C_\psi)f_a, (C_\varphi - C_\psi)f_a\rangle \\ &= \|C_\varphi f_a\|_2^2 + \|C_\psi f_a\|_2^2 - 2\mathrm{Re}\langle C_\varphi f_a, C_\psi f_a\rangle \end{aligned} \tag{4.1}$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the inner product in H^2. As $a \to \lambda$ nontangentially, the first two terms approach $\|\mu_\lambda^{\mathrm{s}}\|$ and $\|\nu_\lambda^{\mathrm{s}}\|$ — this was proved in the course of the proof by Cima and Matheson of Theorem 4.1, but it can also be seen a corollary of Lemma 4.4 below. The third term approaches zero, as we will show.

$$|\langle C_\varphi f_a, C_\psi f_a\rangle| = \left|\int (1-|a|^2)\left(\frac{1}{1-\overline{a}\varphi(\xi)}\frac{1}{1-a\overline{\psi}(\xi)}\right)\mathrm{d}m(\xi)\right|. \tag{4.2}$$

The integral above is taken over the unit circle, and no matter how we divide the unit circle into two (measurable) pieces E and E^{c} (the complement of E in $\partial\mathbb{D}$), we will have

$$\int (1-|a|^2)\left(\frac{1}{1-\overline{a}\varphi(\xi)}\frac{1}{1-a\overline{\psi}(\xi)}\right)\mathrm{d}m(\xi) = \int\limits_E + \int\limits_{E^{\mathrm{c}}},$$

and then we can write

$$\begin{aligned} \left|\int\limits_E + \int\limits_{E^{\mathrm{c}}}\right| &\leqslant \left|\int\limits_E\right| + \left|\int\limits_{E^{\mathrm{c}}}\right| \\ &\leqslant \left(\int\limits_E \frac{1-|a|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi)\int\limits_E \frac{1-|a|^2}{|1-\overline{a}\psi(\xi)|^2}\mathrm{d}m(\xi)\right)^{\frac{1}{2}} \\ &\quad + \left(\int\limits_{E^{\mathrm{c}}} \frac{1-|a|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi)\int\limits_{E^{\mathrm{c}}} \frac{1-|a|^2}{|1-\overline{a}\psi(\xi)|^2}\mathrm{d}m(\xi)\right)^{\frac{1}{2}} \end{aligned} \tag{4.3}$$

by the Cauchy-Schwarz inequality. The idea here is to split the unit circle into two pieces, one where μ_λ^{s} is small, so the contribution from the first integral in the products above is small, and the other where ν_λ^{s} is small, so the contribution from the second integral is small. In order to proceed, we will need the following lemma.

LEMMA 4.4. *Given any open arc $E \subset \partial\mathbb{D}$, for which μ_λ does not have an atom at the endpoints, we have*

$$\int_E \frac{1-|a|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi) \to \mu_\lambda^{\mathrm{s}}(E) \text{ as } a \to \lambda \text{ nontangentially.}$$

Proof. We prove this by using the decomposition

$$\int_E \frac{1-|a|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi) = \int_E \frac{1-|\overline{a}\varphi(\xi)|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi) - |a|^2 \int_E \frac{1-|\varphi(\xi)|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi). \tag{4.4}$$

First we consider the second term on the right as $a \to \lambda$ nontangentially. We use the fact that

$$\frac{(1-|\varphi(\xi)|^2)^{\frac{1}{2}}}{1-\overline{a}\varphi(\xi)} - \frac{(1-|\varphi(\xi)|^2)^{\frac{1}{2}}}{1-\overline{\lambda}\varphi(\xi)} = \frac{(1-|\varphi(\xi)|^2)^{\frac{1}{2}}}{(1-\overline{\lambda}\varphi(\xi))} \frac{(\overline{a}-\overline{\lambda})\varphi(\xi)}{(1-\overline{a}\varphi(\xi))},$$

which tells us that the two L^2 fractions from the left-hand side differ by an L^2 function that approaches 0 pointwise for m-a.e. ξ, and which has norm that approaches 0 as $a \to \lambda$ nontangentially (by dominated convergence, since $\left|\frac{(\overline{a}-\overline{\lambda})\varphi(\xi)}{(1-\overline{a}\varphi(\xi))}\right| \leqslant \frac{|a-\lambda|}{1-|a|}$, which stays bounded as $a \to \lambda$ nontangentially). Thus

$$\int_E \frac{1-|\varphi(\xi)|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi) \to \int_E \frac{1-|\varphi(\xi)|^2}{|\lambda-\varphi(\xi)|^2}\mathrm{d}m(\xi) \quad \text{as } a \to \lambda \text{ nontangentially.} \tag{4.5}$$

For the first term on the right of (4.4), consider the measure $\frac{1-|\overline{a}\varphi(\xi)|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m$, which we will call μ_a. We will show that $\mu_a \to \mu_\lambda$ weak* as $a \to \lambda$. This is true since for any Poisson kernel $P(\xi, w)$ (for $w \in \mathbb{D}$),

$$\begin{aligned} \int P(\xi,w)\mathrm{d}\mu_a(\xi) &= \int P(\xi,w)\frac{1-|\overline{a}\varphi(\xi)|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi) = \frac{1-|\overline{a}\varphi(w)|^2}{|1-\overline{a}\varphi(w)|^2} \\ &\to \frac{1-|\varphi(w)|^2}{|1-\overline{\lambda}\varphi(w)|^2} = \operatorname{Re}\left(\frac{\lambda+\varphi(w)}{\lambda-\varphi(w)}\right) = \int P(\xi,w)\mathrm{d}\mu_\lambda(\xi). \end{aligned}$$

This then tells us that for any continuous function g on $\partial\mathbb{D}$,

$$\int g(\xi)\mathrm{d}\mu_a(\xi) \to \int g(\xi)\mathrm{d}\mu_\lambda(\xi). \tag{4.6}$$

Let $E = (a, b)$ be the arc on the unit circle from $\mathrm{e}^{\mathrm{i}a}$ to $\mathrm{e}^{\mathrm{i}b}$, whose endpoints are not atoms of μ_λ. Now consider a family of continuous functions $\{g_n\}$ which take values in $[0, 1]$ and are 0 off E and 1 on the subarc $E_n = (a + \frac{1}{n}, b - \frac{1}{n})$ of E (for n big enough for this to be nontrivial).

$$\begin{aligned}\left|\int \chi_E(\xi)\mathrm{d}\mu_a(\xi) - \int \chi_E(\xi)\mathrm{d}\mu_\lambda(\xi)\right| \leqslant &\left|\int \chi_E(\xi)\mathrm{d}\mu_a(\xi) - \int g_n(\xi)\mathrm{d}\mu_a(\xi)\right| \\ &+ \left|\int g_n(\xi)\mathrm{d}\mu_a(\xi) - \int g_n(\xi)\mathrm{d}\mu_\lambda(\xi)\right| \\ &+ \left|\int g_n(\xi)\mathrm{d}\mu_\lambda(\xi) - \int \chi_E(\xi)\mathrm{d}\mu_\lambda(\xi)\right|.\end{aligned}$$

We can choose n big enough that the first and third terms on the right above are arbitrarily small, since each can be considered an integral over the region $F_n = (a, a + \frac{1}{n}) \cup (b - \frac{1}{n}, b)$ of a continuous function with values in $[0, 1]$ with respect to the measures μ_a or μ_λ — the measure μ_a is absolutely continuous, and μ_λ has no atom at $\mathrm{e}^{\mathrm{i}a}$ or $\mathrm{e}^{\mathrm{i}b}$, the endpoints of the arc E. The second term on the right approaches zero as $a \to \lambda$ by (4.6). We thus get, as $a \to \lambda$

$$\int \chi_E(\xi)\mathrm{d}\mu_a(\xi) \to \int \chi_E(\xi)\mathrm{d}\mu_\lambda(\xi),$$

i.e.,

$$\int_E \mathrm{d}\mu_a(\xi) \to \int_E \mathrm{d}\mu_\lambda(\xi) = \mu_\lambda(E).$$

We now get the lemma by combining equation (4.4) with the convergences of each piece of the right-hand side proved above. It should be noted that we can extend this easily to any $E \in \partial\mathbb{D}$ that is a finite union of arcs, each as in the lemma.

Given $\varepsilon > 0$, we now use the fact that $\mu_\lambda^{\mathrm{s}} \perp \nu_\lambda^{\mathrm{s}}$ to choose some open set $F \subset \partial\mathbb{D}$ such that $\mu_\lambda^{\mathrm{s}}(F) \geqslant \|\mu_\lambda^{\mathrm{s}}\| - \varepsilon$ while at the same time $\nu_\lambda^{\mathrm{s}}(F) < \varepsilon$. F is a union of open arcs, so we can pick a finite collection of these arcs, whose union we will call E, with $\mu_\lambda^{\mathrm{s}}(F) - \mu_\lambda^{\mathrm{s}}(E) < \varepsilon$, and, by shrinking them a small amount if necessary, we can be sure that none of these arcs has an endpoint at which μ_λ has an atom. E^{c} is now a finite union of closed arcs (we can ignore any possible single point sets), and has the property that $\mu_\lambda^{\mathrm{s}}(E^{\mathrm{c}}) < 2\varepsilon$. With this E, we can see that by the extended version of the lemma, as $a \to \lambda$ nontangentially, the right hand side of the inequality (4.3) is at most

$$(\mu_\lambda^{\mathrm{s}}(E)\nu_\lambda^{\mathrm{s}}(E))^{\frac{1}{2}} + (\mu_\lambda^{\mathrm{s}}(E^{\mathrm{c}})\nu_\lambda^{\mathrm{s}}(E^{\mathrm{c}}))^{\frac{1}{2}} < \|\mu_\lambda^{\mathrm{s}}\|^{1/2}\varepsilon^{\frac{1}{2}} + \|\nu_\lambda^{\mathrm{s}}\|^{1/2}(2\varepsilon)^{\frac{1}{2}}.$$

Since this holds for any $\varepsilon > 0$, we conclude that as $a \to \lambda$ nontangentially, $\langle C_\varphi f_a, C_\psi f_a\rangle \to 0$, and the theorem is proved. ∎

If for some $\lambda \in \partial\mathbb{D}$, μ_λ^{s} and ν_λ^{s} are not mutually singular, but are different, then we can still get a lower estimate for $\|C_\varphi - C_\psi\|_{\mathrm{e}}^2$. If E is any arc (or finite set of arcs) in $\partial\mathbb{D}$, with $\mu_\lambda^{\mathrm{s}}(E) \neq \nu_\lambda^{\mathrm{s}}(E)$, then we can use equations (4.1), (4.2), and the Cauchy-Schwarz inequality, with the integrals all taken over the set E to get

$$\begin{aligned}
\|C_\varphi - C_\psi\|_{\mathrm{e}}^2 &\geqslant \|(C_\varphi - C_\psi) f_a\|_2^2 \\
&\geqslant \int\limits_E \frac{1-|a|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi) + \int\limits_E \frac{1-|a|^2}{|1-\overline{a}\psi(\xi)|^2}\mathrm{d}m(\xi) \\
&\quad - 2\biggl(\int\limits_E \frac{1-|a|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi) \int\limits_E \frac{1-|a|^2}{|1-\overline{a}\psi(\xi)|^2}\mathrm{d}m(\xi)\biggr)^{\frac{1}{2}} \\
&\geqslant \biggl(\biggl(\int\limits_E \frac{1-|a|^2}{|1-\overline{a}\varphi(\xi)|^2}\mathrm{d}m(\xi)\biggr)^{\frac{1}{2}} - \biggl(\int\limits_E \frac{1-|a|^2}{|1-\overline{a}\psi(\xi)|^2}\mathrm{d}m(\xi)\biggr)^{\frac{1}{2}}\biggr)^2.
\end{aligned}$$

By the lemma, this last difference approaches $(\mu_\lambda^{\mathrm{s}}(E)^{1/2} - \nu_\lambda^{\mathrm{s}}(E)^{1/2})^2$ as $a \to \lambda$ nontangentially. This gives us

Theorem 4.5. *If φ and ψ are holomorphic self-maps of the disk which have corresponding measures μ_λ and ν_λ, $\lambda \in \partial\mathbb{D}$, then for any λ and any set E that is a finite union of arcs in $\partial\mathbb{D}$ whose endpoints do not contain atoms of the measures μ_λ or ν_λ, we have*

$$\|C_\varphi - C_\psi\|_{\mathrm{e}} \geqslant |\mu_\lambda^{\mathrm{s}}(E)^{\frac{1}{2}} - \nu_\lambda^{\mathrm{s}}(E)^{\frac{1}{2}}|.$$

In particular, unless $\mu_\lambda^{\mathrm{s}} = \nu_\lambda^{\mathrm{s}}$ for all λ, we can find such a set E for which the right side of the above is positive, and we will thus have

$$\|C_\varphi - C_\psi\|_{\mathrm{e}} > 0.$$

We can generalize the methods above to give a lower bound on the essential norm of a linear combination of composition operators.

Theorem 4.6. *Let $\varphi_1, \dots, \varphi_n$ be holomorphic self-maps of the disk, with corresponding measures $\mu_{1,\lambda}, \dots, \mu_{n,\lambda}$. If for some $\lambda \in \partial\mathbb{D}$, the measures $\mu_{1,\lambda}^{\mathrm{s}}, \dots, \mu_{n,\lambda}^{\mathrm{s}}$ are mutually singular, then*

$$\Bigl\|\sum_{j=1}^n a_j C_{\varphi_j}\Bigr\|_{\mathrm{e}}^2 \geqslant |a_1|^2\|\mu_{1,\lambda}^{\mathrm{s}}\| + \dots + |a_n|^2\|\mu_{n,\lambda}^{\mathrm{s}}\|.$$

Proof. First we write

$$\Bigl\|\sum_{j=1}^n a_j C_{\varphi_j}\Bigr\|_{\mathrm{e}}^2 \geqslant \Bigl\|\Bigl(\sum_{j=1}^n a_j C_{\varphi_j}\Bigr) f_a\Bigr\|_2^2$$

and

$$\Big\|\Big(\sum_{j=1}^{n}a_jC_{\varphi_j}\Big)f_a\Big\|_2^2=\Big\langle\Big(\sum_{j=1}^{n}a_jC_{\varphi_j}\Big)f_a,\Big(\sum_{j=1}^{n}a_jC_{\varphi_j}\Big)f_a\Big\rangle$$
$$=|a_1|^2\|C_{\varphi_1}f_a\|_2^2+\cdots+|a_n|^2\|C_{\varphi_n}f_a\|_2^2+\sum_{i\neq j}a_i\overline{a_j}\langle C_{\varphi_i}f_a,C_{\varphi_j}f_a\rangle.$$

The first terms on the right approach $|a_1|^2\|\mu^{\mathrm{s}}_{1,\lambda}\|+\cdots+|a_n|^2\|\mu^{\mathrm{s}}_{n,\lambda}\|$ as $a\to\lambda$ nontangentially, and the remaining terms approach zero, as we showed earlier. ∎

5. COMPACTNESS OF THE DIFFERENCE OF COMPOSITION OPERATORS

In [8], MacCluer gives several theorems regarding the essential norms of composition operator differences. One of them, [8], Theorem 2.2, tells us

THEOREM 5.1. ([8]) *Let $\varphi:\mathbb{D}\to\mathbb{D}$ and suppose that φ has finite angular derivative at a point $\mathrm{e}^{\mathrm{i}\theta}\in\partial\mathbb{D}$. Let $\psi:\mathbb{D}\to\mathbb{D}$ be holomorphic and consider C_φ and C_ψ acting on H^2. Then, unless both*

$$\psi(\mathrm{e}^{\mathrm{i}\theta})=\varphi(\mathrm{e}^{\mathrm{i}\theta})$$

and

$$\psi'(\mathrm{e}^{\mathrm{i}\theta})=\varphi'(\mathrm{e}^{\mathrm{i}\theta}),$$

we have $\|C_\varphi-C_\psi\|_{\mathrm{e}}^2\geqslant|\varphi'(\mathrm{e}^{\mathrm{i}\theta})|^{-1}$.

From this theorem, we can get the immediate corollary.

COROLLARY 5.2. *If $C_\varphi-C_\psi$ is a compact operator (on H^2), then φ and ψ must have angular derivatives at the same places on the unit circle, and at those places the values of the angular derivatives must be the same.*

From our point of view, we can use Theorem 4.5 to get the following.

THEOREM 5.3. *If $C_\varphi-C_\psi$ is a compact operator, then we must have $\mu^{\mathrm{s}}_\lambda=\nu^{\mathrm{s}}_\lambda$ for all $\lambda\in\partial\mathbb{D}$.*

This theorem is a generalization of the previous one, since if $\mu^{\mathrm{s}}_\lambda=\nu^{\mathrm{s}}_\lambda$ for all $\lambda\in\partial\mathbb{D}$, then φ and ψ have the same angular derivatives — wherever any μ^{s}_λ or ν^{s}_λ has an atom, with magnitude the inverse of the magnitude of the atom.

The angular derivative condition mentioned above, though necessary for compactness of the difference of two composition operators, is, by this last theorem, not sufficient. The situation is seemingly quite similar to the compactness question for a single composition operator, in which the Aleksandrov measure condition was both necessary and sufficient for compactness. In this case, however, the converse to Theorem 5.3 is unknown. We need to have an upper bound to $\|C_\varphi - C_\psi\|_{\mathrm{e}}^2$ of a form similar to the lower bound to prove the converse, which would then give us an answer to a question raised in [16].

CONJECTURE 5.4. Given holomorphic self-maps of the disk φ and ψ, with associated measures μ_λ and ν_λ, $C_\varphi - C_\psi$ is a compact operator on H^2 if and only if $\mu_\lambda^{\mathrm{s}} = \nu_\lambda^{\mathrm{s}}$ for all $\lambda \in \partial\mathbb{D}$.

6. OTHER CONSEQUENCES OF THE MAIN THEOREMS

The Aleksandrov measure approach to the essential norm of a composition operator leads to several interesting inequalities and equalities, both by generalizing some theorems, as we have shown above, and by providing new proofs to some theorems already known. We will begin by presenting new proofs for some theorems by Shapiro and Sundberg in [16], Theorem 2.3, Corollary 2.4.

THEOREM 6.1. *For φ a holomorphic self-map of the disk, let $E = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$. Then $\|C_\varphi\|_{\mathrm{e}}^2 \geqslant m(E)$.*

Proof. We can relate the size of E to properties of the corresponding family $\{\mu_\lambda\}$. The average (over all $\lambda \in \partial\mathbb{D}$) of the norms of the measures $\mu_\lambda^{\mathrm{a.c.}}$ is given by $\iint \frac{1-|\varphi(\zeta)|^2}{|\lambda-\varphi(\zeta)|^2}\mathrm{d}m(\zeta)\mathrm{d}m(\lambda)$. When we switch the order of integration, we can carry out the inside integral for each λ. For any ζ in E, the integrand (and hence the integral) is zero, whereas for any ζ not in E, the integral is the that of the Poisson kernel for the point $\varphi(\zeta)$ (with $|\varphi(\zeta)| < 1$), which has value 1. Therefore we have $\iint \frac{1-|\varphi(\zeta)|^2}{|\lambda-\varphi(\zeta)|^2}\mathrm{d}m(\lambda)\mathrm{d}m(\zeta) = 1 - m(E)$. The average value of the norm of μ_λ is given by $\int \frac{1-|\varphi(0)|^2}{|\lambda-\varphi(0)|^2}\mathrm{d}m(\lambda)$, which is equal to 1. So we find that the average value of $\|\mu_\lambda^{\mathrm{s}}\|$ is just $1-(1-m(E)) = m(E)$. Since $\|C_\varphi\|_e^2 = \sup\limits_{\lambda\in\partial\mathbb{D}} \|\mu_\lambda^{\mathrm{s}}\| \geqslant \mathrm{average}\|\mu_\lambda^{\mathrm{s}}\|$, the theorem is proved. ∎

Using Theorem 4.3, we can get a new proof for the similar lower bound for the essential norms of the difference of two composition operators. The previous theorem is an immediate corollary of the following.

THEOREM 6.2. ([16]) *Let $\varphi \neq \psi$ be holomorphic self-maps of the disk. Then*

$$\|C_\varphi - C_\psi\|_{\mathrm{e}}^2 \geqslant m(E_\varphi) + m(E_\psi),$$

where $E_\varphi = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$ and $E_\psi = \{\zeta \in \partial\mathbb{D} : |\psi(\zeta)| = 1\}$.

Proof. For m-a.e. $\lambda \in \partial\mathbb{D}$, μ_λ^{s} is singular to ν_λ^{s} , since if they are not mutually singular for a.e. λ, then we use the fact that $\varphi(\zeta) = \lambda$ for μ_λ^{s} a.e.-ζ and $\psi(\zeta) = \lambda$ for ν_λ^{s} a.e.-ζ to deduce that $\left\{\lambda \in \partial\mathbb{D} : \varphi^{-1}(\lambda) \cap \psi^{-1}(\lambda) \text{ is nonempty}\right\}$ has positive measure, thus $\{\zeta \in \partial\mathbb{D} : \varphi(\zeta) = \psi(\zeta)\}$ has positive measure, which is impossible for two unequal holomorphic functions. We can thus use Theorem 4.3 to tell us that

$$\begin{aligned}
\|C_\varphi - C_\psi\|_{\mathrm{e}}^2 &\geqslant \sup_{\lambda\in\partial\mathbb{D}, \mu_\lambda^{\mathrm{s}} \perp \nu_\lambda^{\mathrm{s}}} \left(\|\mu_\lambda^{\mathrm{s}}\| + \|\nu_\lambda^{\mathrm{s}}\|\right) \\
&\geqslant \operatorname*{avg}_{\lambda\in\partial\mathbb{D}} \left(\|\mu_\lambda^{\mathrm{s}}\| + \|\nu_\lambda^{\mathrm{s}}\|\right) \\
&= \operatorname*{avg}_{\lambda\in\partial\mathbb{D}} \|\mu_\lambda^{\mathrm{s}}\| + \operatorname*{avg}_{\lambda\in\partial\mathbb{D}} \|\nu_\lambda^{\mathrm{s}}\| = m(E_\varphi) + m(E_\psi).
\end{aligned}$$

The last step follows by the argument in the proof of Theorem 6.1. ∎

Indeed, with the use of Theorem 4.6, we can even get a new proof of the theorem from [16], Theorem 2.3, which gives a similar lower bound for the essential norm of a linear combination of composition operators.

These theorems, as pointed out in [16], provide a generalization of a theorem of Berkson ([1]), and can be used to show that if $m(E_\varphi) > 0$, then C_φ is isolated in the space of composition operators acting on H^2. This is clear, since its distance in operator norm, and, in fact, its essential distance, from any other composition operator is, by the theorem, bounded below by $m(E_\varphi)^{1/2}$.

A different sort of lower bound for the essential norm of a composition operator is given by Cowen in [3], Theorem 2.4, along with a similar upper bound, under the added condition of the continuity of φ' on $\overline{\mathbb{D}}$. It also appears in [13], Theorem 3.3. It is proved by Cima and Matheson in [2] as a corollary of the formula for the essential norm of a composition operator. It is presented here to show further uses of the Aleksandrov measure approach to essential norm inequalities.

THEOREM 6.3. ([3]) *Let* $\delta(\omega) = \sum \left\{|\varphi'(\zeta)|^{-1} : \zeta \in \partial\mathbb{D} \text{ and } \varphi(\zeta) = \omega\right\}$. *Then*

$$\|C_\varphi\|_{\mathrm{e}}^2 \geqslant \sup\left\{\delta(\omega) : \omega \in \partial\mathbb{D}\right\}.$$

Proof. ([2]) This theorem is an immediate consequence of Theorem 4.1, since the measure μ_ω has atoms precisely at those points $\zeta \in \partial\mathbb{D}$ with $\varphi(\zeta) = \omega$, for which φ has an angular derivative, and the magnitude of each atom is just the reciprocal of the absolute value of the angular derivative. Thus the sum of the magnitudes of the atoms of μ_ω is exactly $\delta(\omega)$, so this is certainly $\leqslant \|\mu_\omega^{\mathrm{s}}\|$. The theorem follows. ∎

It should be noted here that we also get information about when we have equality in the theorem. If φ is univalent, or even of bounded valence, then each measure μ_ω has a singular part which consists of either at most a single atom (in the univalent case), or some number of atoms (bounded by the valence). The measure μ_ω can have a nonatomic singular part only if φ has infinite valence near some ω. If there is no nonatomic singular part of μ_ω, then we have equality in Theorem 6.3. Thus we see that though bounded valence for φ is sufficient for equality, it is not necessary.

Finally, we can obtain very easily an exact expression for the essential norm of the composition operator generated by an inner function. We get the same answer, of course, as found by Shapiro in [13], Theorem 2.5, but with a different simple proof.

Theorem 6.4. *If φ is an inner function, then*

$$\|C_\varphi\|_{\mathrm{e}} = \left[\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right]^{\frac{1}{2}}.$$

Proof. This follows from our formula for the essential norm of a composition operator, since for an inner function φ, all of the measures μ_λ are singular. We already know that

$$\|\mu_\lambda\| = \mathrm{Re}\left(\frac{\lambda+\varphi(0)}{\lambda-\varphi(0)}\right) = \frac{1-|\varphi(0)|^2}{|\lambda-\varphi(0)|^2}$$

(this was proven in our list of properties of the μ_λ). The largest value of $\frac{1-|\varphi(0)|^2}{|\lambda-\varphi(0)|^2}$ is taken when the denominator is as small as possible, i.e., λ is the boundary point closest to $\varphi(0)$. We thus have

$$\begin{aligned}\|C_\varphi\|_{\mathrm{e}}^2 &= \sup_{\lambda\in\partial\mathbb{D}} \|\mu_\lambda^{\mathrm{s}}\| = \sup_{\lambda\in\partial\mathbb{D}} \|\mu_\lambda\| = \sup_{\lambda\in\partial\mathbb{D}} \frac{1-|\varphi(0)|^2}{|\lambda-\varphi(0)|^2}\\ &= \frac{1-|\varphi(0)|^2}{(1-|\varphi(0)|)^2} = \frac{1+|\varphi(0)|}{1-|\varphi(0)|}. \quad \blacksquare\end{aligned}$$

REFERENCES

1. E. Berkson, Composition operators isolated in the uniform operator topology, *Proc. Amer. Math. Soc.* **81**(1981), 230–232.
2. J. Cima, A. Matheson, Essential norms of composition operators and Aleksandrov measures, *Pacific J. Math.*, to appear.
3. C. Cowen, Composition operators on H^2, *J. Operator Theory* **9**(1983), 77–106.

4. C. Cowen, C. Pommerenke, Inequalities for the angular derivative of an analytic function in the unit disk, *J. London Math. Soc. (2)* **26**(1982), 271–289.
5. P. Duren, *Theory of H^p Spaces*, Academic Press, New York 1970.
6. O. Frostman, Sur les produits de Blaschke, *Kungl. Fysiog. Sällsk. i Lund Förh.* **12**(1942), 169–182.
7. J.E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc. (3)* **23**(1925), 481–519.
8. B.D. MacCluer, Components in the space of composition operators, *Integral Equations Operator Theory* **12**(1989), 725–738.
9. B.D. MacCluer, J.H. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* **4**(1986), 878–906.
10. D. Sarason, Composition operators as integral operators, in *Analysis and Partial Differential Equations*, Marcel Dekker, New York, 1990, pp. 545–565.
11. D. Sarason, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley & Sons, Inc., New York 1995.
12. J.E. Shapiro, Relative angular derivatives, preprint.
13. J.H. Shapiro, The essential norm of a composition operator, *Ann. of Math. (2)* **125** (1987), 375–404.
14. J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York 1993.
15. J.H. Shapiro, C. Sundberg, Compact composition operators on L^1, *Proc. Amer. Math. Soc.* **108**(1990), 443–449.
16. J.H. Shapiro, C. Sundberg, Isolation amongst the composition operators, *Pacific J. Math.* **145**(1990), 117–152.
17. J.H. Shapiro, P.D. Taylor, Compact, nuclear, and Hilbert-Schmidt composition operators on H^2, *Indiana Univ. Math. J.* **23**(1973), 471–496.

JONATHAN E. SHAPIRO
Department of Mathematics
Northwestern University
Evanston, IL 60208
U.S.A.

E-mail: shapiro@math.nwu.edu

Received February 24, 1997; revised September 1, 1997.