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ON EXTENSIONS OF HERMITIAN FUNCTIONS WITH A FINITE NUMBER OF NEGATIVE SQUARES

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ABSTRACT. We study extensions of a Hermitian function f with a finite number of negative squares given on a finite interval to the whole real axis. We associate to f an (in general degenerated) inner product space and a symmetric relation and use the theory of selfadjoint extensions in order to describe all extensions of f.

KEYWORDS: *Hermitian function with negative squares, selfadjoint (symmetric) relation.*

AMS SUBJECT CLASSIFICATION: 42A82, 47A57, 47B50.

1. INTRODUCTION AND PRELIMINARIES

In [10] M.G. Krein studied positive definite extensions to the whole real axis of a continuous positive definite function defined on the interval $[-2a, 2a]$. He generalized some of his results to continuous hermitian functions with a finite number of negative squares in [11]. His study has been continued e.g. in [5], [13], [16], [15] and, more recently, in [3], [7], [17].

In order to formulate the considered problems properly, we have to introduce some notation. Denote by $\mathcal{C}[-2a, 2a]$ and $\mathcal{C}(\mathbb{R})$ the set of continuous complex valued functions defined on the intervall $[-2a, 2a]$ and $\mathbb{R}$, respectively.

DEFINITION 1.1. Let $0 < a \leqslant \infty$. We write $f \in \mathcal{P}_{\kappa,a}$ if f is a continuous hermitian function, i.e. if $f \in \mathcal{C}[-2a, 2a]$ ($\mathcal{C}(\mathbb{R})$) and $f(-t) = \overline{f(t)}$ for $t \in [-2a, 2a]$ ($t \in \mathbb{R}$), and if the kernel $f(t - s)$ has κ negative squares.

Explicitly this means that for all choices of $n \in \mathbb{N}$ and $t_i \in (-a, a)$, $i = 1, \dots, n$ the quadratic form

$$Q(\xi_1, \dots, \xi_n) = \sum_{i,j=1}^{n} f(t_j - t_i)\xi_i\overline{\xi_j}$$

has at most κ negative squares, and that for some choice of n and $t_1, \dots, t_n$ it has exactly κ negative squares. For abbreviation we write $\mathcal{P}_\kappa$ instead of $\mathcal{P}_{\kappa,\infty}$.

It turns out (see [5]) that a function $f \in \mathcal{P}_{\kappa_0,a}$ admits at least one extension to the whole real axis which is contained in $\mathcal{P}_{\kappa_0}$. In fact, in $\mathcal{P}_{\kappa_0}$, there exists either exactly one or infinitely many extensions of f. However, it is not clear if there exist extensions of f in $\mathcal{P}_\kappa$ with $\kappa > \kappa_0$.

If $f \in \mathcal{P}_{\kappa_0,a}$ has infinitely many extensions in $\mathcal{P}_{\kappa_0}$, it was shown by M.G. Krein, H. Langer and others (see [3], [5], [13], [16]) that the extensions of f in $\mathcal{P}_\kappa$ correspond to certain selfadjoint operators acting in Pontryagin spaces. Under some additional conditions besides the fact that f has infinitely many extensions in $\mathcal{P}_{\kappa_0}$, e.g. if f has a so called accelerant (see [16]) or if $\kappa = \kappa_0$ (see [5]) the extensions of f in $\mathcal{P}_\kappa$ are parametrized by a formula of the type

$$\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}\tilde{f}(t)\,\mathrm{d}t = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \quad \operatorname{Im} z > h, \tag{1.1}$$

where $h \geqslant 0$ and where $w_{ij}(z)$, $i, j = 1, 2$ are entire functions, with a parameter function $\tau(z)$ (see also [15]). The set of parameter functions depends on κ.

In this paper we show that there exists a number $\Delta(f) \in \mathbb{N} \cup \{0, \infty\}$, such that:

(i) $\Delta(f) = 0$ if and only if f has infinitely many extensions in $\mathcal{P}_{\kappa_0}$.

(ii) If $0 < \Delta(f) < \infty$ then f has exactly one extension in $\mathcal{P}_{\kappa_0}$, no extensions in $\mathcal{P}_\kappa$ for $\kappa_0 < \kappa < \kappa_0 + \Delta(f)$ and infinitely many extensions in $\mathcal{P}_\kappa$ for $\kappa \geqslant \kappa_0 + \Delta(f)$.

(iii) $\Delta(f) = \infty$ if and only if f has exactly one extension in $\mathcal{P}_{\kappa_0}$, and no extensions in any set $\mathcal{P}_\kappa$ with $\kappa > \kappa_0$.

Using some results of [8], we give a parametrization of the extensions $\tilde{f} \in \mathcal{P}_\kappa$, $\kappa \geqslant \kappa_0$, of $f \in \mathcal{P}_{\kappa_0,a}$ by a formula similar to (1.1). The set of parameter functions $\tau(z)$ depends on $\Delta(f)$ and κ. The classical case (i.e. $\Delta(f) = 0$) is also covered, some of our results are new even for $\Delta(f) = 0$.

In Section 2 we assume that the function $f \in \mathcal{P}_{\kappa_0,a}$ has an extension $\tilde{f} \in \mathcal{P}_\kappa$ for some $\kappa > \kappa_0$. A certain inner product space $\mathcal{L}(f, \tilde{f})$ and a symmetric operator

$S_{\tilde f}$ is associated to each extension $\tilde f \in \mathcal{P}_\kappa$ of f with $\kappa > \kappa_0$. It is shown that $\mathcal{L}(f,\tilde f)$ and $S_{\tilde f}$ are unique up to isometric isomorphisms. In Section 3 we introduce the notion of a defining set for a function $f \in \mathcal{P}_{\kappa_0,a}$. With a defining set, a model, consisting of a space $\mathcal{H}$ and an operator S, is associated. Also some properties of $\mathcal{H}$ and S are investigated. It turns out in Section 4 that f admits an extension $\tilde f \in \mathcal{P}_\kappa$ for some $\kappa > \kappa_0$ if and only if there exists a defining set. The number $\Delta(f)$ is then the number of elements of a minimal defining set. The model, $\mathcal{H}$ and S, can be identified with $\mathcal{L}(f,\tilde f)$ and $S_{\tilde f}$. It is shown that the extensions of f correspond to selfadjoint extensions of S and are parametrized by (1.1).

Throughout this paper we use the notion of linear relations in Pontryagin spaces, in particular some results concerning symmetric relations provided in [4]. For general notation and elementary facts concerning Pontryagin spaces and their linear operators see [6], concerning functions in $\mathcal{P}_{\kappa,a}$ see [17].

In the remaining part of this section we will fix some notations and recall some results which are often used in the sequel.

For an inner product space $\mathcal{L}$ denote by $\mathrm{Ind}_-\mathcal{L}$ the maximal dimension of a negative subspace of $\mathcal{L}$, and let $\mathrm{Ind}_0\mathcal{L} = \dim \mathcal{L}^\circ$, where $\mathcal{L}^\circ$ denotes the isotropic part of $\mathcal{L} : \mathcal{L}^\circ = \mathcal{L} \cap \mathcal{L}^\perp$.

Definition 1.2. Let $0 < a \leqslant \infty$, $f \in \mathcal{P}_{\kappa_0,a}$ and let f_x, $x \in (-a,a)$, be formal elements. Denote by $\mathcal{L}(f)$ the inner product space

$$\mathcal{L}(f) = \Big\{ \sum_{i=1}^{n} \alpha_i f_{x_i} \mid \alpha_i \in \mathbb{C},\, x_i \in (-a,a) \Big\},$$

endowed with the inner product given by

$$[f_x, f_y]_f = f(y-x).$$

The completion of the quotient space $\mathcal{L}(f)/\mathcal{L}(f)^\circ$ will be denoted by $\mathcal{H}(f)$.

Note that by definition the elements $f_x \in \mathcal{L}(f)$ are linearly independent. Moreover, we have

$$\mathrm{Ind}_-\mathcal{H}(f) = \mathrm{Ind}_-\mathcal{L}(f) = \kappa_0.$$

Via the embedding $g \mapsto g(x) = [g, f_x]$ we may regard $\mathcal{H}(f)$ as a subspace of $\mathcal{C}[-a,a]$. By this embedding, the element f_x corresponds to the right shift of f by $x : f_x(t) = f(t-x)$.

In the remaining part of this section let $f \in \mathcal{P}_{\kappa_0,a}$ be fixed. Choose a maximal negative subspace $\mathcal{L}_-$ of $\mathcal{L}(f)$. A fundamental symmetry J of $\mathcal{L}(f)$ is associated with $\mathcal{L}_-$ by

$$Jg = \begin{cases} -g, & g \in \mathcal{L}_-; \\ g, & g \perp \mathcal{L}_-. \end{cases}$$

Then

$$\|g\|_f = [Jg, g]_f \quad \text{for } g \in \mathcal{L}(f) \tag{1.2}$$

is a seminorm on $\mathcal{L}(f)$. We will drop the indices at inner products and norms whenever no confusion can occur.

The mapping

$$V'_x(f) : \sum_{i=1}^{n} \alpha_i f_{y_i} \mapsto \sum_{i=1}^{n} \alpha_i f_{y_i+x} \tag{1.3}$$

defined for those elements of $\mathcal{L}(f)$ with $y_i, y_i + x \in (-a, a)$, $i = 1, \ldots, n$, induces a partial isometry on $\mathcal{H}(f)$, denoted by $V_x(f)$, with domain

$$\mathcal{D}(V_x(f)) = \overline{\langle f_y \mid y, y + x \in (-a, a)\rangle}.$$

Hereby $\langle f_y \mid y, y + x \in (-a, a)\rangle$ denotes the span of $\{f_y \mid y, y + x \in (-a, a)\}$ in $\mathcal{H}(f)$.

The operators $V_x(f)$ are continuous with respect to $\|\cdot\|_f$ (see [5]), they are unitary if and only if $a = \infty$. Moreover, we have $(V_x(f)g)(y) = [V_x(f)g, f_y] = g(y - x)$.

Clearly, the operators $V_x(f)$ satisfy the semigroup property, i.e. $V_x(f)V_y(f) = V_{x+y}(f)$. The infinitesimal generator of this semigroup is the closure $\frac{1}{i}A(f)$ of the operator $\frac{1}{i}A'(f)$ defined by

$$A'(f)g = \mathrm{i} \lim_{x \searrow 0} \frac{V_x(f)g - g}{x}, \quad g \in \mathcal{D}(A'(f)), \tag{1.4}$$

where

$$\mathcal{D}(A'(f)) = \Big\{ g \in \bigcup_{x>0} \mathcal{D}(V_x(f)) \subseteq \mathcal{H}(f) \mid \lim_{x \searrow 0} \frac{V_x(f)g - g}{x} \text{ exists} \Big\}.$$

If we consider $\mathcal{H}(f)$ as a subspace of $\mathcal{C}(-a, a)$ we have, due to [3] and [4].

Lemma 1.3. *$A(f)$ is a symmetric operator with equal defect numbers. Its adjoint is given by*

$$(A(f)^* g)(x) = -\mathrm{i}g'(x),$$

hence

$$\ker(A(f)^* - z) = \begin{cases} \langle \mathrm{e}^{\mathrm{i}zx} \rangle & \text{if } \mathrm{e}^{\mathrm{i}zx} \in \mathcal{H}(f); \\ \{0\} & \text{if } \mathrm{e}^{\mathrm{i}zx} \notin \mathcal{H}(f); \end{cases}$$

and $A(f)$ has defect numbers $(0,0)$ (i.e. is selfadjoint) or $(1,1)$. The operator $A(f)$ is not selfadjoint if and only if $\ker(A(f)^* - z) \neq \{0\}$ for at least $\kappa_0 + 1$ points of $\mathbb{C}^+$. In this case we have

$$\ker(A(f) - z) = 0 \quad \textit{for } z \in \mathbb{C} \setminus \mathbb{R}.$$

If $a = \infty$, $A(f)$ is selfadjoint.

Note that $A(f)$ is real with respect to the involution $g^+(x) = \overline{g(-x)}$, i.e. $A(f)g^+ = (A(f)g)^+$.

Definition 1.4. The function $f \in \mathcal{P}_{\kappa_0,a}$ is called *extendable*, if it has an extension in some set $\mathcal{P}_\kappa$ with $\kappa > \kappa_0$, and it is called *determining*, if it has a unique extension in $\mathcal{P}_{\kappa_0}$.

Remark 1.5. It follows from the considerations in [3] together with Lemma 2.1 below that $f \in \mathcal{P}_{\kappa_0,a}$ admits extensions $\tilde{f} \in \mathcal{P}_\kappa$ with $\kappa > \kappa_0$ if f is not determining. This shows that f is extendable if and only if f admits more than one extension in $\bigcup_{\kappa \geqslant \kappa_0} \mathcal{P}_\kappa$.

Denote by $\mathcal{N}_\kappa$ the set of all functions τ meromorphic in $\mathbb{C} \setminus \mathbb{R}$, such that $\tau(\overline{z}) = \overline{\tau(z)}$ and that the Nevanlinna kernel

$$N_\tau(z,w) = \frac{\tau(z) - \overline{\tau(w)}}{z - \overline{w}}$$

has κ negative squares. As usual the set $\mathcal{N}_0$ is augmented by $\{\infty\}$. The following result is proved in [5].

Proposition 1.6. *The function $f \in \mathcal{P}_{\kappa_0,a}$ is determining if and only if $A(f)$ is selfadjoint. If f is not determining the extensions $\tilde{f} \in \mathcal{P}_{\kappa_0}$ of f are parametrized by the formula*

$$\mathrm{i} \int_0^\infty \mathrm{e}^{\mathrm{i}zt} \tilde{f}(t)\,\mathrm{d}t = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \quad \operatorname{Im} z > h_{A(f)}$$

where the parameter $\tau(z)$ runs through the Nevanlinna class $\mathcal{N}_0$. Here the matrix $W(z)$ is a resolvent matrix of $A(f)$, and $h_{A(f)} \geqslant 0$ is such that the spectrum of any selfadjoint extension of $A(f)$, acting in a Pontryagin space with negative index κ_0, is contained in the strip $\{z \mid |\operatorname{Im} z| \leqslant h_{A(f)}\}$.

For the notion of a resolvent matrix see [14], the existence of a number $h_{A(f)}$ with the above properties is proved in [12].

2. EXTENSIONS OF FUNCTIONS IN $\mathcal{P}_{\kappa_0,a}$

Throughout this section let $f \in \mathcal{P}_{\kappa_0,a}$ be fixed and assume that an extension $\tilde f \in \mathcal{P}_\kappa$, $\kappa > \kappa_0$, of f is given. Moreover, let $\tilde A = A(\tilde f)$ be as in (1.4).

We call $\tilde f$ a generating element of $\tilde A$ (or, equivalently, call $\tilde A$ an $\tilde f$-minimal operator) if

$$\mathcal{H}(\tilde f) = \overline{\langle \tilde f, (\tilde A - z)^{-1}\tilde f \mid z \in \varrho(\tilde A)\rangle}.$$

Lemma 2.1. *Let $(T_t)_{t\in\mathbb{R}}$ be a strongly continuous group of operators in a Hilbert space $\mathcal{H}$, and let B be its infinitesimal generator. Assume that $\varrho(B)\cap\mathbb{C}^+$ and $\varrho(B)\cap\mathbb{C}^-$ is connected. Let $U \subseteq \varrho(B)$ have an accumulation point in $\mathbb{C}^+$ and in $\mathbb{C}^-$. Then for any element $x \in \mathcal{H}$*

$$\overline{\langle T_t x \mid t \in \mathbb{R}\rangle} = \overline{\langle x, (B-z)^{-1}x \mid z \in U\rangle}.$$

Proof. Let $\mathcal{L} = \overline{\langle x, (B-z)^{-1}x \mid z \in \varrho(B)\rangle}$. We first show that

$$\mathcal{L} = \overline{\langle x, (B-z)^{-1}x \mid z \in U\rangle}.$$

Let $(\,\cdot\,,\cdot\,)$ be the scalar product of $\mathcal{H}$ and assume that $g \perp x$ and $g \perp (B-z)^{-1}x$ for $z \in U$, then

$$((B-z)^{-1}x, g) = 0, \quad z \in U. \tag{2.1}$$

The function $((B-z)^{-1}x, g)$ is holomorphic in $\varrho(B)$. Since $\varrho(B)\cap\mathbb{C}^+$ and $\varrho(B)\cap\mathbb{C}^-$ is connected and U has an accumulation point in both components, (2.1) implies that $g \perp (B-z)^{-1}x$ for all $z \in \varrho(B)$. Hence $g \perp \mathcal{L}$.

The relation $(\gamma > 0)$

$$(B-z)^{-1}x = \mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}T_t x\,\mathrm{d}t, \quad \operatorname{Im} z > \gamma,$$

and the analogous relation for $\operatorname{Im} z < -\gamma$ show that

$$\overline{\langle T_t x \mid t \in \mathbb{R}\rangle} \supseteq \overline{\langle x, (B-z)^{-1}x \mid z \in \varrho(B), |\operatorname{Im} z| > \gamma\rangle} = \mathcal{L}.$$

To prove the converse inclusion consider the Yoshida approximation

$$B_z = \frac{z^2}{i}(B-z)^{-1} - z, \quad z \in \varrho(B),$$

of B. It follows that $(B_z)^n x \in \mathcal{L}$ for all $n \in \mathbb{N}$. Thus

$$\mathrm{e}^{tB_z}x = \sum_{n=0}^\infty \frac{t^n}{n!}(B_z)^n x \in \mathcal{L}$$

and the properties of the Yoshida approximation (see e.g. [18]) imply that for $t > 0$

$$V_t x = \lim_{z\to+\mathrm{i}\infty} \mathrm{e}^{tB_z}x \in \mathcal{L}.$$

The same considerations with $-B$ instead of B show that $V_t x \in \mathcal{L}$ for $t < 0$. ∎

Corollary 2.2. *Whenever $U \subseteq \varrho(\tilde{A})$ has an accumulation point in $\mathbb{C}^+$ and in $\mathbb{C}^-$ we have*

$$\mathcal{H}(\tilde{f}) = \overline{\langle \tilde{f}, (\tilde{A}-z)^{-1}\tilde{f} \mid z \in U\rangle}.$$

In particular the element $\tilde{f}$ is generating for $\tilde{A}$.

Lemma 2.3. *If $\mathcal{L}(f)$ is degenerated, f is determining. If $\tilde{f} \in \mathcal{P}_\kappa$, $\kappa > \kappa_0$, is an extension of f we have $\mathcal{L}(f) \subseteq \mathcal{H}(\tilde{f})$.*

Proof. Let $\widehat{f}$ be an extension of f. Clearly $\mathcal{L}(f) \subseteq \mathcal{L}(\widehat{f})$ by the embedding $f_x \mapsto \widehat{f}_x$.

Assume that $h = \sum_{j=1}^{n} \alpha_j f_{x_j} \in \mathcal{L}(\widehat{f})^\circ \cap \mathcal{L}(f)$, $h \neq 0$, then $\widehat{f}$ satisfies the equation

$$\sum_{j=1}^{n} \alpha_j \widehat{f}(y - x_j) = [h, \widehat{f}_y] = 0 \quad \text{for } y \in \mathbb{R}. \tag{2.2}$$

Hence $\widehat{f}$ is, as the solution of the difference equation (2.2) with the initial condition $\widehat{f}(x) = f(x)$, $x \in (-a, a)$, uniquely determined.

Assume that $\mathcal{L}(f)$ is degenerated and let $\widehat{f} \in \mathcal{P}_{\kappa_0}$ be an extension of f. Since $\mathcal{L}(f)$ and $\mathcal{L}(\widehat{f})$ has the same negative index we must have $\mathcal{L}(f)^\circ \subseteq \mathcal{L}(\widehat{f})^\circ$, thus the previous part of the proof applies and we find that f admits only one extension in $\mathcal{P}_{\kappa_0}$.

Note that, as f has at least one extension to $\mathcal{P}_{\kappa_0}$, the solution of (2.2) is an element of $\mathcal{P}_{\kappa_0}$.

Now let $\tilde{f} \in \mathcal{P}_\kappa$ be an extension of f with $\kappa > \kappa_0$, then the previous considerations show that $\mathcal{L}(\tilde{f})^\circ \cap \mathcal{L}(f) = \{0\}$, hence $\mathcal{L}(f) \subseteq \mathcal{H}(\tilde{f})$. ∎

Definition 2.4. Let $\tilde{f} \in \mathcal{P}_\kappa$, $\kappa > \kappa_0$ be an extension of f. Denote by $\mathcal{L}(f, \tilde{f})$ the closure of $\mathcal{L}(f)$ as a subspace of $\mathcal{H}(\tilde{f})$.

The topology of $\mathcal{L}(f, \tilde{f})$ is in general strictly finer than the topology induced by $[\,\cdot\,,\cdot\,]_{\tilde{f}}$, as $\mathcal{L}(f, \tilde{f})$ is in general not a regular subspace of $\mathcal{H}(\tilde{f})$.

The inner product $[\,\cdot\,,\cdot\,]_f$ as well as a definite norm $\|\cdot\|_f$ of $\mathcal{L}(f)$ are continuous with respect to the norm $\|\cdot\|_{\tilde{f}}$ of $\mathcal{H}(\tilde{f})$. Therefore we may extend both to $\mathcal{L}(f, \tilde{f})$.

Lemma 2.5. *If $g \in \mathcal{L}(f, \tilde{f})^\circ$ and $z \in \varrho(\tilde{A}) \setminus \mathbb{R}$, the relation*

$$[(\tilde{A} - z)^{-1} g, \tilde{f}_x] = C(z, g)\mathrm{e}^{\mathrm{i}zx}, \quad x \in (-a, a)$$

holds with $C(z, g) = [(\tilde{A} - z)^{-1} g, \tilde{f}]$.

If one of the functions $C(z,g)$ and $[(\tilde{A}-z)^{-1}g,g]$ vanishes on a set U which has an accumulation point in $\varrho(\tilde{A})\setminus\mathbb{R}$, then $g=0$.

Proof. Let $g\in\mathcal{L}(f,\tilde{f})^{\circ}$, then

$$g(x)=[g,f_x]=0 \quad \text{for } x\in(-a,a).$$

Let $U_t=V_t(\tilde{f})$ be as in (1.3). As (see [13], [9]) there exists a number $h_{\tilde{A}}\geqslant 0$, such that for $\operatorname{Im} z>h_{\tilde{A}}$ the relation

$$(\tilde{A}-z)^{-1}g(x)=\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}U_t g(x)\,\mathrm{d}t=\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}g(x-t)\,\mathrm{d}t=\mathrm{i}\int_{-\infty}^{x}\mathrm{e}^{\mathrm{i}z(x-t)}g(t)\,\mathrm{d}t,$$

holds, we find for $x\in(-a,a)$

$$(\tilde{A}-z)^{-1}g(x)=\mathrm{i}\,\mathrm{e}^{\mathrm{i}zx}\int_{-\infty}^{-a}\mathrm{e}^{-\mathrm{i}zt}g(t)\,\mathrm{d}t=C(z,g)\mathrm{e}^{\mathrm{i}zx} \tag{2.3}$$

with $C(z,g)=\mathrm{i}\int_{-\infty}^{-a}\mathrm{e}^{-\mathrm{i}zt}g(t)\,\mathrm{d}t$. Substituting $x=0$ shows that

$$C(z,g)=[(\tilde{A}-z)^{-1}g,\tilde{f}]. \tag{2.4}$$

The relations (2.3) and (2.4) extend to $\varrho(\tilde{A})\cap\mathbb{C}^+$ by analyticity. For $\operatorname{Im} z<-h_{\tilde{A}}$ we apply the formulas of [9] to $-\tilde{A}$ and $(U_t')=(U_{-t})$ and find

$$(\tilde{A}-z)^{-1}g(x)=-\mathrm{i}\,\mathrm{e}^{\mathrm{i}zx}\int_a^\infty \mathrm{e}^{\mathrm{i}zt}g(t)\,\mathrm{d}t=C(z,g)\mathrm{e}^{\mathrm{i}zx},$$

where again $C(z,g)=[(\tilde{A}-z)^{-1}g,\tilde{f}]$. These relations extend to $\varrho(\tilde{A})\cap\mathbb{C}^-$.

Assume now that $C(z,g)=0$ for $z\in U$, where U has an accumulation point in $\varrho(\tilde{A})\setminus\mathbb{R}$. Without loss of generality assume that there exists an accumulation point in $\mathbb{C}^+$. Then

$$g\perp\langle\tilde{f},(\tilde{A}-\overline{z})^{-1}\tilde{f}\mid z\in U\cap\mathbb{C}^+\rangle,$$

and a holomorphy argument (compare the proof of Lemma 2.1) shows that

$$g\perp\langle\tilde{f},(\tilde{A}-z)^{-1}\tilde{f}\mid z\in\cap\varrho(\tilde{A})\cap\mathbb{C}^-\rangle.$$

It is proved in [13] that the negative index of $\langle\tilde{f},(\tilde{A}-z)^{-1}\tilde{f}\mid z\in\mathbb{C}^-\cap\varrho(\tilde{A})\rangle$ equals the negative index of $\langle\tilde{f},(\tilde{A}-z)^{-1}\tilde{f}\mid z\in\varrho(\tilde{A})\setminus\mathbb{R}\rangle$. By Corollary 2.2 it is therefore equal to κ. Let $\mathcal{L}_-$ be a κ-dimensional negative subspace contained in $\langle \tilde{f}, (\tilde{A}-z)^{-1}\tilde{f} \mid z \in \mathbb{C}^- \cap \varrho(\tilde{A})\rangle$, then $g \perp \mathcal{L}_-$, and g is neutral, by assumption. This implies $g = 0$.

To prove the remaining assertion assume on the contrary that

$$[(\tilde{A}-z)^{-1}g, g] = 0 \quad \text{for } z \in U.$$

If U has an accumulation point, say, in $\mathbb{C}^+$ we find by analyticity that $[(\tilde{A}-z)^{-1}g, g] = 0$ for $z \in \varrho(\tilde{A}) \cap \mathbb{C}^+$. As $[(\tilde{A}-z)^{-1}g, g]$ is a real function, this implies $[(\tilde{A}-z)^{-1}g, g] = 0$ for all $z \in \varrho(\tilde{A})$. By the resolvent identity, the subspace

$$\langle (\tilde{A}-z)^{-1}g \mid z \in \varrho(\tilde{A})\rangle$$

is neutral. As the functions $(\tilde{A}-z)^{-1}g|(-a,a) = C(z,g)\mathrm{e}^{\mathrm{i}zx}|(-a,a)$ are linearly independent or equal to 0, this subspace has infinite dimension, a contradiction, unless $C(z,g) = 0$ for all but finitely many values of z. The previous part of the proof shows that $g = 0$. ∎

Lemma 2.6. *Let $\tilde{f} \in \mathcal{P}_\kappa$, $\kappa > \kappa_0$, be an extension of f. Then*

$$\mathcal{H}(f) \cong \mathcal{L}(f,\tilde{f})/\mathcal{L}(f,\tilde{f})^\circ$$

via the isomorphism $f_x \mapsto \tilde{f}_x + \mathcal{L}(f,\tilde{f})^\circ$.

Proof. The mapping $f_x \mapsto \tilde{f}_x + \mathcal{L}(f,\tilde{f})^\circ$ induces an isometric relation between the Pontryagin spaces $\mathcal{H}(f)$ and $\mathcal{L}(f,\tilde{f})/\mathcal{L}(f,\tilde{f})^\circ$. Its domain and range are dense in the respective space. Hence it extends to an isomorphism (see [2]). ∎

The following proposition generalizes Lemma 2.3 and gives a necessary and sufficient condition for f to be determining.

Proposition 2.7. *Let $\tilde{f} \in \mathcal{P}_\kappa$, $\kappa > \kappa_0$ be an extension of f. The space $\mathcal{L}(f,\tilde{f})$ is degenerated if and only if f is determining.*

Proof. Assume first that f is not determining. Let $h \in \mathcal{L}(f,\tilde{f})^\circ$ and $\tilde{A} = A(\tilde{f})$, then

$$[(\tilde{A}-z)^{-1}h, \tilde{f}_x] = C(z,h)\mathrm{e}^{\mathrm{i}zx}, \quad x \in (-a,a).$$

As f is not determining $\mathrm{e}^{\mathrm{i}zx} \in \mathcal{H}(f)$ for $z \in \mathbb{C} \setminus \mathbb{R}$ by Lemma 1.3, thus there exist elements $g(z) \in \mathcal{L}(f,\tilde{f})$, such that

$$[g(z), \tilde{f}_x] = C(z,h)\mathrm{e}^{\mathrm{i}zx} \quad \text{for } z \in \varrho(\tilde{A}) \setminus \mathbb{R},\ x \in (-a,a).$$

We find that $(\tilde{A}-z)^{-1}h - g(z)$ is orthogonal to $\mathcal{L}(f)$, hence also to $\mathcal{L}(f,\tilde{f})$. In particular,

$$[(\tilde{A}-z)^{-1}h, h] = [(\tilde{A}-z)^{-1}h - g(z), h] = 0. \tag{2.5}$$

This relation holds for $z \in \varrho(\tilde{A}) \setminus \mathbb{R}$ whenever $C(z,h) \neq 0$. Lemma 2.5 shows that $h = 0$, hence $\mathcal{L}(f,\tilde{f})^{\circ} = \{0\}$.

Assume now on the contrary that $\mathcal{L}(f,\tilde{f})$ is nondegenerated and that f is determining. Then $\mathcal{L}(f,\tilde{f})$ is a regular subspace of $\mathcal{H}(\tilde{f})$ and we find, due to Lemma 2.6

$$\mathcal{H}(f) = \mathcal{L}(f,\tilde{f}) \subseteq \mathcal{H}(\tilde{f}),$$

and clearly $A(f) \subseteq \tilde{A}$. Since $A(f)$ is selfadjoint

$$(\tilde{A}-z)^{-1}\tilde{f} = (A(f)-z)^{-1}\tilde{f} \in \mathcal{H}(f) \quad \text{for } z \in \varrho(\tilde{A}) \cap \varrho(A(f))$$

and it follows from Corollary 2.2 that $\mathcal{H}(f) = \mathcal{H}(\tilde{f})$, a contradiction as $\kappa > \kappa_0$. ∎

Definition 2.8. For $z \in \mathbb{C} \setminus \mathbb{R}$ denote by F_z the functional defined on $\mathcal{L}(f)$ by

$$F_z\Big(\sum_{j=1}^{n} \alpha_j f_{x_j}\Big) = \sum_{j=1}^{n} \alpha_j e^{izx_j}.$$

Proposition 2.9. *Let* $\dim \mathcal{L}(f,\tilde{f})^{\circ} = \delta$. *The functionals* F_z *are continuous on* $\mathcal{L}(f)$ *with respect to* $\|\cdot\|_{\tilde{f}}$ *for all* $z \in -\varrho(\tilde{A})$ *with possible exception of an isolated set. Moreover, the norm* $\|\cdot\|_{\tilde{f}}$ *on* $\mathcal{L}(f)$ *is equivalent to the norm* $\|\cdot\|_{\delta}$ *defined by*

$$\|g\|_{\delta}^2 = \|g\|^2 + |F_{z_1}(g)|^2 + \cdots + |F_{z_\delta}(g)|^2 \tag{2.6}$$

for a suitable choice of (mutually different) numbers $z_1,\dots,z_\delta$. *In fact* $z_1,\dots,z_\delta$ *can be chosen within any open set* U *of* $\mathbb{C}$.

For any $\delta' < \dim \mathcal{L}(f,\tilde{f})^{\circ}$ *the norm* $\|\cdot\|_{\delta'}$ *on* $\mathcal{L}(f)$ *(defined by a formula similar to* (2.6)*) is strictly weaker than* $\|\cdot\|_{\tilde{f}}$.

Proof. Consider first the case that $\delta = 0$. As in this case $\mathcal{L}(f,\tilde{f})$ is regular, the topology of $\mathcal{L}(f,\tilde{f})$ is given by the inner product restricted to $\mathcal{L}(f,\tilde{f})$, which is the same as $[\,\cdot\,,\cdot\,]$. Moreover, Proposition 2.7 shows that f is not determining, i.e. $e^{izx} \in \mathcal{H}(f)$ for $z \in \mathbb{C} \setminus \mathbb{R}$. By Lemma 2.6, $\mathcal{H}(f) = \mathcal{L}(f,\tilde{f})$, thus the functionals F_z are continuous on $\mathcal{L}(f)$ with respect to $\|\cdot\|_{\tilde{f}}$.

Before we proceed recall that $\varrho(\tilde{A})$ is symmetric with respect to the real axis. Assume now that $\delta > 0$ and let $h \in \mathcal{L}(f,\tilde{f})^{\circ}$. Then the functionals F_z for $z \in O$, where

$$O = \{z \in -\varrho(\tilde{A}) \mid C(z,h) \neq 0\},$$

are given by

$$F_z(g) = \left[g, \frac{1}{C(-\overline{z},h)}(\tilde{A}+\overline{z})^{-1}h\right], \quad g \in \mathcal{L}(f),$$

hence are continuous. The extension (by continuity) of F_z to $\mathcal{L}(f,\tilde{f})$ is again denoted by F_z. We find that the norm (2.6) is continuous with respect to $\|\cdot\|_{\tilde{f}}$, if $z_1,\dots,z_\delta \in O$. Note that the complement of O has no accumulation point in $\varrho(\tilde{A}) \setminus \mathbb{R}$ by Lemma 2.5.

Let U be an open set, then $U \cap (\varrho(\tilde{A}) \setminus \mathbb{R}) \neq \emptyset$. We will construct sequences $h_1,\dots,h_\delta \in \mathcal{L}(f,\tilde{f})^{\circ}$ and $z_1,\dots,z_\delta \in U$, such that $\langle h_1,\dots,h_\delta\rangle = \mathcal{L}(f,\tilde{f})^{\circ}$,

$$[h_i,(\tilde{A}+\overline{z_i})^{-1}h_i] \neq 0 \quad \text{and} \quad C(-\overline{z_i},h_i) \neq 0,$$

$$[h_j,(\tilde{A}+\overline{z_i})^{-1}h_i] = 0 \quad \text{for } j > i.$$

Choose $h_1 \in \mathcal{L}(f,\tilde{f})^{\circ}$, then Lemma 2.5 shows that there exists a number $z_1 \in U \cap -\varrho(\tilde{A}) \setminus \mathbb{R}$, such that

$$[h_1,(\tilde{A}+\overline{z_1})^{-1}h_1] \neq 0 \quad \text{and} \quad C(-\overline{z_1},h_1) \neq 0.$$

The kernel D_1 of the functional $[\,\cdot\,,(\tilde{A}+\overline{z_1})^{-1}h_1]$ on $\mathcal{L}(f,\tilde{f})^{\circ}$ has codimension 1. Choose h_2 in this kernel; then another application of Lemma 2.5 shows that there exists $z_2 \in U \cap -\varrho(\tilde{A})$, such that

$$[h_2,(\tilde{A}+\overline{z_2})^{-1}h_2] \neq 0 \quad \text{and} \quad C(-\overline{z_2},h_2) \neq 0.$$

Now consider $D_2 = \ker([\,\cdot\,,(\tilde{A}+\overline{z_1})^{-1}h_1]) \cap \ker([\,\cdot\,,(\tilde{A}+\overline{z_2})^{-1}h_2])$ and proceed inductively.

The space

$$\mathcal{L} = \mathcal{L}(f,\tilde{f}) + \langle (\tilde{A}+\overline{z_i})^{-1}h_i \mid i = 1,\dots,\delta\rangle$$

is a closed subspace of $\mathcal{H}(\tilde{f})$. It is also nondegenerated: assume on the contrary that, for some element $g \in \mathcal{L}(f,\tilde{f})$ and numbers $\lambda_1,\dots,\lambda_\delta$

$$g_1 = g + \sum_{i=1}^{\delta} \lambda_i (\tilde{A}+\overline{z_i})^{-1}h_i$$

is isotropic in $\mathcal{L}$. Multiplication of g_1 with h_j for $j = \delta, \delta-1, \dots, 1$ shows that $\lambda_j = 0$ for all j. It follows that $g_1 = g$, i.e. $g_1 \in \mathcal{L}(f, \tilde{f})^{\circ}$. Due to the construction of $h_1, \dots, h_\delta$ we may write

$$g_1 = \sum_{i=1}^{\delta} \mu_i h_i.$$

Multiplication with $(\tilde{A} + \overline{z_i})^{-1} h_i$ for $i = 1, 2, \dots, \delta$ shows that $\mu_i = 0$ for all i, i.e. $g_1 = 0$. Hence the subspace $\mathcal{L}$ is regular, and thus its topology, and also that of $\mathcal{L}(f, \tilde{f})$, is induced by the inner product $[\,\cdot\,,\cdot\,]_{\tilde{f}}$ restricted to $\mathcal{L}$.

For the definition of the norm $\|\cdot\|_\delta$ choose the points $z_1, \dots, z_\delta$ constructed in the previous paragraph. Let $(x_n) \in \mathcal{L}(f, \tilde{f})$ converge to some element x in the norm (2.6). As the inner product of $\mathcal{H}(\tilde{f})$ coincides with $[\,\cdot\,,\cdot\,]$ on $\mathcal{L}(f, \tilde{f})$ it follows that for $y \in \mathcal{L}(f, \tilde{f})$

$$[x_n, y]_{\tilde{f}} \to [x, y]_{\tilde{f}},$$

and that

$$[x_n, x_n]_{\tilde{f}} \to [x, x]_{\tilde{f}}.$$

Furthermore, we have

$$[x_n, (\tilde{A} + \overline{z_i})^{-1} h_i] = F_{z_i}(x_n) \to F_{z_i}(x) = [x, (\tilde{A} + \overline{z_i})^{-1} h_i].$$

These facts imply that $x_n \to x$ in the norm of $\mathcal{L}$ (see [6]), hence also in $\mathcal{H}(\tilde{f})$. Thus $\|\cdot\|_{\tilde{f}}$ is continuous with respect to $\|\cdot\|_\delta$ and the induced topologies coincide.

If $\delta' < \delta$ choose

$$h \in \mathcal{L}(f, \tilde{f})^{\circ} \cap \bigcap_{i=1}^{\delta'} \ker(F_{z_i}), \quad h \neq 0.$$

Then $\|h\|_{\delta'} = 0$, but $\|h\|_{\tilde{f}} \neq 0$ as $h \neq 0$. If (x_n) is a sequence of elements of $\mathcal{L}(f)$ with $x_n \to h$, we find $\|x_n\|_{\tilde{f}} \to \|h\|_{\tilde{f}} > 0$, but $\|x_n\|_{\delta'} \to 0$, hence $\|\cdot\|_{\delta'}$ is strictly weaker than $\|\cdot\|_{\tilde{f}}$ on $\mathcal{L}(f)$. ∎

Let $S_{\tilde{f}}$ be the restriction of $\tilde{A}$ to $\mathcal{L}(f, \tilde{f})$. As relations we may write

$$S_{\tilde{f}} = \tilde{A} \cap \mathcal{L}(f, \tilde{f})^2.$$

The following theorem asserts that $\mathcal{L}(f, \tilde{f})$ does not depend on $\tilde{f}$. It will turn out later, that $S_{\tilde{f}}$ also does not depend on $\tilde{f}$.

Theorem 2.10. *Let $f \in \mathcal{P}_{\kappa_0,a}$ and let $\tilde{f} \in \mathcal{P}_{\kappa}$, $\tilde{f}_1 \in \mathcal{P}_{\kappa_1}$ be extensions of f with $\kappa, \kappa_1 > \kappa_0$. Then the mapping*

$$\varphi : \tilde{f}_x \mapsto \tilde{f}_{1,x}, \quad x \in (-a, a)$$

induces a bicontinuous linear mapping of $\mathcal{L}(f, \tilde{f})$ onto $\mathcal{L}(f, \tilde{f}_1)$ which is isometric with respect to the inner products $[\,\cdot\,,\cdot\,]_{\tilde{f}}$ and $[\,\cdot\,,\cdot\,]_{\tilde{f}_1}$.

Proof. Since $[\,\cdot\,,\cdot\,]_{\tilde{f}}$ and $[\,\cdot\,,\cdot\,]_{\tilde{f}_1}$ coincide on $\mathcal{L}(f)$, the mapping φ is isometric.

Let $z_1, \ldots, z_\delta$ be as in Proposition 2.9, then the functionals F_z are continuous for z in some open subset O of $\mathbb{C} \setminus \mathbb{R}$. Again due to Proposition 2.9 we may choose points $w_1, \ldots, w_{\delta'} \in O$, such that the norm of $\mathcal{L}(f, \tilde{f}_1)$ is equivalent to

$$\|g\|_{\delta'}^2 = \|g\|^2 + |F_{w_1}(g)|^2 + \cdots + |F_{w_{\delta'}}(g)|^2.$$

As each functional F_{w_i} is continuous with respect to $\|\cdot\|_\delta$ on $\mathcal{L}(f)$ we find

$$\|g\|_{\delta'} \leqslant K\|g\|_\delta, \quad g \in \mathcal{L}(f)$$

hence φ can be extended to $\mathcal{L}(f, \tilde{f})$ by continuity.

The same argument applies to $\varphi^{-1} : \tilde{f}_{1,x} \mapsto \tilde{f}_x$, and therefore φ extends to an isomorphism of $\mathcal{L}(f, \tilde{f})$ onto $\mathcal{L}(f, \tilde{f}_1)$. ∎

Since φ is isometric, it maps the isotropic part of $\mathcal{L}(f, \tilde{f})$ onto the isotropic part of $\mathcal{L}(f, \tilde{f}_1)$. In particular we have:

Corollary 2.11. *The dimension of $\mathcal{L}(f, \tilde{f})^\circ$ is independent of $\tilde{f}$.*

The next proposition shows that an extendable function is in some sense a limit of not determining functions. For $0 < b < a$ denote by $f_{(b)}$ the restriction of f to $[-2b, 2b]$.

Lemma 2.12. *Let $a > 0$ and $f \in \mathcal{P}_{\kappa_0,a}$. Then there exists a number $b(f) < a$, such that $f_{(b)} \in \mathcal{P}_{\kappa_0,b}$ for $b \in (b(f), a]$.*

Proof. A maximal negative subspace of $\mathcal{L}(f)$ is spanned by elements

$$y_i = \sum_{j=1}^{n_i} \lambda_{i,j} f_{x_{i,j}}, \quad i = 1, \ldots, \kappa_0$$

with $x_{i,j} \in (-a, a)$. The assertion follows with $b(f) = \max_{i,j} |x_{i,j}|$. ∎

Proposition 2.13. *Let $f \in \mathcal{P}_{\kappa_0,a}$ be extendable. Then for each $b \in (b(f), a)$ the function $f_{(b)}$ is not determining.*

Proof. Let $\tilde{f} \in \mathcal{P}_\kappa$ with $\kappa > \kappa_0$ be an extension of f and assume on the contrary that $f_{(b)}$ is in the class $\mathcal{P}_{\kappa_0,b}$ and is determining for some value $b(f) < b < a$. Set $c = b + \frac{a-b}{2}$, then we have the following inclusions

$$\mathcal{L}(f_{(b)}, \tilde{f}) \subseteq \mathcal{L}(f_{(c)}, \tilde{f}) \subseteq \mathcal{L}(f, \tilde{f}) \subseteq \mathcal{H}(\tilde{f}).$$

If $U_x = V_x(\tilde{f})$ denotes the family of unitary operators associated with $\tilde{f}$, we have with $\varepsilon = \frac{a-b}{2}$ for $-\varepsilon < x < \varepsilon$

$$U_x \mathcal{L}(f_{(b)}, \tilde{f}) \subseteq \mathcal{L}(f_{(c)}, \tilde{f}) \quad \text{and} \quad U_x \mathcal{L}(f_{(c)}, \tilde{f}) \subseteq \mathcal{L}(f, \tilde{f}).$$

As $f_{(b)}$ is determining, Proposition 2.7 shows that $\mathcal{L}(f_{(b)}, \tilde{f})^\circ \neq \{0\}$. Since

$$\operatorname{Ind}_- \mathcal{L}(f, \tilde{f}) = \operatorname{Ind}_- \mathcal{L}(f_{(b)}, \tilde{f}),$$

an element $h \in \mathcal{L}(f_{(b)}, \tilde{f})^\circ$ is also isotropic in $\mathcal{L}(f, \tilde{f})$. Hence for $g \in \mathcal{L}(f_{(c)}, \tilde{f})$ and $-\varepsilon < x < \varepsilon$ we have

$$[U_x h, g] = [h, U_{-x} g] = 0,$$

i.e. $U_x h \in \mathcal{L}(f_{(c)}, \tilde{f})^\circ$.

For $-a < x < a$ set

$$\sigma(U_x h) = \sup\{y \leqslant 0 \mid (U_x h)(y) \neq 0\}.$$

Note that for $g \in \mathcal{H}(f)$ we have $\sigma(g) > -\infty$ if and only if there exists $y \leqslant 0$, such that $[g, \tilde{f}_y] \neq 0$. If $\sigma(g) = -\infty$ consider

$$\sigma'(g) = \inf\{y \geqslant 0 \mid g(y) \neq 0\}$$

instead. The fact that $\langle \tilde{f}_x \mid x \in \mathbb{R} \rangle$ is dense in $\mathcal{H}(\tilde{f})$ shows that at least one of the numbers $\sigma(h)$ and $\sigma'(h)$ is finite. Assume that $\sigma(h) > -\infty$, then $\sigma(U_x h) = \sigma(h) + x$, as U_x is the right shift and $h(y) = 0$ for $-a < y < a$.

Consider a linear combination

$$\sum_{i=1}^n \lambda_i U_{x_i} h, \quad \lambda_n \neq 0, x_1 < x_2 < \cdots < x_n$$

with $x_i \in (-\varepsilon, \varepsilon)$. Let y be such that

$$\sigma(h) + x_{n-1} < y < \sigma(h) + x_n = \sigma(U_{x_n} h) \quad \text{and} \quad (U_{x_n} h)(y) \neq 0,$$

then

$$\Big[\sum_{i=1}^n \lambda_i U_{x_i} h, \tilde{f}_y \Big] = \sum_{i=1}^n \lambda_i (U_{x_i} h)(y) = \lambda_n (U_{x_n} h)(y) \neq 0.$$

Hence $\sum_{i=1}^n \lambda_i U_{x_i} h \neq 0$, and we find that the elements $U_x h$ for $x \in (-\varepsilon, \varepsilon)$ span an infinite dimensional space. As $U_x h \in \mathcal{L}(f_{(c)}, \tilde{f})^\circ$ this space is neutral, a contradiction. ∎

Proposition 2.13 together with Lemma 2.3 has the following corollary:

Corollary 2.14. *If $\mathcal{L}(f)$ is degenerated, f is not extendable.*

3. THE MODEL SPACE

In Theorem 2.10 we have associated to an extendable function a model, i.e. an inner product space and a symmetric operator. This section is concerned with the construction and investigation of a certain inner product space (symmetric operator) which will turn out in Section 4 to coincide with the above mentioned model.

Let $\|\cdot\|$ be a definite seminorm on $\mathcal{L}(f)$ as in (1.2), and denote by $(\cdot,\cdot)$ the corresponding inner product.

Definition 3.1. The finite set

$$\{z_1, \dots, z_n\} \subseteq \mathbb{C} \setminus \mathbb{R}$$

is called *defining*, if there exists an open set O with $O \cap \mathbb{C}^+ \neq \emptyset$ and $O \cap \mathbb{C}^- \neq \emptyset$, such that for $z \in O$ the functionals F_z are continuous on $\mathcal{L}(f)$ with respect to the seminorm

$$\|g\|_n^2 = \|g\|^2 + |F_{z_1}(g)|^2 + \dots + |F_{z_n}(g)|^2, \quad g \in \mathcal{L}(f). \tag{3.1}$$

Remark 3.2. The seminorm $\|\cdot\|_n$ is induced by the inner product

$$(g_1, g_2)_n = (g_1, g_2) + F_{z_1}(g_1)\overline{F_{z_1}(g_2)} + \dots + F_{z_n}(g_1)\overline{F_{z_n}(g_2)}.$$

Lemma 1.3 together with the first part of Proposition 1.6 has the following corollary:

Corollary 3.3. *The empty set is defining if and only if f is not determining.*

Proof. If the empty set is defining there exist elements $h(z) \in \mathcal{H}(f)$, such that

$$F_z(g) = [g, h(z)], \quad g \in \mathcal{H}(f).$$

We have in particular

$$[f_x, h(z)] = \mathrm{e}^{\mathrm{i}zx},$$

hence $A(f)$ has defect numbers $(1,1)$ by Lemma 1.3. The converse conclusion also follows from Lemma 1.3. ∎

The defining set $\{z_1, \dots, z_\delta\}$ is called *minimal* if no proper subset of $\{z_1, \dots, z_\delta\}$ is defining. For a minimal defining set $\{z_1, \dots, z_\delta\}$ no functional F_{z_j} is continuous with respect to the norm (3.1) constructed with $\{z_1, \dots, z_\delta\} \setminus \{z_j\}$ instead of $\{z_1, \dots, z_\delta\}$.

REMARK 3.4. If f is extendable, Proposition 2.9 shows that there exists a defining set.

Assume throughout the following that there exists a defining set $\{z_1, \dots, z_\delta\}$. Without loss of generality we can assume that it is chosen minimal.

LEMMA 3.5. *If $g \in \mathcal{L}(f)$ and $F_z(g) = 0$ for z in some open set, we have $g = 0$. In particular the seminorm $\|\cdot\|_\delta$ is in fact a norm.*

Proof. Let $g = \sum_{n=1}^{m} \gamma_n f_{x_n}$, then $F_z(g) = \sum_{n=1}^{m} \gamma_n e^{izx_n}$, which is as a function of z the Fourier transform of a discret measure with points of increase $x_1, \dots, x_m$. Thus $F_z(g) = 0$ for all z implies $g = 0$.

Assume that $g \in \mathcal{L}(f)$ and $\|g\|_\delta = 0$. As the functionals F_z are continuous with respect to $\|\cdot\|_\delta$ we find that $F_z(g) = 0$. Hence $g = 0$. ∎

Denote by $\mathcal{H}$ the completion of $\mathcal{L}(f)$ with respect to the norm $\|\cdot\|_\delta$. As, for $g, h \in \mathcal{L}(f)$,

$$|[g, h]| \leqslant \|g\| \cdot \|h\| \leqslant \|g\|_\delta \|h\|_\delta$$

holds, the inner product $[\cdot\,,\cdot]$ of $\mathcal{L}(f)$ can be extended to $\mathcal{H}$ by continuity.

DEFINITION 3.6. The Hilbert space $\mathcal{H}$ with the norm $\|\cdot\|_\delta$ additionally endowed with the inner product $[\cdot\,,\cdot]$ is called *the model space associated to f*.

Note that the topology of $\mathcal{H}$ is in general strictly finer than the topology induced by the inner product.

REMARK 3.7. We will see (Corollary 4.4 below) that $\mathcal{H}$ does not depend on the particular choice of a minimal defining set.

PROPOSITION 3.8. *Consider the inner product space $\langle\mathcal{H}, [\cdot\,,\cdot]\rangle$. We have*

$$\mathrm{Ind}_-\langle\mathcal{H}, [\cdot\,,\cdot]\rangle = \kappa_0,$$

and

$$\mathrm{Ind}_0\langle\mathcal{H}, [\cdot\,,\cdot]\rangle = \delta.$$

Proof. Consider the identity mapping $\iota : \langle \mathcal{L}(f), \|\cdot\|_\delta\rangle \to \langle \mathcal{L}(f), \|\cdot\|\rangle$ and the canonical projection $\pi : \langle \mathcal{L}(f), \|\cdot\|\rangle \longrightarrow \mathcal{H}(f)$. Both mappings are continuous, hence $\pi \circ \iota$ can be extended to a mapping

$$\psi : \langle \mathcal{H}, \|\cdot\|_\delta\rangle \longrightarrow \mathcal{H}(f).$$

As $\pi \circ \iota$ is an isometry with respect to $[\,\cdot\,,\cdot\,]$, and $[\,\cdot\,,\cdot\,]$ is continuous with respect to $\|\cdot\|_\delta$ (the inner product of $\mathcal{H}(f)$ is of course continuous with respect to the norm of $\mathcal{H}(f)$), ψ is also isometric with respect to $[\,\cdot\,,\cdot\,]$. Thus

$$\operatorname{Ind}_-\langle \mathcal{H}, [\,\cdot\,,\cdot\,]\rangle \leqslant \operatorname{Ind}_-\mathcal{H}(f) = \kappa_0,$$

and, in fact, equality holds as $\mathcal{L}(f) \subseteq \mathcal{H}$.

For $0 \leqslant n \leqslant \delta$, let

$$\nu_n = \dim \{g \in \mathcal{H} \mid \|g\|_n = 0\}.$$

Note that $\nu_0 = \operatorname{Ind}_0\langle \mathcal{H}, [\,\cdot\,,\cdot\,]\rangle$ and $\nu_\delta = 0$. If $\delta = 0$ the assertion already follows. If $\delta > 0$ we have $\nu_n \leqslant \nu_{n-1} \leqslant \nu_n + 1$. For if $g, h \in \mathcal{H}$ such that $\|g\|_{n-1} = \|h\|_{n-1} = 0$, but $\|g\|_n, \|h\|_n \neq 0$, we have

$$\|g\|_n^2 = |F_{z_n}(g)|^2, \quad \|h\|_n^2 = |F_{z_n}(h)|^2,$$

and thus for appropriate numbers $\lambda, \mu \in \mathbb{C}$

$$\|\lambda g - \mu h\|_n^2 = \|\lambda g - \mu h\|_{n-1}^2 + |F_{z_n}(\lambda g - \mu h)|^2 = 0,$$

which shows that $\nu_{n-1} \leqslant \nu_n + 1$. The inequality $\nu_n \leqslant \nu_{n-1}$ follows from $\|\cdot\|_{n-1} \leqslant \|\cdot\|_n$.

By the choice of z_δ there exists a sequence $x_s \in \mathcal{L}(f)$ with $\|x_s\|_{\delta-1} \to 0$ and $F_{z_\delta}(x_s) \to 1$. It follows that

$$\|x_s - x_t\|_\delta^2 = \|x_s - x_t\|_{\delta-1}^2 + |F_{z_\delta}(x_s - x_t)|^2 \to 0,$$

i.e. (x_s) is a Cauchy sequence in the norm $\|\cdot\|_\delta$. Let $h_1 = \lim x_s$. As $\|\cdot\|_{\delta-1}$ and F_{z_δ} are continuous with respect to $\|\cdot\|_\delta$ we find $\|h_1\|_{\delta-1} = 0$ and $F_{z_\delta}(h_1) = 1$, in particular $h_1 \neq 0$. Therefore $\nu_{\delta-1} = 1$.

We proceed inductively: let $n \geqslant 2$ and note that $F_{z_{\delta-n+1}}|D_n$, where $D_n = \mathcal{L}(f) \cap \bigcap_{m=\delta-n+2}^{\delta} \ker(F_{z_m})$ is not bounded with respect to $\|\cdot\|_{\delta-n}$, as $F_{z_{\delta-n+1}}$ is not bounded with respect to this seminorm and D_n has finite codimension. Choose a sequence $x_s \in D_n$ with $\|x_s\|_{\delta-n} \to 0$ and $F_{z_{\delta-n+1}}(x_s) \to 1$. We find

$$\|x_s - x_t\|_\delta^2 = \|x_s - x_t\|_{\delta-n}^2 + |F_{z_{\delta-n+1}}(x_s - x_t)|^2 \to 0,$$

hence the limit $h_n = \lim x_s$ exists in $\mathcal{H}$ and we have $\|h_n\|_{\delta-n} = 0$, but $\|h_n\|_{\delta-n+1}^2 = |F_{z_{\delta-n+1}}(h_n)|^2 = 1$. Thus $\nu_{\delta-n} = \nu_{\delta-n+1} + 1$. After δ steps we find that $\nu_0 = \delta$. ∎

Remark 3.9. If $\delta = 0$, i.e. f is not determining, the space $\mathcal{H}$ is nondegenerated, and therefore $\mathcal{H} = \mathcal{H}(f)$. In this case most of the following statements are well known (see [5]). Thus we will not consider the case $\delta = 0$ seperately in the proofs.

The space $\mathcal{H}$ can be embedded canonically into a Pontryagin space $\mathcal{P}_c$. Indeed, let

$$\mathcal{H} = \mathcal{H}_n[\dot{+}]\mathcal{H}^\circ$$

be a decomposition of $\mathcal{H}$ with a closed nondegenerated space $\mathcal{H}_n$, and put

$$\mathcal{P}_c = \mathcal{H}_n[\dot{+}](\mathcal{H}^\circ \dot{+} \mathcal{H}_1)$$

where $\mathcal{H}_1$ is a neutral space and skewly linked to $\mathcal{H}^\circ$. Clearly, $\mathcal{P}_c$ is a Pontryagin space and

$$\operatorname{Ind}_- \mathcal{P}_c = \kappa_0 + \delta.$$

By definition, the inner product of $\mathcal{P}_c$ restricted to $\mathcal{H}$ is equal to $[\,\cdot\,,\cdot\,]$. In fact, the norm of $\mathcal{P}_c$ restricted to $\mathcal{H}$ is equivalent to $\|\cdot\|_\delta$.

In the sequel we will need another lemma.

Lemma 3.10. *Let $\{z_1, \dots, z_\delta\}$ be a minimal defining set, let $h \in \mathcal{H}^\circ$ and assume that $h \in \ker(F_z)$ whenever F_z is continuous, with possible exception of one point z_0. Then $h = 0$.*

Proof. If $z_0 \neq z_i$ for $i = 1, \dots, \delta$ we find

$$\|h\|_\delta^2 = \|h\|^2 + |F_{z_1}(h)|^2 + \cdots + |F_{z_\delta}(h)|^2 = 0,$$

thus $h = 0$.

Assume now that $h \notin \ker(F_{z_0})$, say for $z_0 = z_\delta$. Consider the space $\mathcal{L} = \langle h \rangle^\perp$, where the orthogonal complement has to be understood within $\mathcal{H}$ and with respect to the inner product $(\,\cdot\,,\cdot\,)_\delta$. Let z be such that F_z is continuous and let F_z be represented by

$$F_z(g) = (g, l(z))_\delta \quad \text{for } g \in \mathcal{H},$$

then $l(z) \in \mathcal{L}$ for $z \neq z_\delta$. We show that the norms $\|\cdot\|_{\delta-1}$ and $\|\cdot\|_\delta$ are equivalent on $\mathcal{L}$. This fact will follow once we have proved that $\|\cdot\|_\delta$ is equivalent on $\mathcal{H}$ to the norm induced by

$$(g_1, g_2)_0 = (g_1, g_2) + \sum_{i=1}^{\delta-1} (g_1, l(z_i))_\delta (l(z_i), g_2)_\delta + \frac{\|l(z_\delta)\|_\delta^2}{\|h\|_\delta^2} (g_1, h)_\delta (h, g_2)_\delta, \quad g_1, g_2 \in \mathcal{H}.$$

If P_h and $P_{l(z_\delta)}$ denotes the orthogonal projection onto $\langle h\rangle$ and $\langle l(z_\delta)\rangle$, respectively, we find

$$(g_1, g_2)_0 = ((I - P_{l(z_\delta)} + P_h)g_1, g_2)_\delta.$$

It is proved in [1], p. 96, that

$$\|P_h - P_{l(z_\delta)}\|_\delta = \max\Big(\sup_{g\in\langle l(z_\delta)\rangle, \|g\|_\delta=1} \|(I-P_h)g\|_\delta, \sup_{g\in\langle h\rangle, \|g\|_\delta=1} \|(I-P_{l(z_\delta)})g\|_\delta\Big).$$

As h is not orthogonal to $l(z_\delta)$ this implies that $\|P_h - P_{l(z_\delta)}\|_\delta < 1$, hence $I - P_{l(z_\delta)} + P_h$ is boundedly invertible and we find that the norms $\|\cdot\|_\delta$ and $\|\cdot\|_0$ are equivalent. Denote by P the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$. Lemma 3.5 shows that $h \notin \mathcal{L}(f)$, hence the restriction $P|\mathcal{L}(f)$ is injective. Therefore we can consider $\mathcal{L}$ as the completion of $\mathcal{L}(f)$ with respect to the norm $\|\cdot\|_{\delta-1}$. Since $l(z) \in \mathcal{L}$ for z in some open set which contains points of the upper and lower half plane, we find that the set $\{z_1, \dots, z_{\delta-1}\}$ is defining, a contradiction. Thus $h = 0$. ∎

Consider the semigroup $V'_x : \mathcal{L}(f) \to \mathcal{L}(f)$ of partially defined isometries, given by (1.3). In the following lemmata we investigate some properties of the operators V'_x.

Lemma 3.11. *The operators V'_x, $x \in (-a,a)$, are continuous on $\mathcal{L}(f)$ with respect to $\|\cdot\|_\delta$. In fact for some constant $B > 0$*

$$\|V'_x\|_\delta^2 \leqslant \mathrm{e}^{\max_j(B, 2|z_j|)a} \quad \text{for } |x| < a. \tag{3.2}$$

If $g \in \mathcal{L}(f)$ is in the domain of V'_y for some $y \in (-a,a)$, then the mapping

$$x \mapsto V'_x g$$

is continuous on $[0,y]$, $y > 0$ ($[y,0]$, $y < 0$).

Proof. It is proved in [5] that the operators V'_x considered in the space $\mathcal{H}(f)$ satisfy $\|V'_x\| \leqslant \mathrm{e}^{B|x|}$ for some $B > 0$. If $g = \sum_l \alpha_l f_{y_l}$ we compute

$$\begin{aligned} F_z(V'_x g) &= \sum_l \alpha_l F_z(f_{y_l+x}) = \sum_l \alpha_l \mathrm{e}^{\mathrm{i}z(y_l+x)} \\ &= \mathrm{e}^{\mathrm{i}zx}\Big(\sum_l \alpha_l \mathrm{e}^{\mathrm{i}zy_l}\Big) = \mathrm{e}^{\mathrm{i}zx} F_z(g). \end{aligned} \tag{3.3}$$

We find

$$\begin{aligned} \|V'_x g\|_\delta^2 &= \|V'_x g\|^2 + \sum_j |F_{z_j}(V'_x g)|^2 \leqslant \mathrm{e}^{Bx}\|g\|^2 + \sum_j |\mathrm{e}^{\mathrm{i}z_j x}|^2 |F_{z_j}(g)|^2 \\ &\leqslant \mathrm{e}^{Bx}\|g\|^2 + \max_j |\mathrm{e}^{\mathrm{i}z_j x}|^2 \cdot \sum_j |F_{z_j}(g)|^2 \leqslant \max_j(\mathrm{e}^{Bx}, |\mathrm{e}^{-\mathrm{i}z_j x}|^2)\|g\|_\delta^2. \end{aligned}$$

This shows that

$$\|V_x'\|_\delta^2 \leqslant \mathrm{e}^{\max\limits_j(B,2|z_j|)|x|} \leqslant \mathrm{e}^{\max\limits_j(B,2|z_j|)a} \qquad \text{for } |x|<a.$$

To prove that $x \mapsto V_x'g$ is continuous, it suffices to show that f_x depends continuously on x in the norm of $\mathcal{L}(f)$. Since f is continuous, we have for $x \to x_0$

$$[f_x - f_{x_0}, f_y] = f(y-x) - f(y-x_0) \to 0,$$

$$[f_x - f_{x_0}, f_x - f_{x_0}] = 2f(0) - f(x-x_0) - f(x_0-x) \to 0.$$

The assertion follows from [6]. ∎

Due to Lemma 3.11 we can extend V_x' to $\mathcal{H}$. This extension, also denoted by V_x, has the domain

$$\mathcal{D}(V_x) = \overline{\langle f_y \mid y \in (-a, a-x)\rangle}.$$

It follows from (3.2) that the mapping $x \mapsto V_x g$ is continuous, even for $g \in \mathcal{H}$.

LEMMA 3.12. *Let* $0 \leqslant x < a$, $g \in \mathcal{D}(V_x)$, $z \in \mathbb{C}$ *and assume that* F_z *is continuous. Then*

$$F_z(V_x g) = \mathrm{e}^{\mathrm{i}zx} F_z(g). \tag{3.4}$$

If $g \in \ker(V_x - \lambda)$, *we have* $g \in \ker(F_z)$ *for all* z *such that* F_z *is continuous and* $\mathrm{e}^{\mathrm{i}zx} \neq \lambda$.

Proof. Since $\mathcal{D}(V_x) = \overline{\langle f_y \mid y \in (-a, a-x)\rangle}$ there exists a sequence $g_n = \sum\limits_i \lambda_i^n f_{x_i^n} \in \langle f_y \mid y \in (-a, a-x)\rangle$, such that $g_n \to g$. The relation (3.3) shows that

$$F_z(V_x g_n) = \mathrm{e}^{\mathrm{i}zx} F_z(g_n).$$

Since F_z is continuous, this implies (3.4).

If $V_x g = \lambda g$ we have

$$\lambda F_z(g) = F_z(V_x g) = \mathrm{e}^{\mathrm{i}zx} F_z(g),$$

hence either $\lambda = \mathrm{e}^{\mathrm{i}zx}$ or $F_z(g) = 0$. ∎

DEFINITION 3.13. Denote by $\frac{1}{i}S$ the infinitesimal generator of the semigroup V_x in $\mathcal{H}$, i.e. let S be the closure of the operator

$$S'g = \mathrm{i} \lim_{t \searrow 0} \frac{V_t g - g}{t} \tag{3.5}$$

with domain

$$\mathcal{D}(S') = \Big\{ g \in \bigcup_{t \searrow 0} \mathcal{D}(V_t) \subseteq \mathcal{H} \,\Big|\, \lim_{t \searrow 0} \frac{V_t g - g}{t} \text{ exists} \Big\}.$$

It is proved in [3] that S is densely defined.

Proposition 3.14. *Let $b(f)$ be as in Lemma 2.12. For $b \in (b(f), a)$ the spaces $\mathcal{H}(f_{(b)})$ are regular subspaces of $\mathcal{P}_c$. If $b(f) < b < c < a$ we have, with the obvious identifications,*

$$\mathcal{H}(f_{(b)}) \subseteq \mathcal{H}(f_{(c)}) \subseteq \mathcal{H} \subseteq \mathcal{P}_c.$$

Moreover,

$$\overline{\bigcup_{b(f)<b<a} \mathcal{H}(f_{(b)})} = \mathcal{H}.$$

The corresponding symmetries $S_b = A(f_{(b)})$ satisfy

$$S_b \subseteq S_c \subseteq S.$$

Moreover,

$$\mathcal{H}/\mathcal{H}^\circ = \mathcal{H}(f) \quad \text{and} \quad S/\mathcal{H}^\circ \subseteq A(f). \tag{3.6}$$

Proof. We clearly have

$$\overline{\mathcal{L}(f_{(b)})} \subseteq \overline{\mathcal{L}(f_{(c)})} \subseteq \mathcal{H},$$

where the closure has to be understood with respect to the norm $\|\cdot\|_\delta$ of $\mathcal{H}$.

To prove that $\overline{\mathcal{L}(f_{(b)})}$ is nondegenerated assume on the contrary that $h \in \overline{\mathcal{L}(f_{(b)})}^\circ$ and $h \neq 0$, here the isotropic subspace is understood with respect to the inner product $[\cdot,\cdot]$. If $\mathcal{L}_-$ denotes a maximal negative subspace of $\mathcal{L}(f_{(b)})$, we have $h \perp \mathcal{L}_-$ and of course $h \perp \mathcal{H}^\circ$. Since $\mathrm{Ind}_-\mathcal{P}_c = \kappa_0 + \delta$, $\dim \mathcal{L}_- = \kappa_0$ and $\dim \mathcal{H}^\circ = \delta$ it follows that $h \in \mathcal{H}^\circ$, i.e. we have

$$\overline{\mathcal{L}(f_{(b)})}^\circ \subseteq \mathcal{H}^\circ,$$

hence also $\overline{\mathcal{L}(f_{(b)})}^\circ \subseteq \overline{\mathcal{L}(f_{(b')})}^\circ$ for $b < b'$. Consider the linear space

$$\mathcal{L} = \bigcup_{b\in(b(f),a)} \overline{\mathcal{L}(f_{(b)})}^\circ \subseteq \mathcal{H}^\circ.$$

Note that $\mathcal{L} \neq \{0\}$ and let $\mathcal{L} = \langle h_1, \dots, h_n\rangle$, with $n \leqslant \delta < \infty$. If $h_i \in \overline{\mathcal{L}(f_{(b_i)})}^\circ$ we have $h_i \in \mathcal{D}(V_{\frac{a-b_i}{2}})$ and for $|x| \leqslant \frac{a-b_i}{2}$

$$V_x h_i \in \overline{\mathcal{L}(f_{(\frac{a+b_i}{2})})}^\circ \subseteq \mathcal{L},$$

as $h_i \in \mathcal{H}^\circ$, $V_{\frac{a-b_i}{2}}\overline{\mathcal{L}(f_{(b_i)})} \subseteq \overline{\mathcal{L}(f_{(\frac{a+b_i}{2})})}$ and $V_{\frac{a-b_i}{2}}\overline{\mathcal{L}(f_{(\frac{a+b_i}{2})})} \subseteq \mathcal{H}$. Let $b = \max\limits_i b_i$ and let $\varepsilon = \frac{a-b}{2}$, then

$$\mathcal{L} \subseteq \mathcal{D}(V_x) \quad \text{and} \quad V_x\mathcal{L} \subseteq \mathcal{L} \quad \text{for } -\varepsilon < x < \varepsilon.$$

Let $x \in (-\varepsilon, \varepsilon)$, $x \neq 0$. As $\mathcal{L}$ is finite dimensional there exists an eigenvalue λ of V_x with corresponding eigenvector $h \in \mathcal{L}$, $h \neq 0$. Lemma 3.12 implies that $h \in \ker(F_z)$ if F_z is continuous and $\mathrm{e}^{\mathrm{i}zx} \neq \lambda$. Since $\{z_1, \dots, z_\delta\}$ is minimal the values z_j are distinct. If x is sufficiently small, hence also the values $\mathrm{e}^{\mathrm{i}z_jx}$ are distinct. Together with $\mathcal{L} \subseteq \mathcal{H}^\circ$ this shows that Lemma 3.10 can be applied, which yields $h = 0$, a contradiction. Hence $\overline{\mathcal{L}(f_{(b)})}$ is a regular subspace of $\mathcal{P}_c$.

This shows that the relation $f_x \mapsto f_x + \mathcal{H}^\circ$ yields an isometry between the Pontryagin spaces $\overline{\mathcal{L}(f_{(b)})}$ and $\mathcal{H}(f_{(b)})$. As its domain and range are dense, it extends to an isomorphism and we find

$$\overline{\mathcal{L}(f_{(b)})} = \mathcal{H}(f_{(b)}).$$

As $\mathcal{H}(f_{(b)}) \subseteq \mathcal{H}(f_{(c)}) \subseteq \mathcal{H}$ as regular subspaces we find $S_b \subseteq S_c \subseteq S$.

The mapping $\varphi : f_x + \mathcal{H}^\circ \mapsto f_x$ yields an isometry between the Pontryagin spaces $\mathcal{H}/\mathcal{H}^\circ$ and $\mathcal{H}(f)$, hence extends to an isomorphism. Since the isometry $f_x + \mathcal{H}^\circ \mapsto f_x$ is continuous and intertwines the respective shift operators, we have $S/\mathcal{H}^\circ \subseteq A(f)$. ∎

This result has a number of corollaries.

Corollary 3.15. *Let $b \in (b(f), a)$, then $f_{(b)}$ is not determining.*

Proof. There exists an open set O, such that for $z \in O$ the functionals F_z are continuous on $\mathcal{H}$, hence also on the regular subspace $\mathcal{H}(f_{(b)})$. It follows that $\mathrm{e}^{\mathrm{i}zx}|_{(-b,b)} \in \mathcal{H}(f_{(b)})$, thus $f_{(b)}$ is not determining. ∎

Similar as in Corollary 2.14 we obtain:

Corollary 3.16. *The space $\mathcal{L}(f)$ is nondegenerated.*

Corollary 3.17. *The relation S is symmetric. Hence $\mathcal{R}(S - z)$ is closed for all $z \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. The fact that S is symmetric follows from $S/\mathcal{H}^\circ \subseteq A(f)$, as $\mathcal{H}^\circ$ is isotropic.

The remaining assertion follows from [4], as we can regard S as a symmetric relation in $\mathcal{P}_c$. ∎

Corollary 3.18. *Let $z \in \varrho(A(f))$. Then, for $h \in \mathcal{R}(S-z) \cap \mathcal{H}^\circ$, we have*

$$k \in \mathcal{H}^\circ, \qquad \textit{if } (h;k) \in (S-z)^{-1},$$

and for $h \in \mathcal{D}(S) \cap \mathcal{H}^\circ$ we have

$$k \in \mathcal{H}^\circ, \qquad \textit{if } (h;k) \in S.$$

Proof. Consider the canonical projection π of $\mathcal{H}$ onto $\mathcal{H}(f)$. Then $(h;k) \in (S-z)^{-1}h$ implies that $\pi k = (A(f)-z)^{-1}\pi h = 0$, hence $k \in \mathcal{H}^\circ$. Similar $(h;k) \in S$ implies $\pi k = A(f)\pi h = 0$, hence $k \in \mathcal{H}^\circ$. ∎

Proposition 3.19. *We have*

$$S = \overline{\bigcup_{b \in I} S_b}$$

whenever $I \subseteq (b(f), a)$ has the right endpoint a as accummulation point. In particular

$$\mathcal{R}(S-z) = \overline{\bigcup_{b \in I} \mathcal{R}(S_b - z)},$$

if z is such that $\mathcal{R}(S-z)$ is closed.

Proof. Clearly $S \supseteq \overline{\bigcup_{b \in I} S_b}$. To show the reverse inclusion consider the operator S' as given in (3.5) and let $g \in \mathcal{D}(S')$, $g \in \mathcal{D}(V_t)$ for $t \leqslant t_0$. Then, by definition,

$$-\mathrm{i}S'g = \lim_{\tau \to 0} \frac{V_\tau g - g}{\tau}.$$

For $t < \frac{t_0}{2}$ put

$$g_t = \frac{1}{t}\int_0^t V_s g \,\mathrm{d}s,$$

then $g_t \in \mathcal{D}(V_\tau)$ for $\tau \leqslant \frac{t_0}{2}$, and

$$\lim_{t \to 0} g_t = g.$$

If $\tau < t$ we have

$$\begin{aligned}\frac{V_\tau g_t - g_t}{\tau} &= \frac{1}{t\tau}\left(\int_\tau^{\tau+t} V_s g \,\mathrm{d}s - \int_0^t V_s g \,\mathrm{d}s\right) = \frac{1}{t\tau}\left(\int_t^{t+\tau} V_s g \,\mathrm{d}s - \int_0^\tau V_s g \,\mathrm{d}s\right)\\ &= \frac{1}{t}\left(V_t \frac{1}{\tau}\int_0^\tau V_s g \,\mathrm{d}s - \frac{1}{\tau}\int_0^\tau V_s g \,\mathrm{d}s\right),\end{aligned}$$

hence

$$\lim_{\tau\to 0}\frac{V_\tau g_t - g_t}{\tau} = \frac{V_t g - g}{t}.$$

It follows that for $t < \frac{t_0}{2}$ the relation $g_t \in \mathcal{D}(S')$ and

$$-\mathrm{i}S' g_t = \frac{V_t g - g}{t}$$

holds, and we find

$$\lim_{t\to 0} S' g_t = S' g.$$

Since $\langle f_x \mid x \in (-a, a - t_0)\rangle$ is dense in $\mathcal{D}(V_{t_0})$ there exists a sequence $g_n \in \langle f_x \mid x \in (-a, a-t_0)\rangle$, such that

$$\lim_{n\to\infty} g_n = g.$$

As each g_n is a finite linear combination of elements f_x there exists a number $b_n \in (a - \frac{t_0}{2}, a)$, $b_n \in I$, such that $g_n \in \mathcal{H}(f_{(b_n)})$. Define

$$g_{n,t} = \frac{1}{t}\int_0^t V_s g_n \,\mathrm{d}s,$$

then, as $t < \frac{t_0}{2}$, we have $g_{n,t} \in \mathcal{H}(f_{(b_n)})$, and for $\tau < \frac{t_0}{2}$ we have $g_{n,t} \in \mathcal{D}(V_\tau)$. Moreover, for some number $K > 0$

$$\|g_{n,t} - g_t\|_\delta = \left\| \frac{1}{t}\int_0^t V_s(g_n - g)\,\mathrm{d}s \right\|_\delta \leqslant \max_{s\in[0,t]} \|V_s\|_\delta \cdot \|g_n - g\|_\delta \leqslant K\|g_n - g\|_\delta,$$

where the last inequality follows from Lemma 3.11. Hence

$$\lim_{n\to\infty} g_{n,t} = g_t.$$

A similar computation as above shows that

$$\lim_{\tau\to 0}\frac{V_\tau g_{n,t} - g_{n,t}}{\tau} = \frac{V_t g_n - g_n}{t}$$

holds. As, for $t < \frac{t_0}{2}$ we have $V_t g_n \in \mathcal{H}(f_{(b_n)})$, and for $\tau < \frac{t_0}{2} - t$ we have $g_{n,t} \in \mathcal{D}(V_\tau)$ and $V_\tau g_{n,t} \in \mathcal{H}(f_{b_n})$, we find that $g_{n,t} \in \mathcal{D}(S_{b_n})$ and

$$-\mathrm{i}S_{b_n} g_{n,t} = \frac{V_t g_n - g_n}{t}.$$

It follows that for each $t < \frac{t_0}{2}$

$$\lim_{n\to\infty} S_{b_n} g_{n,t} = S' g_t.$$

Let $\varepsilon > 0$ be given and choose $t > 0$, such that

$$\|g_t - g\|_\delta < \varepsilon \quad \text{and} \quad \|S' g_t - S' g\|_\delta < \varepsilon.$$

Now choose $n \in \mathbb{N}$, such that

$$\|g_{n,t} - g_t\|_\delta < \varepsilon \quad \text{and} \quad \|S_{b_n} g_{n,t} - S' g_t\|_\delta < \varepsilon,$$

which implies that

$$\|g - g_{n,t}\|_\delta < 2\varepsilon \quad \text{and} \quad \|S' g - S_{b_n} g_{n,t}\|_\delta < 2\varepsilon.$$

These facts show that

$$S' \subseteq \overline{\bigcup_{b\in I} S_b},$$

and we find $S = \overline{\bigcup_{b\in I} S_b}$.

Since $\mathcal{R}(S - z)$ is closed and $\varphi : (g, h) \mapsto h - zg$ is a continuous mapping of S onto $\mathcal{R}(S - z)$ we find

$$\mathcal{R}(S - z) = \overline{\bigcup_{b\in I} \mathcal{R}(S_b - z)}. \quad ∎$$

Recall that, although the inner product of $\mathcal{H}$ is in general degenerated, the defect numbers of an operator in $\mathcal{H}$ can be defined as usual (see [7]).

Theorem 3.20. *The relation S is an operator and has defect numbers* $(1, 1)$. *Moreover, each eigenvalue of S is also an eigenvalue of $A(f)$.*

We have

$$\mathcal{R}(S - \overline{z}) = \ker(F_z) \tag{3.7}$$

if F_z is continuous and $\mathcal{R}(S - \overline{z})$ is closed. Moreover, for $h \in \mathcal{H}^\circ$

$$S \cap (\langle h\rangle \times \langle h\rangle) = \{0\}, \tag{3.8}$$

and for $z \in \mathbb{C}$

$$\dim\ker(S - z) \leqslant \dim\ker(A(f) - z). \tag{3.9}$$

If $\mathcal{H}^\circ \neq \{0\}$, *we have for each* $z \in \varrho(A(f))$

$$\mathcal{R}(S - z) + \mathcal{H}^\circ = \mathcal{H}, \tag{3.10}$$

and, for each $h \in \mathcal{H}^\circ$, $h \neq 0$, *the relation*

$$\mathcal{R}(S - z) + \langle h \rangle = \mathcal{H} \tag{3.11}$$

holds for all $z \in \varrho(A(f))$ *with possible exception of finitely many points.*

The relation $S \cap (\mathcal{H}^\circ)^2$ *is in fact an operator, its domain has codimension* 1 *in* $\mathcal{H}^\circ$, *and it has no eigenvalues.*

Proof. First we are concerned with the proof of (3.7). If F_z is continuous (and continuously extended to $\mathcal{P}_c$) there exists an element $k(z) \in \mathcal{P}_c$, such that

$$F_z(g) = [g, k(z)] \quad \text{for } g \in \mathcal{P}_c.$$

Let $b \in (b(f), a)$ and denote by P_b the orthogonal projection of $\mathcal{P}_c$ onto the regular subspace $\mathcal{H}(f_{(b)})$. We have for $x \in (-b, b)$

$$[f_x, P_b k(z)] = [f_x, k(z)] = \mathrm{e}^{\mathrm{i}zx},$$

hence $P_b k(z)$ is a defect element of S_b at z, i.e.

$$\mathcal{R}(S_b - \overline{z}) = \langle P_b k(z) \rangle^\perp = \ker(F_z) \cap \mathcal{H}(f_{(b)}). \tag{3.12}$$

From (3.12) and the fact that $\ker(F_z)$ is closed it follows that

$$\overline{\bigcup_{b \in I} \mathcal{R}(S_b - \overline{z})} \subseteq \ker(F_z).$$

To show the reverse inclusion let $g \in \ker(F_z)$. Let (b_n) be a sequence of numbers $b(f) < b_n < a$, $b_n \in I$, increasing to a, and let $g_n \in \mathcal{L}(f_{b_n})$ be such that $g_n \to g$ as $n \to \infty$. Then, due to (3.12)

$$\begin{aligned} &g_n - \left[g_n, \frac{P_{b_n} k(z)}{\|P_{b_n} k(z)\|_\delta} \right] \frac{P_{b_n} k(z)}{\|P_{b_n} k(z)\|_\delta} \\ &\qquad = g_n - [g_n, k(z)] \frac{P_{b_n} k(z)}{\|P_{b_n} k(z)\|_\delta} \frac{1}{\|P_{b_n} k(z)\|_\delta} \in \mathcal{R}(S_{b_n} - \overline{z}). \end{aligned} \tag{3.13}$$

As $g_n \to g$ we have

$$[g_n, k(z)] \to [g, k(z)] = 0.$$

Assume that $P_{b_n}k(z) \to 0$ as $n \to \infty$. Then we would have $[f, P_{b_n}k(z)] \to 0$, but

$$[f, P_{b_n}k(z)] = [f, k(z)] = 1.$$

Hence, at least for some subsequence of (b_n), the values of $\frac{1}{\|P_{b_n}k(z)\|_\delta}$ remain bounded. This shows that the second term on the left hand side of (3.13) tends to 0, and we find

$$g \in \overline{\bigcup_{b\in I} \mathcal{R}(S_b - \overline{z})}.$$

Thus Proposition 3.19 shows that (3.7) holds.

To prove (3.8) assume on the contrary that $(\lambda h, \mu h) \in S$ for some $h \in \mathcal{H}^\circ$ and λ, μ not both zero. Then

$$(\mu - \overline{z}\lambda)h \in \mathcal{R}(S - \overline{z}),$$

hence $h \in \mathcal{R}(S - \overline{z}) = \ker(F_z)$ with possible exception of one point $z_0 = \overline{(\frac{\mu}{\lambda})}$. Lemma 3.10 shows that $h = 0$, thus (3.8) is proved.

Let φ be the isometry of $\mathcal{H}$ onto $\mathcal{H}(f)$ as given in Proposition 3.14. Then φ maps $\ker(S - z)$ into $\ker(A(f) - z)$. As $\ker(\varphi) = \mathcal{H}^\circ$ and, by (3.8), $\ker(S - z) \cap \mathcal{H}^\circ = \{0\}$ the restriction $\varphi|\ker((S - z))$ is injective. Hence z being an eigenvalue of S implies $z \in \sigma_p(A(f))$ and the relation (3.9) holds. A similar argument shows that S is an operator. Indeed, let $(0, h) \in S$, then $(0, \varphi h) \in A(f)$. As $A(f)$ is an operator we obtain $\varphi h = 0$, i.e. $h \in \mathcal{H}^\circ$. The relation (3.8) shows that $h = 0$.

Let O be as in Definition 3.1. To show that S has defect numbers $(1, 1)$ it suffices to observe that for $z \in \varrho(A(f)) \cap \mathbb{C}^\pm \cap O$

$$\operatorname{codim}(\mathcal{R}(S - z)) = 1 \quad \text{and} \quad \ker(S - z) = \{0\}.$$

To prove (3.10) assume the contrary. This yields that $\mathcal{H}^\circ \subseteq \mathcal{R}(S - z)$, as S has defect numbers $(1, 1)$. Since $z \in \varrho(A(f))$ we find that $(S - z)^{-1}$ is an operator and satisfies, due to Corollary 3.18, $(S - z)^{-1}\mathcal{H}^\circ \subseteq \mathcal{H}^\circ$. Therefore it has a nonzero eigenvector, which contradicts (3.8).

Finally, to prove (3.11), assume on the contrary that some $h \in \mathcal{H}^\circ$, $h \neq 0$ is element of $\mathcal{R}(S - z)$ for infinitely many $z_i \in \varrho(A(f))$, $i \in \mathbb{N}$. Let $h_i = (S - z_i)^{-1}h$, then $h_i \in \mathcal{H}^\circ$. Thus the elements h_i are linearly dependent, say

$$\sum_{i=1}^{n} \lambda_i h_i = 0 \tag{3.14}$$

is a nontrivial vanishing linear combination of minimal lenght. As $h_i \neq 0$ we have $n \geqslant 2$. Since S is an operator, and $h_i \in \mathcal{D}(S)$, we obtain

$$\Big(\sum_{i=1}^{n} \lambda_i\Big) h + \sum_{i=1}^{n} \lambda_i z_i h_i = 0. \tag{3.15}$$

If $\sum_{i=1}^{n} \lambda_i = 0$, we can eliminate from (3.14) and (3.15) the term, say, $i = n$, and obtain a shorter nontrivial vanishing linear combination of the elements h_i, a contradiction. Hence $\sum_{i=1}^{n} \lambda_i \neq 0$, and we obtain $h \in \mathcal{D}(S)$. Repeated application of S to the relation (3.15) shows that $S^j h \in \mathcal{D}(S)$ for each $j \in \mathbb{N}_0$. Clearly $S^j h \in \mathcal{H}^\circ$, hence the space $\langle S^j h\rangle$ is a finite dimensional invariant subspace of S and is contained in $\mathcal{D}(S)$. Thus the operator S has a nonzero eigenvector within $\langle S^j h\rangle$, a contradiction to (3.8). ∎

Since (3.10) implies that for $z \in \varrho(A(f))$ the relation $\mathcal{R}(S/\mathcal{H}^\circ - z) = \mathcal{H}/\mathcal{H}^\circ$ holds, we have

Corollary 3.21. *In the second relation of* (3.6) *in Proposition* 3.14*, in fact equality holds:*

$$S/\mathcal{H}^\circ = A(f).$$

Remark 3.22. In the definition of a defining set (Definition 3.1) we could use the condition

> There exist points $z^\pm \in \varrho(A(f)) \cap \mathbb{C}^\pm$, $z^\pm \notin \{z_1, \dots, z_\delta\} \cup \{\overline{z_1}, \dots, \overline{z_\delta}\}$, such that $F_{z^\pm}$ is continuous with respect to $\|\cdot\|_\delta$.

instead of the condition "There exists an open set ...". Then, except of Corollary 3.3, the results of this section remain in principle valid. We choose the weaker definition to include the case that f is not determining.

4. PARAMETRIZATION OF EXTENSIONS

We start this section by showing that extensions of f correspond to extensions of S.

First let us introduce the notion of the minimal part of a relation. Let A be a selfadjoint relation in a Pontryagin space $\mathcal{P}$, and let $\mathcal{M}$ be a, not necessarily closed, subspace of $\mathcal{P}$. Denote by $\mathcal{L}_\mathcal{M}$ the subspace $\mathcal{L}_\mathcal{M} = \overline{\langle \mathcal{M}, (A-z)^{-1}\mathcal{M} \mid z \in \varrho(A)\rangle}$, and put

$$\mathcal{P}_\mathcal{M} = \mathcal{L}_\mathcal{M}/\mathcal{L}_\mathcal{M}^\circ.$$

The $\mathcal{M}$-minimal part of A is the relation

$$A_{\mathcal{M}} = (A \cap \mathcal{L}_{\mathcal{M}}^2)/\mathcal{L}_{\mathcal{M}}^{\circ}.$$

The relation $A_{\mathcal{M}}$ is again selfadjoint and $\varrho(A_{\mathcal{M}}) \supseteq \varrho(A)$. These facts follow from some results of [4].

Lemma 4.1. *Assume that there exists a minimal defining set and let $\mathcal{H}$ and S be as in the previous section.*

Let A be a selfadjoint relation in a Pontryagin space P with $\varrho(A) \neq \emptyset$, let $\varphi : \mathcal{H} \to \mathcal{P}$ be an isometric mapping, and assume that

$$\varphi S = \{(\varphi a; \varphi b) \in \mathcal{P}^2 \mid (a; b) \in S\} \subseteq A.$$

If there exists a nontrivial subspace $\mathcal{L} \subseteq \mathcal{H}^{\circ}$, such that $\varphi\mathcal{L}$ is invariant under each resolvent $(A - z)^{-1}$, then $\varphi\mathcal{H}$ itself is invariant under each resolvent $(A - z)^{-1}$. In this case we have

$$[(A - z)^{-1} f, f] = [(A(f) - z)^{-1} f, f].$$

Proof. First note that if $\mathcal{H}$ is nondegenerated, i.e. $\delta = 0$, the assumptions of the lemma cannot be satisfied, hence there is nothing to prove.

Choose an element $h \in \mathcal{L}$, $h \neq 0$. Theorem 3.20 shows that for all $z \in \mathbb{C} \setminus \mathbb{R}$, with possible exception of a finite set M, the relation

$$\mathcal{R}(S - z) + \langle h \rangle = \mathcal{H}$$

holds. Since a resolvent $(A - z)^{-1}$ maps $\varphi\mathcal{R}(S - z)$ into $\varphi\mathcal{H}$ and $\langle \varphi h \rangle$ into $\varphi\mathcal{L} \subseteq \varphi\mathcal{H}$, we find that

$$(A - z)^{-1} \varphi\mathcal{H} \subseteq \varphi\mathcal{H}, \quad z \in \varrho(A) \setminus M.$$

As $\varphi\mathcal{H}$ is a closed subspace of $\mathcal{P}$ the invariance of $\varphi\mathcal{H}$ follows for all $z \in \varrho(A)$.

Consider the $\varphi\mathcal{H}$-minimal part of A. Clearly $\mathcal{L}_{\varphi\mathcal{H}} = \varphi\mathcal{H}$, and we obtain from Proposition 3.14 that $\mathcal{P}_{\varphi\mathcal{H}} = \mathcal{H}(f)$. Since $\varphi S/\varphi\mathcal{H}^{\circ} \subseteq A_{\varphi\mathcal{H}}$, it follows from Proposition 3.14 and Theorem 3.20 that $A_{\varphi\mathcal{H}} = A(f)$. Thus

$$[(A - z)^{-1} f, f] = [(A_{\varphi\mathcal{H}} - z)^{-1} f, f] = [(A(f) - z)^{-1} f, f]. \quad ∎$$

Note that, since $\mathcal{H}$ may be degenerated, φ may have a nontrivial kernel.

For a selfadjoint operator $\tilde{A}$ let $h_{\tilde{A}}$ be such that the spectrum of $\tilde{A}$ is contained in the strip $\{z \mid |\operatorname{Im} z| \leqslant h_{\tilde{A}}\}$. The existence of such a number $h_{\tilde{A}}$ is proved e.g. in [6].

Proposition 4.2. *Assume that $\{z_1, \ldots, z_\delta\}$ is a minimal defining set. Let $\tilde{A}$ be a selfadjoint relation in a Pontryagin space P with $\varrho(\tilde{A}) \neq \emptyset$, and assume that $\mathcal{H} \subseteq \mathcal{P}$ and $S \subseteq \tilde{A}$. Then there exists a (unique) function $\tilde{f} : \mathbb{R} \to \mathbb{C}$, such that*

$$\mathrm{i} \int_0^\infty \mathrm{e}^{\mathrm{i}zt} \tilde{f}(t)\, \mathrm{d}t = [(\tilde{A} - z)^{-1} f, f], \quad \operatorname{Im} z > h_{\tilde{A}} \tag{4.1}$$

holds. The function $\tilde{f}$ extends f and is contained in a set $\mathcal{P}_\kappa$ with $\kappa = \kappa_0$ or $\kappa_0 + \delta \leqslant \kappa \leqslant \mathrm{Ind}_-\mathcal{P}$.

If $\tilde{A}$ is f-minimal, the relation $\tilde{A}$ is in fact an operator and we have $\kappa = \mathrm{Ind}_-\mathcal{P}$. Moreover, $\mathcal{P} = \mathcal{H}(\tilde{f})$ and $\tilde{A} = A(\tilde{f})$.

Proof. First note that, due to the fact that S is densely defined in the norm $\|\cdot\|_\delta$ of $\mathcal{H}$, we have

$$\mathcal{H} = \overline{\mathcal{D}(S)} \subseteq \overline{\mathcal{D}(\tilde{A})} \tag{4.2}$$

in the norm of $\mathcal{P}$, hence $f \in \overline{\mathcal{D}(\tilde{A})}$.

Consider now the case that $\tilde{A}$ is f-minimal. This implies that

$$\overline{\mathcal{D}(\tilde{A})} \supseteq \overline{\langle f, (\tilde{A} - z)^{-1} f \mid z \in \varrho(\tilde{A}) \rangle} = \mathcal{P},$$

i.e. $\tilde{A}$ is densely defined and hence an operator. It is shown in [13] (Satz 1.5 and Satz 5.3) that there exists a unique function $\tilde{f}$ defined by (4.1), and that it is an element of $\mathcal{P}_\kappa$ where $\kappa = \mathrm{Ind}_-\mathcal{P}$. Moreover,

$$\tilde{f}(x) = [f, U_x f], \quad x \in \mathbb{R},$$

when U_x is the group of unitary operators generated by $\tilde{A}$. Since $\tilde{A} \supseteq S$ we have (with a similar proof as in [5]) $U_x \supseteq V_x$. Hence, for $x \in (-a, a)$,

$$\tilde{f}(x) = [f, U_x f] = [f, V_x f] = f(x),$$

i.e. $\tilde{f}$ is an extension of f. Since the mapping $\varphi : f_x \mapsto U_x f$ is an isometry of $\mathcal{L}(\tilde{f})$ into $\mathcal{P}$, and $\tilde{A}$ being f-minimal is equivalent to the fact that

$$\mathcal{P} = \overline{\langle U_x f \mid x \in \mathbb{R} \rangle},$$

we find that $\mathcal{H}(\tilde{f}) = \mathcal{P}$. Then clearly $A(\tilde{f}) = \tilde{A}$.

In the following let $\tilde{A}$ be an arbitrary relation. Put $\mathcal{M} = \overline{\mathcal{D}(\tilde{A})}$ and consider the $\mathcal{M}$-minimal part of $\tilde{A}$. Note that

$$\mathcal{L}_{\mathcal{M}} = \overline{\mathcal{D}(\tilde{A})} \quad \text{and} \quad \mathcal{L}_{\mathcal{M}}^{\circ} = \tilde{A}(0)^{\circ}.$$

If $\mathcal{L}_{\mathcal{M}}^{\circ} \cap \mathcal{H} \neq \{0\}$, Lemma 4.1 applies with

$$\mathcal{L} = \mathcal{L}_{\mathcal{M}}^{\circ} \cap \mathcal{H},$$

and shows that $\tilde{f}$ exists and is in fact the unique extension of f in $\mathcal{P}_{\kappa_0}$. If $\mathcal{L}_{\mathcal{M}}^{\circ} \cap \mathcal{H} = \{0\}$ we can regard $\mathcal{H}$ as a subspace of $\mathcal{P}_{\mathcal{M}}$. We have $\mathrm{Ind}_{-}\mathcal{P}_{\mathcal{M}} = \mathrm{Ind}_{-}\mathcal{P} - \dim \tilde{A}(0)^{\circ}$, and clearly $[(\tilde{A} - z)^{-1} f, f] = [(\tilde{A}_{\mathcal{M}} - z)^{-1} f, f]$. Also clearly $S \subseteq \tilde{A}_{\mathcal{M}}$.

Iterate the process described in the above paragraph. Since the negative index of the considered Pontryagin space decreases this process must terminate with either $\tilde{f} \in \mathcal{P}_{\kappa_0}$ or with a relation $\tilde{A}_1$ such that $\tilde{A}_1(0)^{\circ} = \{0\}$ and $[(\tilde{A} - z)^{-1} f, f] = [(\tilde{A}_1 - z)^{-1} f, f]$. Decompose $\tilde{A}_1$ as

$$\tilde{A}_1 = \tilde{A}_{1,s} [\dot{+}] \tilde{A}_{1,\infty},$$

with a selfadjoint operator $\tilde{A}_{1,s}$ acting in $\overline{\mathcal{D}(\tilde{A}_1)}$. Since $\tilde{A}_1 \supseteq S$ and $\overline{\mathcal{D}(S)} = \mathcal{H}$ we have $\overline{\mathcal{D}(\tilde{A}_1)} \supseteq \mathcal{H}$, therefore the operator $\tilde{A}_{1,s}$ extends S.

Due to the above considerations we may restrict our attention to the case of an operator $\tilde{A}$. There exists (see [3]) a group $(U_t)_{t \in \mathbb{R}}$ of unitary operators which has $\tilde{A}$ as its infinitesimal generator, and satisfies $U_t \supseteq V_t$.

Put $\mathcal{M} = \langle f \rangle$ and consider the $\mathcal{M}$-minimal part of $\tilde{A}$. Lemma 2.1 shows that

$$\mathcal{H} = \overline{\langle V_t f \mid t \in (-a, a) \rangle} \subseteq \overline{\langle U_t f \mid t \in \mathbb{R} \rangle} \subseteq \mathcal{L}_{\mathcal{M}}.$$

If $\mathcal{L}_{\mathcal{M}}^{\circ} \cap \mathcal{H} \neq \{0\}$, Lemma 4.1 applies with

$$\mathcal{L} = \mathcal{L}_{\mathcal{M}}^{\circ} \cap \mathcal{H},$$

and shows that $\tilde{f} \in \mathcal{P}_{\kappa_0}$, when $\tilde{f}$ is the function determined by $\tilde{A}$ via (4.1) (see [13]). Otherwise the relation $\tilde{A}_{\mathcal{M}}$ extends S and is f-minimal. Then the first part of this proof applies. ∎

In order to apply Proposition 4.2 we have to show that S admits minimal selfadjoint extension.

Proposition 4.3. *Let S be as above, and assume that $\delta > 0$. Then there exists a selfadjoint extension of S, which acts in a Pontryagin space with negative index $\kappa_0 + \delta$ and is f-minimal.*

Proof. Consider the space $\mathcal{P}_c$ (as constructed in the previous section) and let $z \in \varrho(A(f))$ be fixed. The mapping $V = (S - \overline{z})(S - z)^{-1}$ is an isometry of $\mathcal{R}(S - z)$ onto $\mathcal{R}(S - \overline{z})$. Theorem 3.20 shows that $\mathcal{R}(S - z)^\circ = \mathcal{R}(S - z) \cap \mathcal{H}^\circ$, hence $\dim \mathcal{R}(S - z)^\circ = \delta - 1$, and similar for $\mathcal{R}(S - \overline{z})$. Since the dimensions of the isotropic parts of range and domain of V are equal, V is injective. Let

$$\mathcal{R}(S - z) \cap \mathcal{H}^\circ = \langle h_1, \ldots, h_{\delta-1} \rangle,$$

and put $k_i = V h_i$ for $i = 1, \ldots, \delta - 1$. Then

$$\mathcal{R}(S - \overline{z}) \cap \mathcal{H}^\circ = \langle k_1, \ldots, k_{\delta-1} \rangle.$$

Choose h_δ and k_δ such that

$$\langle h_1, \ldots, h_\delta \rangle = \langle k_1, \ldots, k_\delta \rangle = \mathcal{H}^\circ.$$

Fix complements $\mathcal{H}_{n,z}$ and $\mathcal{H}_{n,\overline{z}}$ of $\mathcal{H}^\circ$ in $\mathcal{H}$, such that $\mathcal{H}_{n,z} \subseteq \mathcal{R}(S - z)$ and $\mathcal{H}_{n,\overline{z}} \subseteq \mathcal{R}(S - \overline{z})$ and choose elements $h_i' \perp \mathcal{H}_{n,z}$, $k_i' \perp \mathcal{H}_{n,\overline{z}}$ for $i = 1, \ldots, \delta$, such that

$$\mathcal{P}_c = \mathcal{H}_{n,z} [\dot{+}] (\langle h_1, \ldots, h_\delta \rangle \dot{+} \langle h_1', \ldots, h_\delta' \rangle),$$

and also

$$\mathcal{P}_c = \mathcal{H}_{n,\overline{z}} [\dot{+}] (\langle k_1, \ldots, k_\delta \rangle \dot{+} \langle k_1', \ldots, k_\delta' \rangle).$$

Moreover, let the bases $\{h_i\}$ and $\{h_i'\}$ (and similar for the k's) be skewly linked, i.e. let

$$[h_i, h_j] = [k_i, k_j] = [h_i', h_j'] = [k_i', k_j'] = 0, \quad [h_i, h_j'] = [k_i, k_j'] = \delta_{ij}.$$

Define an extension U of V by

$$U h_\delta = k_\delta', \quad U h_\delta' = k_\delta,$$

$$U h_i' = k_i', \quad i = 1, \ldots, \delta - 1.$$

It is easily checked that U is unitary.

Let $\tilde{A}$ be the inverse Cayley transform of U, then $\tilde{A}$ is a selfadjoint relation extending S. Due to (4.2) we have

$$\tilde{A}(0) \subseteq \mathcal{H}^\perp = \mathcal{H}^\circ.$$

If $\tilde{A}(0) \neq \{0\}$, Lemma 4.1 applied with $\mathcal{L} = \tilde{A}(0)$ shows that $\mathcal{H}$ is invariant under each resolvent of $\tilde{A}$. Since $U = I + (z - \overline{z})(\tilde{A} - z)^{-1}$ this yields a contradiction to the definition of U. Hence $\tilde{A}$ is an operator.

Put $\mathcal{M} = \langle f \rangle$ and consider the $\mathcal{M}$-minimal part of $\tilde{A}$. The same argument as in the last paragraph of the proof of Proposition 4.2 shows that $\mathcal{L}_{\mathcal{M}} \supseteq \mathcal{H}$, hence $\mathcal{L}_{\mathcal{M}}^{\circ} \subseteq \mathcal{H}^{\circ}$. If $\mathcal{L}_{\mathcal{M}}$ is degenerated, Lemma 4.1 applies with $\mathcal{L} = \mathcal{L}_{\mathcal{M}}^{\circ}$. Similar as in the previous paragraph this leads to a contradiction to the definition of U, thus $\mathcal{L}_{\mathcal{M}}$ is nondegenerated. Together with the fact $\mathcal{H} \subseteq \mathcal{L}_{\mathcal{M}}$, this shows that $\mathcal{L}_{\mathcal{M}} = \mathcal{P}_c$. Hence $\tilde{A}$ is an f-minimal extension of S in the space $\mathcal{P}_c$. ∎

These results have a number of corollaries.

Corollary 4.4. *Let $\{z_1, \ldots, z_\delta\}$ be a minimal defining set and let $\tilde{f} \in \mathcal{P}_\kappa$, $\kappa > \kappa_0$, be an extension of f. Then*

$$\mathcal{H} = \mathcal{L}(f, \tilde{f}) \quad \textit{and} \quad S = S_{\tilde{f}}.$$

Each minimal defining set contains the same number of points.

Proof. If $\delta = 0$ the assertion is clear, since then $\mathcal{L}(f, \tilde{f})$ is a regular subspace. So consider the case $\delta > 0$. By Proposition 4.3 there exists an extension $\tilde{A} \subseteq \mathcal{P}_c^2$ of S which holds the properties assumed in Proposition 4.2. If $\tilde{f}_1$ is the associated extension of f, we have $\mathcal{P}_c = \mathcal{H}(\tilde{f}_1)$. Hence $\mathcal{H} = \mathcal{L}(f, \tilde{f}_1)$. Theorem 2.10 shows that $\mathcal{H} = \mathcal{L}(f, \tilde{f})$. Clearly $S \subseteq S_{\tilde{f}}$. We show that $\mathcal{R}(S_{\tilde{f}} - z) = \mathcal{R}(S - z)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$ with possible exception of an isolated set. This clearly implies that $S_{\tilde{f}} = S$. Assume on the contrary that $\mathcal{R}(S_{\tilde{f}} - z) = \mathcal{H}$ for z in a set M which has an accumulation point in $\mathbb{C} \setminus \mathbb{R}$. For such z we have

$$(\tilde{A} - z)^{-1}\mathcal{H} \subseteq \mathcal{H},$$

in particular $(\tilde{A} - z)^{-1}\tilde{f} \in \mathcal{H}$. Let $h \in \mathcal{H}^{\circ}$, $h \neq 0$, then

$$[(\tilde{A} - z)^{-1}h, f] = 0$$

for $\overline{z} \in M$. Lemma 2.5 yields $h = 0$, a contradiction.

To prove the remaining assertion note that for any minimal defining set $\{z_1, \ldots, z_\delta\}$ we can make the above constructions, hence we obtain that

$$\delta = \mathrm{Ind}_0 \mathcal{H} = \mathrm{Ind}_0 \mathcal{L}(f, \tilde{f})$$

for a certain extension $\tilde{f}$ of f. The assertion now follows from Corollary 2.11. ∎

Denote the number of points contained in some minimal defining set by $\Delta(f)$. If does not exist any defining set put $\Delta(f)=\infty$.

COROLLARY 4.5. *The function f is extendable if and only if $\Delta(f)<\infty$, and f is determining if and only if $\Delta(f)>0$. If $0<\Delta(f)<\infty$, f admits no extensions in a set $\mathcal{P}_\kappa$ with $\kappa_0<\kappa<\kappa_0+\Delta(f)$, but has extensions in $\mathcal{P}_{\kappa_0+\Delta(f)}$.*

Proof. The first assertion follows from the considerations in Corollary 4.4 and Remark 3.4. The second assertion is a restatement of Corollary 3.3.

Let $0<\Delta(f)<\infty$ and assume that $\tilde{f}\in\mathcal{P}_\kappa$ is an extension of f with $\kappa>\kappa_0$. Then, by Corollary 4.4,

$$\kappa=\operatorname{Ind}_-\mathcal{H}(\tilde{f})\geqslant\operatorname{Ind}_-\mathcal{H}+\operatorname{Ind}_0\mathcal{H}=\kappa_0+\Delta(f).$$

The remaining assertion follows from Proposition 4.2 and Proposition 4.3. ∎

Now we are in position to show that extensions of f and extensions of S correspond bijectively.

PROPOSITION 4.6. *Let $f\in\mathcal{P}_{\kappa_0,a}$ be extendable, i.e. assume $\Delta(f)<\infty$, and let $\kappa\geqslant\kappa_0+\Delta(f)$. The relation*

$$\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}\tilde{f}(t)\,\mathrm{d}t=[(\tilde{A}-z)^{-1}f,f],\quad \operatorname{Im}z>h_{\tilde{A}} \tag{4.3}$$

establishes up to unitary equivalence a one-to-one correspondence between the extensions $\tilde{f}\in\mathcal{P}_\kappa$ of f and the selfadjoint operator extensions $\tilde{A}$ of S which act in some Pontryagin space $\mathcal{P}\supset\mathcal{H}$ with $\operatorname{Ind}_-\mathcal{P}=\kappa$ and which are f-minimal.

Proof. We have already proved in Proposition 4.2 that an extension of S leads to an extension of f.

Assume first that $\Delta(f)>0$. Let an extension $\tilde{f}\in\mathcal{P}_\kappa$ of f be given, then

$$\mathcal{H}=\mathcal{L}(f,\tilde{f})\subseteq\mathcal{H}(\tilde{f})\quad\text{and}\quad S\subseteq A(\tilde{f}).$$

The operators $U_x(\tilde{f})$ form a group of unitary operators and satisfy

$$[\tilde{f},U_x\tilde{f}]=[\tilde{f},\tilde{f}_x]=\tilde{f}(x).$$

The relation (4.3) follows from [9]. Moreover, $\operatorname{Ind}_-\mathcal{H}(\tilde{f})=\kappa$ and $A(\tilde{f})$ is f-minimal.

If $\Delta(f)=0$ Proposition 2.7 shows that $\mathcal{L}(f,\tilde{f})=\mathcal{H}(f)$ and $S\subseteq A(\tilde{f})$. Hence, also in this case the assertion follows. Moreover, it follows from [13] that the correspondence given by (4.3) is one-to-one. ∎

In the following we use the results of [8] on the parametrization of generalized resolvents in order to obtain a parametrization of the extensions of f. Proposition 3.8 and Theorem 3.20 show that these results can be applied to the model space $\mathcal{H}$ and the operator S constructed in the previous section.

Definition 4.7. For $\nu, \Delta \in \mathbb{N}_0$, denote by $\mathcal{K}_\nu^\Delta$ the set of all complex valued functions $\tau(z)$, meromorphic in $\mathbb{C} \setminus \mathbb{R}$, which satisfy $\tau(\overline{z}) = \overline{\tau(z)}$ for z in their domain of holomorphy $\varrho(\tau)$, and which are such that the maximal number of negative squares of the quadratic forms ($m \in \mathbb{N}_0$, $z_1, \dots, z_m \in \varrho(\tau)$)

$$Q(\xi_1, \dots, \xi_m; \eta_0, \dots, \eta_{\Delta-1}) = \sum_{i,j=1}^{m} N_\tau(z_i, z_j)\xi_i\overline{\xi_j} + \sum_{k=0}^{\Delta-1}\sum_{i=1}^{m} \Re\left(z_i^k \xi_i \overline{\eta_k}\right)$$

is ν.

Note that $\mathcal{K}_\nu^0 = \mathcal{N}_\nu$. Let us recall that $\mathcal{K}_\nu^\Delta$ contains infinitely many elements if $\nu \geqslant \Delta$ and is empty if $\nu < \Delta$.

Now we obtain from Proposition 4.2 and [8] the following theorem:

Theorem 4.8. *Let $f \in \mathcal{P}_{\kappa_0,a}$ be given and assume that $\Delta(f) < \infty$. The relation*

$$\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}\tilde{f}(t)\,\mathrm{d}t = \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)}, \quad \operatorname{Im} z > h_{\tilde{A}},$$

establishes a bijective correspondence between the extensions $\tilde{f} \in \mathcal{P}_\kappa$, $\kappa \geqslant \kappa_0 + \Delta(f)$, of f and the parameter functions

$$\tau(z) \in \mathcal{K}_{\kappa-\kappa_0}^{\Delta(f)}.$$

If $\Delta(f) > 0$, the unique extension of f in $\mathcal{P}_{\kappa_0}$ corresponds to the parameter function $\tau(z) = \infty$.

The matrix

$$W(z) = \begin{pmatrix} w_{11}(z) & w_{12}(z) \\ w_{21}(z) & w_{22}(z) \end{pmatrix}$$

is a resolvent matrix associated with the model operator S.

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