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ON HOMOGENEOUS CONTRACTIONS

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ABSTRACT. It was proved by B. Bagchi and G. Misra in [1] that if T is a homogeneous contraction such that the restriction $T|\mathcal{D}_T$ of T to the defect space $\mathcal{D}_T$ is of Hilbert-Schmidt class, then T has a constant characteristic function. We show that the assumption on $T|\mathcal{D}_T$ can be relaxed assuming only the compactness of $T|\mathcal{D}_T$. In fact, it turns out that the proof relies solely on the special "decreasing" structure of the spectrum of the absolute value of $T|\mathcal{D}_T$.

KEYWORDS: *Contraction, homogeneous operator, characteristic function, Möbius transformation.*

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Let $\mathcal{M}(\mathbb{D})$ denote the set of all injective, analytic mappings of the open unit disc $\mathbb{D}$ onto itself. It is clear that $\mathcal{M}(\mathbb{D})$ is a group with the composition operation. It is known, furthermore, that any element of $\mathcal{M}(\mathbb{D})$ is of the form $\varphi_{k,a}(z) = k(z-a)/(1-\bar{a}z)$ $(z \in \mathbb{D})$, where $k \in \partial\mathbb{D}$ and $a \in \mathbb{D}$. These mappings are known as the *Möbius transformations* of the unit disc $\mathbb{D}$.

A (bounded, linear) operator T acting on the (complex, separable) Hilbert space $\mathcal{H}$ is called *homogeneous* if $\sigma(T) \subset \mathbb{D}^-$ and $\varphi(T)$ is unitarily equivalent to T, for every $\varphi \in \mathcal{M}(\mathbb{D})$. Homogeneous operators were investigated in several works, see [1], [2], [5], [6] and [7]. In the present paper we shall be concerned with homogeneous contractions.

Given any contraction $T \in \mathcal{L}(\mathcal{H})$, the space $\mathcal{H}$ can be uniquely decomposed into the orthogonal sum $\mathcal{H}_{\mathrm{u}} \oplus \mathcal{H}_{\mathrm{c}}$, reducing for T, such that $T_{\mathrm{u}} = T|\mathcal{H}_{\mathrm{u}}$ is a unitary operator and $T_c = T|\mathcal{H}_c$ is a completely non-unitary (c.n.u.) contraction. Clearly $\varphi(T) = \varphi(T_u) \oplus \varphi(T_c)$, $\varphi \in \mathcal{M}(\mathbb{D})$. Furthermore, for any $\varphi \in \mathcal{M}(\mathbb{D})$ and any contraction $S \in \mathcal{L}(\mathcal{H})$, the Möbius transform $\varphi(S)$ is unitary (c.n.u.) if and only if S is unitary (c.n.u., respectively); see [8], Section I.4.4. Thus *the contraction T is homogeneous if and only if its unitary and c.n.u. parts are both homogeneous.*

It is known that *the only homogeneous unitary operators are the* (*unweighted*) *bilateral shifts* (*of arbitrary multiplicity*). (Applying the functional model of unitary operators, this statement can be derived from the fact that if μ is a positive Borel measure on $\partial\mathbb{D}$ such that $\mu \circ \tau_k$ is equivalent to μ for every $k \in \partial\mathbb{D}$ ($\tau_k(z) = kz$), then μ is equivalent to the Lebesgue measure; see e.g. [4], Proposition V.3.17.)

Let us assume now that $T \in \mathcal{L}(\mathcal{H})$ is a c.n.u. contraction. Let us consider the *characteristic function* Θ_T of T defined by

$$\Theta_T(z) = (-T + zD_{T^*}(I - zT^*)^{-1}D_T)|\mathcal{D}_T \in \mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*}), \quad z \in \mathbb{D},$$

where $D_T = (I - T^*T)^{1/2}, D_{T^*} = (I - TT^*)^{1/2}$ are the defect operators, and $\mathcal{D}_T = (\operatorname{ran} D_T)^-, \mathcal{D}_{T^*} = (\operatorname{ran} D_{T^*})^-$ are the defect spaces of T. It is evident that the mapping $\Theta_T : \mathbb{D} \to \mathcal{L}(\mathcal{D}_T, \mathcal{D}_{T^*})$ is a contraction-valued, analytic function, and $\Theta_T(0) = -T|\mathcal{D}_T$ is a pure contraction, that is $\|Tx\| < \|x\|$ holds for every $0 \neq x \in \mathcal{D}_T$. It is known that the c.n.u. contractions $T_1 \in \mathcal{L}(\mathcal{H}_1)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ are unitarily equivalent if and only if their characteristic functions Θ_{T_1} and Θ_{T_2} *coincide*, that is there exist unitary transformations $Z : \mathcal{D}_{T_1} \to \mathcal{D}_{T_2}$ and $Z_* : \mathcal{D}_{T_2^*} \to \mathcal{D}_{T_1^*}$ such that $\Theta_{T_1}(z) = Z_*\Theta_{T_2}(z)Z$ ($z \in \mathbb{D}$). It can be easily seen that $\Theta_{kT}(z) = k\Theta_T(\bar{k}z)$ ($z \in \mathbb{D}$) holds, for any $k \in \partial\mathbb{D}$. On the other hand, for any $a \in \mathbb{D}$, the characteristic function of the Möbius transform $\varphi_a(T)$ of T (where $\varphi_a = \varphi_{1,a}$) coincides with $\Theta_T \circ \varphi_{-a}$. As an immediate consequence we obtain that *if the characteristic function of the c.n.u. contraction T is constant* (*that is* $\Theta_T(z) = \Theta_T(0)$ *for every* $z \in \mathbb{D}$), *then T is a homogeneous contraction.* (See [8], Section VI.1.3 and [2], Theorem 4.1.)

We recall that a c.n.u. contraction $T \in \mathcal{L}(\mathcal{H})$ of constant characteristic function is a weighted bilateral shift with special operator weights. Namely, the Hilbert space $\mathcal{H}$ splits into the orthogonal sum $\mathcal{H} = M_-(\mathcal{D}_T) \oplus M_+(\mathcal{D}_{T^*})$, where $M_-(\mathcal{D}_T) = \bigvee_{n \geqslant 0} T^{*n}\mathcal{D}_T$, $M_+(\mathcal{D}_{T^*}) = \bigvee_{n \geqslant 0} T^n\mathcal{D}_{T^*}$, and the restrictions $T^*|M_-(\mathcal{D}_T)$ and $T|M_+(\mathcal{D}_{T^*})$ are unilateral shifts with inducing wandering subspaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$, respectively. (See [9], Theorem 1.3 of [1] and Section I.1. of [8].)

It is known that *there are also homogeneous c.n.u. contractions with non-constant characteristic functions*, see [5] and [6]. On the other hand, B. Bagchi and G. Misra have proved in [1], Theorem 2.11 that under the additional condition

$T|\mathcal{D}_T$ being a compact operator of Hilbert-Schmidt class, homogeneity does imply that the characteristic function is constant. Our aim in this paper is to extend the previous statement showing that the condition imposed on the restriction $T|\mathcal{D}_T$ can be relaxed to assuming only that $T|\mathcal{D}_T$ is compact. Actually, it turns out that even this assumption can be essentially weakened.

We shall say that the spectrum $\sigma(Q)$ of a positive operator $Q \in \mathcal{L}(\mathcal{E})$ is of *decreasing type*, if every point $\lambda \in \sigma(Q)$ is isolated from below, that is there exists a positive ε (depending on λ) such that $(\lambda - \varepsilon, \lambda] \cap \sigma(Q) = \{\lambda\}$. Now, we can formulate our main result.

Theorem. *Let $T \in \mathcal{L}(\mathcal{H})$ be a homogeneous c.n.u. contraction. If the spectrum of the absolute value $\big|T|\mathcal{D}_T\big|$ of the restriction $T|\mathcal{D}_T$ is of decreasing type, then the characteristic function of T is constant.*

Corollary. *Let T be a homogeneous c.n.u. contraction. If $T|\mathcal{D}_T$ is compact, then the characteristic function of T is constant.*

The proof of the Theorem relies on the following lemma, claiming the maximum modulus principle for analytic functions taking values in a Hilbert space. Though this statement is certainly well-known, we provide its elementary proof for the sake of reader's convenience.

Lemma. *Let $\mathcal{E}$ be a Hilbert space, and let $\vartheta : \mathbb{D} \to \mathcal{E}$ be an analytic function such that $\|\vartheta(z_0)\| \geqslant \|\vartheta(z)\|$ holds, for every $z \in \mathbb{D}$, with some $z_0 \in \mathbb{D}$. Then ϑ is constant, that is $\vartheta(z) = \vartheta(z_0)$ for any $z \in \mathbb{D}$.*

Proof. Let $e \in \mathcal{E}$ be a unit vector such that $\|\vartheta(z_0)\| = \langle \vartheta(z_0), e\rangle$. Applying the classical maximum modulus principle for the scalar-valued analytic function $f(z) = \langle \vartheta(z), e\rangle$ $(z \in \mathbb{D})$, we obtain that $f(z) = f(z_0)$ for every $z \in \mathbb{D}$. Since $\eta(z) = \vartheta(z) - f(z)e$ is orthogonal to e, we infer that $\|\vartheta(z_0)\|^2 \geqslant \|\vartheta(z)\|^2 = |f(z)|^2 + \|\eta(z)\|^2 = \|\vartheta(z_0)\|^2 + \|\eta(z)\|^2$, whence $\eta(z) = 0$, and so $\vartheta(z) = f(z)e = f(z_0)e = \vartheta(z_0)$ follows. ∎

Proof of the Theorem. We know by the homogeneity of T that, for any $a \in \mathbb{D}$, the function $\Theta_T \circ \varphi_{-a}$ coincides with Θ_T. Hence there exist unitary operators $Z(a) \in \mathcal{L}(\mathcal{D}_T)$ and $Z_*(a) \in \mathcal{L}(\mathcal{D}_{T^*})$ such that $\Theta_T((z+a)/(1+\bar{a}z)) = Z_*(a)\Theta_T(z)Z(a)$, $z \in \mathbb{D}$. Substituting $z = 0$, we obtain that $\Theta_T(a) = Z_*(a)\Theta_T(0)Z(a)$, $a \in \mathbb{D}$. Thus there exist unitary operator-valued functions $Z : \mathbb{D} \to \mathcal{L}(\mathcal{D}_T)$ and $Z_* : \mathbb{D} \to \mathcal{L}(\mathcal{D}_{T^*})$ such that the composition $\Theta_T(z) = Z_*(z)\Theta_T(0)Z(z)$ $(z \in \mathbb{D})$ is an analytic function. We want to show that Θ_T is constant.

We can suppose that $\dim\ker\Theta_T(0) = \dim\ker\Theta_T(0)^*$. Indeed, in the opposite case Θ_T should be replaced by the analytic function $\Theta_T(z) \oplus 0 \in \mathcal{L}(\mathcal{D}_T \oplus \mathcal{F}, \mathcal{D}_{T^*} \oplus \mathcal{F})$ $(z \in \mathbb{D})$, where $\mathcal{F}$ is an infinite dimensional, separable Hilbert space. Our assumption ensures that there exists a unitary transformation $W : \mathcal{D}_T \to \mathcal{D}_{T^*}$ such that $\Theta_T(0) = WQ$ and $Q = |\Theta_T(0)| = \big|T|\mathcal{D}_T\big|$. It is clear that the function $\Theta(z) := W^*\Theta_T(z) = (W^*Z_*(z)W)QZ(z) = V_*(z)QV(z)$ $(z \in \mathbb{D})$ is analytic, where $V : \mathbb{D} \to \mathcal{L}(\mathcal{D}_T)$ and $V_* : \mathbb{D} \to \mathcal{L}(\mathcal{D}_T)$ are unitary operator-valued functions. It is sufficient to show that Θ is constant. Furthermore, considering the equation $V_*(0)^*\Theta(z)V(0)^* = V_*(0)^*V_*(z)QV(z)V(0)^*$ $(z \in \mathbb{D})$ we can assume that $V(0) = V_*(0) = I$.

Since the spectrum $\sigma(Q)$ is of decreasing type, we infer that $\mu := \|Q\|$ is an isolated eigenvalue of Q. We may assume that $\mu > 0$. Let us consider the eigenspace $\mathcal{M} = \ker(Q - \mu I)$. Let $e \in \mathcal{M}$ be any vector, and let us consider the analytic mapping $\vartheta_e : \mathbb{D} \to \mathcal{D}_T, z \mapsto \Theta(z)e$. Since

$$\begin{aligned}\|\vartheta_e(z)\| &= \|V_*(z)QV(z)e\| = \|QV(z)e\| \leqslant \mu\|V(z)e\| = \mu\|e\| \\ &= \|Qe\| = \|V_*(0)QV(0)e\| = \|\Theta(0)e\| = \|\vartheta_e(0)\|\end{aligned}$$

holds for every $z \in \mathbb{D}$, we infer by the Lemma that $\vartheta_e(z) = \vartheta_e(0) = \mu e$ is true, for any $z \in \mathbb{D}$. The equations $\mu\|e\| = \|\vartheta_e(z)\| = \|QV(z)e\|$ and $\|V(z)e\| = \|e\|$ readily yield that $V(z)e \in \mathcal{M}$ $(z \in \mathbb{D}, e \in \mathcal{M})$. Let us suppose that $V(z_0)\mathcal{M} \neq \mathcal{M}$ for some $z_0 \in \mathbb{D}$. Then there exists a vector $0 \neq y \in \mathcal{H} \ominus \mathcal{M}$ such that $V(z_0)y \in \mathcal{M}$. Since

$$\|\vartheta_y(z)\| = \|QV(z)y\| \leqslant \mu\|V(z)y\| = \mu\|y\| = \mu\|V(z_0)y\| = \|QV(z_0)y\| = \|\vartheta_y(z_0)\|$$

for any $z \in \mathbb{D}$, the Lemma implies that the function ϑ_y is constant. We obtain that

$$\mu\|y\| > \|Qy\| = \|\vartheta_y(0)\| = \|\vartheta_y(z_0)\| = \mu\|y\|,$$

what is a contradiction. We conclude therefore that $V(z)\mathcal{M} = \mathcal{M}$ holds, for every $z \in \mathbb{D}$. Given any vector $e \in \mathcal{M}$, in view of the relation $V(z)e \in \mathcal{M}$ we have

$$\mu e = \Theta(z)e = V_*(z)QV(z)e = V_*(z)\mu V(z)e = \mu V_*(z)V(z)e,$$

which means that $V_*(z)\mathcal{M} = \mathcal{M}$ is also valid.

We have obtained that $\mathcal{M}$ is a joint reducing subspace of the family of operators $\{Q\} \cup \{V(z)\}_{z\in\mathbb{D}} \cup \{V_*(z)\}_{z\in\mathbb{D}}$, and that $\Theta(z)|\mathcal{M} = Q|\mathcal{M}$ is true for every $z \in \mathbb{D}$. It follows that the assumptions hold for the restrictions onto the orthogonal complement $\mathcal{H} \ominus \mathcal{M}$ too. Since $\sigma(Q)$ is a well-ordered set (any non-empty subset has a largest element), transfinite induction results in that $\Theta(z) = Q$, for every $z \in \mathbb{D}$. ∎

Remark. In the preceding proof we have used only the fact that $\varphi_a(T)$ is unitarily equivalent to T for every $a \in \mathbb{D}$, for the c.n.u. contraction T; the circular symmetry, that is the unitary equivalence of kT to T for every $k \in \partial\mathbb{D}$, has not been exploited.

On the other hand, if U is a unitary operator then solely circular symmetry already implies that U is a bilateral shift.

The examples exhibited in [5] and [6] for homogeneous contractions T with non-constant characteristic functions are of class C_{00}, that is $\lim\limits_{n\to\infty} \|T^n x\| = \lim\limits_{n\to\infty} \|T^{*n} x\| = 0$ hold for every $x \in \mathcal{H}$. In fact, these are backward weighted shifts with weight sequence $\{w_n = (1+n)^{1/2}(c+n)^{-1/2}\}_{n=0}^{\infty}$, where $c > 1$, belonging to the Cowen-Douglas class $B_1(\mathbb{D})$ introduced in [3].

Question. Does every homogeneous contraction of class C_{11} have a constant characteristic function?

We recall that the contraction T is of class C_{11}, if $\lim\limits_{n\to\infty} \|T^n x\| \neq 0 \neq \lim\limits_{n\to\infty} \|T^{*n} x\|$, for every non-zero vector x. We note that if T is a c.n.u. contraction and Θ_T is constant, then T is the orthogonal sum of a unilateral shift, a backward shift and a C_{11}-contraction.

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REFERENCES

1. B. Bagchi, G. Misra, Constant characteristic functions and homogeneous operators, *J. Operator Theory* **37**(1997), 51–65.
2. D.N. Clark, G. Misra, On homogeneous contractions and unitary representations of $SU(1,1)$, *J. Operator Theory* **30**(1993), 109–122.
3. M.J. Cowen, R.G. Douglas, Complex geometry and operator theory, *Acta Math.* **141**(1978), 187–261.
4. S.A. Gaal, *Linear Analysis and Representation Theory*, Springer Verlag, Berlin–Heidelberg–New York 1973.
5. G. Misra, Curvature and the backward shift operators, *Proc. Amer. Math. Soc.* **91**(1984), 105–107.
6. G. Misra, Curvature and discrete series representation of $SL_2(\mathbf{R})$, *Integral Equations Operator Theory* **9**(1986), 452–459.
7. G. Misra, N.S. Narasimha Sastry, Homogeneous tuples of operators and representations of some classical groups, *J. Operator Theory* **24**(1990), 23–32.

8. B. SZ.-NAGY, C. FOIAŞ, *Harmonic Analysis of Hilbert Space Operators*, North Holland–Akadémiai Kiadó, Amsterdam–Budapest, 1970.
9. R.I. TEODORESCU, Fonctions caractéristiques constantes, *Acta Sci. Math. (Szeged)* **38**(1976), 183–185.

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