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CRISS-CROSS COMMUTIVITY. II

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ABSTRACT. Equality of non zero spectra of reversed products have multivariable analogues for "criss-cross commuting" tuples; some of these multivariable results in turn have single variable consequences.

KEYWORDS: *Criss-cross commutivity, spectrum, exactness, Kato invertibility.*

MSC (2000): 47A10.

We recall ([8], [6]) that two n-tuples of operators $T \in \mathrm{BL}(X,Y)^n$ and $S \in \mathrm{BL}(Y,X)^n$ *criss-cross commute* if

$$T_iS_kT_j = T_jS_kT_i \quad \text{and} \quad S_iT_kS_j = S_jT_kS_i \tag{0.1}$$

for each $i, j, k = 1, 2, \ldots, n$; an immediate consequence is that each of the tuples

$$\begin{aligned} S \cdot T &= (S_1T_1, S_2T_2, \ldots, S_nT_n) \in \mathrm{BL}(X,X)^n \qquad \text{and} \\ T \cdot S &= (T_1S_1, T_2S_2, \ldots, T_nS_n) \in \mathrm{BL}(Y,Y)^n \end{aligned} \tag{0.2}$$

is commutative. Li Shauchan has noticed ([8]) that if S and T criss-cross commute then the tuples $S \cdot T$ and $T \cdot S$ share the same non-zero Taylor spectrum, and we have offered ([6]) some break-down of the argument. In this note we continue these observations, in particular for the inclusions (9.1) of [7] which between them may well make up most of the Taylor spectrum of a general n-tuple. What is amusing is how this multivariable observation feeds back into the single variable situation, enabling us to add a footnote to the rather comprehensive discussion of Barnes ([1]). We also see how what we have called *skew exactness* ([5], [2]) is transmitted to reversed products with criss-cross commutivity.

In single variables we compare the non-zero spectrum of reversed products ST and TS, which means in practise the analysis of the operators $I - ST$ and $I - TS$. For n-tuples $S \cdot T$ and $T \cdot S$ the "non-zero spectrum" consists of all complex n-tuples $\lambda = (\lambda_1, \lambda_1, \ldots, \lambda_n) \neq 0 = (0, 0, \ldots, 0)$: we can without loss of generality always take $\lambda_1 = 1$.

THEOREM 1. *Suppose* (T_1, T_j, T_k) *in* $\mathrm{BL}(X, Y)$ *and* (S_1, S_j, S_k) *in* $\mathrm{BL}(Y, X)$ *criss-cross commute: then there is implication*

$$(I - S_1T_1)^{-1}(0) \cap \bigcap_j (\lambda_j I - S_jT_j)^{-1}(0) \subseteq \sum_k (\lambda_k I - S_kT_k)(X) \tag{1.1}$$

implies

$$(I - T_1S_1)^{-1}(0) \cap \bigcap_j (\lambda_j I - T_jS_j)^{-1}(0) \subseteq \sum_k (\lambda_k I - T_kS_k)(Y), \tag{1.2}$$

and implication

$$\bigcap_j (\lambda_j I - S_jT_j)^{-1}(0) \subseteq (I - S_1T_1)(X) + \sum_k (\lambda_k I - S_kT_k)(X) \tag{1.3}$$

implies

$$\bigcap_j (\lambda_j I - T_jS_j)^{-1}(0) \subseteq (I - T_1S_1)(Y) + \sum_k (\lambda_k I - T_kS_k)(X). \tag{1.4}$$

Proof. If (1.1) holds and if $y \in Y$ is in the left hand side of (1.2) then

$$\begin{aligned} S_1y &\in S_1(I - T_1S_1)^{-1}(0) \cap \bigcap_j S_1(\lambda_j I - T_jS_j)^{-1}(0) \\ &\subseteq (I - S_1T_1)^{-1}(0) \cap \bigcap_j (\lambda_j I - S_jT_j)^{-1}(0), \end{aligned}$$

using at this point the criss-cross commutivity assumption. Now applying (1.1) and the assumption about $y \in Y$,

$$y = T_1S_1y \in T_1 \sum_k (\lambda_k I - S_kT_k)(X) \subseteq \sum_k (\lambda_k I - T_kS_k)T_1X \subseteq \sum_k (\lambda_k I - T_kS_k)Y,$$

using again criss-cross commutivity. Thus (1.2) holds. If instead (1.3) holds and if $y \in Y$ is in the left hand side of (1.3) then

$$\begin{aligned} S_1y \in S_1 \bigcap_j (\lambda_j I - T_jS_j)^{-1}(0) &\subseteq \bigcap_j (\lambda_j I - S_jT_j)^{-1}(0) \\ &\subseteq (I - S_1T_1)X + \sum_k (\lambda_k I - S_kT_k)X, \end{aligned}$$

using again the criss-cross commutivity assumption and the condition (1.3). Thus

$$T_1S_1y \in T_1(I - S_1T_1)X + T_1 \sum_k (\lambda_k I - S_kT_k)X \subseteq (I - T_1S_1)Y + \sum_k (\lambda_k I - T_kS_k)Y,$$

using criss-cross commutivity, and finally

$$y = (I - T_1S_1)y + T_1S_1y. \quad \blacksquare$$

To apply this to the one variable environment we offer a lemma (cf. [1], page 1060):

LEMMA 2. *If $T \in \mathrm{BL}(X,X)$ and $S \in \mathrm{BL}(Y,Y)$ there are polynomials p_m for each $m \in \mathbb{N}$ for which*

$$(I - ST)^m = I - Sp_m(TS)T \in \mathrm{BL}(X,X) \tag{2.1}$$

with

$$Sp_m(TS) = p_m(ST)S \in \mathrm{BL}(Y,X). \tag{2.2}$$

Proof. Inductively

$$p_1(U) = I \quad \text{and} \quad p_{m+1}(U) = I + p_m(U) - Up_m(U). \quad \blacksquare \tag{2.3}$$

Barnes ([1]) shows that if $T \in \mathrm{BL}(X,Y)$ and $S \in \mathrm{BL}(Y,X)$ then $I - ST \in \mathrm{BL}(X,X)$ and $I - TS \in \mathrm{BL}(Y,Y)$ either both or neither have closed range (Theorem 5 of [1]), and either both or neither have generalized inverses (Theorem 4 of [1]). We here extend these observations to what we have called *Kato invertibility* ([3]) and *Kato non-singularity* ([4]), which consist of either the generalized invertibility or the closed range condition together with the *Saphar condition* ([3], [4]): $U \in \mathrm{BL}(X,X)$ is "hyperexact", or has the Saphar condition, iff

$$U^{-1}(0) \subseteq \bigcap_n U^n(X); \tag{2.4}$$

equivalently

$$\bigcup_n U^{-n}(0) \subseteq U(X). \tag{2.5}$$

THEOREM 3. *If $T \in \mathrm{BL}(X,Y)$ and $S \in \mathrm{BL}(Y,X)$ and $m \in \mathbb{N}$ then*

$$(I - ST)^{-1}(0) \subseteq (I - ST)^m X \tag{3.1}$$

if and only if

$$(I - TS)^{-1}(0) \subseteq (I - TS)^m Y. \tag{3.2}$$

Proof. This is easy to see without recourse to criss-cross commutivity; however Lemma 2 shows that we can write, taking $R = p_m(TS)T$,

$$(I - ST)^m = I - SR$$

in such a way that

$$(T_1, T_2) = (T, R) \quad \text{and} \quad (S_1, S_2) = (S, S)$$

criss-cross commute. Indeed it is trivial that

$$S_1 T_j S_2 = S_2 T_j S_1, \quad j = 1, 2,$$

and we notice

$$T_1 S_j T_2 = TSR = TSp_m(TS)T = Tp_m(ST)ST = RST = T_2 S_j T_1, \quad j = 1, 2.$$

Now Theorem 1 applies. $\blacksquare$

We recall that we have described a chain of operators $(S,T) : X \to Y \to Z$ as *skew exact* ([5], Section 10.9 of [2]) if either

$$(ST)^{-1}(0) = T^{-1}(0), \quad \text{equivalently } S^{-1}(0) \cap T(X) = \{0\}, \tag{3.3}$$

or dually

$$(3.4)\qquad (ST)X \supseteq S(Y), \quad \text{equivalently } S^{-1}(0) + T(X) = Y.$$

Stronger "split" versions would be that there is R for which respectively

$$(3.5)\qquad T = RST$$

or

$$(3.6)\qquad S = STR.$$

THEOREM 4. *Suppose* (T_1, T_j, T_k) *in* $\mathrm{BL}(X, Y)$ *and* (S_1, S_j, S_k) *in* $\mathrm{BL}(Y, X)$ *criss-cross commute: then there is implication*

$$(4.1)\qquad (I - S_1T_1)^{-1}(0) \cap \bigcap_j (\lambda_j I - S_jT_j)^{-1}(0) \cap \sum_k (\lambda_k I - S_kT_k)(X) = \{0\}$$

implies

$$(4.2)\qquad (I - T_1S_1)^{-1}(0) \cap \bigcap_j (\lambda_j I - T_jS_j)^{-1}(0) \cap \sum_k (\lambda_k I - T_kS_k)(Y) = \{0\},$$

and implication

$$(4.3)\qquad \bigcap_j (\lambda_j I - S_jT_j)^{-1}(0) + (I - S_1T_1)X + \sum_k (\lambda_k I - S_kT_k)X = X$$

implies

$$(4.4)\qquad \bigcap_j (\lambda_j I - T_jS_j)^{-1}(0) + (I - T_1S_1)X + \sum_k (\lambda_k I - T_kS_k)X = X.$$

Proof. If (4.1) holds and if $y \in Y$ is in the left hand side of (4.2) then, using criss-cross commutivity, $S_1 y \in X$ is in the left hand side of (4.1). Thus by (4.1) $S_1 y = 0$, by assumption $y = T_1S_1 y = 0$, giving (4.2). If (4.3) holds and if $y \in Y$ then with criss-cross commutivity $S_1 y \in X$ is in the left hand side of (4.3): there are $x, z_1, z_k \in X$ for which

$$S_1 y = x + (I - S_1T_1)z_1 + \sum_k (\lambda_k I - S_kT_k)z_k \quad \text{with } \lambda_j x = S_jT_j x.$$

By criss-cross commutivity it follows

$$T_1S_1 y = T_1 x + (I - T_1S_1)T_1 z_1 + \sum_k (\lambda_k I - T_kS_k)T_1 z_k \quad \text{with } \lambda_j T_1 x = T_jS_jT_1 x.$$

Therefore, $T_1S_1 y$, and hence also $y = (I - T_1S_1)y + T_1S_1 y$, is in the left hand side of (4.4). ∎

The criss-cross commutivity cannot be omitted from the assumptions:

EXAMPLE 5. If $X = Y = \ell_2$ and if U and V are the forward and the backward shifts then (3.5) holds and (3.4) fails with $T = I - VU$ and $S = I - UV$, while (3.6) holds and (3.3) fails with $T = I - UV$ and $S = I - VU$.

For the proof notice that $I - VU = 0 \neq I - UV$.

Of course the pairs $(T_1, T_2) = (V, U)$ and $(S_1, S_2) = (U, V)$ do not criss-cross commute. We might also remark on the failure of a sort of dual to Theorem 4:

Example 6. If U and V are the forward and backward shifts on $X = Y = \ell_2$ then

$$(VU)^{-1}(0) \cap (I - VU)(X) = \{0\} \text{ but } (UV)^{-1}(0) \cap (I - UV)(Y) \neq \{0\} \tag{6.1}$$

and

$$(VU)X + (I - VU)^{-1}(0) = X \text{ but } (UV)(Y) + (I - UV)^{-1}(0) \neq Y. \tag{6.2}$$

The proof is clear.

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