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HIGHER ORDER OPERATORS AND GAUSSIAN BOUNDS ON LIE GROUPS OF POLYNOMIAL GROWTH

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ABSTRACT. Let G be a connected Lie group of polynomial growth. We consider m-th order subelliptic differential operators H on G, the semigroups $S_t = \mathrm{e}^{-tH}$ and the corresponding heat kernels K_t. For a large class of H with $m \geqslant 4$ we demonstrate equivalence between the existence of Gaussian bounds on K_t, with "good" large t behaviour, and the existence of "cutoff" functions on G. By results of [14], such cutoff functions exist if and only if G is the local direct product of a compact Lie group and a nilpotent Lie group.

KEYWORDS: *Lie group, heat kernel, higher-order differential operators.*

MSC (2000): 22E30, 35J30.

1. INTRODUCTION

Let $\Delta = -\sum_{i=1}^{d'} A_i^2$ be a sublaplacian on a connected Lie group G of polynomial growth. Here the A_i are right-invariant vector fields corresponding to an algebraic basis $a_1, \ldots, a_{d'}$ of the Lie algebra of G. It is a well-known theorem (see, for example, [20], Section IV.4, or [22], Chapter VIII) that the corresponding heat kernel, and its first derivatives with respect to the A_i, satisfy Gaussian bounds with "good" large time behaviour. It was recently proved in [14] that the second derivatives of the heat kernel have the expected good large time behaviour if, and only if, G is the local direct product of a compact and a nilpotent group. Moreover, it was proved this is equivalent to another analytic condition on G: the existence of a family of cutoff functions of order j for some positive integer $j \geqslant 2$. The latter is defined to be a family $(\eta_R)_{R>0}$ of C^∞ functions on G such that $0 \leqslant \eta_R \leqslant 1$, the support of η_R is contained in $B(R)$, $\eta_R(g) = 1$ if $g \in B(\sigma R)$, and

$$\|A^\alpha \eta_R\|_\infty \leqslant c R^{-j},$$

for some constants $c > 0$, $\sigma \in (0,1)$, all multi-indices α of length j, and all $R > 0$. (If $(\eta_R)_{R>0}$ is of order j, it is automatically of order j' for any $j' < j$; see [14].) Here $B(R)$ is the ball of radius R associated with a canonical distance on G (see below for details). Further, in the case when G is such a local direct product, there is a family $(\eta_R)_{R>0}$ of cutoff functions of order ∞ on G, i.e., $(\eta_R)_{R>0}$ is of order j for every $j \in \mathbb{N}$.

In this paper we obtain analogues of these results for higher-order subelliptic operators on G. Our results are in two directions: in one direction, we prove that for an important class of right-invariant operators, of order 4 or more, the semigroup kernels can satisfy "good" Gaussian bounds only if G has cutoff functions of order 2 or more (and hence G is a local direct product as above.) In the reverse direction, we assume G is a local direct product as above and prove that a certain class of operators with order larger than the "dimension" of G satisfies "good" Gaussian bounds. The latter proof is based on ideas of Davies in [4], but a new feature is the use of cutoff functions in the standard "Davies perturbation" technique. The resulting estimates eliminate the need for any scaling arguments to remove an undesired $e^{\omega t}$ factor from the Gaussian bounds. Such scaling arguments (see for example, Lemma 6 of [2]) are only available if G possesses dilations, i.e., if G is a homogeneous group, and are not available if G is a general nilpotent group. To state our results precisely we introduce more notation.

Generally we adopt the notation of [11], [20], [14] or [13], with small changes. Throughout, G will be a connected Lie group of polynomial growth, and $a_1, \ldots, a_{d'}$ a fixed algebraic basis of the Lie algebra of G. We fix a (bi-invariant) *Haar measure* dg on G. Then G is called a $K \times_l N$ group if it is the local direct product of a connected compact Lie group and a connected nilpotent Lie group. Let $A_i = dL_G(a_i)$, for $i \in \{1, \ldots, d'\}$, be the generators of left translations on the L_p-spaces $L_p(G; dg)$. The set of multi-indices is defined by $J(d') = \bigcup_{j=0}^{\infty} \{1, \ldots, d'\}^j$. If $\alpha = (i_1, \ldots, i_j) \in J(d')$ we say α has length $|\alpha| = j$ and set $A^\alpha = A_{i_1} \cdots A_{i_j}$. (If $j = 0$ then α is the "empty' multi-index and we set $|\alpha|$ to be 0 and A^α to be the identity on L_p.) The reverse multi-index of α is $\alpha_* = (i_j, \ldots, i_1)$. Let $L_{p;j} = \bigcap_{|\alpha|=j} D(A^\alpha)$ be the Sobolev space of j-times differentiable functions in L_p. The seminorm N_j is defined on $L_{2;j}$ by $N_j(\varphi) = \Big(\sum_{|\alpha|=j} \|A^\alpha \varphi\|_2^2 \Big)^{1/2}$. Moreover, $(g,h) \mapsto d(g;h)$ denotes the right-invariant distance associated with the algebraic basis and $g \mapsto |g| = d(g;e)$ the modulus. Then $V(r)$ denotes the Haar measure of the ball $B(r) = \{g \in G : |g| < r\}$. There are integers $D' \geqslant 1$, and $D \geqslant 0$, the local dimension and the dimension at infinity, such that for some $C > 0$,

$$\begin{aligned} C^{-1} r^{D'} \leqslant V(r) \leqslant C r^{D'} \qquad & \text{if } 0 < r \leqslant 1 \\ C^{-1} r^{D} \leqslant V(r) \leqslant C r^{D} \qquad & \text{if } r \geqslant 1. \end{aligned}$$

Set $N = D' \vee D$. In general, c, c', b, b', etc., denote positive constants whose value we allow to change from line to line when convenient.

Throughout, m and n denote positive integers with $m = 2n$. We consider (right-invariant) operators

$$H = \sum_{|\alpha|=m} c_\alpha A^\alpha$$

where $c_\alpha \in \mathbb{C}$, defined on the domain $D(H) = L_{2;m}$ in L_2. If H satisfies the Gårding inequality

$$\mathrm{Re}(H\varphi, \varphi) \geqslant \mu N_n(\varphi)^2 - \lambda \|\varphi\|_2^2 \tag{1.1}$$

for some $\mu > 0$, $\lambda \geqslant 0$ and all $\varphi \in C_c^\infty(G)$, and then by density for all $\varphi \in L_{2;m}$, one can establish local Gaussian bounds on the semigroup kernel K associated with H. Indeed, it follows from [11] that H generates a semigroup $S_t = \mathrm{e}^{-tH}$ in L_2 with a smooth convolution kernel K_t, i.e, $S_t\varphi = K_t * \varphi$ for $\varphi \in L_2$. For each $\alpha \in J(d')$ there exist $c > 0$, $b > 0$, and $\omega \geqslant 0$ such that

$$|(A^\alpha K_t)(g)| \leqslant cV(t)^{-1/m} t^{-|\alpha|/m} \mathrm{e}^{\omega t} \mathrm{e}^{-b(|g|^m/t)^{1/(m-1)}} \tag{1.2}$$

for all $t > 0$ and $g \in G$. We mention also the paper [17] in which similar results were obtained.

If G is a homogeneous group and the vector fields A_i are homogeneous of order 1, then a scaling argument implies that (1.1) is equivalent to the strong Gårding inequality

$$\mathrm{Re}(H\varphi, \varphi) \geqslant \mu N_n(\varphi)^2 \tag{1.3}$$

obtained by setting $\lambda = 0$ in (1.1). Similarly, (1.2) is equivalent to the global Gaussian bounds

$$|(A^\alpha K_t)(g)| \leqslant c\, V(t)^{-1/m} t^{-|\alpha|/m} \mathrm{e}^{-b(|g|^m/t)^{1/(m-1)}} \tag{1.4}$$

obtained by setting $\omega = 0$ in (1.2). We will examine the relationship between the strong Gårding inequality (1.3) and the Gaussian bounds (1.4) for a general polynomial group with no assumption of homogeneity.

If H satisfies (1.3) for $\varphi \in L_{2;m}$ we call H an *m-th order Gårding operator.* In this case the semigroup S extends to a holomorphic semigroup on L_2 with $\|S_z\| \leqslant 1$ for all z in some sector of the complex plane. This may be deduced, for example, from the fact that the associated sesquilinear form h satisfies (1.6) and (1.7) below (see the remarks after (1.7) and the proof of Theorem 1.3 (i) below).

THEOREM 1.1. *Let H be an m-th order Gårding operator with $m \geqslant 4$. If K_t satisfies bounds*

$$|K_t(g)| \leqslant c\, V(t)^{-1/m} \mathrm{e}^{-b(|g|^m/t)^{1/(m-1)}} \tag{1.5}$$

for all $t > 0$, $g \in G$, then G is a $K \times_l N$ group.

The condition $m \geqslant 4$ in this theorem is necessary; indeed $\Delta = -\sum_{i=1}^{d'} A_i^2$ is a second-order Gårding operator which provides a counterexample to the theorem in the $m = 2$ case.

Examples of m-th order Gårding operators include

$$\Delta_m = (-1)^n \sum_{|\alpha|=n} A^{\alpha_*} A^\alpha$$

or, more generally,

$$H = (-1)^n \sum_{|\alpha|=|\beta|=n} c_{\alpha\beta} A^{\beta_*} A^\alpha,$$

where the coefficients $c_{\alpha\beta} \in \mathbb{C}$ satisfy

$$\operatorname{Re} \sum_{|\alpha|=|\beta|=n} c_{\alpha\beta}\xi_\alpha\overline{\xi}_\beta \geqslant \mu \sum_{|\alpha|=n} |\xi_\alpha|^2$$

for all $\xi = (\xi_\alpha)_{|\alpha|=n} \in \mathbb{C}^{(d')^n}$. For such operators the strong Gårding inequality follows by a simple calculation (see [11], Section 1).

Positive Rockland operators are also m-th order Gårding operators. More precisely, let G be a stratified nilpotent group and suppose that the algebraic basis vector fields $A_1, \ldots, A_{d'}$ are homogeneous of degree 1 with respect to the dilation structure of G. If $H = \sum_{|\alpha|=m} c_\alpha A^\alpha$ is a positive Rockland operator on G (homogeneous of degree m) then H is m-th order Gårding. More information on positive Rockland operators, including heat kernel bounds, can be found in [7], [1] and [16].

For another example, let G be nilpotent and $H = \sum_{|\alpha|=m} c_\alpha A^\alpha$ an m-th order operator on G. Without going into details, it is possible to construct a larger "free" nilpotent group $\widetilde{G}$, which is a stratified group, such that H lifts to an operator $\widetilde{H}$ homogeneous of degree m on $\widetilde{G}$. If $\widetilde{H}$ is a positive Rockland operator on $\widetilde{G}$, then H will be an m-th order Gårding operator on G. The strong Gårding inequality for H is a consequence of the fact that all Riesz transforms of H are bounded on $L_2(G)$; see [13], [19]. As special cases, the operators $H = (-1)^n \sum_{i=1}^{d'} A_i^m$ and $H = \Delta^{m/2}$ are m-th order Gårding whenever G is nilpotent.

When G is not a $K \times_l N$ group, however, the class of m-th order Gårding operators no longer includes some important subelliptic operators. For example, when $m \geqslant 4$ we claim that the operator $H = \Delta^{m/2} = \Big(- \sum_{i=1}^{d'} A_i^2 \Big)^{m/2}$ is m-th order Gårding if and only if $G = K \times_l N$. Indeed, H satisfies (1.3) if and only if all of the Riesz transforms $A^\alpha \Delta^{-n/2}$, $|\alpha| = n$, are bounded on L_2, and as shown in [14], when $n \geqslant 2$ this occurs if and only if $G = K \times_l N$. Nevertheless, on any polynomial group G (and with an arbitrary choice of $a_1, \ldots, a_{d'}$), the operator $\Delta^{m/2}$ satisfies m-th order Gaussian bounds (see [21] or [5]).

Our second main result, Theorem 1.2 below, is formulated for operators associated with a sesquilinear form on L_2 satisfying three abstract assumptions inspired by [4].

Let h be a sesquilinear form with domain $L_{2;n}$. We write $h(\varphi)$ for $h(\varphi, \varphi)$. Our first two assumptions are that there are constants $\mu > 0$, $\widetilde{\mu} > 0$, $\nu \geqslant 0$ such that for all $\varphi \in L_{2;n}$,

$$\mu N_n(\varphi)^2 \leqslant \operatorname{Re} h(\varphi) \leqslant \widetilde{\mu}(\|\varphi\|_2^2 + N_n(\varphi)^2) \tag{1.6}$$

$$|\operatorname{Im} h(\varphi)| \leqslant \nu N_n(\varphi)^2. \tag{1.7}$$

It follows that

$$|\operatorname{Im} h(\varphi)| \leqslant \mu^{-1}\nu \operatorname{Re} h(\varphi). \tag{1.8}$$

Thus h is a sectorial form with semiangle $\zeta = \tan^{-1}(\mu^{-1}\nu) \in [0, \pi/2)$, i.e., we have

$$h(\varphi) \in \overline{\Lambda}(\zeta) = \{\, z \in \mathbb{C} : |\arg z| \leqslant \zeta \,\} \cup \{0\}.$$

Moreover, assumption (1.6) then implies that h is a closed form. Let H be the m-sectorial operator associated with the closed sectorial form h, in the sense of [18], Theorem VI.2.1. Then $D(H) \subseteq L_{2;n}$ and

$$(H\varphi, \psi) = h(\varphi, \psi)$$

for all $\varphi \in D(H)$ and $\psi \in L_{2;n}$. The spectrum of H is contained in $\overline{\Lambda}(\zeta)$, and H is the generator of a holomorphic semigroup $S_z = \mathrm{e}^{-zH}$ on L_2 in the open sector $\Lambda(\theta_H) = \{z \in \mathbb{C} - \{0\} : |\arg z| < \theta_H\}$, where $\theta_H = \pi/2 - \zeta$. In addition, $\|S_z\| \leqslant 1$ for all $z \in \Lambda(\theta_H)$ (see [18], p. 280 and Theorem IX.1.24). In the case where h is a symmetric form, i.e., $h(\varphi, \psi) = \overline{h(\psi, \varphi)}$, we can choose $\nu = 0$ and $\zeta = 0$, and H is a nonnegative self-adjoint operator.

Our third assumption, (1.10) below, is an analogue of (3) of [4] or Lemma III.4.5 of [20], and is expressed in terms of a perturbed form defined using cutoff functions. For the definition we suppose G is a $K \times_l N$ group. Fix a family $(\eta_R)_{R>0}$ of cutoff functions of order ∞ on G and define $\psi^l_R = R \cdot R(l)\eta_R$ for $R > 0$ and $l \in G$, where $R(l)$ denotes right translation by l. For each $\alpha \in J(d')$ we have an estimate

$$\|A^\alpha \psi^l_R\|_\infty \leqslant c_\alpha R^{1-|\alpha|} \tag{1.9}$$

because A^α commutes with $R(l)$.

If $\rho \in \mathbb{R}$, we let $\mathrm{e}^{\rho\psi^l_R}$ denote the bounded operator of multiplication by $\mathrm{e}^{\rho\psi^l_R}$ on L_2 and the spaces $L_{2;j}$. Then the perturbed form, operator and semigroup are defined by

$$h_\rho(\varphi) = h(\mathrm{e}^{\rho\psi^l_R}\varphi, \mathrm{e}^{-\rho\psi^l_R}\varphi), \quad H_\rho = \mathrm{e}^{-\rho\psi^l_R} H \mathrm{e}^{\rho\psi^l_R}, \quad S^\rho_z = \mathrm{e}^{-\rho\psi^l_R} S_z \mathrm{e}^{\rho\psi^l_R}$$

for $\varphi \in L_{2;n}$, $\rho \in \mathbb{R}$, $l \in G$, $R > 0$, and $z \in \Lambda(\theta_H) \cup \{0\}$. One finds that $S^\rho_z = \mathrm{e}^{-zH_\rho}$ and $(H_\rho\varphi, \varphi) = h_\rho(\varphi)$ whenever $\varphi \in D(H_\rho)$. Note also that $D(H_\rho) = \mathrm{e}^{-\rho\psi^l_R}(D(H)) \subseteq \mathrm{e}^{-\rho\psi^l_R}(L_{2;n}) = L_{2;n}$.

Our third assumption is that there exist an $\varepsilon \in (0, 1)$ and $C_\varepsilon > 0$ such that

$$|h_\rho(\varphi) - h(\varphi)| \leqslant \varepsilon \,\mathrm{Re}\, h(\varphi) + C_\varepsilon \rho^m \|\varphi\|_2^2 \tag{1.10}$$

for all $\varphi \in L_{2;n}$, $l \in G$, $\rho \in \mathbb{R}^* = \mathbb{R} - \{0\}$, and $R > 0$, subject to the condition $|\rho| \geqslant R^{-1}$. All subsequent estimates involving the perturbed objects are also understood to hold for all $l \in G$, $\rho \in \mathbb{R}^*$, and $R > 0$ subject to the condition $|\rho| \geqslant R^{-1}$, even though for brevity R and l do not appear in our notation.

We remark that assumption (3) of [4] differs crucially from our assumption (1.10) in having $(1+\rho^m)$ in place of ρ^m. The absence of the 1 allows us to avoid an $\mathrm{e}^{\omega t}$ factor which occurs in semigroup estimates in [4]. In [2], Barbatis and Davies obtained an estimate similar to (1.10) when $G = \mathbb{R}^N$ and the form is perturbed by linear functions.

In the following theorem, for a function F on $G \times G$, we use the notations $A^\alpha F$ and $B^\alpha F$ for the A^α derivatives of F with respect to the first and second variables, respectively.

THEOREM 1.2. *Let G be a $K \times_l N$ group and let the form h satisfy the assumptions* (1.6), (1.7), *and* (1.10), *with $m > N$. Then the semigroup generated by the associated operator H has an integral kernel K_t, continuous on $G \times G$ for each $t > 0$, satisfying*

$$|K_t(g\,;h)| \leqslant c\,V(t)^{-1/m} \mathrm{e}^{-b(|gh^{-1}|^m/t)^{1/(m-1)}}$$

for all $t > 0$ and $g, h \in G$.

Moreover, for $\alpha, \beta \in J(d')$ with $|\alpha|, |\beta| < 2^{-1}(m - N)$, the derivatives $A^\alpha B^\beta K_t$ exist and are continuous on $G \times G$, and

$$|A^\alpha B^\beta K_t(g\,;h)| \leqslant c'\,V(t)^{-1/m}\, t^{-(|\alpha|+|\beta|)/m}\, \mathrm{e}^{-b'(|gh^{-1}|^m/t)^{1/(m-1)}}$$

for all $t > 0$ and $g, h \in G$.

The next theorem verifies that Theorem 1.2 applies not only to m-th order Gårding operators but to an important class of operators with variable coefficients in divergence form.

THEOREM 1.3. *Let G be a $K \times_l N$ group. The hypotheses of Theorem* 1.2 *hold if either of the following two conditions is valid:*

(i) *H is an m-th order Gårding operator, with $m > N$;*

(ii) *$H = (-1)^n \sum_{|\alpha|=|\beta|=n} A^{\beta_*} c_{\alpha\beta} A^\alpha$ is an m-th order operator, with $m > N$, associated with the form $h(\varphi, \psi) = \sum_{|\alpha|=|\beta|=n} (c_{\alpha\beta} A^\alpha \varphi, A^\beta \psi)$ where the $c_{\alpha\beta}$ are bounded measurable complex-valued functions on G and*

$$\sum_{|\alpha|=|\beta|=n} \mathrm{Re}\, c_{\alpha\beta}(g) \xi_\alpha \overline{\xi}_\beta \geqslant \mu \sum_{|\alpha|=n} |\xi_\alpha|^2$$

for all $g \in G$ and $\xi \in \mathbb{C}^{(d')^n}$.

Our final result is obtained by combining Theorems 1.1–1.3.

COROLLARY 1.4. *Let H be an m-th order Gårding operator with $m > N$. Then:*

(i) *the convolution kernel satisfies bounds $\|K_t\|_2 \leqslant c\,V(t)^{-1/(2m)}$ and $\|K_t\|_\infty \leqslant c\,V(t)^{-1/m}$ for all $t > 0$;*

(ii) *K_t satisfies* (1.5) *if and only if G is a $K \times_l N$ group.*

Our results indicate that (1.3) and (1.5) are probably not to be expected for many operators on groups which are not of the form $K \times_l N$. Examples ([12]) show that the kernel K_t can have more complicated large t behaviour, for example, it can behave like a Gaussian of order less than m for large t.

2. PROOF OF THEOREM 1.1

To prove Theorem 1.1, by the result of [14] it is sufficient to construct cutoff functions of order $n = m/2$ on G. The method of construction is an extension of the argument on page 14 of [14]. If H is self-adjoint, then K_t is a positive-definite function on G, and using techniques of [10] one can find $\kappa > 0$ such that

$$\operatorname{Re} K_t(g) \geqslant c\, V(t)^{-1/m}$$

for all $g \in G$ and $t > 0$ satisfying $|g| \leqslant \kappa\, t^{1/m}$. For an appropriate $\phi \in C^\infty(\mathbb{R})$, one defines

$$\phi_R(g) = \phi\left(\frac{\operatorname{Re} K_{R^m}(g)}{K_{R^m}(e)}\right)$$

and one can argue that for an appropriate $\rho > 0$, $(\phi_{\rho R})_{R>0}$ is a family of cutoff functions of order n. Since the argument for H self-adjoint is contained in the argument for general H, we omit further details, and turn to the general proof. The proof is in 3 steps.

Step 1. Since H is the generator of a bounded holomorphic semigroup on L_2, one has an estimate $\|HS_t\|_{2\to 2} \leqslant c_1 t^{-1}$ for all $t > 0$. Also, a standard quadrature argument using the Gaussian bounds (1.5) gives $\|S_t\|_{2\to\infty} = \|K_t\|_2 \leqslant c_2 V(t)^{-1/(2m)}$. Therefore

$$\|HK_t\|_2 = \|HS_t\|_{2\to\infty} \leqslant \|S_{t/2}\|_{2\to\infty} \|HS_{t/2}\|_{2\to 2} \leqslant c\, t^{-1} V(t)^{-1/(2m)}.$$

Now by (1.3), whenever $|\alpha| = n$,

$$\|A^\alpha K_t\|_2^2 \leqslant \mu^{-1} \operatorname{Re}(HK_t, K_t) \leqslant \mu^{-1} \|HK_t\|_2 \|K_t\|_2 \leqslant c\, t^{-1} V(t)^{-1/m}.$$

The interpolating inequality

$$\|A^\alpha \varphi\|_2 \leqslant c_j (\|\varphi\|_2)^{1-|\alpha|/j} \, (N_j(\varphi))^{|\alpha|/j} \tag{2.1}$$

holds for all $\varphi \in L_{2;j}$ and $\alpha \in J(d')$ with $0 \leqslant |\alpha| \leqslant j$, where $j \in \mathbb{N}$ (see [14], equation (25), or [20], Lemma III.3.3). Applying this with $j = n$ we obtain

$$\|A^\alpha K_t\|_2 \leqslant c\, t^{-|\alpha|/m} V(t)^{-1/(2m)}$$

whenever $0 \leqslant |\alpha| \leqslant n$.

Define $L_t = K_{t/2} * K^*_{t/2}$ where $K^*_t(g) = \overline{K_t(g^{-1})}$ is the kernel of the adjoint semigroup S^*_t. (If H is self-adjoint, $L_t = K_t$.) Since $A^\alpha L_t = (A^\alpha K_{t/2}) * K^*_{t/2}$ and $\|K^*_{t/2}\|_2 = \|K_{t/2}\|_2$,

$$\|A^\alpha L_t\|_\infty \leqslant \|A^\alpha K_{t/2}\|_2 \|K^*_{t/2}\|_2 \leqslant c\, t^{-|\alpha|/m} V(t)^{-1/m}$$

for $0 \leqslant |\alpha| \leqslant n$. Finally, since L_t is a convolution of two Gaussian bounded kernels we can easily obtain Gaussian bounds

$$|L_t(g)| \leqslant c\, V(t)^{-1/m} e^{-b'(|g|^m/t)^{1/(m-1)}}$$

for all $t > 0$, $g \in G$ (see for example [8], Lemma 2.2).

STEP 2. In this step we prove lower bounds

$$\operatorname{Re} L_t(g) \geqslant c\, V(t)^{-1/m} \tag{2.2}$$

valid for all $t > 0$ and $g \in G$ such that $|g| \leqslant \kappa t^{1/m}$, for some constant $\kappa > 0$. The technique is that of [10].

First, it follows straightforwardly from the definition of L_t that it is a positive-definite function on G, i.e.,

$$\int_G \int_G \mathrm{d}g\, \mathrm{d}h L_t(gh^{-1}) \phi(g) \overline{\phi(h)} \geqslant 0$$

for all $\phi \in C_{\mathrm{c}}(G)$. As a consequence,

$$L_t(e) \geqslant 2\rho \operatorname{Re} L_t(h) - \rho^2 L_t(e) \tag{2.3}$$

for all $t > 0$, $h \in G$ and $\rho \in \mathbb{R}$ (see [10] and Chapter 3 of [15]).

LEMMA 2.1. *There exists $r > 0$ such that*

$$\int_{B(rt^{1/m})} \mathrm{d}h \operatorname{Re} L_t(h) \geqslant 1/2 \tag{2.4}$$

for all $t > 0$.

Proof. Since H and its adjoint H^* are pure m-th order operators, $\int K_t = \int K_t^* = 1$ for all $t > 0$ (see for example [20], p. 216). Then an easy calculation shows that $\int L_t = 1$. Also, it is standard that the Gaussian bounds on L_t imply an estimate

$$\Big| \int_{|h| \geqslant rt^{1/m}} \mathrm{d}h \operatorname{Re} L_t(h) \Big| \leqslant \int_{|h| \geqslant rt^{1/m}} \mathrm{d}h |L_t(h)| \leqslant c\, \mathrm{e}^{-b' r^{m/(m-1)}}$$

for all $r > 0$ and $t > 0$. Therefore, by writing

$$\int_{B(rt^{1/m})} \mathrm{d}h \operatorname{Re} L_t(h) = \operatorname{Re} \int_G \mathrm{d}h L_t - \int_{|h| \geqslant rt^{1/m}} \mathrm{d}h \operatorname{Re} L_t(h),$$

one deduces (2.4) for all sufficiently large r. ∎

Fix $r > 0$ such that (2.4) holds. Integrating (2.3) over $B(rt^{1/m})$ and dividing by $V(rt^{1/m})$ gives

$$L_t(e) \geqslant 2\rho\, V(rt^{1/m})^{-1} \int_{B(rt^{1/m})} \operatorname{Re} L_t - \rho^2 L_t(e) \geqslant \rho\, c_r^{-1} V(t)^{-1/m} - a\rho^2 V(t)^{-1/m}$$

for all $\rho > 0$, $t > 0$, where we have used (2.4), an estimate $V(rt^{1/m}) \leqslant c_r V(t)^{1/m}$ and the upper bound $L_t(e) \leqslant aV(t)^{-1/m}$. Then maximizing over ρ yields the lower bound

$$L_t(e) \geqslant c' V(t)^{-1/m}$$

for all $t > 0$. As a consequence of the bounds on $\|A_i L_t\|_\infty$ from Step 1, $|L_t(g) - L_t(e)| \leqslant c'' V(t)^{-1/m} |g| t^{-1/m}$ for all $t > 0$, $g \in G$. Now (2.2) follows easily through the inequality $\operatorname{Re} L_t(g) \geqslant L_t(e) - |L_t(g) - L_t(e)|$.

STEP 3. Now we complete the proof of Theorem 1.1.

From Steps 1 and 2, there exist constants $c_1, b > 0$ so that

$$\frac{\operatorname{Re} L_{R^m}(g)}{L_{R^m}(e)} \leqslant c_1 \mathrm{e}^{-b(|g|/R)^{m/(m-1)}} \tag{2.5}$$

for all $g \in G$ and $R > 0$. Also there exists a $c_2 > 0$ such that

$$\frac{\operatorname{Re} L_{R^m}(g)}{L_{R^m}(e)} \geqslant c_2 \tag{2.6}$$

whenever $g \in G$ and $R > 0$ with $|g| \leqslant \kappa R$, where κ is as in Step 2. Let $\varphi \in C^\infty(\mathbb{R})$ with $0 \leqslant \varphi \leqslant 1$ such that $\varphi(x) = 1$ for all $x \geqslant c_2$ and $\varphi(x) = 0$ for all $x \leqslant (1/2)c_2$. For $R > 0$, define $\varphi_R \in C^\infty(G)$ by

$$\varphi_R(g) = \varphi\left(\frac{\operatorname{Re} L_{R^m}(g)}{L_{R^m}(e)}\right).$$

Then $0 \leqslant \varphi_R \leqslant 1$, and by (2.6), $\varphi_R(g) = 1$ if $|g| \leqslant \kappa R$.

Next choose $\tau > 0$ large enough so that $\tau > \kappa$ and $c_1 \mathrm{e}^{-b\tau^{m/(m-1)}} < (1/2)c_2$. If $\tau' \in (\kappa, \tau)$ is sufficiently close to τ we have $c_1 \mathrm{e}^{-b\tau'^{m/(m-1)}} < (1/2)c_2$ and hence by (2.5), $\varphi_R(g) = 0$ whenever $|g| \geqslant \tau' R$. Therefore the support of φ_R is contained in $B(\tau R)$.

When $|\alpha| = n$, a straightforward calculation gives

$$(A^\alpha \varphi_R)(g) = \sum \varphi^{(l)}\left(\frac{\operatorname{Re} L_{R^m}(g)}{L_{R^m}(e)}\right) \prod_{p=1}^{l} \frac{(A^{\beta_p}(\operatorname{Re} L_{R^m}))(g)}{L_{R^m}(e)}$$

where the sum is over a subset of all $l \in \{1, \ldots, n\}$ and $\beta_1, \ldots, \beta_l$ in $J(d')$ with $|\beta_p| \geqslant 1$ for all p and $|\beta_1| + \cdots + |\beta_l| = n$. Combining the equality $A^{\beta_p}(\operatorname{Re} L_{R^m}) = \operatorname{Re}(A^{\beta_p} L_{R^m})$, the bounds

$$\|\operatorname{Re}(A^{\beta_p} L_{R^m})\|_\infty \leqslant \|A^{\beta_p} L_{R^m}\|_\infty \leqslant c\, R^{-|\beta_p|}\, V(R^m)^{-1/m}$$

from Step 1, together with the lower bound $L_{R^m}(e) \geqslant c' V(R^m)^{-1/m}$, we obtain an estimate $\|A^\alpha \varphi_R\|_\infty \leqslant c R^{-n}$ for all $R > 0$, whenever $|\alpha| = n$. Finally, define $\eta_R = \varphi_{\tau^{-1}R}$. It follows easily from the properties of the φ_R that $(\eta_R)_{R>0}$ is a family of cutoff functions of order n. ∎

REMARK 2.2. By modifying the above proof it is possible to prove Theorem 1.1 under the assumption of pointwise bounds on K_t which have a much slower decay on G than Gaussian bounds. To be specific, it is enough to assume *Poisson bounds* as defined in [6]:

$$|K_t(g)| \leqslant V(t)^{-1/m} P(|g|^m/t)$$

where $P : [0, \infty) \to (0, \infty)$ is a continuous, bounded and decreasing function which satisfies

$$\lim_{r \to \infty} r^{N+\delta} P(r^m) = 0$$

for some $\delta > 0$. The modified proof requires integral estimates for Poisson bounds found in the statement and proof of Proposition 2.1 of [6].

3. PROOF OF THEOREM 1.2

Our proof is similar in structure to proofs in [4] and [2]. We concentrate on proving the bounds on K_t, and sketch in the final remarks of this section how the proof can be extended to obtain bounds on the derivatives $A^\alpha B^\beta K_t$. In Lemma 3.2 below, using the Sobolev embedding of Lemma 3.1, we derive uniform bounds on the kernel. We remark that these two lemmas hold for any polynomial group G, since the proofs do not use the existence of cutoff functions. However in the subsequent derivation of Gaussian bounds, the requirement that G be a $K \times_l N$ group, and assumption (1.10), are crucial.

Lemma 3.1 is a generalization to polynomial groups of a standard Sobolev embedding theorem for $\mathbb{R}^N$. In fact, when $G = \mathbb{R}^N$ the lemma is equivalent to Lemma 16 of [4]. On the other hand, on a general unimodular Lie group there is a local version of the lemma which holds whenever $m > D'$ but with the restriction that $\lambda \in (0,1]$ (see [20], Theorem IV.5.8 and its proof, or one can use a Laplace transform argument and the bounds (1.2)). When G is polynomial, to prove the lemma we will use the fact ([20], Section IV.4, or [22], Chapter VIII) that the heat kernel p_t of $\Delta = -\sum_{i=1}^{d'} A_i^2$ satisfies Gaussian bounds

$$|p_t(g)| \leqslant c\, V(t)^{-1/2} \mathrm{e}^{-b(|g|^2/t)}$$

for all $t > 0$ and $g \in G$.

Lemma 3.1. *If m, n are positive integers such that $m = 2n > N$, there exists $c_m > 0$ such that*

$$\|\varphi\|_\infty \leqslant c_m V(\lambda)^{-1/m} \left(\|\varphi\|_2 + \lambda N_n(\varphi)\right)$$

for all $\lambda > 0$ and $\varphi \in L_{2;n}$.

Proof. The bounds on p_t imply bounds $\|\mathrm{e}^{-t\Delta}\|_{2\to\infty} \leqslant c\, V(t)^{-1/4}$. Using a volume inequality $V(t\lambda^{2/n})^{-1/4} \leqslant c\,(1 + t^{-N/4}) V(\lambda)^{-1/m}$, valid for all $\lambda > 0$, $t > 0$, and the Laplace transformation,

$$\begin{aligned}\|(1+\lambda^{2/n}\Delta)^{-n/2}\|_{2\to\infty} &\leqslant \Gamma(n/2)^{-1} \int_0^\infty \mathrm{d}t \mathrm{e}^{-t}\, t^{-1} t^{n/2}\, \|\mathrm{e}^{-t\lambda^{2/n}\Delta}\|_{2\to\infty} \\ &\leqslant c\, V(\lambda)^{-1/m} \Big(\int_0^\infty \mathrm{d}t \mathrm{e}^{-t}\, t^{-1} t^{n/2}\, (1 + t^{-N/4}) \Big)\end{aligned}$$

where the last integral converges because $n > N/2$. Then for $\varphi \in C_c^\infty(G)$, using spectral theory

$$\begin{aligned}\|\varphi\|_\infty &\leqslant c\, V(\lambda)^{-1/m} \|(1+\lambda^{2/n}\Delta)^{n/2}\varphi\|_2 \leqslant c'\, V(\lambda)^{-1/m} \|(1+\lambda\Delta^{n/2})\varphi\|_2 \\ &\leqslant c'\, V(\lambda)^{-1/m} \big(\|\varphi\|_2 + \lambda(\Delta^n\varphi,\varphi)^{1/2}\big).\end{aligned}$$

But Δ^n is a pure m-th order operator, i.e., it is of the form $\sum_{|\alpha|=m} b_\alpha A^\alpha$. Since for $|\alpha| = m$ we can write $(A^\alpha\varphi,\varphi) = (-1)^{|\beta|}(A^\gamma\varphi, A^{\beta_*}\varphi)$ where $\alpha = \beta\gamma$ and $|\beta| = |\gamma| = n$, it follows that $|(\Delta^n\varphi,\varphi)| \leqslant c\, N_n(\varphi)^2$. ∎

LEMMA 3.2. *For each $t > 0$ the operator $S_t = \mathrm{e}^{-tH}$ has an integral kernel $K_t \in L_\infty(G \times G)$ and*

$$|K_t(g\,;h)| \leqslant c\,V(t)^{-1/m}$$

for all $t > 0$ and $g, h \in G$.

Proof. Let $\psi \in L_2$. For any $t > 0$, $S_t\psi \in D(H) \subseteq L_{2;n}$, so we have the Sobolev inequality

$$\|S_t\psi\|_\infty \leqslant c\,V(t^{1/2})^{-1/m}\big(\|S_t\psi\|_2 + t^{1/2}N_n(S_t\psi)\big).$$

But $\|S_t\psi\|_2 \leqslant \|\psi\|_2$ and

$$N_n(S_t\psi)^2 \leqslant \mu^{-1}\mathrm{Re}\,h(S_t\psi) \leqslant \mu^{-1}\|HS_t\psi\|_2\|S_t\psi\|_2 \leqslant c\,t^{-1}\|\psi\|_2^2.$$

Hence $\|S_t\|_{2\to\infty} \leqslant c\,V(t)^{-1/(2m)}$. Next, the adjoint H^* of H is the operator associated with the form h^*, where $h^*(\varphi,\psi) = \overline{h(\psi,\varphi)}$ (see [18], Theorem VI.2.5). Since h^* clearly satisfies (1.6), (1.7) (and (1.10)) whenever h does, we obtain

$$\|S_t\|_{1\to 2} = \|S_t^*\|_{2\to\infty} \leqslant c\,V(t)^{-1/(2m)}$$

and hence $\|S_t\|_{1\to\infty} \leqslant c\,V(t)^{-1/m}$. The statement of the lemma follows. ∎

LEMMA 3.3. (i) *For ε, C_ε as in (1.10), there is $\theta_\varepsilon \in (0,\theta_H)$ such that*

$$\|S^\rho_{r\mathrm{e}^{\mathrm{i}\theta}}\|_{2\to 2} \leqslant \mathrm{e}^{C_\varepsilon\rho^m r}$$

for all $r > 0$, $\theta \in [-\theta_\varepsilon,\theta_\varepsilon]$, $\rho \in \mathbb{R}^*$. In particular, there is $k > 0$ such that for all $t > 0$, $\rho \in \mathbb{R}^*$,

$$\|S^\rho_t\|_{2\to 2} \leqslant \mathrm{e}^{k\rho^m t}.$$

(ii) *There is $k' > 0$ such that*

$$\|H_\rho S^\rho_t\|_{2\to 2} \leqslant c\,t^{-1}\mathrm{e}^{k'\rho^m t}$$

for all $t > 0$, $\rho \in \mathbb{R}^*$.

Proof. It follows from (1.8) that there is $\theta_\varepsilon \in (0,\theta_H)$ such that whenever $\theta \in [-\theta_\varepsilon,\theta_\varepsilon]$, $\varphi \in L_{2;n}$,

$$\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}h(\varphi)) = \cos\theta\mathrm{Re}\,h(\varphi) - \sin\theta\mathrm{Im}\,h(\varphi) \geqslant \varepsilon\,\mathrm{Re}\,h(\varphi). \tag{3.1}$$

Given $\psi \in L_2$, $\theta \in [-\theta_\varepsilon,\theta_\varepsilon]$, define $\psi_r = S^\rho_{r\mathrm{e}^{\mathrm{i}\theta}}\psi$ for $r > 0$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\|\psi_r\|_2^2 &= -\mathrm{e}^{\mathrm{i}\theta}(H_\rho\psi_r,\psi_r) - \mathrm{e}^{-\mathrm{i}\theta}(\psi_r,H_\rho\psi_r) = -2\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}h_\rho(\psi_r))\\ &= -2\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}h(\psi_r)) + 2\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}(h(\psi_r) - h_\rho(\psi_r))) \leqslant 2\,C_\varepsilon\,\rho^m\|\psi_r\|_2^2\end{aligned}$$

where in the last inequality we used (3.1) and (1.10). Solving the differential inequality yields $\|\psi_r\|_2 \leqslant \mathrm{e}^{C_\varepsilon\rho^m r}\|\psi\|_2$, and statement (i) follows.

Statement (ii) follows from statement (i) and the Cauchy integral formula as in the proof of Lemma 2.38 of [3]. ∎

Lemma 3.4. *There is $k'' > 0$ such that whenever $|\alpha| = n$,*

$$\|A^\alpha S_t^\rho\|_{2\to 2} \leqslant c\, t^{-1/2} \mathrm{e}^{k''\rho^m t}$$

for all $t > 0$ and $\rho \in \mathbb{R}^*$.

Proof. Since $\varepsilon \in (0,1)$, equation (1.10) implies an estimate

$$\operatorname{Re} h(\varphi) \leqslant c \operatorname{Re} h_\rho(\varphi) + c\,\rho^m \|\varphi\|_2^2 \tag{3.2}$$

for all $\varphi \in L_{2;n}$. For any $\psi \in L_2$, $t > 0$, $S_t^\rho \psi \in D(H_\rho) \subseteq L_{2;n}$. Applying (1.6), (3.2) and Lemma 3.3, one finds

$$\begin{aligned} \|A^\alpha S_t^\rho \psi\|_2^2 &\leqslant \mu^{-1} \operatorname{Re} h(S_t^\rho \psi) \leqslant c' \operatorname{Re} h_\rho(S_t^\rho \psi) + c' \rho^m \|S_t^\rho \psi\|_2^2 \\ &\leqslant c' \|H_\rho S_t^\rho \psi\|_2 \|S_t^\rho \psi\|_2 + c' \rho^m \|S_t^\rho \psi\|_2^2 \\ &\leqslant \big(c\, t^{-1} \mathrm{e}^{k'\rho^m t} \mathrm{e}^{k\rho^m t} + c\rho^m \mathrm{e}^{2k\rho^m t}\big) \|\psi\|_2^2. \end{aligned}$$

The statement of the lemma follows immediately. ∎

Now we complete the proof of the Gaussian bounds on K_t. For $\psi \in L_2$, applying the Sobolev inequality and Lemmas 3.3 and 3.4 gives

$$\|S_t^\rho \psi\|_\infty \leqslant c\, V(t^{1/2})^{-1/m} \big(\|S_t^\rho \psi\|_2 + t^{1/2} N_n(S_t^\rho \psi)\big) \leqslant c\, V(t)^{-1/(2m)} \mathrm{e}^{k\rho^m t} \|\psi\|_2$$

for some $k > 0$. Thus $\|S_t^\rho\|_{2\to\infty} \leqslant c\, V(t)^{-1/(2m)} \mathrm{e}^{k\rho^m t}$. Arguing by duality as in the proof of Lemma 3.2, we find that there is a $k > 0$ such that $\|S_t^\rho\|_{1\to\infty} \leqslant c\, V(t)^{-1/m} \mathrm{e}^{k\rho^m t}$. Since S_t^ρ has the kernel

$$K_t^\rho(g\,; h) = \mathrm{e}^{-\rho \psi_R^l(g)} K_t(g\,; h) \mathrm{e}^{\rho \psi_R^l(h)},$$

we obtain bounds

$$|K_t(g\,; h)| \leqslant c\, V(t)^{-1/m} \mathrm{e}^{k\rho^m t - \rho(\psi_R^l(h) - \psi_R^l(g))}$$

uniformly for all $t > 0$, $g, h, l \in G$, $\rho \in \mathbb{R}^*$, and $R > 0$ such that $|\rho| \geqslant R^{-1}$. Setting $l = h^{-1}$ and $R = |gh^{-1}|$ and noting that $\psi_R^{h^{-1}}(h) = |gh^{-1}|$, $\psi_R^{h^{-1}}(g) = 0$ yields

$$|K_t(g\,; h)| \leqslant c\, V(t)^{-1/m} \mathrm{e}^{k\rho^m t - \rho|gh^{-1}|}$$

whenever $\rho > 0$ and g, h are such that $|gh^{-1}| \geqslant \rho^{-1}$. Now the function $0 < \rho \mapsto k\rho^m t - \rho|gh^{-1}|$ has the minimum $-b(|gh^{-1}|^m/t)^{1/(m-1)}$, where $b > 0$ depends only on k and m, and this minimum is attained when

$$\rho = \rho_0 = (km)^{-1/(m-1)} (|gh^{-1}|/t)^{1/(m-1)}.$$

Thus we have the Gaussian bounds of Theorem 1.2 under the condition that $|gh^{-1}| \geqslant \rho_0^{-1}$, or equivalently, $|gh^{-1}| \geqslant (km)^{1/m} t^{1/m}$. But in the sector consisting of those g, h and $t > 0$ for which $|gh^{-1}| \leqslant (km)^{1/m} t^{1/m}$, the Gaussian bounds are equivalent to the bounds of Lemma 3.2. Thus the desired bounds are proved. ∎

Gaussian bounds and continuity for the kernels $A^\alpha B^\beta K_t$, where $|\alpha|,|\beta| < 2^{-1}(m-N)$, are obtained by combining the ideas of the above proof with standard techniques for dealing with derivatives and Hölder derivatives of kernels, found for example in [9]. We only sketch the proofs.

The proof of the bounds on $A^\alpha B^\beta K_t$ is based on the Sobolev inequalities

$$\|A^\alpha\varphi\|_\infty \leqslant c_{m,\alpha} V(\lambda)^{-1/m}\lambda^{-|\alpha|/n}\left(\|\varphi\|_2 + \lambda N_n(\varphi)\right) \tag{3.3}$$

valid for positive integers m,n with $m = 2n$, $\alpha \in J(d')$, $\lambda > 0$, and $\varphi \in L_{2;n}$ whenever $|\alpha| < 2^{-1}(m-N)$. (One can derive (3.3) by substituting $A^\alpha\varphi$ for φ and $n-|\alpha|$ for n in Lemma 3.1, and applying (2.1).) Reasoning as in the proof of Lemma 3.2, but applying (3.3) in place of Lemma 3.1, one obtains bounds

$$\|A^\alpha S_t A^{\beta *}\|_{1\to\infty} \leqslant c\,V(t)^{-1/m}t^{-(|\alpha|+|\beta|)/m}$$

when $|\alpha|$, $|\beta| < 2^{-1}(m-N)$. This yields uniform bounds on the mixed derivatives $\|A^\alpha B^\beta K_t\|_{L_\infty(G\times G)} \leqslant cV(t)^{-1/m}t^{-(|\alpha|+|\beta|)/m}$. To obtain Gaussian bounds, one applies (3.3) and Lemmas 3.3 and 3.4 to obtain

$$\|A^\alpha S^\rho_t\|_{2\to\infty} \leqslant c\,V(t)^{-1/(2m)}t^{-|\alpha|/m}\mathrm{e}^{k\rho^m t},$$

where $|\alpha| < 2^{-1}(m-N)$. This leads, via the identity (3.6) below, to bounds

$$\|\mathrm{e}^{-\rho\psi^l_R}A^\alpha S_t \mathrm{e}^{\rho\psi^l_R}\|_{2\to\infty} \leqslant c\,V(t)^{-1/(2m)}t^{-|\alpha|/m}\mathrm{e}^{k\rho^m t}$$

and then to

$$\|\mathrm{e}^{-\rho\psi^l_R}A^\alpha S_t A^{\beta *}\mathrm{e}^{\rho\psi^l_R}\|_{1\to\infty} \leqslant c\,V(t)^{-1/m}t^{-(|\alpha|+|\beta|)/m}\mathrm{e}^{k\rho^m t}.$$

These bounds yield Gaussian bounds on $A^\alpha B^\beta K_t$ outside a sector.

Finally, the continuity, in fact the Hölder continuity, of the kernels $A^\alpha B^\beta K_t$ on $G\times G$ is a consequence of bounds

$$\|(I-\widetilde{L}(l,s))A^\alpha B^\beta K_t\|_\infty \leqslant c\left(|l|^\sigma + |s|^\sigma\right)t^{-(|\alpha|+|\beta|)/m}\,V(t)^{-1/m} \tag{3.4}$$

for all $t > 0$, $l,s \in G$, where $\widetilde{L}$ denotes left translation on $G\times G$, and $\sigma \in (0,1)$ satisfies $|\alpha|+\sigma < 2^{-1}(m-N)$, $|\beta|+\sigma < 2^{-1}(m-N)$. The derivation of (3.4) is again similar to the proof of Lemma 3.2, but one now begins with the Sobolev inequality

$$\sup_{0\neq l\in G}|l|^{-\sigma}\|(I-L(l))A^\alpha\varphi\|_\infty \leqslant c_{m,\alpha,\sigma}V(\lambda)^{-1/m}\lambda^{-(|\alpha|+\sigma)/n}\left(\|\varphi\|_2 + \lambda N_n(\varphi)\right) \tag{3.5}$$

for $\lambda > 0$, $\varphi \in L_{2;n}$, where $|\alpha|+\sigma < 2^{-1}(m-N)$. One can obtain (3.5) in the case $|\alpha| = 0$ by a Laplace transform argument based on the bounds $\sup_{0\neq l\in G}|l|^{-\sigma}\|(I-L(l))\mathrm{e}^{-t\Delta}\|_{2\to\infty} \leqslant c_\sigma\,V(t)^{-1/4}\,t^{-\sigma/2}$, and the case of general α follows by substituting $A^\alpha\varphi$ for φ. We omit further details of the proof of (3.4) and refer to [9] for a similar proof.

4. PROOF OF THEOREM 1.3 AND COROLLARY 1.4

We first prove part (ii) of Theorem 1.3. We need to show that the form

$$h(\varphi,\psi)=\sum_{|\alpha|=|\beta|=n}(c_{\alpha\beta}A^\alpha\varphi,A^\beta\psi),$$

with $\varphi,\psi\in L_{2;n}$, satisfies (1.6), (1.7) and (1.10). The first inequality of (1.6) follows from the condition on the $c_{\alpha\beta}$. For (1.7), we note that

$$\operatorname{Im}h(\varphi)=\sum_{|\alpha|=|\beta|=n}(I_{\alpha\beta}A^\alpha\varphi,A^\beta\varphi)$$

where $I_{\alpha\beta}=(1/(2\mathrm{i}))(c_{\alpha\beta}-\overline{c}_{\beta\alpha})$ so that

$$|\operatorname{Im}h(\varphi)|\leqslant\sum_{|\alpha|=|\beta|=n}\|I_{\alpha\beta}\|_\infty\|A^\alpha\varphi\|_2\|A^\beta\varphi\|_2\leqslant\Big(\sum_{|\alpha|=|\beta|=n}\|I_{\alpha\beta}\|_\infty^2\Big)^{1/2}N_n(\varphi)^2.$$

A similar estimate holds for the second inequality of (1.6) with $R_{\alpha\beta}=(1/2)(c_{\alpha\beta}+\overline{c}_{\beta\alpha})$ replacing $I_{\alpha\beta}$. To complete the proof of (ii) we will prove:

PROPOSITION 4.1. *There exists a $K>0$ such that assumption* (1.10) *holds for any $\varepsilon\in(0,1]$, with $C_\varepsilon=K\varepsilon^{-(m-1)}$.*

Proof. The relation

$$\mathrm{e}^{-\rho\psi}A_i\mathrm{e}^{\rho\psi}\varphi=A_i\varphi+\rho(A_i\psi)\varphi$$

is straightforward to establish. It may be iterated to show that there exist integer constants $c_{k,\gamma_1,\ldots,\gamma_k,\delta}$ such that

$$\mathrm{e}^{-\rho\psi}A^\alpha\mathrm{e}^{\rho\psi}\varphi=A^\alpha\varphi+\sum c_{k,\gamma_1,\ldots,\gamma_k,\delta}\,\rho^k\,(A^{\gamma_1}\psi)\cdots(A^{\gamma_k}\psi)(A^\delta\varphi)\tag{3.6}$$

for all $\alpha\in J(d')$, $\varphi,\psi\in C^\infty(G)$, and $\rho\in\mathbb{R}$. The sum is over $k\in\mathbb{N}$ and multi-indices $\gamma_1,\ldots,\gamma_k,\delta$ satisfying $|\gamma_j|\geqslant1$ for all $j\in\{1,\ldots,k\}$ and $|\gamma_1|+\cdots+|\gamma_k|+|\delta|=|\alpha|$. Now, it suffices to prove (1.10) for $\varphi\in C_c^\infty(G)$. We have

$$h(\varphi)=\int\sum_{|\alpha|=|\beta|=n}c_{\alpha\beta}(A^\alpha\varphi)\overline{(A^\beta\varphi)}$$

$$h_\rho(\varphi)=\int\sum_{|\alpha|=|\beta|=n}c_{\alpha\beta}\big(\mathrm{e}^{-\rho\psi_R^l}A^\alpha\mathrm{e}^{\rho\psi_R^l}\varphi\big)\overline{\big(\mathrm{e}^{\rho\psi_R^l}A^\beta\mathrm{e}^{-\rho\psi_R^l}\varphi\big)}$$

and using (3.6) it follows that $h_\rho(\varphi)-h(\varphi)$ is a sum of constant multiples of terms T of the form

$$T=\rho^k\int c_{\alpha\beta}(A^{\gamma_1}\psi_R^l)\cdots(A^{\gamma_k}\psi_R^l)(A^{\delta_1}\varphi)\overline{(A^{\delta_2}\varphi)}$$

where $k\in\mathbb{N}$, $\gamma_1,\ldots,\gamma_k,\delta_1,\delta_2$ are in $J(d')$ with $|\gamma_j|\geqslant1$ for all j, $|\delta_1|,|\delta_2|\leqslant n$ and $|\gamma_1|+\cdots+|\gamma_k|+|\delta_1|+|\delta_2|=m$. Now $c_{\alpha\beta}\in L_\infty$ and by (1.9),

$$\|(A^{\gamma_1}\psi_R^l)\cdots(A^{\gamma_k}\psi_R^l)\|_\infty\leqslant c\,R^{-(|\gamma_1|+\cdots+|\gamma_k|-k)}\leqslant c'\,|\rho|^{|\gamma_1|+\cdots+|\gamma_k|-k}$$

because $|\rho| \geqslant R^{-1}$ and $|\gamma_1| + \cdots + |\gamma_k| - k \geqslant 0$. Hence

$$|T| \leqslant c\,|\rho|^r \|A^{\delta_1}\varphi\|_2 \|A^{\delta_2}\varphi\|_2$$

where $r = |\gamma_1| + \cdots + |\gamma_k|$. Note that $0 < r \leqslant m$ and $|\delta_1| + |\delta_2| + r = m$. Thus $|T| \leqslant c\rho^m \|\varphi\|_2^2$ in the case $r = m$.

If $0 < r < m$, one applies (2.1) with $j = n$ and $\alpha = \delta_i$, $i = 1, 2$, and then applies (1.6) to deduce that

$$|T| \leqslant c\,|\rho|^r \left(\|\varphi\|_2^2\right)^{r/m} (\mathrm{Re}\, h(\varphi))^{1-(r/m)} \leqslant \varepsilon \mathrm{Re}\, h(\varphi) + c'\varepsilon^{-(m-r)/r}\rho^m \|\varphi\|_2^2$$

for all $\varepsilon > 0$, by a standard ε, ε^{-1} argument. Since $\varepsilon^{-(m-r)/r} \leqslant \varepsilon^{-(m-1)}$ when $0 < \varepsilon \leqslant 1$, these estimates on T complete the proof of the proposition. ∎

Now we prove part (i) of Theorem 1.3.

Let $H = \sum\limits_{|\alpha|=m} c_\alpha A^\alpha$ be an m-th order Gårding operator so $D(H) = L_{2;m}$. We first show that H is the m-sectorial operator associated with a sectorial form satisfying (1.6) and (1.7). Define

$$h(\varphi, \psi) = (H\varphi, \psi) \tag{3.7}$$

for $\varphi, \psi \in L_{2;m}$. One easily verifies (1.6) and (1.7) for $\varphi \in L_{2;m}$ (see the last step in the proof of Lemma 3.1). It follows that h is a closable sectorial form and that the domain of the closure is $L_{2;n}$. We continue to denote the closure by h: then (1.6) and (1.7) hold for $\varphi \in L_{2;n}$. Let $\widetilde{H}$ be the m-sectorial operator associated with h, as in Section 1. It follows from (3.7) and Corollary VI.2.4 of [18] that $\widetilde{H}$ is an extension of H. But $\widetilde{H}$ and H are both semigroup generators, and hence $\widetilde{H} = H$.

Finally, we verify (1.10) and in fact show that C_ε can be chosen to have the same form as in Proposition 4.1. For $\varphi \in C_c^\infty(G)$

$$h_\rho(\varphi) - h(\varphi) = \sum_{|\alpha|=m} c_\alpha \big((e^{-\rho\psi_R^l} A^\alpha e^{\rho\psi_R^l} - A^\alpha)\varphi, \varphi \big)$$

is, by (3.6), a sum of constant multiples of terms

$$T' = \rho^k \left(A^\delta \varphi, (A^{\gamma_1}\psi_R^l) \cdots (A^{\gamma_k}\psi_R^l)\varphi \right)$$

where $k \in \mathbb{N}$, $|\gamma_j| \geqslant 1$ for all j, and $|\gamma_1| + \cdots + |\gamma_k| + |\delta| = m$. Note $|\delta| < m$. If $|\delta| \leqslant n$, T' can be estimated just like T in the proof of Proposition 4.1. If $|\delta| > n$, let $\delta = \delta_1\delta_2$ where $|\delta_2| = n$, $|\delta_1| < n$. One uses the identity $(A^\delta\varphi, \chi) = (-1)^{|\delta_1|}(A^{\delta_2}\varphi, A^{\delta_1 *}\chi)$ and then expands T' as a sum of constant multiples of terms

$$T'' = \rho^k (A^{\delta_2}\varphi, (A^{\beta_1}\psi_R^l) \cdots (A^{\beta_k}\psi_R^l)(A^{\delta_3}\varphi))$$

where $|\beta_j| \geqslant 1$ for all j, $|\delta_2| = n$, $|\delta_3| < n$, and $|\beta_1| + \cdots + |\beta_k| + |\delta_2| + |\delta_3| = m$. Then T'' can be estimated just like T above, and the proof of Theorem 1.3 is complete. ∎

Corollary 1.4 follows easily from our previous results. For (i), we have $\|K_t\|_2 = \|S_t\|_{2\to\infty}$ and $\|K_t\|_\infty = \|S_t\|_{1\to\infty}$, so the required estimates follow from the proof of Lemma 3.2. For (ii), we may assume that $d \geqslant 2$, where d is the vector space dimension of the Lie algebra of G. Now $D' \geqslant d$ (see [11], Section 6, or [22], Chapter V) so $m \geqslant 4$, and the result follows by combining Theorems 1.1, 1.2, and 1.3 (i).

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