J. OPERATOR THEORY
46(2001), 173–181

SCHATTEN CLASS COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES OF THE DISK

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Communicated by Norberto Salinas

Abstract. If φ is an analytic self-map of the open unit disk $\mathbb{D}$ with bounded valence and $2 \leqslant p < +\infty$, we show that the composition operator C_φ, acting on the weighted Bergman L^2 space of $\mathbb{D}$ with radial weight $(1-|z|^2)^\alpha$ $(\alpha > -1)$, belongs to the Schatten class S_p if and only if

$$\int\limits_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{p(\alpha+2)/2} \mathrm{d}\lambda(z) < \infty,$$

where $\mathrm{d}\lambda$ is the Möbius invariant measure of $\mathbb{D}$.

Keywords: *Composition operator, Bergman space, Schatten class, bounded valence.*

MSC (2000): 47B38, 47B10.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Throughout the paper we fix a real parameter $\alpha > -1$ and consider the weighted Bergman space $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ consisting of analytic functions f in $\mathbb{D}$ with

$$\|f\|^2 = \int\limits_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}A_\alpha(z) < +\infty,$$

where

$$\mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \,\mathrm{d}A(z),$$

and $\mathrm{d}A$ is the normalized area measure on $\mathbb{D}$. The space $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$, when equipped with the obvious inner product, is a Hilbert space with reproducing kernel

$$K_\alpha(z,w) = \frac{1}{(1-z\overline{w})^{2+\alpha}}.$$

We will study composition operators on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$. More specifically, for any analytic $\varphi : \mathbb{D} \to \mathbb{D}$ we define the composition operator C_φ by $C_\varphi f = f \circ \varphi$. Each C_φ is a bounded linear operator on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$. Furthermore, it is shown in [5] that the operator C_φ is compact on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ if and only if

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|\varphi(z)|^2} = 0.$$

The main result of this paper is the following characterization of Schatten class (we will define Schatten classes in the next section) composition operators on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ induced by functions of bounded valence. Here we say that an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ is of *bounded valence* if there exists a positive integer N such that for every $z \in \mathbb{D}$ the set $\varphi^{-1}(z)$ contains at most N points.

THEOREM 1.1. *Suppose* $2 \leqslant p < +\infty$ *and* $\varphi : \mathbb{D} \to \mathbb{D}$ *is an analytic function of bounded valence. Then* C_φ *belongs to the Schatten class* S_p *of* $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ *if and only if*

$$\int_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{p(\alpha+2)/2} \mathrm{d}\lambda(z) < +\infty,$$

where

$$\mathrm{d}\lambda(z) = (1-|z|^2)^{-2}\,\mathrm{d}A(z)$$

is the Möbius invariant measure on $\mathbb{D}$.

Schatten class composition operators on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ (as well as on the Hardy space) are characterized in [4] in terms of the L^p integrability of Nevanlinna type counting functions with respect to the Möbius invariant measure. Our result here is more explicit and should be more applicable.

The special case $p = 2$ of our main result has been well known (see [4] and references there) and follows easily from the classical characterization of Hilbert-Schmidt integral operators. Also, the "only if part" of our main theorem is essentially proved in [7] and [4]. In fact, Section 7 of [4] proves, in the special case $\alpha = 0$, that the condition

$$\int_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{p(\alpha+2)/2} \mathrm{d}\lambda(z) < \infty$$

is necessary when $p \geqslant 2$ and sufficient when $p \leqslant 2$ for C_φ to be in S_p, without the bounded valence condition on φ; the same proof works for all $-1 < \alpha < \infty$. So our contribution here is the sufficiency of the above integral condition when $p \geqslant 2$ under the assumption that φ be of bounded valence. We are able to supply the missing part here using an interpolation argument.

We suspect our theorem holds for all $p > 2/(\alpha+2)$. In fact, we know that the "if part" of our theorem is true in the full range $0 < p < \infty$; we mentioned in the previous paragraph that this part holds for $0 < p \leqslant 2$ even without the bounded valence condition. On the other hand, the "only if part" of the theorem is false when $0 < p \leqslant 2/(\alpha+2)$, as the involved integrals clearly diverge.

We are not sure if the bounded valence condition here can be removed altogether, although the condition is clearly not needed for $p = 2$. We point out that our theorem still holds if the bounded valence condition is replaced by the condition that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^2\int_{\mathbb{D}}\frac{|\varphi'(w)|^2\,\mathrm{d}A(w)}{|1-\overline{z}\varphi(w)|^4}<+\infty.$$

It is easy to see that this inequality holds if and only if the composition operator C_φ maps the Dirichlet space into itself. The paper [5] contains several results on composition operators on Dirichlet and Hardy spaces that definitely rely on inequalities of the type above. The paper [2] contains our main result under the additional assumption that the symbol function is second differentiable on the closed disk (but without the bounded valence condition).

2. Relationship to Toeplitz operators

It has long been known that composition operators are closely related to Toeplitz operators on weighted Bergman spaces. One such connection, based on suitably defined counting functions, is used in [7] and [4] to characterize Schatten class composition operators on weighted Bergman spaces (as well as on the Hardy space) in terms of these counting functions.

We will make use of a different (but still well-known) connection here, involving Toeplitz operators induced by positive *measures* but defined on the same space on which the composition operator acts. This will enable us to describe Schatten class composition operators on weighted Bergman spaces by a much more explict condition.

Let μ be a finite positive Borel measure on $\mathbb{D}$. Then the Toeplitz operator

$$T_\mu : L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)\to L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$$

is densely defined on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ by

$$T_\mu f(z)=\int_{\mathbb{D}}K_\alpha(z,w)f(w)\,\mathrm{d}\mu(w).$$

To estimate the "size" of T_μ we define the *Berezin symbol* $\widetilde{\mu}$ of μ as follows:

$$\widetilde{\mu}(z)=\int_{\mathbb{D}}\frac{(1-|z|^2)^{2+\alpha}}{|1-z\overline{w}|^{2(2+\alpha)}}\,\mathrm{d}\mu(w),\qquad z\in\mathbb{D}.$$

Recall that a positive linear operator T on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ is in the trace class if

$$\mathrm{tr}(T)=\sum_{n=1}^{\infty}\langle Te_n,e_n\rangle<+\infty$$

for some (or all) orthonormal basis $\{e_n\}$ of $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$. In general, if $0<p<+\infty$ and T is a bounded linear operator on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$, then we say that T belongs to the Schatten class S_p if $(T^*T)^{p/2}$ is in the trace class.

The space S_1 is usually called the *trace class*, and S_2 is usually called the *Hilbert-Schmidt class*. It is well known that T belongs to S_p if and only if T^* does. See [7] for basic properties of Schatten classes.

We will need the following result from the theory of Toeplitz operators on weighted Bergman spaces.

LEMMA 2.1. *Suppose* $0 < p < +\infty$ *and* μ *is a finite positive Borel measure on* $\mathbb{D}$. *Then:*

(i) T_μ *is bounded on* $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ *if and only if* $\widetilde{\mu}$ *is bounded on* $\mathbb{D}$*;*
(ii) T_μ *is compact on* $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ *if and only if* $\widetilde{\mu}$ *vanishes on* $\partial\mathbb{D}$*;*
(iii) T_μ *is in* S_p *of* $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ *if and only if* $\widetilde{\mu}$ *is in* $L^p(\mathbb{D}, \mathrm{d}\lambda)$.

Proof. This can be found in [6] and [7] in its present form for the range $1 \leqslant p < +\infty$. It is observed in [4] that the result can be extended to the range $0 < p < +\infty$ (at least in the case of the unit disk) by making some adjustments in [3]. ∎

To describe our desired connection between composition operators and Toeplitz operators we need the notion of pull-back measures. Let μ be any Borel measure on $\mathbb{D}$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ be any analytic function. Then we can define another Borel measure ν on $\mathbb{D}$ by

$$\nu(E) = \mu(\varphi^{-1}(E)), \quad E \subset \mathbb{D};$$

we denote this new measure ν by $\mu \circ \varphi^{-1}$ and call it the *pull-back* measure of μ induced by φ.

LEMMA 2.2. *Suppose* $\varphi : \mathbb{D} \to \mathbb{D}$ *is analytic and* C_φ *is the composition operator on* $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$. *Then* $C^*_\varphi C_\varphi = T_\mu$, *where* $\mu = \mathrm{d}A_\alpha \circ \varphi^{-1}$.

Proof. It is easy to check that

$$C^*_\varphi C_\varphi f(z) = \int_{\mathbb{D}} K_\alpha(z, \varphi(w)) f \circ \varphi(w) \, \mathrm{d}A_\alpha(w), \quad f \in L^2_{\mathrm{a}}(\mathrm{d}A_\alpha).$$

The desired result then follows from a change of variables. ∎

The above connection between composition operators and Toeplitz operators was observed in [3], and possibly earlier.

3. PROOF OF THE THEOREM

We are going to need the following well-known estimate on several occasions later on.

LEMMA 3.1. *For any* $a > -1$ *and* $b > 0$ *there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^a \, \mathrm{d}A(w)}{|1 - z\overline{w}|^{2+a+b}} \leqslant \frac{C}{(1 - |z|^2)^b}$$

for all $z \in \mathbb{D}$.

Proof. See page 53 of [7]. ∎

For a bounded linear operator T on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ the *Berezin symbol* of T is the function $\widetilde{T}$ on $\mathbb{D}$ defined by

$$\widetilde{T}(z) = \langle Tk_z^{(\alpha)}, k_z^{(\alpha)}\rangle, \quad z \in \mathbb{D}.$$

Here $\langle\ ,\ \rangle_\alpha$ is the inner product in $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ and $k_z^{(\alpha)}$ are the normalized reproducing kernels of $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$:

$$k_z^{(\alpha)}(w) = \frac{(1-|z|^2)^{\frac{\alpha+2}{2}}}{(1-\overline{z}w)^{\alpha+2}}, \quad w \in \mathbb{D}.$$

LEMMA 3.2. *Suppose $1 \leqslant p \leqslant +\infty$. If T is a positive operator on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ belonging to the Schatten class S_p, then $\widetilde{T}$ is in $L^p(\mathbb{D}, \mathrm{d}\lambda)$.*

Proof. This follows easily from Propositions 6.3.2 and 6.3.3 in [7]. ∎

Now suppose the composition operator

$$C_\varphi : L^2_{\mathrm{a}}(\mathrm{d}A_\alpha) \to L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$$

belongs to the Schatten class S_p of $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$, where $2 \leqslant p < +\infty$. Then the operator $T = C_\varphi C_\varphi^*$ belongs to $S_{p/2}$ of $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$. It is easy to compute that

$$\widetilde{T}(z) = \|C_\varphi^* k_z^{(\alpha)}\|^2 = \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha+2}.$$

Combining this with Lemma 3.2, we conclude that

$$\int_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{p(\alpha+2)}{2}} \mathrm{d}\lambda(z) < +\infty.$$

This proves the "only if" part of the main theorem (without using the assumption that φ is of bounded valence); and this proof is well-known to experts in the field.

To prove the "if part" of the theorem, we are going to make use of the relation

$$T_\mu = C_\varphi^* C_\varphi, \quad \mu = \mathrm{d}A_\alpha \circ \varphi^{-1},$$

and employ the technique of complex interpolation. So let us assume φ is of bounded valence and

$$\int_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\frac{p(\alpha+2)}{2}} \mathrm{d}\lambda(z) < +\infty,$$

where $2 \leqslant p < +\infty$, and proceed to show that the composition operator C_φ on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ must belong to the Schatten class S_p. Equivalently, we must show that the Toeplitz operator T_μ belongs to $S_{p/2}$.

Let Φ be the Berezin symbol of the Toeplitz operator T_μ. Changing the variable of integration again gives

$$\Phi(z) = (1-|z|^2)^{\alpha+2} \int_{\mathbb{D}} \frac{\mathrm{d}A_\alpha(w)}{|1-\overline{z}\varphi(w)|^{2(\alpha+2)}}, \quad z \in \mathbb{D}.$$

According to Lemma 2.1, it suffices for us to show that the function Φ above is in $L^{p/2}(\mathbb{D}, \mathrm{d}\lambda)$. We replace $p/2$ by p and rephrase the desired estimate as the following.

LEMMA 3.3. *If* $1 \leqslant p < +\infty$, $\varphi : \mathbb{D} \to \mathbb{D}$ *is of bounded valence, and*

$$M = \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p(\alpha+2)} \mathrm{d}\lambda(z) < +\infty,$$

then Φ *is in* $L^p(\mathbb{D}, \mathrm{d}\lambda)$.

Proof. The case $p = 1$ is a direct consequence of Lemma 3.1 and Fubini's theorem; and we do not need the assumption that φ is of finite valence in this case. So we assume $1 < p < +\infty$ in what follows.

Choose a constant $C > 0$ such that for all analytic functions f in $\mathbb{D}$

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^{\alpha} \, \mathrm{d}A(z) \leqslant C \left[|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{\alpha+2} \, \mathrm{d}A(z) \right].$$

This is a well-known estimate and follows easily from the Taylor expansion of f and standard estimates for the Gamma function. Using the inequality above we then obtain

$$\Phi(z) \leqslant C_\alpha \frac{(1 - |z|^2)^{\alpha+2}}{|1 - \overline{z}\varphi(0)|^{2(\alpha+2)}} + C_\alpha F(z),$$

where $C_\alpha = (\alpha + 1)C$ and

$$F(z) = (1 - |z|^2)^{\alpha+2} \int_{\mathbb{D}} \frac{|\varphi'(w)|^2 (1 - |w|^2)^{\alpha+2} \, \mathrm{d}A(w)}{|1 - \overline{z}\varphi(w)|^{2(\alpha+3)}}, \quad z \in \mathbb{D}.$$

Obviously, we only need to show that $F \in L^p(\mathbb{D}, \mathrm{d}\lambda)$.

Write

$$\frac{1}{p} = \frac{1 - \theta}{1} + \frac{\theta}{\infty},$$

where

$$\theta = 1 - \frac{1}{p}.$$

For any complex parameter ζ with $0 \leqslant \operatorname{Re} \zeta \leqslant 1$ we define

$$F_\zeta(z) = (1 - |z|^2)^{\alpha+2} \int_{\mathbb{D}} \frac{(1 - |w|^2)^{p(\alpha+2)(1-\zeta)} |\varphi'(w)|^2 \, \mathrm{d}A(w)}{|1 - \overline{z}\varphi(w)|^{\alpha+4+p(\alpha+2)(1-\zeta)}}, \quad z \in \mathbb{D}.$$

Clearly F_ζ depends on the parameter ζ analytically in the closed strip $0 \leqslant \operatorname{Re} \zeta \leqslant 1$.

It is easy to check that $F_\theta = F$.

If $\operatorname{Re} \zeta = 0$, then Fubini's theorem together with Lemma 3.1 shows that

$$\int_{\mathbb{D}} |F_\zeta(z)| \, \mathrm{d}\lambda(z) \leqslant C \int_{\mathbb{D}} \frac{(1 - |w|^2)^{p(\alpha+2)} |\varphi'(w)|^2 \, \mathrm{d}A(w)}{(1 - |\varphi(w)|^2)^{2+p(\alpha+2)}},$$

where C is a positive constant independent of ζ. According to (the generalized) Schwarz lemma,

$$(1 - |w|^2)|\varphi'(w)| \leqslant 1 - |\varphi(w)|^2$$

for all $w \in \mathbb{D}$; see page 136 of [1]. It follows that

$$\int_{\mathbb{D}} |F_\zeta(z)| \,\mathrm{d}\lambda(z) \leqslant CM$$

whenever $\operatorname{Re}\zeta = 0$.

If $\operatorname{Re}\zeta = 1$, then

$$|F_\zeta(z)| \leqslant (1-|z|^2)^{\alpha+2} \int_{\mathbb{D}} \frac{|\varphi'(w)|^2 \,\mathrm{d}A(w)}{|1-\overline{z}\varphi(w)|^{\alpha+4}}.$$

For any $z \in \mathbb{D}$ let $n_\varphi(z)$ denote the number of points in $\varphi^{-1}(z)$. Then a change of variable argument gives

$$|F_\zeta(z)| \leqslant (1-|z|^2)^{\alpha+2} \int_{\varphi(\mathbb{D})} \frac{n_\varphi(w) \,\mathrm{d}A(w)}{|1-\overline{z}w|^{\alpha+4}}$$

for all $z \in \mathbb{D}$ and $\operatorname{Re}\zeta = 1$. Since φ is of bounded valence, the function n_φ is bounded. Combining this with Lemma 3.1, we conclude that there exists a constant $C > 0$ such that $|F_\zeta(z)| \leqslant C$ for all $z \in \mathbb{D}$ and all ζ with $\operatorname{Re}\zeta = 1$.

From the well-known complex interpolation of L^p spaces (see [7] for example) we conclude that the function F, and hence Φ, belongs to $L^p(\mathbb{D}, \mathrm{d}\lambda)$. This completes the proof of Lemma 3.3 as well as that of the main theorem. ∎

4. FURTHER REMARKS

From the proof of Lemma 3.3 we see that the main theorem remains valid if the bounded valence conditon on φ is replaced by the assumption that the function

$$\Psi(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(w)|^2 \,\mathrm{d}A(w)}{|1-\overline{z}\varphi(w)|^4}, \quad z \in \mathbb{D},$$

is bounded on $\mathbb{D}$. Rewrite Ψ as

$$\Psi(z) = (1-|z|^2)^2 \int_{\mathbb{D}} \frac{n_\varphi(w) \,\mathrm{d}A(w)}{|1-\overline{z}w|^4}.$$

Then we see that Ψ is the Berezin symbol (corresponding to $\alpha = 0$) of the measure $n_\varphi \,\mathrm{d}A$. Applying Lemma 2.1 to this measure (in the special case $\alpha = 0$), we conclude that Ψ is bounded if and only if there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} |f(z)|^2 n_\varphi(z) \,\mathrm{d}A(z) \leqslant C \int_{\mathbb{D}} |f(z)|^2 \,\mathrm{d}A(z)$$

for all analytic functions f in $\mathbb{D}$. Replacing f by f' and changing the variable of integration again, we realize that Ψ is bounded if and only if

$$\int_{\mathbb{D}} |f'(\varphi(z))|^2 |\varphi'(z)|^2 \,\mathrm{d}A(z) \leqslant C \int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}A(z)$$

for all analytic f in $\mathbb{D}$.

Let D_2 denote the Dirichlet space of analytic functions f in $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f'(z)|^2 \, \mathrm{d}A(z) < +\infty.$$

Then we see that the function Ψ is bounded if and only if the composition operator C_φ is bounded on D_2. By a standard argument involving the closed graph theorem we conclude that Ψ is bounded on D_2 if and only if the composition operator C_φ maps the Dirichlet space into itself.

Coming back to composition operators on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$, we observe that

$$C_\varphi C_\varphi^* f(z) = \int_{\mathbb{D}} K_\alpha(\varphi(z), \varphi(w)) f(w) \, \mathrm{d}A_\alpha(w), \quad z \in \mathbb{D},$$

for all $f \in L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$. It follows that the operator $C_\varphi C_\varphi^*$ is Hilbert-Schmidt on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ if and only if

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha (1-|w|^2)^\alpha \, \mathrm{d}A(z) \, \mathrm{d}A(w)}{|1-\varphi(z)\,\overline{\varphi(w)}|^{2(\alpha+2)}} < +\infty.$$

Equivalently, this condition characterizes compositions operators on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ belonging to the Schatten class S_4. It is natural to ask for a characterization of composition operators on $L^2_{\mathrm{a}}(\mathrm{d}A_\alpha)$ belonging to S_p in terms of a similar condition.

The referee pointed out that it is possible to prove our main theorem without the use of complex interpolation. Instead, one can use Schur's test (see [7]) for the boundedness of integral operators on L^p spaces. In fact, one can rewrite the function F from the proof of Lemma 3.3 as follows:

$$F(z) = \int_{\mathbb{D}} H(z, w) h(w) \, \mathrm{d}\mu(w),$$

where

$$H(z, w) = \frac{(1-|z|^2)^{\alpha+2}(1-|\varphi(w)|^2)^{\alpha+4}}{|1-\overline{z}\varphi(w)|^{2(\alpha+3)}}$$

and

$$h(w) = \left(\frac{1-|w|^2}{1-|\varphi(w)|^2}\right)^{\alpha+2}, \quad \mathrm{d}\mu(z) = \frac{|\varphi'(z)|^2 \, \mathrm{d}A(z)}{(1-|\varphi(z)|^2)^2}.$$

By Schwarz lemma, $\mu \leqslant \lambda$. With the assumptions of Lemma 3.3 one then checks that

$$\int_{\mathbb{D}} H(z, w) \, \mathrm{d}\mu(w) \leqslant C$$

and

$$\int_{\mathbb{D}} H(z, w) \, \mathrm{d}\lambda(z) \leqslant C$$

for some constant C. By Schur's test (see [7] for a proof of the special case when the two measures coincide; the general case follows from the same proof), the integral operator with kernel $H(z,w)$ and measure $\mathrm{d}\mu$ maps $L^p(\mathbb{D},\mathrm{d}\mu)$ boundedly into $L^p(\mathbb{D},\mathrm{d}\lambda)$. In particular, since $h \in L^p(\mathbb{D},\mathrm{d}\mu)$, one must have $F \in L^p(\mathbb{D},\mathrm{d}\lambda)$. Note that the boundedness of n_φ, or, more generally, the boundedness of an integral similar to Ψ, is needed in the proof of

$$\int_{\mathbb{D}} H(z,w)\,\mathrm{d}\lambda(z) \leqslant C.$$

A whole family of such integrals is obtained if, instead of the simple Schur's test used above, one uses the more complex tests

$$\int_{\mathbb{D}} H(z,w)\beta(w)^q\,\mathrm{d}\mu(w) \leqslant C\gamma(z)^q$$

and

$$\int_{\mathbb{D}} H(z,w)\gamma(z)^p\,\mathrm{d}\lambda(z) \leqslant C\beta(w)^p,$$

where $\beta(w)$ is power of $1-|\varphi(w)|^2$ and $\gamma(z)$ is the same power of $1-|z|^2$. All these conditions are equivalent to Ψ being bounded.

REFERENCES

1. L. Ahlfors, *Complex Analysis*, 3rd edition, McGraw-Hill, New York 1979.
2. S.Y. Li, Trace ideal criteria for composition operators on Bergman spaces, *Amer. J. Math.* **117**(1995), 1299–1323.
3. D. Luecking, Trace ideal criteria for Toeplitz operators, *J. Funct. Anal.* **73**(1987), 345–368.
4. D. Luecking, K. Zhu, Composition operators belonging to the Schatten ideals, *Amer. J. Math.* **114**(1992), 1127–1145.
5. B. MacCluer, J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* **38**(1986), 878–906.
6. K. Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, *J. Operator Theory* **20**(1988), 329-357.
7. K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York 1990.

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Received January 14, 1999; revised September 13, 1999.