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OPERATOR-NORM CONVERGENCE OF THE TROTTER PRODUCT FORMULA FOR HOLOMORPHIC SEMIGROUPS

VINCENT CACHIA and VALENTIN A. ZAGREBNOV

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Abstract. We study the error estimates in operator norm for the Trotter product formula. It is shown that some recent results in this direction can be extended to non-self-adjoint generators and to the case of a Banach space for a class of holomorphic semigroups.

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1. INTRODUCTION

Recently the operator-norm convergence and the error bound estimates were obtained for the Trotter and the Trotter-Kato product formulae in the abstract context of semigroups in a Hilbert space with self-adjoint generators ([12], [7], [9], [10]), as well as for the Schrödinger semigroups ([3], [4], [5], [6]).

Let A and B be non-negative self-adjoint operators on the Hilbert space $\mathcal{H}$. Then some of these abstract results can be formulated as follows (cf. [9]): if B is A-bounded with a relative bound $a < 1$ and $A \geqslant I$, then for the Trotter formula one gets the operator-norm estimate

$$\big\|(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n})^n - \mathrm{e}^{-t(A+B)}\big\| \leqslant C\frac{\ln n}{n} \tag{1.1}$$

uniformly in t for any bounded interval $[0, T]$.

The aim of the present note is to extend the class of semigroups for which we have the operator-norm convergence of the Trotter product formula. This extension is accomplished in two directions: we show that under certain conditions on the generators $-A$ and $-B$ the operator-norm convergence takes place in a Banach space, which was the initial context for the Trotter formula ([16]). As a consequence we obtain a generalization of the results [7], [9], [10] in Hilbert space to some class of non-self-adjoint generators: $-A$ generates a holomorphic semigroup, and $-B$ is A-small with a relative bound equal to zero (cf. [7]).

For these results the properties of holomorphic semigroups are essential. So we begin with some preliminaries concerning bounded, holomorphic semigroups, and fractional powers of generators.

2. PRELIMINARIES

2.1. BOUNDED SEMIGROUPS. As a starting point we recall the definition of a one-parameter strongly continuous semigroup (or simply semigroup), then we state some of its properties.

DEFINITION 2.1. A family $Q(t)_{t\geqslant 0}$ of bounded linear operators on a Banach space $\mathcal{B}$ is called a *one-parameter strongly continuous semigroup* if it satisfies the conditions:

(i) $Q(0) = I$;
(ii) $Q(s+t) = Q(s)Q(t)$ for all $s, t \geqslant 0$;
(iii) $\lim_{t\to 0} Q(t)x = x$ for all $x \in \mathcal{B}$.

The immediate consequences of this definition are (see e.g. [13], Section 13.34):

• There are constants $C_A \geqslant 1$ and $\gamma_A \in \mathbb{R}$, depending on the generator of the semigroup, such that $\|Q(t)\| \leqslant C_A \mathrm{e}^{\gamma_A t}$ for all $t \geqslant 0$.

• $t \mapsto Q(t)$ is a strongly continuous function from $[0, +\infty)$ onto the algebra of bounded linear operators on $\mathcal{B}$.

• There exists a closed densely defined linear operator $-A$ on $\mathcal{B}$ with domain $\mathcal{D}(A)$, called the *generator of the semigroup*, such that $\lim_{t\to 0}(Q(t)x - x)/t = -Ax$ for any $x \in \mathcal{D}(A)$, i.e. by convention $Q(t) = \mathrm{e}^{-tA}$.

• The resolvent of the generator satisfies the estimate $\|R_{-\lambda}(A)\| = \|(A + \lambda)^{-1}\| \leqslant C_A/(\mathrm{Re}(\lambda) - \gamma_A)$ for all λ with $\mathrm{Re}(\lambda) > \gamma_A$, thus the open half plane with $\mathrm{Re}(z) < -\gamma_A$ is contained into the resolvent set of A, defined as $\rho(A) = \{z \in \mathbb{C} : \|R_z(A)\| < +\infty\}$.

REMARK 2.2. If $\gamma_A \leqslant 0$, $Q(t)$ is called a *bounded semigroup* (otherwise, $Q(t)$ is called a *quasi-bounded semigroup of type* $\gamma_A > 0$). For any strongly continuous semigroup, we can construct a bounded semigroup by adding some constant $\eta \geqslant \gamma_A$ to its generator. Let $\widetilde{A} = A + \eta$, then for the semigroup $\widetilde{Q}(t)$ generated by $\widetilde{A}$, one has $\|\widetilde{Q}(t)\| \leqslant C_A$, $t \geqslant 0$, and the open half-plane $\mathrm{Re}(\lambda) < \eta - \gamma_A$ is included into the resolvent set $\rho(\widetilde{A})$ of $\widetilde{A}$. So it is not restrictive to suppose that the considered semigroup $Q(t)$ is bounded and that $\{z \in \mathbb{C}\colon \mathrm{Re}(z) \leqslant 0\} \subseteq \rho(A)$.

REMARK 2.3. If $\|Q(t)\| \leqslant 1$, $t \geqslant 0$, the semigroup is called a *contraction semigroup*. It is clear that the method of the preceding remark does not permit to construct a contraction semigroup from a bounded semigroup in general, since the constant C_A does not change. Below we need a characterization of generators of contraction semigroups (see e.g. [2], Theorem 2.24). We note by $\mathcal{B}^*$ the dual space of the Banach space $\mathcal{B}$.

DEFINITION 2.4. Let A be an operator in a Banach space $\mathcal{B}$. A is said to be *accretive* if for all pairs $\{u,\varphi\} \in \mathcal{D}(A) \times \mathcal{B}^*$ with $\|u\| = 1$, $\|\varphi\| = 1$, $(u,\varphi) = 1$, one has $\operatorname{Re}(Au,\varphi) \geqslant 0$.

PROPOSITION 2.5. *An operator $-A$ with dense domain in a Banach space $\mathcal{B}$ is the generator of a contraction semigroup if and only if A is accretive and the range of $\lambda + A$ equals $\mathcal{B}$ for some $\lambda > 0$.*

The following observations will be useful for the sequel, see Section 3.

LEMMA 2.6. *Let $Q(t)$ be a bounded semigroup with boundedly invertible generator $-A$; then for all $t \geqslant 0$, and for any $n \in \mathbb{N}$, we have:*

$$\Big(Q(t) - \sum_{k=0}^{n} \frac{(-tA)^k}{k!}\Big)A^{-n-1} = -\int_0^t \Big(Q(\tau) - \sum_{k=0}^{n-1} \frac{(-\tau A)^k}{k!}\Big)A^{-n}\,\mathrm{d}\tau, \tag{2.1}$$

$$\Big\|\Big(Q(t) - \sum_{k=0}^{n} \frac{(-tA)^k}{k!}\Big)A^{-n-1}\Big\| \leqslant C_A \frac{t^{n+1}}{(n+1)!}. \tag{2.2}$$

Proof. We proceed by induction, and we first prove (cf. [13]):

$$(Q(t) - I)x = -\int_0^t Q(\tau)Ax\,\mathrm{d}\tau, \quad \forall x \in \mathcal{D}(A). \tag{2.3}$$

By virtue of semigroup properties one has for $\varepsilon > 0$

$$\int_0^t Q(s)\frac{Q(\varepsilon) - I}{\varepsilon}\,\mathrm{d}s = \int_0^t \frac{Q(s+\varepsilon) - Q(s)}{\varepsilon}\,\mathrm{d}s = \int_t^{t+\varepsilon} \frac{Q(s)}{\varepsilon}\,\mathrm{d}s - \int_0^{\varepsilon} \frac{Q(s)}{\varepsilon}\,\mathrm{d}s$$
$$= (Q(t) - I)\frac{1}{\varepsilon}\int_0^{\varepsilon} Q(s)\,\mathrm{d}s.$$

Moreover we have:

$$\lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon}\int_0^{\varepsilon} Q(s)x\mathrm{d}s = x, \qquad \forall x \in \mathcal{B},$$
$$\lim_{\varepsilon\to 0} \frac{Q(\varepsilon) - I}{\varepsilon}x = -Ax, \qquad \forall x \in \mathcal{D}(A).$$

This proves (2.3), and since A is boundedly invertible, we obtain (2.1) for $n = 0$. Furthermore, since $Q(t)$ is bounded by C_A, we obtain the estimate (2.2) for $n = 0$.

Suppose that (2.1) and (2.2) are true for some n, then a simple calculation leads to (2.1) for $n + 1$. Hence, using the representation (2.1) and (2.2) for n to estimate the integrand, we obtain (2.2) for $n + 1$. This completes the proof by induction. ∎

Similarly, we obtain a representation for a restricted development of $(I + A)^{-1}$.

LEMMA 2.7 *Let A be as in Lemma* 2.6. *Then for any $n \geqslant 0$:*

$$(I+A)^{-1}A^{-n-1} = \left(\sum_{k=0}^{n}(-A)^k\right)A^{-n-1} + (-1)^{n+1}(I+A)^{-1}. \tag{2.4}$$

Proof. For $n = 0$, the representation (2.4) follows from the resolvent formula:

$$(I+A)^{-1} - A^{-1} = -(I+A)^{-1}A^{-1}. \tag{2.5}$$

Suppose that (2.4) holds for an integer n; then:

$$(I+A)^{-1}A^{-n-2} = \left(\sum_{k=0}^{n}(-A)^k\right)A^{-n-2} + (-1)^{n+1}(I+A)^{-1}A^{-1}. \tag{2.6}$$

Applying (2.5) to the last term of (2.6) we get the representation (2.4) for $n+1$, and thus for any n by induction. ∎

LEMMA 2.8. *If $Q(t)$ is a bounded semigroup with boundedly invertible generator $-A$ then:*

$$\left\| \frac{1}{t^2}\left((I+tA)^{-1} - Q(t)\right)A^{-2} \right\| \leqslant \frac{3C_A}{2}, \quad \forall t > 0. \tag{2.7}$$

Proof. By Lemma 2.6 one gets

$$\left\| (Q(t) - I + tA)\frac{1}{t^2}A^{-2} \right\| \leqslant \frac{C_A}{2}. \tag{2.8}$$

On the other hand by Lemma 2.7, we have

$$\left\| ((I+tA)^{-1} - I + tA)\frac{1}{t^2}A^{-2} \right\| \;=\; \|(I+tA)^{-1}\| \leqslant C_A. \tag{2.9}$$

Here the last estimate follows from $(I+tA)^{-1} = (1/t)R_{-1/t}(A)$ and $\|R_{-\lambda}(A)\| \leqslant C_A/(\lambda+\delta)$, $\delta \geqslant 0$, for bounded semigroups with boundedly invertible generators (see Remark 2.2). Hence (2.7) follows from (2.8) and (2.9). ∎

2.2. HOLOMORPHIC SEMIGROUPS.

Now let $U(z)$ be a family of operators with z taking values in the sector of the complex plane:

$$S_\omega = \{z \in \mathbb{C} : z \neq 0 \text{ and } |\arg(z)| < \omega\} \tag{2.10}$$

where $0 < \omega \leqslant \pi/2$.

DEFINITION 2.9. (cf. [2]) We define a *bounded holomorphic semigroup of angle ω on a Banach space $\mathcal{B}$* to be a family of bounded operators $U(z)$ where $z \in S_\omega$, and satisfying the following conditions:

(i) $U(z_1)U(z_2) = U(z_1 + z_2)$ for all $z_1, z_2 \in S_\omega$.
(ii) If $0 < \varepsilon < \omega$, then $\|U(z)\| \leqslant M_\varepsilon$ for all $z \in S_{\omega-\varepsilon}$ and some $M_\varepsilon < \infty$.
(iii) $U(z)$ is an analytic function of $z \in S_\omega$.
(iv) If $x \in \mathcal{B}$, and $0 < \varepsilon < \omega$, then $\lim_{z\to 0} U(z)x = x$ provided $z \in S_{\omega-\varepsilon}$.

Let $\sigma(A) = \mathbb{C} \setminus \rho(A)$ denote the spectrum of A. We mention the following characterization of generators of holomorphic semigroups (see e.g. [2], Theorems 2.33 and 2.34):

PROPOSITION 2.10. *The operator $-A$ in a Banach space $\mathcal{B}$ is the generator of a bounded holomorphic semigroup of angle $\omega \leqslant \pi/2$ if and only if A is a closed operator with a dense domain $\mathcal{D}(A)$ such that:*

$$\exists\, 0 < \omega \leqslant \frac{\pi}{2}, \quad \sigma(A) \subseteq \Big\{ w \in \mathbb{C} : |\arg(w)| \leqslant \frac{\pi}{2} - \omega \Big\} \tag{2.11}$$

and

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \geqslant 0,\ \forall w \in S_{\omega-\varepsilon+\pi/2}, \quad \|(w + A)^{-1}\| \leqslant N_\varepsilon |w|^{-1}. \tag{2.12}$$

We shall use below the following property which is an alternative characterization of these semigroups (see e.g. [2], Theorems 2.38, 2.39, and [8], Chapter IX, Remark 1.20):

PROPOSITION 2.11. *If $U(z)$ is a bounded holomorphic semigroup of angle ω with generator $-A$, then for all $z \in S_\omega$, $n \in \mathbb{N}$, $U(z)\mathcal{B} \subseteq \mathcal{D}(A^n)$, and there are positive constants $C'_A, C_A^{(n)}$ such that:*

$$\left\| \frac{\mathrm{d}U(t)}{\mathrm{d}t} \right\| = \|AU(t)\| \leqslant \frac{C'_A}{t} \quad \textit{and} \quad \left\| \frac{\mathrm{d}^n U(t)}{\mathrm{d}t^n} \right\| = \|A^n U(t)\| \leqslant \frac{C_A^{(n)}}{t^n}. \tag{2.13}$$

In fact, these estimates are valid for complex values of t in any sector S_θ for $0 < \theta < \omega$ with constants depending on θ.

REMARK 2.12. Similarly to strongly continuous semigroups, a family $U(z)$, $z \in S_\omega$ is called a *quasi-bounded holomorphic semigroup of angle* ω if there exists a constant $\gamma > 0$ such that $\mathrm{e}^{-\gamma z}U(z)$ is a bounded holomorphic semigroup of angle ω.

Finally, the class of semigroups we shall consider below is:

DEFINITION 2.13. We say that $U(z)$, $z \in S_\omega$, is a *holomorphic contraction semigroup of angle* ω if it is a bounded holomorphic semigroup of angle ω, and its restriction on $\mathbb{R}^+$ is a contraction semigroup.

Notice that this class of semigroups is not empty:

(i) Let $U(t)$, $t \in \mathbb{R}$, be a contraction semigroup with generator $-A$ on a Banach space $\mathcal{B}$, such that $U(t)\mathcal{B} \subseteq \mathcal{D}(A)$ for $t > 0$. If $\|AU(t)\| \leqslant ct^{-1}$ for some $c > 0$ and all $t > 0$, then there exists $\omega > 0$ such that $U(t)$ may be analytically continued to a bounded holomorphic semigroup of angle ω; see [2], Theorem 2.39.

(ii) Let A be a sectorial operator on a Hilbert space $\mathcal{H}$, i.e. its numerical range $\{(u, Au) : u \in \mathcal{D}(A) \text{ and } \|u\| = 1\} \subseteq S_{\pi/2-\omega}$, $0 < \omega \leqslant \pi/2$. If $-A$ is closed, then it is the generator of a holomorphic contraction semigroup of angle ω; see [8], Chapter IX, Theorem 1.24.

(iii) Let $-A$ be the generator of a holomorphic semigroup on a Banach space, if A is accretive, then $-A$ generates a holomorphic contraction semigroup.

2.3. FRACTIONAL POWERS OF GENERATORS.

For generators of bounded semigroups, it is possible to define fractional powers (see e.g. [17], Chapter IX, Section 11). We need below some properties of these operators.

Following [17], we define the fractional power $0 < \alpha < 1$ of the generator A of a bounded semigroup $Q(t)$ by the integral (when it is well defined)

$$A^{\alpha}x = \frac{1}{\Gamma(-\alpha)} \int_{0}^{\infty} \lambda^{-\alpha-1}(Q(\lambda) - I)x \,\mathrm{d}\lambda, \tag{2.14}$$

where λ^{α} is chosen to be positive for $\lambda > 0$. Notice that for any $x \in \mathcal{D}(A)$, the integral (2.14) is convergent, thus $\mathcal{D}(A) \subseteq \mathcal{D}(A^{\alpha})$. We set also $A^0 = I$ and for any $\alpha > 0$, if $[\alpha]$ denotes the integer part of α, we put $A^{\alpha} = A^{\alpha-[\alpha]}A^{[\alpha]}$.

PROPOSITION 2.14. *For each $\alpha \in [0, 1]$, there exists a constant C_{α}, depending only on C_A and α, such that, for all $\mu > 0$,*

$$\left\| A^{\alpha}(A + \mu)^{-1} \right\| \leqslant \frac{C_{\alpha}}{\mu^{1-\alpha}}. \tag{2.15}$$

Proof. For $\alpha = 0$ or $\alpha = 1$, the result follows directly from the estimate of the resolvent. Let $0 < \alpha < 1$ and $x \in \mathcal{B}$ (notice that $\operatorname{Ran}(A+\mu)^{-1} = \mathcal{D}(A) \subseteq \mathcal{D}(A^{\alpha})$), then

$$A^{\alpha}(A + \mu)^{-1}x = \frac{1}{\Gamma(-\alpha)} \int_{0}^{\infty} \lambda^{-\alpha-1}(Q(\lambda) - I)(A + \mu)^{-1}x \,\mathrm{d}\lambda. \tag{2.16}$$

We divide the integral (2.16) in two parts: $0 < \lambda \leqslant \mu^{-1}$ and $\lambda > \mu^{-1}$, and use the representation (2.3):

$$\begin{aligned} A^{\alpha}(A + \mu)^{-1}x = {} & \frac{1}{\Gamma(-\alpha)} \int_{0}^{\mu^{-1}} \lambda^{-\alpha-1} \int_{0}^{\lambda} -Q(t)(I - \mu(A + \mu)^{-1})x \,\mathrm{d}t \,\mathrm{d}\lambda \\ & + \frac{1}{\Gamma(-\alpha)} \int_{\mu^{-1}}^{\infty} \lambda^{-\alpha-1}(Q(\lambda) - I)(A + \mu)^{-1}x \,\mathrm{d}\lambda. \end{aligned}$$

Now by the estimate of the resolvent $\|(A + \mu)^{-1}\| \leqslant C_A/\mu$ for all $\mu > 0$ one obtains:

$$\begin{aligned} \|A^{\alpha}(A + \mu)^{-1}x\| & \leqslant \frac{C_A(1 + C_A)\|x\|}{\Gamma(-\alpha)} \Bigg(\int_{0}^{\mu^{-1}} \lambda^{-\alpha} \,\mathrm{d}\lambda + \frac{1}{\mu} \int_{\mu^{-1}}^{\infty} \lambda^{-\alpha-1} \,\mathrm{d}\lambda \Bigg) \\ & \leqslant \frac{C_A(1 + C_A)\mu^{\alpha-1}}{\Gamma(-\alpha)\alpha(1 - \alpha)} \,\|x\|. \end{aligned}$$

Setting $C_{\alpha} = C_A(1 + C_A)/(\alpha\Gamma(-\alpha)(1 - \alpha))$ we get the estimate (2.15). ∎

PROPOSITION 2.15. ([15], Lemma 2.3.5) *$\mathcal{D}((A + \delta)^{\alpha}) = \mathcal{D}(A^{\alpha})$ for all $\delta > 0$ and $0 < \alpha < 1$.*

PROPOSITION 2.16. *Let $U(t)$ be a bounded holomorphic semigroup with generator $-A$; then for any real $\alpha > 0$, we have*

$$\sup_{t>0} \|t^{\alpha}A^{\alpha}U(t)\| = M_{\alpha} < \infty. \tag{2.17}$$

Proof. Let $0 < \alpha < 1$: by $\mathcal{D}(A) \subseteq \mathcal{D}(A^\alpha)$ one gets $\mathcal{D}(A^\alpha U(t)) = \mathcal{B}$. Hence by (2.14) we have

$$A^\alpha U(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \lambda^{-\alpha-1}(U(t+\lambda) - U(t))\,\mathrm{d}\lambda. \tag{2.18}$$

Now we split the integral (2.18) in two parts: $0 < \lambda < t$ and $\lambda > t$, and we use the estimate of the derivative of the holomorphic semigroup (see Proposition 2.11) to obtain

$$\|U(t+\lambda) - U(t)\| \leqslant \lambda \sup_{t \leqslant \tau \leqslant t+\lambda} \|U'(\tau)\| \leqslant \lambda \frac{C'_A}{t}. \tag{2.19}$$

This leads to the estimate

$$\begin{aligned}\|A^\alpha U(t)\| &\leqslant \frac{1}{\Gamma(-\alpha)} \Big(\int_0^t \lambda^{-\alpha} \frac{C'_A}{t}\,\mathrm{d}\lambda + \int_t^\infty 2C_A \lambda^{-\alpha-1}\,\mathrm{d}\lambda \Big) \\ &\leqslant \frac{t^{-\alpha}}{\Gamma(-\alpha)} \left(\frac{C'_A}{1-\alpha} + \frac{2C_A}{\alpha} \right).\end{aligned}$$

Therefore we get (2.17) for $0 < \alpha < 1$ by setting $M_\alpha = \Gamma(-\alpha)^{-1}(C'_A/(1-\alpha) + 2C_A/\alpha)$.

For integer powers α, (2.17) follows directly from Proposition 2.11. Notice that by Proposition 2.11, $\mathrm{Ran}(U(t)) \subseteq \mathcal{D}(A^n)$ for $t > 0$. Then result (2.17) follows for any non-integer $\alpha > 1$, from $\mathcal{D}(A^\alpha = A^{\alpha-[\alpha]}A^{[\alpha]}) \supset \mathcal{D}(A^{[\alpha]+1})$, representation (2.18), and the estimate (2.13) of derivatives of order $[\alpha] + 1$. ∎

3. TROTTER PRODUCT FORMULA

3.1. IN A BANACH SPACE. The aim of this section is to prove the operator-norm convergence of the Trotter product formula for holomorphic contraction semigroups in a Banach space. We require that the operator $-A$ generates a holomorphic contraction semigroup, and that:

(H1) $-B$ is the generator of a contraction semigroup;
(H2) there is a real $\alpha \in [0, 1)$ such that $\mathcal{D}(A^\alpha) \subseteq \mathcal{D}(B)$ and $\mathcal{D}(A^*) \subseteq \mathcal{D}(B^*)$.

Notice that we can suppose A boundedly invertible; if it is not the case, let consider $A + \eta$ for some $\eta > 0$, and we have $\mathcal{D}((A+\eta)^\alpha) = \mathcal{D}(A^\alpha) \subseteq \mathcal{D}(B)$ by Proposition 2.15.

REMARK 3.1. We note that the last assumption (introduced by Ichinose and Tamura in [7]) implies that B is relatively bounded with respect to A with the relative bound equals to zero. Indeed, for $\eta > 0$ by $\mathcal{D}(A + \eta) \subseteq \mathcal{D}(A^\alpha) \subseteq \mathcal{D}(B)$ and by Proposition 2.14 one gets (A is supposed boundedly invertible):

$$\|B(A+\eta)^{-1}\| \leqslant \|BA^{-\alpha}\|\,\|A^\alpha(A+\eta)^{-1}\| \leqslant \frac{C_\alpha}{\eta^{1-\alpha}} \|BA^{-\alpha}\|. \tag{3.1}$$

Since the operators A^α and B are closed, the inclusions (H2) are equivalent to A^α-boundedness of B and the A^*-boundedness of B^*. In particular, $\|BA^{-\alpha}\| \leqslant d$ and $\|B^* A^{*-1}\| \leqslant d'$ for some $d, d' > 0$. Therefore for any $x \in \mathcal{D}(A) \subseteq \mathcal{D}(B)$, we have the estimate

$$\|Bx\| \leqslant \frac{C_\alpha \|BA^{-\alpha}\|}{\eta^{1-\alpha}} \|Ax\| + \eta^\alpha C_\alpha \|BA^{-\alpha}\| \, \|x\| \tag{3.2}$$

and the relative bound in (3.2) can be chosen arbitrarily small by the shift $\eta > 0$. For such perturbations we can prove the following result (cf. [8], Chapter IX, Corollary 2.5):

LEMMA 3.2. *Let* e^{-tA} *be a holomorphic contraction semigroup of angle* ω *on* $\mathcal{B}$*, and suppose* B *satisfies* (H1) *and* (H2)*. Then the algebraic sum* $-(A+B)$ *with* $\mathcal{D}(A+B) = \mathcal{D}(A)$ *is the generator of a holomorphic contraction semigroup.*

Proof. First we verify the conditions of Proposition 2.10. Let $\varepsilon > 0$; by (3.2) we have for $|\arg(z)| < \omega + \pi/2 - \varepsilon$

$$\|B(A+z)^{-1}\| \leqslant \frac{C_\alpha \|BA^{-\alpha}\|}{\eta^{1-\alpha}} \|A(A+z)^{-1}\| + \eta^\alpha C_\alpha \|BA^{-\alpha}\| \, \|(A+z)^{-1}\|, \tag{3.3}$$

which leads to

$$\|B(A+z)^{-1}\| \leqslant \frac{C_\alpha \|BA^{-\alpha}\|}{\eta^{1-\alpha}} (1 + N_\varepsilon) + \eta^\alpha C_\alpha \|BA^{-\alpha}\| \frac{N_\varepsilon}{|z|}. \tag{3.4}$$

Therefore the Neumann series for $(A+B+z)^{-1}$ converges if the right hand side of (3.4) is smaller than 1. Thus we can choose η such that the first term in the estimate (3.4) becomes smaller than 1. Then we obtain:

$$\|(A+B+z)^{-1}\| \leqslant \frac{M}{|z-\gamma|} \tag{3.5}$$

for $|\arg(z)| < \omega + \pi/2 - \varepsilon$, where M and γ are some positive constants. By Proposition 2.10 we conclude that $-(A+B)$ generates a quasi-bounded holomorphic semigroup of angle $\omega - \varepsilon$.

On the other hand, A and B are accretive, thus $A+B$ is accretive. But for $x < 0$, $|x|$ sufficiently large ($|x| > \gamma$), x is in the resolvent set of $A+B$, hence we conclude that $-(A+B)$ generates a contraction semigroup, by Proposition 2.5. ∎

The proof of the main theorem of this section involves three technical lemmata. For the two first we suppose only that B and B^* are A-bounded, i.e. there are positive constants a and b such that:

$$\forall x \in \mathcal{D}(A) \subseteq \mathcal{D}(B), \quad \|Bx\| \leqslant a\|Ax\| + b\|x\|, \tag{3.6}$$

$$\forall \varphi \in \mathcal{D}(A^*) \subseteq \mathcal{D}(B^*), \quad \|B^*\varphi\| \leqslant a\|A^*\varphi\| + b\|\varphi\|. \tag{3.7}$$

If A is boundedly invertible, then we can set $b = 0$ with the relative bound $a + b\|A^{-1}\|$ instead of a.

Lemma 3.3. *Let $-A$, boundedly invertible, and $-B$ be generators of bounded semigroups. Let B and B^* be A-bounded as in* (3.6), (3.7) *and suppose that the operator $-H = -(A+B)$ with $\mathcal{D}(H) = \mathcal{D}(A)$ is the boundedly invertible generator of a bounded semigroup. Then there exists a constant L_1 such that for all $\tau \geqslant 0$:*

$$\left\|A^{-1}(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A} - \mathrm{e}^{-\tau(A+B)})\right\| \leqslant L_1\tau, \tag{3.8}$$

$$\left\|(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A} - \mathrm{e}^{-\tau(A+B)})A^{-1}\right\| \leqslant L_1\tau. \tag{3.9}$$

Proof. By virtue of the identity

$$\begin{aligned}A^{-1}(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A} - \mathrm{e}^{-\tau(A+B)}) &= A^{-1}(\mathrm{e}^{-\tau B} - I)\mathrm{e}^{-\tau A} + A^{-1}(\mathrm{e}^{-\tau A} - I)\\ &\quad + A^{-1}HH^{-1}(I - \mathrm{e}^{-\tau H})\end{aligned}$$

and Lemma 2.6 we get (3.8):

$$\begin{aligned}\|A^{-1}(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A} - \mathrm{e}^{-\tau H})\| &\leqslant \left\|\int_0^\tau \mathrm{d}s\, A^{-1}B\mathrm{e}^{-sB}\right\| + \left\|A^{-1}\left(\mathrm{e}^{-\tau A} - I\right)\right\|\\ &\quad + \|A^{-1}H\|\,\|H^{-1}(I - \mathrm{e}^{-\tau H})\|\\ &\leqslant \|A^{-1}B\|C_B\tau + C_A\tau + \|A^{-1}H\|C_H\tau.\end{aligned}$$

Finally, we remark that (3.7) implies the boundedness of the closed operator $A^{-1}B$, and that $\|A^{-1}H\| \leqslant \|I + A^{-1}B\| \leqslant 1 + a + b\|A^{-1}\|$. To prove (3.9) one has to use (3.6), and the same line of reasoning as above to put finally $L_1 = C_B a' + C_A + C_H(1 + a')$ where $a' = a + b\|A^{-1}\|$. ∎

Lemma 3.4. *Let A, B and $H = A + B$ be the same as in Lemma* 3.3. *Then there exists a constant L_2 such that for all $\tau \geqslant 0$:*

$$\left\|A^{-1}(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A} - \mathrm{e}^{-\tau(A+B)})A^{-1}\right\| \leqslant L_2\tau^2, \tag{3.10}$$

$$\left\|A^{-1}(\mathrm{e}^{-\tau A}\mathrm{e}^{-\tau B} - \mathrm{e}^{-\tau(A+B)})A^{-1}\right\| \leqslant L_2\tau^2. \tag{3.11}$$

Proof. By virtue of

$$\begin{aligned}\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A} - \mathrm{e}^{-\tau H} &= \left(I - \mathrm{e}^{-\tau B}\right)\left(I - \mathrm{e}^{-\tau A}\right) + \left(\mathrm{e}^{-\tau A} - (I + \tau A)^{-1}\right)\\ &\quad + \left(\mathrm{e}^{-\tau B} - (I + \tau B)^{-1}\right) + \left((I + \tau H)^{-1} - \mathrm{e}^{-\tau H}\right)\\ &\quad + \tau H(I + \tau H)^{-1} - \tau A(I + \tau A)^{-1} - \tau B(I + \tau B)^{-1}\end{aligned}$$

and the identity

$$\begin{aligned}&A^{-1}\left(\tau H(I + \tau H)^{-1} - \tau A(I + \tau A)^{-1} - \tau B(I + \tau B)^{-1}\right)A^{-1}\\ &\qquad = \tau^2\left((I + \tau A)^{-1} + A^{-1}B(I + \tau B)^{-1}BA^{-1} - A^{-1}H(I + \tau H)^{-1}HA^{-1}\right),\end{aligned}$$

we get the representation

$$\begin{aligned}A^{-1}(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A} - \mathrm{e}^{-\tau H})A^{-1} &= A^{-1}(I - \mathrm{e}^{-\tau B})(I - \mathrm{e}^{-\tau A})A^{-1}\\ &\quad + \left(\mathrm{e}^{-\tau A} - (I + \tau A)^{-1}\right)A^{-2} + A^{-1}\left(\mathrm{e}^{-\tau B} - (I + \tau B)^{-1}\right)A^{-1}\\ &\quad + A^{-1}H\left((I + \tau H)^{-1} - \mathrm{e}^{-\tau H}\right)H^{-2}HA^{-1} + \tau^2(I + \tau A)^{-1}\\ &\quad + \tau^2A^{-1}B(I + \tau B)^{-1}BA^{-1} - \tau^2A^{-1}H(I + \tau H)^{-1}HA^{-1},\end{aligned}$$

which gives the following estimate:

$$\frac{1}{\tau^2}\|A^{-1}(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A}-\mathrm{e}^{-\tau H})A^{-1}\| \leqslant \frac{1}{\tau}\|A^{-1}B\|\left\|\int_0^\tau \mathrm{d}s\mathrm{e}^{-sB}\right\|\frac{1}{\tau}\|(I-\mathrm{e}^{-\tau A})A^{-1}\|$$
$$+\frac{1}{\tau^2}\|(\mathrm{e}^{-\tau A}-(I+\tau A)^{-1})A^{-2}\|$$
$$+\frac{1}{\tau^2}\|A^{-1}B\|\left\|\int_0^\tau \mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\mathrm{e}^{-s_2 B}-\tau^2(I+\tau B)^{-1}\right\|\|BA^{-1}\|$$
$$+\frac{1}{\tau^2}\|A^{-1}H\|\,\|((I+\tau H)^{-1}-\mathrm{e}^{-\tau H})H^{-2}\|\,\|HA^{-1}\|$$
$$+1+\|A^{-1}B\|\,\|BA^{-1}\|+\|A^{-1}H\|\,\|HA^{-1}\|.$$

Now Lemmata 2.6 and 2.8 together with (3.6), (3.7) imply (3.10): we can take $L_2=a'C_AC_B+3C_A/2+3C_Ba'^2/2+3C_H(1+a')^2/2+1+a'^2+(1+a')^2$ with $a'=a+b\|A^{-1}\|$. Similarly one gets (3.11). ∎

In the next lemma we need the Ichinose-Tamura condition (H2) and contraction semigroups.

LEMMA 3.5. *Let $-A$ be the boundedly invertible generator of a holomorphic contraction semigroup. If $-B$ is the generator of a contraction semigroup and there exists $\alpha\in[0,1)$ such that $\mathcal{D}(A^\alpha)\subseteq\mathcal{D}(B)$, then for any $k\geqslant 1$ and $\tau>0$:*

$$\|(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A})^kA\| \leqslant \frac{L_3}{\tau^\alpha}+\frac{C'_A}{k\tau},\quad \alpha>0, \tag{3.12}$$

$$\|(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A})^kA\| \leqslant \widetilde{L}_3(1+\ln k)+\frac{C'_A}{k\tau},\quad \alpha=0. \tag{3.13}$$

Proof.

$$\begin{aligned}\big\|(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A})^kA\big\| &\leqslant \big\|\big((\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A})^k-\mathrm{e}^{-k\tau A}\big)A\big\|+\big\|\mathrm{e}^{-k\tau A}A\big\|\\ &\leqslant \left\|\sum_{j=0}^{k-1}\left(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A}\right)^{k-1-j}\left(\mathrm{e}^{-\tau B}-I\right)\mathrm{e}^{-\tau A}\mathrm{e}^{-j\tau A}A\right\|+\big\|\mathrm{e}^{-k\tau A}A\big\|\\ &\leqslant \sum_{j=0}^{k-1}\left\|\int_0^\tau \mathrm{d}s\,\mathrm{e}^{-sB}BA^{-\alpha}\right\|\|A^\alpha\mathrm{e}^{-(j+1)\tau A}A\|+\|\mathrm{e}^{-k\tau A}A\|.\end{aligned}$$

Notice that the second inequality is in particular due to contractiveness of e^{-tA} and e^{-tB}, and to equation (2.3) of Lemma 2.6. From the hypothesis $\mathcal{D}(A^\alpha)\subseteq\mathcal{D}(B)$ we deduce (cf. Remark 3.1) that $\|BA^{-\alpha}\|\leqslant d$. By Propositions 2.11 and 2.16 we get respectively:

$$\|\mathrm{e}^{-k\tau A}A\|\leqslant\frac{C'_A}{k\tau}\quad\text{and}\quad\|A^{1+\alpha}\mathrm{e}^{-(j+1)\tau A}\|\leqslant\frac{M_\alpha}{((j+1)\tau)^{1+\alpha}}.$$

Therefore, we conclude that:

$$\big\|(\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A})^k A\big\| \leqslant \frac{M_\alpha d}{\tau^\alpha}\sum_{j=0}^{k-1}\frac{1}{(j+1)^{1+\alpha}} + \frac{C_A'}{k\tau}.$$

Since $\alpha > 0$, this gives the annonced result (3.12) with

$$L_3 = dM_\alpha \sum_{j=1}^{\infty}(1/j)^{1+\alpha},$$

and (3.13) for $\alpha = 0$ with $\widetilde{L}_3 = \|B\|C_A'$. ∎

Since $\mathcal{D}(A^\alpha) \subseteq \mathcal{D}(B)$ implies $\mathcal{D}(A^{\alpha'}) \subseteq \mathcal{D}(B)$ for $\alpha' \geqslant \alpha$, the estimate (3.12) is valid in fact for any $\alpha' \geqslant \alpha$.

THEOREM 3.6. *Let* e^{-tA} *be a holomorphic contraction semigroup. If* $-B$ *is the generator of a contraction semigroup, and there exists* $\alpha \in [0,1)$ *such that* $\mathcal{D}(A^\alpha) \subseteq \mathcal{D}(B)$ *and* $\mathcal{D}(A^*) \subseteq \mathcal{D}(B^*)$*, then there are constants* $M_1, M_2, \widetilde{M}_2, \eta > 0$*, such that for any* $t \geqslant 0$ *and* $n > 2$*:*

$$\big\|(\mathrm{e}^{-tB/n}\mathrm{e}^{-tA/n})^n - \mathrm{e}^{-t(A+B)}\big\| \leqslant (M_1 + M_2 t^{1-\alpha})\mathrm{e}^{\eta t}\frac{\ln n}{n^{1-\alpha}}, \quad \alpha > 0, \tag{3.14}$$

$$\big\|(\mathrm{e}^{-tB/n}\mathrm{e}^{-tA/n})^n - \mathrm{e}^{-t(A+B)}\big\| \leqslant (M_1 + \widetilde{M}_2 t)\mathrm{e}^{\eta t}\frac{2(\ln n)^2}{n}, \quad \alpha = 0. \tag{3.15}$$

Proof. Since B satisfies (H1) and (H2), by Lemma 3.2 the operator $-H = -(A+B)$ is generator of a holomorphic contraction semigroup. If the operator A has no bounded inverse, let $\widetilde{A} = A + \eta$ and $\widetilde{H} = \widetilde{A} + B$ for some $\eta > 0$ (see Remark 2.2). Then they are boundedly invertible. As we indicated at the beginning of the section, these changes of generators do not modify the domain inclusions. If we want to obtain $\|B\widetilde{A}^{-1}\| < 1$ then by the estimate (3.1) we have to choose a sufficiently large shift η. This gives us the estimate $\|\widetilde{A}\widetilde{H}^{-1}\| = \|(I + B\widetilde{A}^{-1})^{-1}\| \leqslant 1/(1-a)$ where we set $a = \|B\widetilde{A}^{-1}\|$.

Now we put $\tau = t/n$, $\widetilde{U}(t) = \mathrm{e}^{-t\widetilde{H}}$, and $\widetilde{T}(\tau) = \mathrm{e}^{-\tau B}\mathrm{e}^{-\tau\widetilde{A}}$. To estimate the left-hand side of (3.14) we use

$$(\mathrm{e}^{-tB/n}\mathrm{e}^{-tA/n})^n - \mathrm{e}^{-t(A+B)} = (\widetilde{T}^n(\tau) - \widetilde{U}^n(\tau))\mathrm{e}^{t\eta}$$

and the identity:

$$\begin{aligned}\widetilde{T}(\tau)^n - \widetilde{U}(\tau)^n &= \sum_{m=0}^{n-1}\widetilde{T}(\tau)^{n-m-1}(\widetilde{T}(\tau) - \widetilde{U}(\tau))\widetilde{U}(\tau)^m \\ &= \widetilde{T}(\tau)^{n-1}\widetilde{A}\widetilde{A}^{-1}(\widetilde{T}(\tau) - \widetilde{U}(\tau)) + (\widetilde{T}(\tau) - \widetilde{U}(\tau))\widetilde{A}^{-1}\widetilde{A}\widetilde{H}^{-1}\widetilde{H}\widetilde{U}(\tau)^{n-1} \\ &\quad + \sum_{m=1}^{n-2}\widetilde{T}(\tau)^{n-m-1}\widetilde{A}\widetilde{A}^{-1}(\widetilde{T}(\tau) - \widetilde{U}(\tau))\widetilde{A}^{-1}\widetilde{A}\widetilde{H}^{-1}\widetilde{H}\widetilde{U}(\tau)^m,\end{aligned}$$

which implies:

$$
\begin{aligned}
&\big\|\widetilde{T}(\tau)^n - \widetilde{U}(\tau)^n\big\| \\
&\quad \leqslant \|\widetilde{T}(\tau)^{n-1}\widetilde{A}\|\,\|\widetilde{A}^{-1}(\widetilde{T}(\tau)-\widetilde{U}(\tau))\| + \|(\widetilde{T}(\tau)-\widetilde{U}(\tau))\widetilde{A}^{-1}\|\,\|\widetilde{A}\widetilde{H}^{-1}\|\,\|\widetilde{H}\widetilde{U}(\tau)^{n-1}\| \\
&\qquad + \sum_{m=1}^{n-2}\|\widetilde{T}(\tau)^{n-m-1}\widetilde{A}\|\,\|\widetilde{A}^{-1}(\widetilde{T}(\tau)-\widetilde{U}(\tau))\widetilde{A}^{-1}\|\,\|\widetilde{A}\widetilde{H}^{-1}\|\,\|\widetilde{H}\widetilde{U}(\tau)^{m}\|.
\end{aligned}
$$

Hence by Lemmata 3.3, 3.4, and 3.5 (it is at this point that we use the hypothesis of contractiveness), and Proposition 2.11 we obtain the estimate :

$$
\begin{aligned}
\|\widetilde{T}(\tau)^n - \widetilde{U}(\tau)^n\| &\leqslant \left(\frac{L_3}{\tau^\alpha} + \frac{C'_A}{(n-1)\tau}\right)L_1\tau + \frac{L_1}{1-a}\frac{C'_H}{n-1} \\
&\quad + \sum_{m=1}^{n-2}\left(L_3\tau^{1-\alpha} + \frac{C'_A}{n-m-1}\right)\frac{L_2}{1-a}\frac{C'_H}{m} \\
&\leqslant L_3L_1\frac{t^{1-\alpha}}{n^{1-\alpha}} + \frac{L_1}{n-1}\left(C'_A + \frac{C'_H}{1-a}\right) + \frac{L_3L_2C'_H}{1-a}\frac{t^{1-\alpha}}{n^{1-\alpha}}\sum_{m=1}^{n-2}\frac{1}{m} \\
&\quad + \frac{L_2C'_HC'_A}{1-a}\sum_{m=1}^{n-2}\frac{1}{n-m-1}\cdot\frac{1}{m} \\
&\leqslant L_3L_1\frac{t^{1-\alpha}}{n^{1-\alpha}} + \frac{L_1}{n-1}\left(C'_A + \frac{C'_H}{1-a}\right) \\
&\quad + 2\frac{L_3L_2C'_H}{1-a}t^{1-\alpha}\frac{\ln n}{n^{1-\alpha}} + 4\frac{L_2C'_HC'_A}{1-a}\frac{\ln n}{n}.
\end{aligned}
\tag{3.16}
$$

Here we used:

$$
\sum_{m=1}^{n-1}\frac{1}{(n-m)m} = \frac{2}{n}\sum_{m=1}^{n-1}\frac{1}{m} \leqslant \frac{2}{n}(1+\ln(n-1)) \leqslant 4\frac{\ln n}{n}.
$$

The estimate (3.16) implies the announced result (3.14) for $\alpha > 0$, with $M_1 = 4L_1\left(C'_A + \frac{C'_H}{1-a}\right) + 4\frac{L_2C'_HC'_A}{1-a}$ and $M_2 = 2L_3L_1 + 2\frac{L_3L_2C'_H}{1-a}$. In a similar way for $\alpha = 0$ one gets:

$$
\begin{aligned}
\|\widetilde{T}(\tau)^n - \widetilde{U}(\tau)^n\| &\leqslant \widetilde{L}_3(1+\ln(n-1))L_1\frac{t}{n} + \frac{L_1C'_A}{n-1} + \frac{L_1C'_H}{1-a}\frac{1}{n-1} \\
&\quad + \sum_{m=1}^{n-2}\left(\widetilde{L}_3\frac{t}{n}(1+\ln(n-m-1)) + \frac{C'_A}{n-m-1}\right)\frac{L_2}{1-a}\frac{C'_H}{m}.
\end{aligned}
$$

This estimate gives (3.15) with $\widetilde{M}_2 = 2\widetilde{L}_3L_1 + 2\frac{\widetilde{L}_3L_2C'_H}{1-a}$. ∎

COROLLARY 3.7. *Let* e^{-tA} *be a holomorphic contraction semigroup. If* $-B$ *is the generator of a contraction semigroup, and there exists* $\alpha \in [0,1)$ *such that* $\mathcal{D}((A^\alpha)^*) \subseteq \mathcal{D}(B^*)$ *and* $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ *(and* $\mathcal{D}(A^*) \subseteq \mathcal{D}(B^*)$ *if* $\mathcal{B}$ *is not reflexive),* *then there are constants* $M_3, M_4, \widetilde{M}_4, \eta > 0$*, such that for any* $t \geqslant 0$ *and* $n > 2$*:*

$$\big\|(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n})^n - \mathrm{e}^{-t(A+B)}\big\| \leqslant (M_3 + M_4 t^{1-\alpha})\mathrm{e}^{\eta t}\frac{\ln n}{n^{1-\alpha}}, \quad \alpha > 0, \tag{3.17}$$

$$\big\|(\mathrm{e}^{-tA/n}\mathrm{e}^{-tB/n})^n - \mathrm{e}^{-t(A+B)}\big\| \leqslant (M_3 + \widetilde{M}_4 t)\mathrm{e}^{\eta t}\frac{2(\ln n)^2}{n}, \quad \alpha = 0. \tag{3.18}$$

Proof. Let $\widetilde{F}(\tau) = \mathrm{e}^{-\tau\widetilde{A}}\mathrm{e}^{-\tau B}$. Then by the same arguments as in the proof of Theorem 3.6, one gets:

$$\begin{aligned}\widetilde{U}(\tau)^n - \widetilde{F}(\tau)^n &= \sum_{m=0}^{n-1}\widetilde{U}(\tau)^{n-m-1}(\widetilde{U}(\tau)-\widetilde{F}(\tau))\widetilde{F}(\tau)^m \\ &= \widetilde{U}(\tau)^{n-1}\widetilde{H}\widetilde{H}^{-1}\widetilde{A}\widetilde{A}^{-1}(\widetilde{U}(\tau)-\widetilde{F}(\tau)) + (\widetilde{U}(\tau)-\widetilde{F}(\tau))\widetilde{A}^{-1}\widetilde{A}\widetilde{F}(\tau)^{n-1} \\ &\quad + \sum_{m=1}^{n-2}\widetilde{U}(\tau)^{n-m-1}\widetilde{H}\widetilde{H}^{-1}\widetilde{A}\widetilde{A}^{-1}(\widetilde{U}(\tau)-\widetilde{F}(\tau))\widetilde{A}^{-1}\widetilde{A}\widetilde{F}(\tau)^m.\end{aligned}$$

Notice that Lemmata 3.3 and 3.4 hold for $\widetilde{F}(\tau)$. By a simple modification of Lemma 3.5, where one uses $\|\widetilde{A}^{-\alpha}B\| = \|B^*(\widetilde{A}^{-\alpha})^*\| < \infty$, we get:

$$\begin{aligned}\big\|\widetilde{A}(\mathrm{e}^{-\tau\widetilde{A}}\mathrm{e}^{-\tau B})^k\big\| &\leqslant \frac{L_4}{\tau^\alpha} + \frac{C'_A}{k\tau}, \quad \alpha > 0, \\ \big\|\widetilde{A}(\mathrm{e}^{-\tau\widetilde{A}}\mathrm{e}^{-\tau B})^k\big\| &\leqslant \widetilde{L}_4(1+\ln k) + \frac{C'_A}{k\tau}, \quad \alpha = 0.\end{aligned}$$

These ingredients ensure the estimates (3.17) and (3.18). ∎

Corollary 3.8. *Under the same conditions as in Theorem* 3.6*, we have the operator-norm convergence of a symmetrized Trotter formula, i.e. there exist* $M_5, M_6, \widetilde{M}_6, \eta > 0$*, such that for any* $t \geqslant 0$ *and* $n > 2$*:*

$$\big\|(\mathrm{e}^{-tA/2n}\mathrm{e}^{-tB/n}\mathrm{e}^{-tA/2n})^n - \mathrm{e}^{-t(A+B)}\big\| \leqslant (M_5 + M_6 t^{1-\alpha})\mathrm{e}^{\eta t}\frac{\ln n}{n^{1-\alpha}}, \ \alpha > 0 \tag{3.19}$$

$$\big\|(\mathrm{e}^{-tA/2n}\mathrm{e}^{-tB/n}\mathrm{e}^{-tA/2n})^n - \mathrm{e}^{-t(A+B)}\big\| \leqslant (M_5 + \widetilde{M}_6 t)\mathrm{e}^{\eta t}\frac{2(\ln n)^2}{n}, \ \alpha = 0. \tag{3.20}$$

Proof. Since Lemmata 3.3, 3.4, and 3.5 can be easily extend to the symmetrized product $\mathrm{e}^{-\tau A/2}\mathrm{e}^{-\tau B}\mathrm{e}^{-\tau A/2}$, the proof of the Theorem 3.6 carries through verbatim to obtain (3.19) and (3.20). ∎

3.2. In a Hilbert space. Up to our knowledge, all preceding results concerning operator-norm convergence of the Trotter or Trotter-Kato product formulae require semibounded self-adjoint generators $-A$ and $-B$ in a Hilbert space ([3]–[7], [9]–[12]). The results of Section 3.1 are evidently valid in a Hilbert space $\mathcal{B} = \mathcal{H}$. This permits to compare them with the above results concerning the operator-norm convergence of the Trotter formula.

In the case of a Hilbert space, we have generalized the operator-norm convergence to non-self-adjoint (moreover non-normal) semigroups. This indicates that normality is not indispensable for the operator-norm convergence of the Trotter formula. In fact, the evident advantage is technical: one uses the spectral theorem, which is widely exploited in [5]–[7], [9]–[12]. On the other hand notice that non-negative self-adjoint operators generate holomorphic semigroups of angle $\pi/2$, contractive on the entire half-plane (moreover these holomorphic semigroups have a strong continuous extension on $\mathrm{i}\mathbb{R}$, which is a unitary group). Since they are continuous in the operator-norm topology for $t > 0$, the holomorphic contraction semigroups seem to be the natural generalization of the self-adjoint ones, in order to prove the operator-norm convergence of the Trotter formula.

Notice that our generalization to the non-self-adjoint semigroups is obtained under Ichinose-Tamura hypothesis (H1) and (H2). They are stronger than those of [9] where, instead of 0, B has relative bound less than 1 with respect to A. In [10], these conditions are extended to $\mathcal{D}(A^{\alpha}) \subseteq \mathcal{D}(B^{\alpha})$ for some $1/2 < \alpha < 1$, which does not imply that B is relatively bounded by A nor A by B. Hiroshi Tamura ([14]) has recently shown that the error bound in [10] is optimal, and that operator-norm convergence cannot be extended to $\alpha \leqslant 1/2$ without supplementary conditions ([11]).

Finally, we note that the error bounds we found are less optimal than those of [9] and [10].

4. CONCLUSION

We generalize the Trotter-Chernoff results ([16], [1]) to the operator-norm convergence in a Banach space assuming B with zero A-bound and e^{-tA} holomorphic. This shows that the hypothesis of self-adjointness in the case of a Hilbert space has only a technical importance. On the other hand the operator-norm topology is "natural" for holomorphic semigroups, which leads to think that it is an essential hypothesis. We would like also to remark that the contractiveness assumption is not as superfluous as one could suppose. For demonstration we address the reader to an instructive example by Trotter ([16]) where it is called "the norm condition".

Our results could find applications in quantum mechanics and in the theory of Schrödinger semigroups ([3]–[5], [7]). Let $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$ be the Hilbert space of square integrable functions on $\mathbb{R}^d$, $H = -\Delta + \lambda V$ where Δ denotes the Laplace operator and V a real potential with a complex coupling constant λ. If V is relatively bounded with respect to $(-\Delta)^{\alpha}$ for some $\alpha \in [0, 1)$, then these operators satisfy the hypothesis of the Theorem 3.6. However, for real coupling constants λ the results of [3]–[7] or [9]–[10] are more efficient.

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VINCENT CACHIA
Centre de Physique Théorique
UPR 7061
CNRS Luminy, Case 907
F–13288 Marseille Cedex 9
FRANCE

and

Université de la Méditerranée
(Aix-Marseille II*)*
(Allocataire-moniteur)
FRANCE
E-mail: cachia@cpt.univ-mrs.fr

VALENTIN A. ZAGREBNOV
Centre de Physique Théorique
UPR 7061
CNRS Luminy, Case 907
F–13288 Marseille Cedex 9
FRANCE

and

Université de la Méditerranée
(Aix-Marseille II*)*
FRANCE
E-mail: zagrebnov@cpt.univ-mrs.fr

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