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NORMALITY, NON-QUASIANALYTICITY AND INVARIANT SUBSPACES

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Abstract. We prove that some classes of functions, defined on a closed set in the complex plane with planar Lebesgue measure zero, are non-quasianalytic. We particularly treat the Carleman classes and classes of functions having asymtotically holomorphic continuation. Combining this with Dyn'kin's functional calculus based on the Cauchy-Pompeiu formula, we establish the existence of invariant subspaces for operators for which a part of the spectrum is of planar Lebesgue measure zero, provided that the resolvent has a moderate growth near this part of the spectrum.

Keywords: *Invariant subspaces, operators, non-quasianalytic classes, Carleman classes.*

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1. INTRODUCTION

Let E be a compact set in the complex plane $\mathbb{C}$ with planar Lebesgue measure zero and let M be a non-increasing function on $(0,+\infty)$ with $M(0+)=+\infty$.

Consider a Banach space A of continuous functions defined on E, continuously embedded in $C(E)$, where $C(E)$ is the Banach space of all continuous functions defined on E. Suppose that A contains the constants and that for every $\lambda\in\mathbb{C}\setminus E$ the function $r_\lambda(z)=\frac{1}{\lambda-z}$, $z\in E$, belongs to A and $\|r_\lambda\|_A\leqslant M(d(\lambda,E))$.

In this paper we determine certain hypotheses concerning the function M which imply that A is non-quasianalytic, that is, for every $\zeta\in E$ and every neighborhood V of ζ in E, there exists a function $f\in A$ such that $f(\zeta)=1$ and $f\equiv 0$ on $E\setminus V$.

As we may expect, this condition depends on the geometrical properties of E. In order to state our result, let us introduce a function related to E which will play a decisive role throughout this paper. We set

$$\theta_E(x)=m_2(\{z\in\mathbb{C}:d(z,E)<x\}),\quad x>0,$$

where m_2 denotes a Lebesgue planar measure. The function θ_E is continuous and increasing. Since we assume that $m_2(E) = 0$, we have $\theta_E(0+) = 0$. We prove that, under the condition

$$\int_0 \left(\ln \ln M \circ \theta_E^{-1}(x^2)\right)^{1/2} \,\mathrm{d}x < +\infty \tag{1.1}$$

the Banach space A is non-quasianalytic.

The key to the proof of this theorem is the normality of some classes of holomorphic functions which we obtain as a simple consequence of Lomonosov, Ljubich, Matsaev ([25]) and Domar's ([12]) results (see Subsection 2.1).

We give an application of the above result concerning non-quasianalyticity to classes of functions having asymptotically holomorphic continuations and to the Carleman classes $C_E(M_n)$. We obtain in particular, that if E is a rectifiable arc and if the sequence $\left(\frac{M_n}{n!}\right)_{n\geqslant 0}$ is log-convex then the condition

$$\sum_{n\geqslant 1} \left(M_n^{1/n} \ln M_n\right)^{-1/2} < +\infty$$

is sufficient for the non-quasianalyticity of $C_E(M_n)$. On the other hand, it is known that the condition

$$\sum_{n\geqslant 1} M_n^{-1/n} < +\infty \tag{1.2}$$

is necessary ([6] and [10]). Note also that the classical Denjoy-Carleman theorem asserts that if $E = [-1, 1]$ and if $(M_n)_{n\geqslant 0}$ is log-convex then condition (1.2) is necessary and sufficient for the non-quasianalyticity of $C_{[-1,1]}(M_n)$ ([27]).

This work was also motivated by the invariant subspace problem. Let X be a Banach space and let $T \in \mathcal{L}(X)$, where $\mathcal{L}(X)$ stands for the algebra of bounded operators acting on X. A closed subspace Y of X is called *invariant* for $T \in \mathcal{L}(X)$ if $TY \subset Y$ and nontrivial if $\{0\} \subsetneqq Y \subsetneqq X$. The subspace Y is called *hyperinvariant* for T if it is invariant for every operator commuting with T. The spectrum of T is denoted by $\mathrm{Sp}(T)$. Lyubich and Matsaev proved in [26] that if there exists an open set O in $\mathbb{C}$ and a smooth arc E such that $\mathrm{Sp}(T) \cap O = E \cap O$ and if the function

$$M(x) = \sup\{\|(z - T)^{-1}\| : d(z, E) \geqslant x, z \in O\}, \quad x > 0, \tag{1.3}$$

satisfies the Levinson condition, that is:

$$\int_0 \ln \ln M(x) \,\mathrm{d}x < +\infty,$$

then T has a nontrivial hyperinvariant subspace.

The condition "E is a smooth arc" may be replaced by the weaker condition "E is a Lipschitz arc" (see [30]). We extend this result to operators with spectrum not necessarily as "smooth" provided that the resolvent grows moderately near the spectrum. More precisely, suppose that there exist an open set O and a perfect compact set E in $\mathbb{C}$ such that $m_2(E) = 0$, $\mathrm{Sp}(T) \cap O \subset E \cap O$ and $\mathrm{Sp}(T) \cap O$

contains at least two distinct points. If the function defined by (1.3) satisfies condition (1.1) and if θ_E satisfies the condition

$$\int_0 \left(\ln\ln \frac{1}{\theta_E^{-1}(x^2)}\right)^{1/2} \mathrm{d}x < +\infty$$

then T has a nontrivial hyperinvariant subspace.

To obtain this theorem we combine the functional calculus based on the Cauchy-Pompeiu formula, introduced by Dyn'kin in [14], and a result concerning the non-quasianalyticity of classes of functions having asymptotically holomorphic continuations.

This paper is organized as follows:

In Section 2 we establish the normality of some classes of subharmonic and holomorphic functions and afterwards we give a general method showing how to get the non-quasianalyticity of some Banach spaces of continuous functions.

In Section 3 we give two examples of non-quasianalytic classes of functions: the classes of functions having asymptotically holomorphic continuations and the Carleman classes.

Section 4 is devoted to the existence of invariant subspaces and to operators which are decomposable.

We finish with an appendix where we investigate the properties of the function θ_E and we give an estimate of this function in some special cases.

2. NORMALITY AND NON-QUASIANALYTICITY

2.1. ON NORMALITY. Let Ω be an open subset of the complex plane $\mathbb{C}$ and let F be a measurable function on Ω with values in $[1, +\infty]$. The distribution function of F is defined by

$$\lambda(t) = m_2(\{z \in \Omega : F(z) > t\}), \quad t > 0,$$

where m_2 denotes a Lebesgue planar measure. The decreasing rearrangement of F is the function defined by

$$F^*(x) = \inf\{t : \lambda(t) \leqslant x\}, \quad x > 0.$$

Clearly F^* is a decreasing non-negative function which is continuous on the right. Moreover for every $a > 0$, the set $\{x > 0 : F^*(x) > a\}$ is an interval with length equal to $m_2(\{z \in \Omega : F(z) > a\})$, which means that the two functions F and F^* have the same distribution functions.

We denote by $\mathrm{SH}_F(\Omega)$ the set of all non-negative subharmonic functions defined on Ω, such that

$$u(z) \leqslant F(z), \quad z \in \Omega.$$

Domar showed in [11] and [12] that under some conditions on F (or F^*), $\mathrm{SH}_F(\Omega)$ is a normal family.

The following theorem is proved in more general setting in [12].

THEOREM 2.1. (Domar) *If for some* $\delta > 0$,

$$\int_0^\delta (\ln F^*(x^2))^{1/2}\,\mathrm{d}x < +\infty, \tag{2.1}$$

then $\mathrm{SH}_F(\Omega)$ *is a normal family.*

REMARK 2.2. It is shown in [12] that under certain regularity conditions on F^*, the inequality (2.1) is sharp. It is also shown that if $F = G \circ \rho$ where G is a measurable function from $\mathbb{R}$ to $[1, +\infty]$ and ρ a C^1 mapping of rank 1 from Ω to $\mathbb{R}$, then the inequality (2.1) can be replaced by the following weaker condition

$$\int_0^\delta \ln F^*(x)\,\mathrm{d}x < +\infty. \tag{2.2}$$

The particular case $F(z) = G(y)$, $z = x + \mathrm{i}y$, where G is a decreasing function, is known as the Levinson-Sjöberg theorem and the inequality (2.2) becomes

$$\int_0^\delta \ln G(y)\,\mathrm{d}y < +\infty.$$

We consider here the case $F(z) = G(d(z,E))$, G being a decreasing function on $(0, +\infty)$, E a closed subset of $\mathbb{C}$ and $d(z,E)$ the distance of z to E.

We use Theorem 2.1 to derive a condition related to the geometry of E, sufficient for the normality of $\mathrm{SH}_F(\Omega)$. The geometrical property of E in which we are interested may be evaluated by the behaviour of the function θ_E at 0. Recall that θ_E is defined by the formula:

$$\theta_E(x) = m_2(\{z \in \mathbb{C} : d(z,E) < x\}), \quad x > 0.$$

It is clear that θ_E is an increasing and positive function on $(0, +\infty)$ with $\theta_E(0+) = m_2(E)$. Furthermore, by Proposition 5.1 in the Appendix, θ_E is continuous.

Now we state the following result.

THEOREM 2.3. *Let E be a compact subset of $\mathbb{C}$ with planar Lebesgue measure zero and let $F(z) = G(d(z,E))$, $z \in \Omega$, G being a non-increasing function on $(0, +\infty)$ with values in $[1, +\infty]$. If for some $\delta > 0$,*

$$\int_0^\delta \left(\ln G \circ \theta_E^{-1}(x^2)\right)^{1/2}\,\mathrm{d}x < +\infty, \tag{2.3}$$

then $\mathrm{SH}_F(\Omega)$ *is a normal family.*

Proof. Consider the distribution function λ of F and take $x > 0$. We have

$$\begin{aligned}
\lambda(G \circ \theta_E^{-1}(x)) &= m_2(\{z \in \Omega : F(z) > G \circ \theta_E^{-1}(x)\}) \\
&= m_2(\{z \in \Omega : G(d(z,E)) > G \circ \theta_E^{-1}(x)\}) \\
&\leqslant m_2(\{z \in \Omega : d(z,E) < \theta_E^{-1}(x)\}) \\
&\leqslant m_2(\{z \in \mathbb{C} : d(z,E) < \theta_E^{-1}(x)\}) = x.
\end{aligned}$$

Since $F^*(x) = \inf\{a : \lambda(a) \leqslant x\}$, we get

$$F^*(x) \leqslant G \circ \theta_E^{-1}(x).$$

We obtain from the last inequality and from (2.3) that

$$\int_0^\delta (\ln F^*(x^2))^{1/2}\,\mathrm{d}x < +\infty.$$

Now the proof follows from Theorem 2.1. ∎

The following statement may be obtained easily from a result announced in [25].

Corollary 2.4. *Let E be a compact subset of $\mathbb{C}$ with planar Lebesgue measure zero and let M be a non-increasing function on* $(0, +\infty)$ *with values in* $[\mathrm{e}, +\infty]$. *If for some $\delta > 0$,*

$$\int_0^\delta \left(\ln\ln M \circ \theta_E^{-1}(x^2)\right)^{1/2}\,\mathrm{d}x < +\infty, \tag{2.4}$$

then the family

$$\mathcal{H}_M(\Omega) = \{f : f \textit{ holomorphic in } \Omega \textit{ and } |f(z)| \leqslant M(d(z,E)), z \in \Omega\}$$

is normal.

Proof. It is well known that if f is holomorphic then the function $\ln^+|f| = \max\{0, \ln|f|\}$ is subharmonic. So

$$\{\ln^+|f| : f \in \mathcal{H}_M(\Omega)\} \subset \mathrm{SH}_F(\Omega),$$

where $F(z) = \ln M(d(z,E))$, $z \in \Omega$. The proof, now, is a direct consequence of Theorem 2.3 applied to the function $G = \ln M$. ∎

Remark 2.5. If E is not bounded, Theorem 2.3 and Corollary 2.4 remain true provided that E satisfies the condition (2.3) or (2.4) locally. This means that for every $z \in E$ there exists a bounded neighborhood V of z such that if we set $K = \overline{V} \cap E$ then θ_K satisfies the corresponding hypothesis.

2.2. On non-quasianalyticity.

We first define the meaning of non-quasianalyticity. For this, let E be a closed subset of $\mathbb{C}$ and let $\mathcal{F}$ be a family of functions defined on E. $\mathcal{F}$ is called *non-quasianalytic* if for every $\zeta \in E$ and every open subset Ω of $\mathbb{C}$ with $\zeta \in \Omega$, there exists a function $f \in \mathcal{F}$ such that $f(\zeta) \neq 0$ and $f|E \setminus \Omega \equiv 0$.

The Banach algebras of functions that are non-quasianalytic are also called regular algebras (see [21], Chapter VIII, Section 5). Examples of non-quasianalytic families of functions are given in the next section. We give here a general scheme showing how to get non-quasianalyticity.

Let A be a Banach space of functions defined on a closed subset E of $\mathbb{C}$. We suppose that A satisfies the following conditions:

(i) The constant functions are contained in A.

(ii) For every $z \in E$, the pointwise evaluation $\tau_z : f \to f(z)$ is continuous from A to $\mathbb{C}$.

(iii) For every $\lambda \in \mathbb{C} \setminus E$, the functions r_λ defined by $r_\lambda(z) = \frac{1}{\lambda - z}$, $z \in E$, belong to A and the map $\lambda \to r_\lambda$ is continuous.

Now we state a theorem concerning the non-quasianalyticity of A. The idea of the proof is based on an argument used in [15] and [17] and called the *duality principle of Matsaev.*

Theorem 2.6. *Let A be a Banach space of functions on E satisfying the above conditions* (i)–(iii)*. Suppose that E is compact with planar Lebesgue measure zero and that*

$$\|r_\lambda\|_A = \mathrm{O}\Big(\frac{1}{h(d(\lambda, E))}\Big), \quad d(\lambda, E) \to 0,$$

where h is a non-decreasing function on $(0, +\infty)$ *with* $h(0+) = 0$*. If for sufficiently small $\delta > 0$,*

$$\int\limits_0^\delta \Big(\ln\ln \frac{1}{h \circ \theta_E^{-1}(x^2)}\Big)^{1/2} \mathrm{d}x < +\infty, \tag{2.5}$$

then A is non-quasianalytic.

Proof. Let $\zeta \in E$ and Ω be an open subset of $\mathbb{C}$ with $\zeta \in \Omega$. Without loss of generality we may assume that Ω is bounded. We denote by $\mathcal{C}_h(\overline{\Omega})$ the linear space of all continuous functions f on $\overline{\Omega}$, endowed with the norm

$$\|f\|_h = \sup_{z \in \overline{\Omega}} |f(z)| h(d(z, E)),$$

and by $\mathcal{F}_h(\Omega)$ the linear subspace of $\mathcal{C}_h(\overline{\Omega})$ consisting of functions that are holomorphic in Ω. Note that the unit ball of $\mathcal{F}_h(\Omega)$, according to the norm $\|\cdot\|_h$ is the set $\mathcal{H}_{1/h}(\Omega)$ defined in the Corollary 2.4. Let

$$L_\zeta(f) = f(\zeta), \quad f \in \mathcal{F}_h(\Omega).$$

Because of the normality of $\mathcal{H}_{1/h}(\Omega)$, by Corollary 2.4, L_ζ defines a continuous linear functional on $\mathcal{F}_h(\Omega)$ endowed with the norm $\|\cdot\|_h$. The Hahn-Banach Theorem ensures that there exists a continuous linear functional on $\mathcal{C}_h(\overline{\Omega})$, say Λ_ζ, which extends L_ζ. So by the representation theorem of Riesz, there exists a regular measure μ, with support in $\overline{\Omega}$, such that

$$\int\limits_{\overline{\Omega}} \frac{\mathrm{d}|\mu|(z)}{h(d(z, E))} < +\infty, \tag{2.6}$$

and

$$\Lambda_\zeta f = \int\limits_{\overline{\Omega}} f(z)\, \mathrm{d}\mu(z), \quad f \in \mathcal{C}_h(\overline{\Omega}).$$

Let

$$\varphi = \int\limits_{\overline{\Omega}} \mathrm{d}\mu(\lambda) + \int\limits_{\overline{\Omega}} (\zeta - \lambda) r_\lambda \, \mathrm{d}\mu(\lambda).$$

Since Ω is bounded, it follows from the assumption (iii) and inequality (2.6) that

$$\int_{\overline{\Omega}} \|(\zeta - \lambda) r_\lambda\|_A \, \mathrm{d}|\mu|(\lambda) < +\infty.$$

Hence $\varphi \in A$. The pointwise evaluations are continuous on A, which implies that for $z \in E$,

$$\varphi(z) = \int_{\overline{\Omega}} \mathrm{d}\mu(\lambda) + \int_{\overline{\Omega}} (\zeta - \lambda) r_\lambda(z) \, \mathrm{d}\mu(\lambda) = (\zeta - z) \int_{\overline{\Omega}} \frac{\mathrm{d}\mu(\lambda)}{\lambda - z}.$$

For a given $z \in E \setminus \overline{\Omega}$, the function $\lambda \to \frac{1}{\lambda - z}$ belongs to $\mathcal{F}_h(\Omega)$ and thus we obtain from the definition of L_ζ that $\varphi(z) = 1$. Let $\psi = 1 - \varphi$; clearly $\psi \in A$, $\psi(\zeta) = 1$ and $\psi(z) = 0$ if $z \in E \setminus \overline{\Omega}$. ∎

REMARK 2.7. Suppose that E is not bounded and A is a Banach space of continuous functions on E vanishing at infinity. Assume that A satisfies the condition (ii) and (iii) cited above. The conclusion of Theorem 2.6 remains true if we suppose that E satisfies the condition (2.5) locally (see Remark 2.5). Indeed the Banach space A is non-quasianalytic iff $A + \mathbb{C}$ is and we reproduce the proof of Theorem 2.6 for $A + \mathbb{C}$.

3. EXAMPLES OF NON-QUASIANALYTIC CLASSES OF FUNCTIONS

3.1. ALGEBRA OF ASYMPTOTICALLY HOLOMORPHIC FUNCTIONS. Let E be a closed subset of $\mathbb{C}$ and let h be a non-decreasing function on $(0, +\infty)$ with $h(0+) = 0$. Let $C_0(\mathbb{C})$ be the space of continuous functions on $\mathbb{C}$ vanishing at infinity. We denote by $\mathcal{D}_h(E)$ the space of all functions f defined on $\mathbb{C}$ such that f and $\overline{\partial} f$ belong to $C_0(\mathbb{C})$ and such that

$$|\overline{\partial} f(z)| = \mathrm{o}(h(d(z, E))), \quad d(z, E) \to 0.$$

Here $z = x + \mathrm{i}y$, $\overline{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \frac{1}{\mathrm{i}}\frac{\partial}{\partial y}\Big)$ and $\overline{\partial} f$ is taken in the sense of the distribution theory. We set

$$\|f\|_{\mathcal{D}_h(E)} = \|f\|_\infty + \sup_{z \in \mathbb{C} \setminus E} \frac{|\overline{\partial} f(z)|}{h(d(z, E))},$$

where $\|\cdot\|_\infty$ stands for the supremum norm on $\mathbb{C}$. The space $\mathcal{D}_h(E)$ endowed with the pointwise product and the norm $\|\cdot\|_{\mathcal{D}_h(E)}$ is a commutative Banach algebra.

Consider the space $\mathcal{Q}_h(E) = \mathcal{D}_h(E)|E = \mathcal{D}_h(E)/I(E)$, where $I(E)$ is the closed ideal of $\mathcal{D}_h(E)$ consisting of those elements that vanish on E; $\mathcal{Q}_h(E)$ endowed with the quotient norm of $\mathcal{D}_h(E)/I(E)$ is a Banach algebra. It is clear that $\mathcal{Q}_h(E)$ possesses a unit if and only if E is compact.

Algebras like $\mathcal{D}_h(E)$ and $\mathcal{Q}_h(E)$ were first introduced by Dyn'kin in [14] in order to define a functional calculus $f(T)$, where T is an operator and f a "smooth" function not necessarily holomorphic in a neighborhood of the spectrum of T. More precisely Dyn'kin has considered the class $\mathcal{Q}'_h(E)$ consisting of functions in $\mathcal{Q}_h(E)$ having a C^1 extension in $\mathcal{D}_h(E)$. He has shown in [13] and [17] that if $E = \mathbb{R}$

or $E=\mathbb{T}$, where $\mathbb{T}$ is the unit circle, then up to some regularity condition on h, $\mathcal{Q}'_h(E)$ is non-quasianalytic if and only if for sufficiently small $\delta>0$

$$\int_0^\delta \ln\ln\frac{1}{h(t)}\,\mathrm{d}t<+\infty. \tag{3.1}$$

This result remains true for $\mathcal{Q}_h(\mathbb{R})$ or $\mathcal{Q}_h(\mathbb{T})$. Note that the non-quasianalyticity of $\mathcal{Q}_h(\mathbb{T})$ is equivalent to the existence of a nonzero function $f\in\mathcal{Q}_h(\mathbb{T})$ which vanishes with all its derivatives at some points $z\in\mathbb{T}$ (see [13]).

We give here a condition related to the geometry of E, sufficient for the non-quasianalyticity of $\mathcal{Q}_h(E)$. On the other hand, if E is a rectifiable arc, we prove that the divergence of the integral on the left side of the inequality (3.1) implies that $\mathcal{Q}_h(E)$ is quasianalytic.

First of all, we state some elementary properties of $\mathcal{Q}_h(E)$. Note that these properties were established in [5] in the case where E is the lower half-plane, $E=\{z\in\mathbb{C}:\operatorname{Im}z\leqslant 0\}$.

PROPOSITION 3.1. *Let E be a compact subset of $\mathbb{C}$. For $\lambda\in\mathbb{C}\setminus E$ we set $r_\lambda(z)=\frac{1}{\lambda-z}$, $z\in E$. Then:*

(i) *For every $\lambda\in\mathbb{C}\setminus E$, $r_\lambda\in\mathcal{Q}_h(E)$ and for every $t\in(0,1)$,*

$$\|r_\lambda\|_{\mathcal{Q}_h(E)}=\mathrm{O}\Big(\frac{1}{d(\lambda,E)^2h(td(\lambda,E))}\Big),\quad d(z,E)\to 0.$$

(ii) *The space of finite linear combinations of r_λ, $\lambda\in\mathbb{C}\setminus E$, is dense in $\mathcal{Q}_h(E)$.*

(iii) *The set of characters of $\mathcal{Q}_h(E)$ can be identified with E, by the map $E\ni z\to\chi_z$, where $\chi_z(f)=f(z)$, $f\in\mathcal{Q}_h(E)$.*

Proof. (i) See the proof of [5], Lemma 3.1.

(ii) We consider for every $\varepsilon>0$ two open subsets U_ε and V_ε of $\mathbb{C}$ such that $E\subset V_\varepsilon$, $\overline{V_\varepsilon}\subset U_\varepsilon$ and $d(E,\partial U_\varepsilon)\leqslant\varepsilon$; ∂U_ε stands for the boundary of U_ε. Take a C^∞ function χ_ε such that $\chi_\varepsilon\equiv 1$ on V_ε and $\chi_\varepsilon\equiv 0$ on $\mathbb{C}\setminus U_\varepsilon$.

Let $f\in\mathcal{D}_h(E)$. By the Cauchy-Green formula we have

$$\begin{aligned}(\chi_\varepsilon f)(z)&=-\frac{1}{\pi}\int_{\mathbb{C}}\frac{\overline{\partial}(\chi_\varepsilon f)(\zeta)}{\zeta-z}\,\mathrm{d}m_2(\zeta)\\&=-\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(\zeta)\overline{\partial}\chi_\varepsilon(\zeta)}{\zeta-z}\,\mathrm{d}m_2(\zeta)-\frac{1}{\pi}\int_{\mathbb{C}}\frac{\overline{\partial}f(\zeta)\chi_\varepsilon(\zeta)}{\zeta-z}\,\mathrm{d}m_2(\zeta)=g_\varepsilon(z)+k_\varepsilon(z).\end{aligned}$$

The function $f\overline{\partial}\chi_\varepsilon$ is continuous with compact support, which implies that its Cauchy transform, g_ε, is also continuous and satisfies $g_\varepsilon(z)\underset{|z|\to+\infty}{\longrightarrow}0$. Since the support of $\overline{\partial}\chi_\varepsilon$ is contained in $\overline{U_\varepsilon}\setminus V_\varepsilon$, we have

$$g_\varepsilon(z)=-\frac{1}{\pi}\int_{\overline{U_\varepsilon}\setminus V_\varepsilon}f(\zeta)\overline{\partial}\chi_\varepsilon(\zeta)r_\zeta(z)\,\mathrm{d}m_2(\zeta),\quad z\in\mathbb{C}$$

and we see that g_ε is holomorphic in V_ε, $g_\varepsilon \in \mathcal{D}_h(E)$ and $g_\varepsilon|E$ is a limit of finite linear combinations of r_ζ, $\zeta \in \mathbb{C}\setminus V_\varepsilon$.

It is clear that $k_\varepsilon = \chi_\varepsilon f - g_\varepsilon \in \mathcal{D}_h(E)$. Furthermore, using the fact that the support of χ_ε is contained in $\overline{U_\varepsilon}$, we obtain

$$\begin{aligned}\|f|E - g_\varepsilon|E\|_{\mathcal{Q}_h(E)} &= \|k_\varepsilon|E\|_{\mathcal{Q}_h(E)} \leqslant \|k_\varepsilon\|_{\mathcal{D}_h(E)} \\ &\leqslant \frac{1}{\pi}\sup_{z\in U_\varepsilon\setminus E}\frac{|\overline{\partial} f(z)|}{h(d(z,E))}\int\limits_{U_\varepsilon\setminus E}\frac{\mathrm{d}m_2(\zeta)}{|\zeta - z|} + \sup_{z\in U_\varepsilon\setminus E}\frac{|\overline{\partial} f(z)|}{h(d(z,E))} \\ &\leqslant \Big(\frac{1}{\pi}(2\pi m_2(U_\varepsilon))^{1/2}+1\Big)\sup_{z\in U_\varepsilon\setminus E}\frac{|\overline{\partial} f(z)|}{h(d(z,E))}.\end{aligned}$$

So $\|f|E - g_\varepsilon|E\|_{\mathcal{Q}_h(E)} \to 0$ as $\varepsilon \to 0$ and then $f|E$ is a limit of finite linear combinations of r_ζ, $\zeta \in \mathbb{C}\setminus E$.

For the proof of (iii) see [5], Lemma 3.2. ∎

Theorem 3.2. *Suppose that E is a compact set with planar Lebesgue measure zero and such that*

$$\int\limits_0^{\sqrt{\theta_E(\mathrm{e}^{-1})}}\Big(\ln\ln\frac{1}{\theta_E^{-1}(x^2)}\Big)^{1/2}\,\mathrm{d}x < +\infty. \tag{3.2}$$

If (2.5) *holds then $\mathcal{Q}_h(E)$ is non-quasianalytic.*

Proof. Let $t \in (0,1)$ and $h_1(x) = x^2h(tx)$, $x>0$; clearly h_1 is an increasing function. It follows from (2.5), (3.2) and part (i) of Remark 5.3 that for sufficiently small $\delta > 0$,

$$\int\limits_0^{\delta}\Big(\ln\ln\frac{1}{h_1\circ\theta_E^{-1}(x^2)}\Big)^{1/2}\,\mathrm{d}x < +\infty.$$

Moreover by part (i) of Proposition 3.1 we have for $\lambda \in \mathbb{C}\setminus E$, $r_\lambda \in \mathcal{Q}_h(E)$ and

$$\|r_\lambda\|_{\mathcal{Q}_h(E)} = \mathrm{O}\Big(\frac{1}{h_1(d(z,E))}\Big),\quad d(z,E)\to 0.$$

It follows from Theorem 2.6 that $\mathcal{Q}_h(E)$ is non-quasianalytic. ∎

Remark 3.3. (i) Under the hypothesis of Theorem 3.2 the algebra $\mathcal{Q}_h(E)$ is normal in the sense of [21], Chapter VIII, Section 5 and thus we may construct a partition of unity on E, with functions in $\mathcal{Q}_h(E)$, subordinate to any covering of E.

(ii) The normality or non-quasianalyticity can be linked to the problem of weighted polynomial approximation. Let Ω be a bounded simply connected domain. Let ω be a positive measurable function defined on Ω and bounded away from zero locally. For $1 \leqslant p < +\infty$, we denote by $L^p_{\mathrm{a}}(\Omega,\omega\,\mathrm{d}m_2)$ the space of analytic functions such that

$$\|f\|_{p,\omega} = \Big(\int\limits_\Omega |f(z)|^p\omega^p(z)\Big)^{1/p} < +\infty.$$

Let $H^p_{\mathrm{a}}(\Omega, \omega\, \mathrm{d}m_2)$ be the closure of polynomials in $L^p_{\mathrm{a}}(\Omega, \omega\, \mathrm{d}m_2)$ for the norm $\|f\|_{p,\omega}$.

The inner boundary of Ω is defined by $\partial_i\Omega := \partial\Omega \setminus \partial\Omega_\infty$, where Ω_∞ is the unbounded component of $\mathbb{C}\setminus\overline{\Omega}$. In the following we suppose that $\partial_i\Omega$ is not empty. Let $\omega(z) = h(d(z, \partial_i\Omega))$, $z \in \Omega$, where h is a non-decreasing function on $(0, +\infty)$ with $h(0^+) = 0$. In the case where $\partial_i\Omega$ is a "smooth" arc, Brennan proves in [6] and [7] that if (3.1) holds then $L^p_{\mathrm{a}}(\Omega, \omega\, \mathrm{d}m_2) \neq H^p_{\mathrm{a}}(\Omega, \omega\, \mathrm{d}m_2)$. By Theorem 3.2 and the same arguments as in the proof of [6], Theorem 3.5, one can verify the following results: set $E = \partial_i\Omega$ and suppose that $m_2(E) = 0$. If (2.5) and (3.2) hold then $L^p_{\mathrm{a}}(\Omega, \omega\, \mathrm{d}m_2) \neq H^p_{\mathrm{a}}(\Omega, \omega\, \mathrm{d}m_2)$.

(iii) It follows from the above theorem and Proposition 5.4 that if E is a rectifiable arc then the condition $\int\limits_0 \left(\ln\ln \frac{1}{h(x^2)}\right)^{1/2} \mathrm{d}x < +\infty$ is sufficient for the non-quasianalyticity of $\mathcal{Q}_h(E)$. The next theorem proves that the condition $\int\limits_0 \ln\ln \frac{1}{h(x)}\, \mathrm{d}x < +\infty$ is necessary, provided that h satisfies some regularity conditions.

Suppose that E is perfect and h is such that for every $n \in \mathbb{N}$, $h(x) = \mathrm{O}(x^n)$ $(x \to 0)$. Let $f \in \mathcal{Q}_h(E)$. We use also f to denote an extension of f to $\mathbb{C}$ belonging to $\mathcal{D}_h(E)$. By the Cauchy-Pompeiu integral formula,

$$f(z) = \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial\Delta} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - \frac{1}{\pi} \int\limits_{\Delta} \frac{\overline{\partial} f(\zeta)}{\zeta - z}\, \mathrm{d}m_2(\zeta), \quad z \in E,$$

where Δ is an appropriate open set in $\mathbb{C}$, containing E and such that $d(E, \partial\Delta) < 1$, where $\partial\Delta$ is the boundary of Δ; f is a C^∞ function on E and we have for $n \geqslant 1$, $z \in E$,

$$f^{(n)}(z) = \frac{n!}{2\mathrm{i}\pi} \int\limits_{\partial\Delta} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}\zeta - \frac{n!}{\pi} \int\limits_{\Delta} \frac{\overline{\partial} f(\zeta)}{(\zeta - z)^{n+1}}\, \mathrm{d}m_2(\zeta)$$

and

$$|f^{(n)}(z)| \leqslant \text{const } M_n,$$

where $M_n = n! \sup\limits_{0<r<1} (h(r)/r^n)$. If h is such that

$$\sigma \mapsto \ln \frac{1}{h(\mathrm{e}^{-\sigma})} \text{ is a convex function for } \sigma \geqslant 0, \tag{3.3}$$

then the condition

$$\int\limits_0 \ln\ln \frac{1}{h(x)}\, \mathrm{d}x = +\infty, \tag{3.4}$$

implies that

$$\sum_{n\geqslant 1} M_n^{-1/n} = +\infty$$

(see [17]). From these observations and [6], Theorem 3.2 the following statement follows:

THEOREM 3.4. *Suppose that E is a rectifiable arc and h satisfies the above conditions* (3.3) *and* (3.4). *If $f \in \mathcal{Q}_h(E)$ vanishes together with all its derivatives $f^{(n)}$, $n \geqslant 1$, at some $z \in E$, then f vanishes identically on E.*

3.2. CARLEMAN CLASSES. Let E be a compact subset of $\mathbb{C}$ and let $(M_n)_{n\geqslant 0}$ be a sequence of positive reals. We say that a continuous function f defined on E belongs to $C_E(M_n)$ if there exist a sequence of functions $f^{(n)}$, $n \geqslant 0$, defined on E and a constant $c > 0$ such that $f^{(0)} = f$ and for every integers $0 \leqslant k \leqslant n$,

$$f^{(k)}(\zeta) = f^{(k)}(z) + f^{(k+1)}(z)\frac{\zeta - z}{1!} + \cdots + f^{(n)}(z)\frac{(\zeta - z)^{n-k}}{(n-k)!} + R_{n,k}(\zeta, z),$$

where

$$|R_{n,k}(\zeta, z)| \leqslant cM_{n+1}\frac{|\zeta - z|^{n-k+1}}{(n-k+1)!}, \quad \zeta, z \in E. \tag{3.5}$$

The norm $\|f\|_{C_E(M_n)}$ of f will be the sum of the supremum of f on E and the infimum of the constants c satisfying (3.5). $C_E(M_n)$ endowed with the norm $\|\cdot\|_{C_E(M_n)}$ becomes a Banach space.

If E is perfect, the sequence $(f^{(n)})_{n\geqslant 1}$ is uniquely defined and we have for $n \geqslant 1$ and $z \in E$,

$$f^{(n)}(z) = \lim_{E\ni\zeta\to z} \frac{f^{(n-1)}(\zeta) - f^{(n-1)}(z)}{\zeta - z}.$$

We associate, to the sequence $(M_n)_{n\geqslant 0}$, the function h defined by:

$$h(r) = \inf_{n\geqslant 0} \frac{M_n}{n!} r^n, \quad r \geqslant 0. \tag{3.6}$$

THEOREM 3.5. *Suppose that E is perfect with planar Lebesgue measure zero and let h be the function associated to $(M_n)_{n\geqslant 0}$ by* (3.6). *If* (2.5) *and* (3.2) *hold then $C_E(M_n)$ is non-quasianalytic.*

Proof. We shall use Theorem 2.6 and for this we need to prove that $r_\lambda \in C_E(M_n)$ and give an estimate of $\|r_\lambda\|_{C_E(M_n)}$ for $\lambda \in \mathbb{C} \setminus E$, where $r_\lambda(z) = \frac{1}{\lambda - z}$, $z \in E$.

We do this with the help of the following formula which we take from [16]:

$$\frac{1}{(\lambda - z)^{k+1}} = \frac{k+1}{\pi(\varepsilon d)^{2k+1}} \int\limits_{\{w:|w-\lambda|<\varepsilon d\}} \frac{\overline{(\lambda - w)}^k}{(w - z)}\, \mathrm{d}m_2(w), \tag{3.7}$$

where k is non-negative integer, $0 < \varepsilon < 1$ and $d = d(\lambda, E)$. Fix two integers k and n with $0 \leqslant k \leqslant n$. We get from (3.7)

$$r_\lambda^{(k)}(z) = \frac{k!}{(\lambda - z)^{k+1}} = \frac{(k+1)!}{\pi(\varepsilon d)^{2k+1}} \int\limits_{\{w:|w-\lambda|<\varepsilon d\}} \frac{\overline{(\lambda - w)}^k}{(w - z)}\, \mathrm{d}m_2(w)$$

and for $i \geqslant 0$

$$r_\lambda^{(k+i)}(z) = \frac{(k+1)!i!}{\pi(\varepsilon d)^{2k+1}} \int\limits_{\{w:|w-\lambda|<\varepsilon d\}} \frac{\overline{(\lambda - w)}^k}{(w-z)^{i+1}} \, \mathrm{d}m_2(w).$$

By a simple computation we obtain

$$\begin{aligned} r_\lambda^{(k)}(\zeta) &- \sum_{i=0}^{n-k} r_\lambda^{(k+i)}(z) \frac{(\zeta - z)^i}{i!} \\ &= \frac{(k+1)!}{\pi(\varepsilon d)^{2k+1}} (\zeta - z)^{n-k+1} \int\limits_{\{w:|w-\lambda|<\varepsilon d\}} \frac{\overline{(\lambda - w)}^k}{(w-\zeta)(w-z)^{n-k+1}} \, \mathrm{d}m_2(w) \\ &= k!(\varepsilon d)(\zeta - z)^{n-k+1} \sum_{j=0}^{k} \frac{(n-j)!}{(n-k)!(k-j)!} \, \frac{1}{(\lambda-\zeta)^{j+1}(\lambda - z)^{n-j+1}}. \end{aligned}$$

So

$$\Big| r_\lambda^{(k)}(\zeta) - \sum_{i=0}^{n-k} r_\lambda^{(k+i)}(z) \frac{(\zeta - z)^i}{i!} \Big| \leqslant \frac{\varepsilon}{d^{n+1}} |\zeta - z|^{n-k+1} \frac{k!}{(n-k)!} \sum_{j=0}^{k} \frac{(n-j)!}{(k-j)!}.$$

Using the equality

$$(n-k+1) \sum_{j=0}^{k} \frac{(n-j)!}{(k-j)!} = \frac{(n+1)!}{k!},$$

we obtain

$$\begin{aligned} \Big| r_\lambda^{(k)}(\zeta) - \sum_{i=0}^{n-k} r_\lambda^{(k+i)}(z)(\zeta - z)^{n-k+1} \Big| &\leqslant \frac{\varepsilon}{d^{n+1}} |\zeta - z|^{n-k+1} \frac{(n+1)!}{(n-k+1)!} \\ &\leqslant \frac{\varepsilon}{(n-k+1)!} \frac{M_{n+1}}{h(d)} |\zeta - z|^{n-k+1}. \end{aligned}$$

Therefore

$$\|r_\lambda\|_{C_E(M_n)} \leqslant \frac{1}{d} + \frac{\varepsilon}{h(d)},$$

and so

$$\|r_\lambda\|_{C_E(M_n)} = \mathrm{O}\Big(\frac{1}{dh(d)}\Big), \quad d \to 0.$$

It follows from Theorem 2.6 that $C_E(M_n)$ is non-quasianalytic. ∎

We shall now investigate conditions on $(M_n)_{n \geqslant 0}$ that ensure the convergence of the integral

$$\int\limits_0 \Big(\ln \ln \frac{1}{h \circ \theta_E^{-1}(x^2)} \Big)^{1/2} \mathrm{d}x$$

(by which we mean the convergence of $\int\limits_0^\delta$ for sufficiently small δ) and therefore the non-quasianalyticity of $C_E(M_n)$. For this, we need to introduce the Legendre envelope of a given function.

Let p be a non-negative and non-increasing function defined on $(0,+\infty)$ such that $p(y)\to+\infty$ as $y\to 0$. The *lower Legendre envelope* of p is the function defined by

$$q(x)=\inf_{y>0}(p(y)+xy),\quad x>0. \tag{3.8}$$

The following lemma may be obtained by the same arguments used in the proof of [3], Lemma 1 (see also [23]).

Lemma 3.6. *Let p be a function as above and q its lower Legendre envelope. Then the following two conditions are equivalent:*

(i) $\int\limits_0 \left(\ln p(y^2)\right)^{1/2}\,\mathrm{d}y<+\infty$.

(ii) $\int\limits_1^{+\infty}\left(\frac{q(x)}{x^3\ln x}\right)^{1/2}\,\mathrm{d}x<+\infty$.

Consider a sequence $(M_n)_{n\in\mathbb{N}}$ of positive reals and put $m_n=\frac{M_n}{n!}$, $n\geqslant 0$. We suppose that $(m_n)_{n\geqslant 0}$ is log-convex that is

$$m_n^2\leqslant m_{n-1}m_{n+1},\quad n\geqslant 1. \tag{3.9}$$

We associate to $(M_n)_{n\in\mathbb{N}}$ the functions defined on $(0,+\infty)$ by

$$h(x)=\inf_{n\geqslant 0}m_nx^n \tag{3.10}$$

and

$$k(x)=\sup_{n\geqslant 0}\frac{x^n}{M_n}. \tag{3.11}$$

Let us make two observations which will be used below. The first one is that:

$$h(x)=\frac{M_n}{n!}x^n\quad\text{if } x\in\Big[(n+1)\frac{M_n}{M_{n+1}},n\frac{M_{n-1}}{M_n}\Big]$$

and

$$k(x)=\frac{x^n}{M_n}\qquad\text{if } x\in\Big[\frac{M_n}{M_{n-1}},\frac{M_{n+1}}{M_n}\Big].$$

The second observation is the following: Put $p(x)=\ln\frac{1}{h(x)}$, $x>0$ and let q be its lower Legendre envelope defined by (3.8). By a simple computation we get

$$\ln k(x)\leqslant q(x),$$

and for $x\in\big[\frac{M_n}{M_{n-1}},\frac{M_{n+1}}{M_n}\big]$,

$$\exp(-q(x))=\sup_{y>0}(h(y)\mathrm{e}^{-xy})\geqslant h(y_0)\mathrm{e}^{-xy_0}\geqslant\frac{M_n}{n!}y_0^n\mathrm{e}^{-xy_0}\geqslant\frac{1}{\mathrm{e}k(\mathrm{e}x)},$$

where y_0 is the maximum of $\frac{n}{x}$ and $(n+1)\frac{M_n}{M_{n+1}}$.

So

$$\ln k(x)\leqslant q(x)\leqslant 1+\ln k(\mathrm{e}x),\quad x\geqslant\frac{M_1}{M_0}. \tag{3.12}$$

We formulate now the following statement:

LEMMA 3.7. *Suppose that $(m_n)_{n\geqslant 0}$ is log-convex. The following conditions are equivalent:*

(i) $\int\limits_0 \Big(\ln\ln \frac{1}{h(x^2)}\Big)^{1/2} \mathrm{d}x < +\infty$.

(ii) $\int\limits_1^{+\infty} \Big(\frac{\ln k(x)}{x^3 \ln x}\Big)^{1/2} \mathrm{d}x < +\infty$.

(iii) $\sum\limits_{n\geqslant 1} \big(M_n^{1/n} \ln M_n\big)^{-1/2} < +\infty$.

Proof. Let $p(x) = \ln \frac{1}{h(x)}$, $x > 0$ and q its lower Legendre envelope. The equivalence between (i) and (ii) follows from (3.12) and Lemma 3.6. For the proof of the equivalence between (ii) and (iii) we need the following inequality

$$\frac{1}{(a\ln a)^{1/2}} - \frac{1}{(b\ln b)^{1/2}} \leqslant \int\limits_a^b \frac{1}{(x^3\ln x)^{1/2}}\,\mathrm{d}x \leqslant \frac{2}{(a\ln a)^{1/2}} - \frac{2}{(b\ln b)^{1/2}}, \tag{3.13}$$

where $\mathrm{e} < a \leqslant b$. We set $a_n = \mathrm{e}M_n^{1/n}$ and $A_n = (a_n \ln a_n)^{-1/2}$, $n \geqslant 0$. We observe that for $x \geqslant a_n$, $\ln k(x) \geqslant n$ and since $(m_n)_{n\geqslant 0}$ is log-convex, we have $a_n \underset{n\to+\infty}{\longrightarrow} +\infty$. Thus for some integer $n_0 \geqslant 1$,

$$\begin{aligned}\int\limits_{a_{n_0}}^{+\infty} \Big(\frac{\ln k(x)}{x^3\ln x}\Big)^{1/2}\mathrm{d}x &\geqslant \sum_{n\geqslant n_0} n^{1/2} \int\limits_{a_n}^{a_{n+1}} \frac{\mathrm{d}x}{(x^3\ln x)^{1/2}}\\ &\geqslant \sum_{n\geqslant n_0} n^{1/2}(A_n - A_{n+1}) \geqslant {n_0}^{1/2}A_{n_0} + \frac{1}{2}\sum_{n\geqslant n_0+1}\frac{A_n}{n^{1/2}}\end{aligned}$$

which proves the implication (ii) $\Rightarrow$ (iii).

For the converse we set $b_n = M_n/M_{n-1}$ and $B_n = (b_n \ln b_n)^{-1/2}$, $n \geqslant 0$. We get from the log-convexity of $(m_n)_{n\geqslant 0}$ that there exists a constant $c > 1$ such that $b_n \geqslant c a_n$, $n \geqslant 0$. Using the inequality

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(\frac{-2}{(x\ln x)^{1/2}}\Big) \geqslant \frac{1}{(x^3\ln x)^{1/2}}, \quad x \geqslant 1,$$

and integration by parts, we obtain for some $n_0 \geqslant 1$,

$$\int\limits_{b_{n_0}}^{\infty} \Big(\frac{\ln k(x)}{x^3\ln x}\Big)^{1/2}\mathrm{d}x \leqslant \sum_{n\geqslant n_0}\Big[-2\Big(\frac{\ln k(x)}{x\ln x}\Big)^{1/2}\Big]_{b_n}^{b_{n+1}} + \sum_{n\geqslant n_0}\int\limits_{b_n}^{b_{n+1}} \frac{\mathrm{d}(\ln k(x))}{(x\ln x \ln k(x))^{1/2}}.$$

For $x \in [b_n, b_{n+1}]$, we have $k(x) = x^n/M_n$, $\mathrm{d}(\ln k(x)) = (n/x)\,\mathrm{d}x$ and $\ln k(x) \geqslant n\ln c$. These observations and (3.13) give

$$\begin{aligned}\int\limits_{b_{n_0}}^{\infty} \Big(\frac{\ln k(x)}{x^3\ln x}\Big)^{1/2}\mathrm{d}x &\leqslant \mathrm{Const}\sum_{n\geqslant n_0} n^{1/2}\int\limits_{b_n}^{b_{n+1}}\frac{\mathrm{d}x}{(x^3\ln x)^{1/2}}\\ &\leqslant \mathrm{Const}\sum_{n\geqslant n_0} n^{1/2}(B_n - B_{n+1}) \leqslant \mathrm{Const}\sum_{n\geqslant n_0}\frac{B_n}{n^{1/2}} \leqslant \mathrm{Const}\sum_{n\geqslant n_0}\frac{A_n}{n^{1/2}}.\end{aligned}$$

This proves the implication (iii) $\Rightarrow$ (ii). ∎

Theorem 3.8. *Let E be a perfect and compact set in $\mathbb{C}$ and $(M_n)_{n\geqslant 0}$ be a sequence of positive reals such that $\left(\frac{M_n}{n!}\right)_{n\geqslant 0}$ is log-convex. If $\theta_E(x) = \mathrm{O}(x^\alpha)$ $(x \to 0)$, where α is a positive constant, and if*

$$\sum_{n\geqslant 1} \left(n^{1-\alpha} M_n^{\alpha/n} \ln M_n\right)^{-1/2} < +\infty, \tag{3.14}$$

then $C_E(M_n)$ is non-quasianalytic.

Proof. Let $m_n = M_n/n!$, $n \geqslant 0$ and let h be the function associated to $(M_n)_{n\geqslant 0}$ defined by (3.10). We set $r_n = m_{n-1}/m_n$, $n \geqslant 1$. Since $(m_n)_n$ is log-convex the sequence $(r_n)_n$ decreases and we can easily verify that $h(x) = m_n x^n$ if $x \in [r_{n+1}, r_n]$ and $m_n = \sup\limits_{x>0} h(x)/x^n$, for $n \geqslant 0$. Set $\widetilde{M}_n = n! \sup\limits_{x>0} \frac{h\circ\theta_E^{-1}(x)}{x^n}$, $n \geqslant 0$. We have for some constant $c > 0$

$$\widetilde{M}_n = n! \sup_{x>0} \frac{h(x)}{\theta_E(x)^n} \geqslant c^n n! \sup_{x>0} \frac{h(x)}{x^{\alpha n}} \geqslant c^n n! \frac{h(r_k)}{r_k^{\alpha n}},$$

where $k = [\alpha n]$ is the integral part of αn. Clearly $0 \leqslant \alpha - k/n \leqslant 1/n$, and using the log-convexity of $(m_n)_n$ we get easily $M_0^{1/k}/r_k M_k^{1/k} \geqslant 1/k$.

It follows from these inequalities that

$$\widetilde{M}_n^{1/n} \geqslant \mathrm{Const} \frac{k^{1-\alpha} M_k^{\alpha/k}}{\left(r_k M_k^{1/k}\right)^{\alpha - k/n}} \geqslant \mathrm{Const}\, k^{1-\alpha} M_k^{\alpha/k}.$$

The last inequality combined with (3.14) gives

$$\sum_{n\geqslant 1} (\widetilde{M}_n^{1/n} \ln \widetilde{M}_n)^{-1/2} < +\infty.$$

It follows then from Lemma 3.7 that the function $\widetilde{h}$ defined by the formula

$$\widetilde{h}(x) = \inf_{n\geqslant 0} \frac{\widetilde{M}_n}{n!} x^n, \quad x > 0,$$

satisfies the condition

$$\int_0 \left(\ln\ln \frac{1}{\widetilde{h}(x^2)}\right)^{1/2} \mathrm{d}x < +\infty.$$

It is easily seen that $\widetilde{h}(x) \leqslant h \circ \theta_E^{-1}(x)$, $x > 0$. So h and θ_E satisfy (2.5) and (3.2), which implies by Theorem 3.5 that $C_E(M_n)$ is non-quasianalytic. ∎

Remark 3.9. (i) In the case when E is a rectifiable arc, we have by Proposition 5.4, $\theta_E(x) = \mathrm{O}(x)(x \to +\infty)$ and then the condition $\sum\limits_{n\geqslant 1} \left(M_n^{1/n} \ln M_n\right)^{-1/2} < +\infty$ implies that $C_E(M_n)$ is non-quasianalytic. On the other hand, by a theorem of Davies reported in [6], Theorem 3.2 we know that the condition $\sum\limits_{n\geqslant 1} M_n^{-1/n} < +\infty$ is necessary.

The above theorem may also be applied when E is the graph of a real function satisfying the Hölder condition with exponent $\alpha \in (0,1]$, since we know in this case that $\theta_E(x) = \mathrm{O}(x^\alpha)(x \to 0)$ (see Proposition 5.5).

(ii) For a general set E, perfect and compact, we may check, by the same arguments in the above proof, that $C_E(M_n)$ is non-quasianalytic under the condition

$$\sum_{n \geqslant 1} \left(A_n^{1/n} \ln A_n\right)^{-1/2} < +\infty,$$

where $A_n = n! \sup\limits_{k \geqslant 1} \frac{M_k}{k!(\theta_E(kM_{k-1}/M_k))^n} \left(kM_{k-1}/M_k\right)^k$.

4. ON THE EXISTENCE OF HYPERINVARIANT SUBSPACES

Let X be a Banach space, $x \in X$ and $T \in \mathcal{L}(X)$. The *local resolvent* $\rho(T,x)$ of T in x consist of the complex numbers λ for which there exists an open set $V \ni \lambda$ and an analytic function F from V to X, such that

$$(\mu - T)F(\mu) = x, \quad \mu \in V.$$

The *local spectrum* of T in X is the closed set $\mathrm{Sp}(T,x) = \mathbb{C} \setminus \rho(T,x)$.

We say that T satisfies the *single valued extension property* (S.V.E.P) if for an arbitrary open subset V of $\mathbb{C}$, the function that vanishes identically on V is the unique function F from V to X satisfying

$$(z - T)F(z) \equiv 0, \quad z \in V.$$

In this case, the function $z \longrightarrow (z-T)^{-1}x$ possesses a unique maximal analytic extension function, which we denote by $R_{x,T}$; $R_{x,T}$ is defined on $\rho(T,x)$ and is called the local resolvent of T in x.

Observe that if T does not satisfy the S.V.E.P, then T possesses an eigenvalue and consequently it admits a nontrivial hyperinvariant subspace of T. Throughout the Subsection 4.1 and 4.2 we assume always that T satisfies the S.V.E.P, which is not a restriction for the problem of the existence of invariant subspaces.

4.1. Dyn'kin functional calculus.

The functional calculus which we use was introduced by Dyn'kin in [14]. It is based on the Cauchy-Pompeiu formula. Indeed, let $x \in X$ and U an open subset of $\mathbb{C}$ containing $\mathrm{Sp}(T,x)$. If f is a continuous function on U such $\overline{\partial} f$ is continuous, $\overline{\partial} f \equiv 0$ on $\mathrm{Sp}(T,x)$ and

$$|\overline{\partial} f(z)| \, \|R_{x,T}(z)\| \underset{d(z,\mathrm{Sp}(T,x)) \to 0}{\longrightarrow} 0, \tag{4.1}$$

then we may define $f(T)x$ by the formula

$$f(T)x = \frac{1}{2\mathrm{i}\pi} \int_{\partial\Delta} f(\zeta) R_{x,T}(\zeta)\,\mathrm{d}\zeta - \frac{1}{\pi} \int_{\Delta \setminus \mathrm{Sp}(T,x)} \overline{\partial} f(\zeta) R_{x,T}(\zeta)\,\mathrm{d}m_2(\zeta) \tag{4.2}$$

where Δ is an open subset of $\mathbb{C}$, such that $\overline{\Delta} \subset U$, $\mathrm{Sp}(T,x) \subset \Delta$ and for which the boundary $\partial\Delta$ is a finite union of disjoints piecewise C^1 Jordan curves. By the

Cauchy-Pompeiu formula the definition of $f(T)x$ does not depend on the choice of Δ.

We will call a set Δ as above an admissible domain. We recall that as in Subsection 4.1, $\overline{\partial} f$ is taken in the distribution sense. If f is a function defined on $F \subset \mathbb{C}$, we set

$$\operatorname*{supp}_F(f) = \overline{\{z \in F : f(z) \neq 0\}}.$$

Now we state some properties satisfied by the above functional calculus.

PROPOSITION 4.1. *Let U be an open subset of $\mathbb{C}$ containing $\mathrm{Sp}(T,x)$ and let f, g be two continuous functions on U satisfying* (4.1) *and such that $\overline{\partial} f \equiv \overline{\partial} g \equiv 0$ on $\mathrm{Sp}(T,x)$. Then:*

(i) $\mathrm{Sp}(T, f(T)x) \subset \mathrm{Sp}(T,x) \cap \operatorname*{supp}_U(f)$.

(ii) $(gf)(T)x = g(T)(f(T)x)$.

Proof. (i) If $\lambda \notin \mathrm{Sp}(T,x)$, we set $f_\lambda(z) = f(z)/(\lambda - z)$ for $z \in U \setminus \{\lambda\}$. The function f_λ satisfies (4.1) and then we may define $f_\lambda(T)x$ by the formula (4.2). It is easy to see that the function $\lambda \to f_\lambda(T)x$ is analytic in $\mathbb{C} \setminus \mathrm{Sp}(T,x)$.

For an admissible domain Δ such that $\lambda \notin \overline{\Delta}$, we have

$$\begin{aligned}(\lambda - T) f_\lambda(T)x = \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial\Delta} \frac{f(\zeta)}{\lambda - \zeta} (\lambda - T) R_{x,T}(\zeta)\, \mathrm{d}\zeta \\ - \frac{1}{\pi} \int\limits_{\Delta \setminus \mathrm{Sp}(T,x)} \frac{\overline{\partial} f(\zeta)}{\lambda - \zeta} (\lambda - T) R_{x,T}(\zeta)\, \mathrm{d}m_2(\zeta).\end{aligned}$$

We observe that

$$(\lambda - T) R_{x,T}(\zeta) = (\lambda - \zeta) R_{x,T}(\zeta) + (\zeta - T) R_{x,T}(\zeta) = (\lambda - \zeta) R_{x,T}(\zeta) + x,$$

and then we get

$$(\lambda - T) f_\lambda(T)x = f(T)x + \Bigg(\frac{1}{2\mathrm{i}\pi} \int\limits_{\partial\Delta} \frac{f(\zeta)}{\lambda - \zeta}\, \mathrm{d}\zeta - \frac{1}{\pi} \int\limits_{\Delta \setminus \mathrm{Sp}(T,x)} \frac{\overline{\partial} f(\zeta)}{\lambda - \zeta}\, \mathrm{d}m_2(\zeta) \Bigg) x = f(T)x.$$

The last inequality holds since $\lambda \notin \overline{\Delta}$. So $R_{f(T)x,T}(\lambda) = f_\lambda(T)x$, and then $\mathrm{Sp}(f(T)x, T) \subset \mathrm{Sp}(T,x)$.

For $\lambda \notin \operatorname*{supp}_U(f)$, we see that the function f_λ has a continuous extension to U which vanishes on $U \setminus \operatorname*{supp}_U(f)$ and by an analoguous argument to the previous one, we verify that $\mathrm{Sp}(T, f(T)x) \subset \operatorname*{supp}_U(f)$.

(ii) We will first show that $g(T)(f(T)x)$ is well defined. Let Δ be an admissible domain in U and $\lambda \in \Delta \setminus \mathrm{Sp}(T,x)$. Denote by $\overline{D}(\lambda, \varepsilon)$ the closed disk with center λ and $\varepsilon > 0$. For ε sufficiently small, $\Delta \setminus \overline{D}(\lambda, \varepsilon)$ is also an admissible domain and we have

$$f_\lambda(T)x = \frac{1}{2\mathrm{i}\pi} \int\limits_{\partial(\Delta \setminus \overline{D}(\lambda,\varepsilon))} f_\lambda(\zeta) R_{x,T}(\zeta)\, \mathrm{d}\zeta - \frac{1}{\pi} \int\limits_{\Delta \setminus (\overline{D}(\lambda,\varepsilon) \cup \mathrm{Sp}(T,x))} \overline{\partial} f_\lambda(\zeta) R_{x,T}(\zeta)\, \mathrm{d}m_2(\zeta).$$

Passing to the limit $\varepsilon \to 0$, we obtain

$$R_{f(T)x,T}(\lambda) = f_\lambda(T)x$$
$$= f(\lambda)R_{x,T}(\lambda) + \frac{1}{2i\pi}\int_{\partial\Delta} f_\lambda(\zeta)R_{x,T}(\zeta)\,d\zeta - \frac{1}{\pi}\int_{\Delta\setminus \mathrm{Sp}(T,x)} \overline{\partial} f_\lambda(\zeta)R_{x,T}(\zeta)\,dm_2(\zeta).$$

For λ close to $\mathrm{Sp}(T,x)$, we get from the last equality that

$$\|R_{f(T)x,T}(\lambda)\| \leqslant |f(\lambda)|\,\|R_{x,T}(\lambda)\| + C,$$

where C is a constant independent from λ. So $|\overline{\partial}g(\lambda)|\,\|R_{f(T)x,T}(\lambda)\| \to 0$ as $d(\lambda, \mathrm{Sp}(T,x)) \to 0$, and, since $\mathrm{Sp}(f(T)x,T) \subset \mathrm{Sp}(T,x)$, we see that $|\overline{\partial}g(\lambda)|\,\|R_{f(T)x,T}(\lambda)\| \to 0$ as $d(\lambda, \mathrm{Sp}(T, f(T)x)) \to 0$. Thus $g(T)(f(T)x)$ is well defined.

Now the proof of the equality $(gf)(T)x = g(T)(f(T)x)$ follows easily from the last integral expression of $R_{f(T)x,T}(\lambda)$ and the Fubini theorem. ∎

Let h be an increasing function defined on $[0,+\infty)$ with $h(0+) = 0$ and satisfying the condition

$$\inf_{n\geqslant 0}\sup_{0<s<1} h(r)\Big(\frac{r}{s}\Big)^n \leqslant \frac{h(r)}{r}, \quad 0<r<1. \tag{4.3}$$

Note that by [18], Lemma 3 the last inequality holds if the function

$$t \to \ln\frac{1}{h(e^{-t})} \quad \text{is convex for } t>0. \tag{4.4}$$

Observe also that when h is differentiable, then (4.4) is equivalent to the fact that the function $x \to x\frac{h'(x)}{h(x)}$ is non-increasing for $0<x<1$.

For an integer $k \geqslant 0$ we put $h_k(r) = r^k h(r)$, $r>0$. It is clear that for $k\geqslant 1$, h_k satisfies (4.3) and we have precisely,

$$\inf_{n\geqslant 0}\sup_{0<s<1} h_k(r)\Big(\frac{r}{s}\Big)^n \leqslant h_{k-1}(r), \quad 0<r<1. \tag{4.5}$$

Let O be an open subset of $\mathbb{C}$ and let $x\in X$ be such that

$$\|R_{x,T}(z)\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big), \quad z\in O, d(z,E)\to 0. \tag{4.6}$$

If $\widetilde{f} \in \mathcal{D}_h(E)$ is such that $\mathop{\mathrm{supp}}_{\mathbb{C}}(\widetilde{f}) \subset O$ then $\widetilde{f}$ satisfies (4.1) and thus we may define the vector $\widetilde{f}(T)x$ by the formula (4.2). Note that the equality (4.6) implies, in particular, that $\mathrm{Sp}(T,x)\cap O \subset E$.

Dyn'kin showed in [14], Theorem 3, the following unicity theorem.

Theorem 4.2. *Suppose that E is perfect and with planar Lebesgue measure zero. If $\widetilde{f}\in\mathcal{D}_{h_2}(E)$ is such that $\mathrm{supp}_{\mathbb{C}}(\widetilde{f}) \subset O$ and $\widetilde{f}|E\cap O = 0$, then $\widetilde{f}(T)x = 0$.*

This theorem permits us to define a local functional calculus for a class of functions in $\mathcal{Q}_{h_3}(E)$, in the following manner:

Let $f \in \mathcal{Q}_{h_3}(E)$ such that $\operatorname{supp}_E(f) \subset E \cap O$ and let $\widetilde{f} \in \mathcal{D}_{h_2}(E)$ such that $\widetilde{f}|E = f$. Note that the existence of $\widetilde{f} \in \mathcal{D}_{h_2}(E)$ is guaranteed by Lemma 4.3 stated below. We set

$$f(T)x = \widetilde{f}(T)x,$$

and we see that by Theorem 4.2, $f(T)x$ does not depend on the choice of the extension $\widetilde{f}$ of f. Thus $f(T)x$ is well defined.

LEMMA 4.3. *Suppose that E is perfect.*

(i) *Let $\widetilde{f} \in \mathcal{D}_{h_{k+1}}(E)$. If χ is a bounded C^1 function on $\mathbb{C}$ such that $\chi \equiv 1$ on a neighborhood of $\operatorname{supp}_E(\widetilde{f}|E)$, then $\chi\widetilde{f} \in \mathcal{D}_{h_k}(E)$.*

(ii) *If $f \in \mathcal{Q}_{h_{k+1}}(E)$ is such that $\operatorname{supp}_E(f) \subset O \cap E$, then there exists $\widetilde{f} \in \mathcal{D}_{h_k}(E)$ such that $\widetilde{f}|E = f$ and $\operatorname{supp}_{\mathbb{C}}(\widetilde{f}) \subset O$.*

Proof. (i) Set $F = \overline{E \setminus \operatorname{supp}_E(\widetilde{f}|E)}$. We suppose that F is nonempty, otherwise the result is obvious. Clearly F is perfect and $\widetilde{f} \in \mathcal{D}_{h_{k+1}}(F)$. Since $\widetilde{f}_{|F} = 0$, it follows from [16], Theorem 2, and inequality (4.5) that

$$|\widetilde{f}(z)| = \mathrm{O}(d(z,F)|\ln d(z,F)|h_k(d(z,F))) = \mathrm{o}(h_k(d(z,F))), \quad d(z,F) \to 0. \tag{4.7}$$

We check now that $\chi\widetilde{f} \in \mathcal{D}_{h_k}(E)$. For $z \notin \operatorname{supp}_{\mathbb{C}}(\overline{\partial}\chi)$,

$$|\overline{\partial}(\chi\widetilde{f})(z)| = |\chi(z)|\,|\overline{\partial}\widetilde{f}(z)| = \mathrm{o}(h_k(d(z,E))), \quad d(z,E) \to 0.$$

On the other hand, if $z \in \operatorname{supp}_{\mathbb{C}}(\overline{\partial}\chi)$ and if $d(z,E) < d(\operatorname{supp}_{\mathbb{C}}(\overline{\partial}\chi), \operatorname{supp}_E(\widetilde{f}|E))$ then $d(z,E) < d(z, \operatorname{supp}_E(\widetilde{f}|E))$ and since $E = \operatorname{supp}_E(\widetilde{f}|E) \cup F$ we have $d(z,E) = d(z,F)$. This observation and (4.7) imply that

$$\begin{aligned} |\overline{\partial}(\chi\widetilde{f})(z)| &\leqslant |\overline{\partial}\chi(z)\widetilde{f}(z)| + |\chi(z)\overline{\partial}\widetilde{f}(z)| = \mathrm{o}(h_k(d(z,F)) + h_k(d(z,E))) \\ &= \mathrm{o}(h_k(d(z,E))), \quad z \in \operatorname{supp}_{\mathbb{C}} \overline{\partial}\chi,\ d(z,E) \to 0. \end{aligned}$$

Finally we have $\chi\widetilde{f} \in \mathcal{D}_{h_k}(E)$ as claimed.

(ii) Let $f \in \mathcal{Q}_{h_{k+1}}(E)$ such that $\operatorname{supp}_E(f) \subset O$ and let $\widetilde{g} \in \mathcal{D}_{h_{k+1}}(E)$ such that $\widetilde{g}|E = f$. Take a C^1 bounded function χ on $\mathbb{C}$ such that $\chi \equiv 1$ on a neighborhood of $\operatorname{supp}_E(f)$ and $\operatorname{supp}_{\mathbb{C}}(\chi) \subset O$. The previous assertion proves that the function $\widetilde{f} = \chi\widetilde{g}$ possesses the desired properties. ∎

4.2. SPECTRAL PROPERTIES. In this subsection we use the notation of the previous one. We recall that:

(i) E denotes a compact perfect set with planar Lebesgue measure zero.
(ii) h an increasing function satisfying (4.3).
(iii) O an open subset of $\mathbb{C}$ and x an element of X satisfying (4.6).

As we may expect, we have the following result.

Lemma 4.4. *Let $f, g \in \mathcal{Q}_{h_3}(E)$ such that $\operatorname{supp}_E(f)$ and $\operatorname{supp}_E(g)$ are contained in O. Then*

(i) $\operatorname{Sp}(T, f(T)x) \subset \operatorname{supp}_E(f) \cap \operatorname{Sp}(T, x)$.

(ii) $(gf)(T)x = g(T)(f(T)x)$.

Proof. (i) The inclusion $\operatorname{Sp}(T, f(T)x) \subset \operatorname{Sp}(T, x)$ follows immediately from the assertion (i) of Proposition 4.1. For the proof of the inclusion $\operatorname{Sp}(T, f(T)x) \subset \operatorname{supp}_E(f)$, take $\lambda \notin \operatorname{supp}_E(f)$ and $\widetilde{f} \in \mathcal{D}_{h_3}(E)$ with $\widetilde{f}|E = f$. Let χ be a bounded C^1 function on $\mathbb{C}$ such that $\operatorname{supp}_{\mathbb{C}}(\chi) \subset O$, $\chi \equiv 1$ on a neighborhood of $\operatorname{supp}_{\mathbb{C}}(f)$ and $\chi \equiv 0$ on a neighborhood of λ. By part (i) of Lemma 4.3 , $\widetilde{g} := \chi\widetilde{f} \in \mathcal{D}_{h_2}(E)$. Furthermore $\operatorname{supp}_{\mathbb{C}}(\widetilde{g}) \subset O$ and $\lambda \notin \operatorname{supp}_{\mathbb{C}}(\widetilde{g})$ which implies by part (i) of Proposition 4.1 that $\lambda \notin \operatorname{Sp}(T, \widetilde{g}(T)x)$. Since $\widetilde{g}|E = f$ we have $f(T)x = \widetilde{g}(T)x$ and so $\lambda \notin \operatorname{Sp}(T, f(T)x)$. This finishes the proof of (i). The proof of (ii) is a direct consequence of part (ii) of Proposition 4.1. ∎

Lemma 4.5. *Suppose that* (2.5) *and* (3.2) *hold. If $f \in \mathcal{Q}_{h_3}(E)$ is such that $\operatorname{supp}_E(f) \subset O$, then*

$$\overline{\operatorname{Sp}(T, x) \setminus Z_E(f)} \subset \operatorname{Sp}(T, f(T)x),$$

where $Z_E(f) = \{z \in E : f(z) = 0\}$.

Proof. Let $\lambda \in \operatorname{Sp}(T, x)$ be such that $f(\lambda) \neq 0$. By (ii) of Lemma 4.3, there exists $\widetilde{f} \in \mathcal{D}_{h_2}(E)$ with $\widetilde{f}|E = f$ and $\operatorname{supp}_{\mathbb{C}}(\widetilde{f}) \subset O$. There exists also an open disk $\mathbb{D}$ with center λ such that $\overline{\mathbb{D}} \subset O$ and $\widetilde{f}(z) \neq 0$ for every $z \in \overline{\mathbb{D}}$. Take another open disk $\mathbb{D}_0$ concentric with $\mathbb{D}$ such that $\overline{\mathbb{D}}_0 \subset \mathbb{D}$. It follows from Theorem 3.2 that the algebra $\mathcal{Q}_{h_3}(E)$ is non-quasianalytic and so there exists $\varphi \in \mathcal{Q}_{h_3}(E)$ such that $\varphi \equiv 1$ on $\mathbb{D}_0 \cap E$ and $\varphi \equiv 0$ on $E \setminus \mathbb{D}$ (see [21], Chapter VIII, Section 5). Let $\widetilde{\varphi} \in \mathcal{D}_{h_3}(E)$ with $\widetilde{\varphi}|E = \varphi$ and let χ be a C^∞ smooth function with $\chi \equiv 1$ on $\mathbb{D}_0$ and $\chi \equiv 0$ on $\mathbb{C} \setminus \mathbb{D}$. By (ii) of Lemma 4.3, the function $\widetilde{\psi} = \chi\widetilde{\varphi} \in \mathcal{D}_{h_2}(E)$. Put $\widetilde{g_1}(z) = \widetilde{\psi}(z)/\widetilde{f}(z)$ if $z \in \mathbb{D}$ and $\widetilde{g_1}(z) = 0$ if $z \in \mathbb{C} \setminus \mathbb{D}$. It is easily seen that $\widetilde{g_1} \in \mathcal{D}_{h_2}(E)$ and $\operatorname{supp}_{\mathbb{C}}(\widetilde{g_1}) \subset O$. The function $\widetilde{g_2} := 1 - \widetilde{f}\widetilde{g_1} = 1 - \widetilde{\psi}$ is continuous on $\mathbb{C}$ and $\overline{\partial}\widetilde{g_2} = -\overline{\partial}\widetilde{\psi}$ is also continuous and satisfies (4.1), which permits us to define $\widetilde{g_2}(T)x$. We get $x = \widetilde{\psi}(T)x + \widetilde{g_2}(T)x$ and so

$$\operatorname{Sp}(T, x) \subset \operatorname{Sp}(T, \widetilde{\psi}(T)x) \cup \operatorname{Sp}(T, \widetilde{g_2}(T)x).$$

Since $\lambda \notin \operatorname{supp}_{\mathbb{C}}(\widetilde{g_2})$, we can assert by (i) of Proposition 4.1 that $\lambda \notin \operatorname{Sp}(T, \widetilde{g_2}(T)x)$, which implies that $\lambda \in \operatorname{Sp}(T, \widetilde{\psi}(T)x)$. By (ii) of Proposition 4.1, we obtain

$$\widetilde{\psi}(T)x = \widetilde{g_1}(T)(\widetilde{f}(T)x) = \widetilde{g_1}(T)(f(T)x)$$

and we deduce from Lemma 4.1 that $\lambda \in \operatorname{Sp}(T, f(T)x)$. Since $\operatorname{Sp}(T, f(T)x)$ is closed, we conclude that the desired inclusion holds. ∎

4.3. REGULARITY OF THE MAJORANT FUNCTION. In the previous subsections we have defined a local functional calculus for $T \in \mathcal{L}(X)$ and $x \in X$ submitted to a condition of the type

$$\|R_{x,T}(z)\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big), \quad d(z,E) \to 0$$

where h is an increasing function satisfying the condition (4.3), that is

$$\inf_{n\geqslant 0} \sup_{0<s<1} h(s)\Big(\frac{r}{s}\Big)^n \leqslant \frac{h(r)}{r}, \quad 0<r<1.$$

We show here that this condition is not restrictive. We will use this fact later on. As observed in Subsection 4.1 the condition (4.3) holds if the function

$$t \to \ln \frac{1}{h(\mathrm{e}^{-t})} \qquad \text{is convex for } t>0.$$

In the following statement we adapt [13], Lemma 3, to our situation. The proof needs a slight modification and for the sake of completeness we include it here.

PROPOSITION 4.6. *Let α be a function on $(0,+\infty)$ with $\alpha(0+)=0$ and h be a non-decreasing function with $h(0+)=0$. Suppose that*

$$\int_0 \Big(\ln\ln\frac{1}{\alpha(x^2)}\Big)^{1/2} \mathrm{d}x < +\infty \quad \textit{and} \quad \int_0 \Big(\ln\ln\frac{1}{h\circ\alpha(x^2)}\Big)^{1/2} \mathrm{d}x < +\infty.$$

Then for every $s>1$, there exists an increasing function h_s satisfying (4.3) *such that $h_s(x) \leqslant h(x)$, $0<x<1$ and*

$$\int_0 \Big(\ln\ln\frac{1}{h_s(s\alpha(x^2))}\Big)^{1/2} \mathrm{d}x < +\infty.$$

Proof. Without loss of generality we may assume that h is continuous and $h(x)\leqslant 1$, $x>0$. Set, for $x>0$,

$$\psi(x) = \int_{x/s}^{1} \frac{|\ln h_0(\lambda)|}{\lambda}\,\mathrm{d}\lambda \quad \text{and} \quad h_s(x) = \exp\Big(-\frac{1}{\ln s}\psi(x)\Big).$$

We have for $x>0$,

$$\psi(x) \geqslant \int_{x/s}^{x} \frac{|\ln h(\lambda)|}{\lambda}\,\mathrm{d}\lambda \geqslant \ln(s)|\ln h(x)|$$

and thus $h_s(x) \leqslant h(x)$.

On the other hand we get $\psi(x) \leqslant \ln(s/x)|\ln h(x/s)|$, $0<x<1$. Therefore, for sufficiently small $\delta>0$,

$$\int_0^{\delta} \Big(\ln\ln\frac{1}{h_s(s\alpha(x^2))}\Big)^{1/2}\mathrm{d}x \leqslant \int_0^{\delta}\Big(-\ln\ln s + \ln\ln\frac{1}{\alpha(x^2)} + \ln\ln\frac{1}{h\circ\alpha(x^2)}\Big)^{1/2}\mathrm{d}x$$

$$\leqslant \mathrm{const} + \int_0^{\delta}\Big(\ln\ln\frac{1}{\alpha(x^2)}\Big)^{1/2}\mathrm{d}x + \int_0^{\delta}\Big(\ln\ln\frac{1}{h(\alpha(x^2))}\Big)^{1/2}\mathrm{d}x < +\infty.$$

Since the function $x \to x\psi'(x) = \ln h(x/s)$ is non-decreasing, we get that the function $t \to \ln \frac{1}{h_s(\mathrm{e}^{-t})}$ is convex for $t > 0$ and that h satisfies (4.3). Finally we see that h_s has the required properties. ∎

4.4. Existence of hyperinvariant subspaces. We formulate now a result on the existence of hyperinvariant subspaces. We denote by X^* the dual space of X and by T^* the adjoint operator of T.

Theorem 4.7. *Suppose that T and T^* satisfies the* S.V.E.P. *Assume that there exist a perfect compact set E with planar Lebesgue measure zero, an open set O, a non-decreasing function h on $(0, +\infty)$, $x_0 \in X$, $y_0 \in X^*$ and two distinct complex numbers λ_1, λ_2 such that:*

(i) $\lambda_1 \in \mathrm{Sp}(T, x_0) \cap O$ *and* $\lambda_2 \in \mathrm{Sp}(T^*, y_0) \cap O$.

(ii) $\|R_{x_0,T}(z)\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big)$, $z \in O$, $d(z, E) \to 0$.

(iii) $\|R_{y_0,T^*}(z)\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big)$, $z \in O$, $d(z, E) \to 0$.

If (2.5) *and* (3.2) *hold then T possesses a hyperinvariant subspace.*

Proof. By Proposition 4.6 and part (i) of Remark 5.3 we may assume that h satisfies (4.3). Let V be an open neighborhood of λ_2 such that $\overline{V} \subset O$ and $\lambda_1 \notin \overline{V}$. Let F be the space of all $x \in X$ such that the function $z \to (z - T)^{-1}x$ has an analytic extension to V that is, $F = \{x \in X : V \subset \rho(x, T)\}$. We shall prove that $\overline{F}$, the closure of F, is a nontrivial closed hyperinvariant subspace of T. Let $x \in F$ and let A be a bounded operator on X that commutes with T. It is easy to see that $\mathrm{Sp}(T, Ax) \subset \mathrm{Sp}(T, x)$ and that $Ax \in F$. Hence $\overline{F}$ is hyperinvariant and it remain to prove that $\overline{F}$ is nontrivial.

Theorem 3.2 implies that the algebra $\mathcal{Q}_{h_3}(E)$ is non-quasianalytic, where $h_3(r) = r^3 h(r)$, $r > 0$. It follows that there exist two functions f and g in $\mathcal{Q}_{h_3}(E)$ such that $f(\lambda_1) \neq 0$, $g(\lambda_2) \neq 0$, $\underset{E}{\mathrm{supp}}(f) \subset E \setminus \overline{V}$ and $\underset{E}{\mathrm{supp}}(g) \subset V$. It follows from Lemma 4.5, that the sets $\mathrm{Sp}(T, f(T)x_0)$ and $\mathrm{Sp}(T^*, g(T^*)y_0)$ are both non empty, and thus $f(T)x_0 \neq 0$ and $g(T^*)y_0 \neq 0$. By (i) of Lemma 4.4, $\mathrm{Sp}(T, f(T)x_0) \subset \underset{E}{\mathrm{supp}}(f)$ and thus $f(T)x_0 \in F$. So $\{0\} \subsetneqq F$.

On the other hand, take $x \in F$ and consider the function

$$\phi(z) = \langle (z - T)^{-1}x, g(T^*)y_0 \rangle$$

defined for instance on $\mathbb{C} \setminus \mathrm{Sp}(T)$. Clearly ϕ has an analytic extension to $\mathbb{C} \setminus \mathrm{Sp}(T, x)$. Writing

$$\phi(z) = \langle x, (z - T^*)^{-1} g(T^*)y_0 \rangle$$

we see that ϕ has also an analytic extension to $\mathbb{C} \setminus \mathrm{Sp}(T^*, g(T^*)y_0)$. By (i) of Lemma 4.4, we have $\mathrm{Sp}(T^*, g(T^*)y_0) \subset \underset{E}{\mathrm{supp}}(g)$. So $\mathrm{Sp}(T, x) \cap \mathrm{Sp}(T^*, g(T^*)y_0) \neq \emptyset$ and then ϕ admits an extension which is an entire function. Since $\phi(z) \to 0$ as $|z| \to +\infty$, we conclude that ϕ vanishes identically and consequently $(z - T^*)^{-1} g(T^*)y_0$ is orthogonal to $\overline{F}$ for all $z \in \mathbb{C} \setminus \mathrm{Sp}(T^*, g(T^*)y_0)$. We have seen above that $g(T^*)y_0 \neq 0$ and then, for all $z \in \mathbb{C} \setminus \mathrm{Sp}(T^*)$, $(z - T^*)^{-1} g(T^*)y_0 \neq 0$. Hence $F \subsetneqq X$, which finishes the proof. ∎

Corollary 4.8. *Suppose that there exist a perfect compact set E with planar Lebesgue measure zero, and an open set O such that* $\mathrm{Sp}(T) \cap O$ *contains at least two distinct points. Assume that there exists a non-decreasing function h on* $(0, +\infty)$ *such*

$$\|(z-T)^{-1}\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big) \quad z \in O,\ d(z,E) \to 0.$$

If θ_E and h satisfies (2.5) *and* (3.2), *then T has a nontrivial hyperinvariant subspace.*

Proof. Without loss of generality we may assume that T and T^* satisfy the S.V.E.P. Since

$$\mathrm{Sp}(T) = \mathrm{Sp}(T^*) = \bigcup_{x \in X} \mathrm{Sp}(T,x) = \bigcup_{y \in X^*} \mathrm{Sp}(T^*, y)$$

we can find $x_0 \in X$ and $y_0 \in X^*$ satisfying the hypothesis of Theorem 4.7, from which the proof follows. ∎

4.5. Decomposable operators. Lyubich and Matsaev proved in [26] that if T is a bounded operator with real spectrum and if the function

$$M(x) = \sup_{|z| \geqslant x} \|(z-T)^{-1}\|, \quad x > 0,$$

satisfies the Levinson condition:

$$\int_0^\delta \ln\ln M(x)\,\mathrm{d}x < +\infty, \quad \text{for sufficiently small } \delta > 0,$$

then T is decomposable. They also proved that the Levinson condition is the best possible. For this, they considered the multiplication operator by z acting on a Hilbert space of functions related to a quasianalytic Carleman class $C_{[-1,1]}(M_n)$ (see the introduction of [26]). Recently A. Atzmon and M. Sodin have constructed an operator on a Hilbert space, such neither T nor T^* is decomposable ([2]).

Definition 4.9. Let $T \in \mathcal{L}(X)$ and $x \in X$. T is said to be *decomposable* if for a given open cover $(U_i)_{1 \leqslant i \leqslant n}$ of $\mathrm{Sp}(T)$, there exist a closed invariant subspaces $X_1, \ldots, X_n$ of T such that:

(i) $X = X_1 + X_2 + \cdots + X_n$,

(ii) $\mathrm{Sp}(T|X_i) \subset U_i$, $i = 1, \ldots, n$, where $T|X_i$ is the restriction of T to X_i.

If T satisfies the S.V.E.P, we say that T is *decomposable at x* if for a given open cover $(U_i)_{1 \leqslant i \leqslant n}$ of $\mathrm{Sp}(T,x)$, there exist $x_1, \ldots, x_n \in X$ such that

(i) $x = x_1 + x_2 + \cdots + x_n$,

(ii) $\mathrm{Sp}(T, x_i) \subset U_i$, $i = 1, \ldots, n$.

Let us first recall that $(U_i)_{1 \leqslant i \leqslant n}$ is called an open cover of a set $F \subset \mathbb{C}$ if each U_i is open in $\mathbb{C}$ and $F \subset \bigcup_{1 \leqslant i \leqslant n} U_i$. We have the following result:

Proposition 4.10. *Suppose that T satisfies the* S.V.E.P *and is decomposable at every $x \in X$. Assume that there exist a perfect compact set E with planar Lebegue measure zero and a non-decreasing function h on $(0,+\infty)$ such that*

$$\|(z-T)^{-1}\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big), \quad d(z,E) \to 0.$$

If (2.5) *holds, then T is decomposable.*

Proof. Let $(U_i)_{1\leqslant i\leqslant n}$ be an open cover of $\mathrm{Sp}(T)$. There exists an open cover $(V_i)_{1\leqslant i\leqslant n}$ of $\mathrm{Sp}(T)$ such that $\overline{V_i} \subset U_i$, $i = 1,\dots,n$. Let $X_i = \{x \in X : \mathrm{Sp}(T,x) \subset \overline{V_i}\}$. It is clear that X_i is a linear subspace of X. Decomposability of T at every $x \in X$ implies that $X = X_1 + \dots + X_n$. It remains to prove that each X_i is closed. For this let $(x_n)_n$ be a sequence in X_i converging to $x \in X$. Take $l \in X^*$ and set

$$\varphi_n(z) = \langle R_{x_n,T}(z), l\rangle.$$

Each function φ_n has an analytic extension to $\rho(T,x_n)$ and in particular to $\mathbb{C}\setminus\overline{V_i}$. Moreover, since $(x_n)_n$ is bounded, there exists a constant $c>0$ such that

$$|\varphi_n(z)| \leqslant c\|(z-T)^{-1}\|, \quad z \in \mathbb{C}\setminus(\overline{V_i}\cup \mathrm{Sp}(T,x)).$$

Thus

$$|\varphi_n(z)| \leqslant \frac{\mathrm{const}}{h(d(z,E))}, \quad z \in \mathbb{C}\setminus(\overline{V_i}\cup E).$$

By Corollary 2.4 the family $\{\varphi_n : n \in \mathbb{N}\}$ is normal, therefore there exists a subsequence of $(\varphi_n)_n$ which converges uniformly on every compact subset of $\mathbb{C}\setminus\overline{V_i}$ to an analytic function φ. Clearly

$$\varphi(z) = \langle (z-T)^{-1}x, l\rangle \quad \text{for } z \in \mathbb{C}\setminus(\overline{V_i}\cup \mathrm{Sp}(T)).$$

So for every $l \in X^*$ the function $z \to \langle (z-T)^{-1}x, l\rangle$ has an analytic continuation to $\mathbb{C}\setminus\overline{V_i}$. It follows from Lemma 4.11 stated below that the function $z \to (z-T)^{-1}x$ has also an analytic continuation to $\mathbb{C}\setminus\overline{V_i}$. Hence $x \in X_i$, which proves that X_i is closed. ∎

The following lemma needed above may be proved by the same argument used in the proof of [1], Lemma 2.4.

Lemma 4.11. *Let E and U be respectively a compact and an open subset of $\mathbb{C}$ such that the interior of E is empty and $E\cap U \neq \emptyset$. Let F be an analytic function from $U\setminus E$ to X. Assume that for every $l \in X^*$, the function $z \to \langle F(z), l\rangle$ has an analytic continuation to U. Then F has also an analytic continuation to U.*

Theorem 4.12. *Let E be a perfect compact subset of $\mathbb{C}$ with planar Lebesgue measure zero and let h be a non-decreasing function on $(0,+\infty)$ such that* (2.5) *and* (3.2) *hold.*

(i) *If T satisfies the* S.V.E.P *and $x \in X$ is such that*

$$\|R_{x,T}(z)\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big), \quad d(z,E) \to 0,$$

then T is decomposable at x.

(ii) *If the resolvent of* T *satisfies*

$$\|(z-T)^{-1}\| = \mathrm{O}\Big(\frac{1}{h(d(z,E))}\Big), \quad d(z,E) \to 0,$$

then T *is decomposable.*

Proof. (i) By Proposition 4.6 and part (i) of Remark 5.3 we may assume that h satisfies (4.3). Let $(U_i)_{0\leqslant i\leqslant n}$ be an open cover of $\mathrm{Sp}(T,x)$. We can find a set U_0 such that $U_0 \cap \mathrm{Sp}(T,x) = \emptyset$ and $E \subset \bigcup\limits_{0\leqslant i\leqslant n} U_i$. By Theorem 3.2 the algebra $\mathcal{Q}_{h_3}(E)$ is nonquasianalytic; so we may construct a partition of unity $f_0,\dots,f_n$ in $\mathcal{Q}_{h_3}(E)$ subordinate to the covering $(U_i\cap E)_{0\leqslant i\leqslant n}$ of E ([21], Chapter VIII, Section 5). We obtain $x = x_0 + x_1 + \dots + x_n$, where $x_i = f_i(T)x$, $i = 0,\dots,n$. From Lemma 4.4 we get:

$$\mathrm{Sp}(T,x_i) \subset \operatorname*{supp}_E(f_i) \cap \mathrm{Sp}(T,x) \subset U_i \cap \mathrm{Sp}(T,x), \quad i = 0,\dots,n.$$

Since $\mathrm{Sp}(T,x)\cap U_0 = \emptyset$, we have $\mathrm{Sp}(T, f_0(T)x) = \emptyset$ which implies that $x_0 = 0$. So that $x_1,\dots,x_n$ have the required conditions.

(ii) The fact that $\mathrm{Sp}(T,x) \subset E$ and $m_2(E) = 0$ implies that the interior of $\mathrm{Sp}(T,x)$ is empty and therefore T satisfies the S.V.E.P. The proof of (ii) follows from the part (i) and Proposition 4.10. ∎

REMARK 4.13. Let φ be a continuous function on $\mathbb{T}$ non identically constant and suppose ω_φ denotes its modulus of continuity:

$$\omega_\varphi(t) = \sup_{|\zeta_1-\zeta_2|\leqslant t} |\varphi(\zeta_1) - \varphi(\zeta_2)|, \quad t > 0.$$

The Toeplitz operator with symbol φ is defined on the usual Hardy space H^2 by $T_\varphi : f \to P_+\varphi f$, $f \in H^2$, where P_+ is the orthogonal projection from L^2 onto H^2 (see also [29]).

Peller showed in [30] that if there exist an open disk $\mathbb{D}$ and a Lipschitz Jordan arc E such that

$$\varphi(\mathbb{T})\cap\mathbb{D} \neq \emptyset \quad\text{and}\quad \varphi(\mathbb{T})\cap\mathbb{D} \subset E\cap\mathbb{D} \tag{4.8}$$

then the condition

$$\int\limits_0 \frac{\omega_\varphi(t)}{t\ln 1/t}\,\mathrm{d}t < +\infty \tag{4.9}$$

implies that T_φ admits a nontrivial hyperinvariant subspace. It is also proved in [31] that if the set E is assumed to be C^2 then the above result remains true under the weaker condition

$$\int\limits_0 \frac{\omega_\varphi^2(t)}{t\ln 1/t}\,\mathrm{d}t < +\infty. \tag{4.10}$$

For the proof of these results, Peller proceeds as follows: he proves that the condition (4.9) or (4.10) ensures that the resolvent of T_φ satisfies the Levinson condition and concludes through the Ljubich-Matsaev theorem. Using Corollary 4.8

and the arguments of [30] for the estimate of the resolvent of T_φ one may prove the following:

Suppose that there exist a disk $\mathbb{D}$ and a compact perfect set E satisfying (4.8) and such that $m_2(E) = 0$. Suppose also that for every $\lambda \in \mathbb{D} \setminus E$, there exists a continuous determination α_λ of the argument of $\varphi - \lambda$ such that

$$\sup_{\lambda \in \mathbb{D}\setminus E} \sup_{z\in\mathbb{T}} |\alpha_\lambda| < +\infty.$$

If

$$\int_0 \frac{(\theta_E \circ \omega_\varphi(t))^{1/2}}{t \ln 1/t (\ln\ln 1/t)^{1/2}} < +\infty,$$

then T_φ admits a nontrivial hyperinvariant subspace.

It should be noted that if there exists $\lambda \in \mathbb{C} \setminus f(\mathbb{T})$ such that $f - \lambda$ has no continuous branch of the argument, then T_φ admits a non trivial hyperinvariant subspace.

5. APPENDIX

The function θ_E associated with the compact set E plays a decisive part in this paper. For this reason we investigate some of its properties. Recall first that θ_E is defined by

$$\theta_E(x) = m_2(\{z \in \mathbb{C} : d(z, E) < x\}), \quad x > 0,$$

where m_2 is planar Lebesgue measure.

Proposition 5.1. *θ_E is an increasing continuous function.*

Proof. The monotonicity of θ_E is obvious. For the continuity it suffices to prove that for every $r > 0$, $m_2(C_r) = 0$, where $C_r = \{z \in \mathbb{C} : d(z, E) = r\}$. Suppose $w \in C_r$ and $z \in E$, is such that $d(z, w) = r$. For $\varepsilon > 0$, we denote by $B(w, \varepsilon)$ the ball with center w and radius ε. We have

$$\overline{\lim_{\varepsilon\to 0}} \frac{m_2(C_r \cap B(w,\varepsilon))}{m_2(B(w,\varepsilon))} \leqslant \overline{\lim_{\varepsilon\to 0}} \frac{m_2(B(w,\varepsilon) \setminus B(z,r))}{m_2(B(w,\varepsilon))} = \frac{1}{2}.$$

Therefore w is not a point of density. Since w is an arbitrary point in C_r we conclude that $m_2(C_r) = 0$. ∎

Let $N_E(\varepsilon)$ be the smallest number of balls of radius ε that cover E and let $M_E(\varepsilon)$ be the largest number of disjoint balls with radius ε and centers in E. It is easy to see that

$$M_E(2\varepsilon) \leqslant N_E(\varepsilon) \leqslant M_E\Big(\frac{\varepsilon}{2}\Big). \tag{5.1}$$

The following result shows the relationship between N_E and θ_E.

PROPOSITION 5.2. *There exist positive constants c and C such that*

$$cN_E(\varepsilon) \leqslant \frac{\theta_E(\varepsilon)}{\varepsilon^2} \leqslant CN_E(\varepsilon), \quad \varepsilon > 0.$$

Proof. Let $B_1, \ldots, B_{M_E(\varepsilon/2)}$ be disjoint balls with radius $\varepsilon/2$ and centers in E. We clearly have

$$\bigcup_{1\leqslant i\leqslant M_E(\varepsilon/2)} B_i \subset \Big\{z \in E : d(z,E) < \frac{\varepsilon}{2}\Big\}.$$

Thus

$$\frac{\pi}{4}\varepsilon^2 M_E\Big(\frac{\varepsilon}{2}\Big) \leqslant \theta_E\Big(\frac{\varepsilon}{2}\Big) \leqslant \theta_E(\varepsilon).$$

It follows from (5.1) that

$$\frac{\pi}{4}\varepsilon^2 N_E(\varepsilon) \leqslant \theta_E(\varepsilon).$$

Consider now $B_1, \ldots, B_{N_E(\varepsilon)}$ balls of radius ε that cover E. Denote by B_i' the concentric ball with B_i of radius 2ε. We have

$$\{z \in E : d(z,E) < \varepsilon\} \subset \bigcup_{1\leqslant i\leqslant N_E(\varepsilon)} B_i'$$

and so

$$\theta_E(\varepsilon) \leqslant 4\pi\varepsilon^2 N_E(\varepsilon). \qquad \blacksquare$$

REMARK 5.3. (i) It follows from Proposition 5.2 that if $\theta_E(0+) = 0$, which means that $m_2(E) = 0$, then there exists a constant $a > 0$ such that for all $t \in (0,1)$ and $x > 0$ we have

$$\theta_E^{-1}(tx) \leqslant a\sqrt{t}\theta_E^{-1}(x).$$

Indeed if $0 < \varepsilon_1 < \varepsilon_2$ then $\frac{\theta_E(\varepsilon_1)}{\varepsilon_1^2} \geqslant \frac{c}{C}\frac{\theta_E(\varepsilon_2)}{\varepsilon_2^2}$. If we set $\varepsilon_1 = \theta_E^{-1}(tx)$ and $\varepsilon_2 = \theta_E^{-1}(x)$, we get the desired inequality with $a = \sqrt{C/c}$. This observation is of use in Subsection 3.1.

(ii) The lower and upper box-counting dimension of E are given by

$$\underline{\dim} E = \varliminf_{\varepsilon\to 0} \frac{\ln N_E(\varepsilon)}{\ln 1/\varepsilon} \quad \text{and} \quad \overline{\dim} E = \varlimsup_{\varepsilon\to 0} \frac{\ln N_E(\varepsilon)}{\ln 1/\varepsilon}.$$

It follows from the above proposition that the box-counting dimension, may be related to θ_E by the following equalities:

$$\underline{\dim} E = 2 + \varlimsup_{\varepsilon\to 0} \frac{\ln \theta_E(\varepsilon)}{\ln 1/\varepsilon} \quad \text{and} \quad \overline{\dim} E = 2 + \varliminf_{\varepsilon\to 0} \frac{\ln \theta_E(\varepsilon)}{\ln 1/\varepsilon},$$

(see also [20], Proposition 3.2).

From these equalities and [20], Example 3.3, we get that if E is the Cantor set then for every $s < 2 - \ln 2/\ln 3$ we have $\theta_E(\varepsilon) = \mathrm{O}(\varepsilon^s)$, $\varepsilon \to 0$.

We say that E is a rectifiable arc if there exists a continuous function $\varphi : [a,b] \to \mathbb{C}$ such that $\varphi([a,b]) = E$ and

$$l = \sup \sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| < +\infty$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \cdots < x_n = b$ and all integers $n \geqslant 1$. The number l is the length of E corresponding to the parametrisation φ of E.

The following statement may be well known but we are not able to find a precise reference.

Proposition 5.4. *If E is a rectifiable arc not reduced to a single point, then there exist positive constants c and C such that*

$$c\varepsilon \leqslant \theta_E(\varepsilon) \leqslant C\varepsilon, \quad 0 < \varepsilon < 1.$$

Proof. Recall first that the diameter of E is the quantity

$$\operatorname{diam} E = \sup\{|z - w| : z, w \in E\}.$$

Let $B_1, \ldots, B_{M_E(\varepsilon/2)}$ be disjoint balls of radius $\varepsilon/2$ and centers in E. We have for $0 < \varepsilon < \operatorname{diam} E$, $\varepsilon(M_E(\varepsilon/2) - 1) \leqslant l$, where l is the length of E. So $\varepsilon M_E(\varepsilon/2) \leqslant 2l$. It follows from (5.1) that $N_E(\varepsilon) \leqslant 2l/\varepsilon$. Therefore, the second inequality follows from Proposition 5.2.

For the proof of the first inequality we need to introduce the 1-dimensional Hausdorff measure of E. A countably or finite many sets $(U_k)_k$ is said to be an ε-cover of E if $E \subset \bigcup_k U_k$ and for all k, $\operatorname{diam} U_k \leqslant \varepsilon$. We set

$$H_1(E) = \lim_{\varepsilon \to 0^+} \inf \sum_k \operatorname{diam} U_k,$$

where the infimum is taken over all ε-covers of E. By Lemma 3.4 of [19], we have $H_1(E) \geqslant \operatorname{diam} E > 0$. Clearly $(1/2)H_1(E) \leqslant 2\varepsilon N_E(\varepsilon)$ for ε sufficiently small. It follows from Proposition 5.2 that for some positive constant c, $c\varepsilon \leqslant \theta(\varepsilon)$, $\varepsilon > 0$, which finishes the proof. ∎

Let f be a real function defined on an interval I. The graph of f in $\mathbb{C}$ is

$$\operatorname{graph}(f) = \{t + \mathrm{i}f(t) : t \in I\}.$$

Proposition 5.5. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function satisfying the Hölder condition:*

$$|f(u) - f(v)| \leqslant c|u - v|^\alpha, \quad 0 \leqslant u, v \leqslant 1,$$

where $c > 0$ and $0 < \alpha \leqslant 1$. Then

$$\theta_E(\varepsilon) = \mathrm{O}(\varepsilon^\alpha), \quad \varepsilon \to 0,$$

where $E = \operatorname{graph}(f)$.

Proof. In the proof of [20], Proposition 11.2, it is shown that $N_E(\varepsilon) = \mathrm{O}(\varepsilon^{\alpha-2})$, $\varepsilon \to 0$. The proof follows now from Proposition 5.2. ∎

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