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COHOMOLOGY OF TOPOLOGICAL GRAPHS AND CUNTZ-PIMSNER ALGEBRAS

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ABSTRACT. We compute the sheaf cohomology of a groupoid built from a local homeomorphism of a locally compact space X. In particular, we identify the twists over this groupoid, and its Brauer group. Our calculations refine those made by Kumjian, Muhly, Renault and Williams in the case X is the path space of a graph, and the local homeomorphism is the shift. We also show how the C^*-algebra of a twist may be identified with the Cuntz-Pimsner algebra constructed from a certain C^*-correspondence.

KEYWORDS: *Groupoid cohomology, Cuntz-Pimsner algebras, Hilbert modules, Hilbert bimodules, and C^*-correspondence.*

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1. INTRODUCTION

Let X be a second countable, locally compact, Hausdorff space and let $\sigma : X \to X$ be a local homeomorphism (not necessarily surjective). The pair (X, σ) will be fixed throughout this note. In [5], Theorem 1, the first author showed how to build an r-discrete groupoid with Haar system from (X, σ) as follows:

$$\Gamma = \Gamma(X,\sigma) = \{(x,m,y) \in X \times \mathbb{Z} \times X \mid \text{there are } k, l \geqslant 0 \text{ such that } \sigma^k(x) = \sigma^l(y) \text{ and } k - l = m\}.$$

The groupoid operations are given by the formulae:

$$r(x,m,y) = x, \quad s(x,m,y) = y, \quad (x,m,y)(y,n,z) = (x,m+n,z), \quad (x,m,y)^{-1} = (y,-m,x).$$

The Haar system is given by the counting measures on the sets $r^{-1}(x)$, $x \in X$. (See [2] and [3] also.) As noted in [5], groupoids of the form Γ generalize transformation groupoids associated with homeomorphisms. Indeed, if σ is a homeomorphism, then Γ is its transformation group groupoid. For another example to keep in mind in this note, suppose that E is a graph with no sinks as in [18], suppose that $X = E^\infty$, the infinite path space, and suppose that $\sigma : X \to X$ is the unilateral shift,

$$\sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Then σ is a local homeomorphism, and $\Gamma = \mathcal{G}_E$ — the groupoid studied in [18].

The aim of this note is twofold. First we use the long exact sequence of [15], 3.7 to compute the sheaf cohomology of Γ. This computation allows us to identify explicitly all (circle) twists over Γ in the sense of [14]. These are extensions Λ of Γ by the groupoid $X \times \mathbb{T}$. Roughly speaking, twists over groupoids generalize 2-cocycles over groups and the (restricted) groupoid C^*-algebra of a twist (to be defined below) is the groupoid analogue of the C^*-algebra of a group twisted by a 2-cocycle. Our principal result in this direction, Theorem 2.2, which is proved in the next section, asserts that for any sheaf of abelian groups A on which Γ acts, $Z^n_\Gamma(A)$ is naturally isomorphic to $H^n(X, A)$, where Z^n_Γ is the n^{th} right derived functor of the cocycle functor and $H^n(X, A)$ is the usual sheaf cohomology of X with values in A. Further, our computation of the sheaf cohomology of Γ allows us to refine the calculations made in [16], Proposition 11.8 that show that the Brauer group of Γ vanishes when X is the path space of a graph (with no sinks). In Section 4, we give some other examples that illustrate this in settings that are of current interest in operator algebra.

Our second objective is to use the calculations of Section 2 to show that for each twist Λ over Γ, the (restricted) groupoid C^*-algebra $C^*(\Gamma; \Lambda)$ is naturally isomorphic to a Cuntz-Pimsner algebra (see [22]) constructed from the data used to build Λ. A bit more explicitly, first note that Λ is a bona fide groupoid with Haar system in its own right and we may therefore form its C^*-algebra, $C^*(\Lambda)$. It is a completion of the space of continuous, complex-valued, compactly supported functions on Λ, $C_c(\Lambda)$. The circle $\mathbb{T}$ acts on Λ in the obvious way, and the restricted groupoid C^*-algebra of Λ, $C^*(\Gamma; \Lambda)$, is defined to be the closure in $C^*(\Lambda)$ of $\{f \in C_c(\Lambda) \mid f(z\lambda) = zf(\lambda),\ z \in \mathbb{T}\}$. For the notion of Cuntz-Pimsner algebras, recall from [22] and [19] that a C^**-correspondence* $\mathcal{E}$ over a C^*-algebra A is a (right) Hilbert C^*-module $\mathcal{E}$ over A that is endowed with a C^*-representation φ of A into the space of continuous, adjointable module maps on $\mathcal{E}$, $\mathcal{L}(\mathcal{E})$. We shall assume that our correspondences are faithful, meaning that φ is injective. However, we do not assume that they are full, meaning that the closed span of $\langle \mathcal{E}, \mathcal{E} \rangle$, which is an ideal in A, is, in fact, all of A. Also, in order to lighten the notation, we shall usually not mention φ unless it helps to clarify some point. Given a correspondence $\mathcal{E}$ one can build a C^*-algebra from $\mathcal{E}$, denoted $\mathcal{O}_\mathcal{E}$, that is a simultaneous generalization of a crossed product of A determined by an automorphism and of a Cuntz-Krieger algebra. In [22], $\mathcal{O}_\mathcal{E}$ is called the C^*-algebra of $\mathcal{E}$, while in [19] and elsewhere, $\mathcal{O}_\mathcal{E}$ is called the *Cuntz-Pimsner algebra* associated with $\mathcal{E}$. The faithfulness of φ guarantees that $\mathcal{O}_\mathcal{E}$ is non-zero. In [22], Pimsner usually assumes that his modules $\mathcal{E}$ are full. However, this is not necessary for our purposes and we choose to use what he calls the *augmented* algebra in [22], Remark 1.2.3 instead of his $\mathcal{O}_\mathcal{E}$. We shall not need anything about the actual construction of $\mathcal{O}_\mathcal{E}$ in this note and we shall only use a few of the properties of these algebras. We shall therefore refer the reader to [22] and [19] for most details.

If X is compact it was shown by the first author in [6], 3.3 that $C^*(\Gamma)$ may be identified with the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ where $\mathcal{H}$ is the C^*-correspondence over $C(X)$, denoted $\ell^2(\sigma)$, naturally associated to σ. His analysis works even when X is locally compact. In this case, $\ell^2(\sigma)$ is defined to be the completion of the pre-Hilbert C^*-module $C_{\mathrm{c}}(X)$ over $C_0(X)$ defined by the formulae:

$$\xi \cdot f(x) = \xi(x) f(\sigma(x)) \quad \text{and} \quad \langle \xi, \eta \rangle(x) = \sum_{\sigma(y)=x} \overline{\xi(y)} \eta(y),$$

$\xi, \eta \in C_{\mathrm{c}}(X)$, $f \in C_0(X)$. The fact that σ is a local homeomorphism, coupled with the fact that ξ and η both lie in $C_{\mathrm{c}}(X)$ guarantee that the sum defining the inner product is finite. The algebra $C_0(X)$ acts on $\ell^2(\sigma)$ to the left via the formula

$$f \cdot \xi(x) = f(x)\xi(x),$$

and with respect to this left action $\ell^2(\sigma)$ becomes a C^*-correspondence over $C_0(X)$.

As we shall show in Section 3, twists over Γ are naturally associated to line bundles over X. If T is such a line bundle, then we may build the associated twist Λ_T over Γ and we may "twist" the correspondence $\ell^2(\sigma)$ by T to obtain a new C^*-correspondence $\mathcal{H}_T$ over $C_0(X)$. We shall show in Theorem 3.3 that there is a natural isomorphism between $C^*(\Gamma; \Lambda_T)$ and the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{H}_T}$. If the map σ is a homeomorphism, then the correspondences $\ell^2(\sigma)$ and $\mathcal{H}_T$ are "invertible", meaning that they are Hilbert bimodules in the sense used by Abadie, Eilers, and Exel in [1] and by others. In this event, the presentation in [1] shows how to realize $\mathcal{O}_{\mathcal{H}_T}$ as a Fell bundle over $\mathbb{Z}$. Our analysis in the general setting provides a way of thinking about $\mathcal{O}_{\mathcal{H}_T}$ in terms of a twist over Γ — a kind of a Fell bundle over Γ.

One benefit of our isomorphism theorem, Theorem 3.3, that is under development, is that we will be able to apply the results from [20] and [8] to give conditions implying that $C^*(\Gamma; \Lambda)$ is simple. We also will be able to apply technology developed in [5] to calculate the K-theory of $C^*(\Gamma; \Lambda)$.

2. COHOMOLOGY CALCULATIONS

A Γ-sheaf is simply a sheaf over the unit space X of Γ on which Γ acts. We shall view a sheaf A over X both as an étale space over X with abelian group fibers and as a functor from the category of open subsets of X to the category of abelian groups satisfying the usual relations. For an *arbitrary* r-discrete groupoid Γ and Γ-sheaf A one has the following long exact sequence (by [14], 3.7):

$$\begin{aligned} 0 \to H^0(\Gamma, A) \to H^0(\Gamma^0, A) \xrightarrow{d} Z^0_\Gamma(A) \to H^1(\Gamma, A) \to \cdots \\ \to H^{n-1}(\Gamma^0, A) \xrightarrow{d} Z^{n-1}_\Gamma(A) \to H^n(\Gamma, A) \to H^n(\Gamma^0, A) \to Z^n_\Gamma(A) \to \cdots \end{aligned} \tag{2.1}$$

where $H^n(\Gamma, A)$ denotes the n^{th} equivariant cohomology of Γ with coefficients in A (cf. [9]), $H^n(\Gamma^0, A)$ denotes the usual sheaf cohomology of the unit space Γ^0 with coefficients in A (we use the same symbol for a Γ-sheaf and its underlying sheaf) and Z^n_Γ denotes the n^{th} right derived functor of the cocycle functor. The cocycle functor $Z_\Gamma : \mathrm{Ab}(\Gamma) \to \mathrm{Ab}$, where $\mathrm{Ab}(\Gamma)$ is the category of Γ-sheaves and Ab is the category of abelian groups, is defined as follows: Given a Γ-sheaf A, the abelian group $Z_\Gamma(A)$ consists of all continuous functions $f : \Gamma \to A$ such that $f(\gamma) \in A_{r(\gamma)}$ (i.e. f is a continuous section of $r^*(A)$) and $f(\gamma_1\gamma_2) = f(\gamma_1)+\gamma_1 f(\gamma_2)$ for all $(\gamma_1, \gamma_2) \in \Gamma^2$. Thus, $Z_\Gamma(A)$ is the usual group of one-cocycles or crossed homomorphisms with values in the bundle A.

When $\Gamma = \Gamma(X, \sigma)$, we have $\Gamma^0 = X$ (under the identification $(x, 0, x) \mapsto x$) and we shall show that there is an isomorphism $Z^n_\Gamma(A) \simeq H^n(X, A)$ for any Γ-sheaf A.

Note that $H^0(X, A) = S(A)$, the group of continuous sections of A, and that $H^n(X, \cdot)$ is the n^{th} right derived functor of S (when it is regarded as a functor from the category of sheaves of abelian groups over X, $\mathrm{Ab}(X)$ to Ab). Our first goal is to show that the functors S and Z_Γ are naturally isomorphic. Define a map $\varphi_A : Z_\Gamma(A) \to S(A)$ by $\varphi_A(f)(x) = f(x, 1, \sigma(x))$.

Lemma 2.1. *The map φ_A defines a natural isomorphism between Z_Γ and S.*

Proof. It is easy to check that φ_A defines a natural transformation between the functors Z_Γ and S and that φ_A is a homomorphism. It remains to show that it is bijective. The map $x \mapsto (x, 1, \sigma(x))$ defines a homeomorphism from X onto a clopen subset of Γ, which we denote by X'. The bijectivity of φ_A is equivalent to the assertion that every continuous section on X' has a unique continuous extension to Γ that satisfies the cocycle property. But this is straightforward: For $\gamma = (x, k - l, y) \in \Gamma$ (with $\sigma^k(x) = \sigma^l(y)$) one has the factorization $\gamma = \xi_1 \cdots \xi_k \eta_l^{-1} \cdots \eta_1^{-1}$ where $\xi_i = (\sigma^{i-1}(x), 1, \sigma^i(x))$ for $i = 1, \ldots, k$ and $\eta_j = (\sigma^{j-1}(y), 1, \sigma^j(y))$ for $j = 1, \ldots, l$. Note that $\xi_i, \eta_j \in X'$ and that the extension is uniquely determined by the cocycle property $f(\gamma_1\gamma_2 \cdots \gamma_n) = f(\gamma_1) + \gamma_1 f(\gamma_2) + \cdots + \gamma_1\gamma_2 \cdots \gamma_{n-1} f(\gamma_n)$ and the fact that $f(\gamma^{-1}) = -\gamma^{-1} f(\gamma)$. It follows that φ_A is indeed bijective and so defines a natural isomorphism between the functors Z_Γ and S. ∎

Theorem 2.2. *The map φ_A induces an isomorphism between $Z^n_\Gamma(A)$ and $H^n(X, A)$.*

Proof. Given a Γ-sheaf B, there is a Γ-sheaf $Q(B)$ (see [15], 1.6) which is flabby as a sheaf over X (a sheaf is flabby or flasque if any continuous section defined on an open subset of X may be extended continuously to all of X). Hence, for any Γ-sheaf A there is an injective resolution (in $\mathrm{Ab}(\Gamma)$) $A \to Q^0 \to Q^1 \to Q^2 \to \cdots$, which is flabby when regarded as a resolution in $\mathrm{Ab}(X)$. By applying the functors Z_Γ and S to the complex Q^* and invoking Lemma 2.1, one obtains the diagram:

$$\begin{array}{ccccccc} Z_\Gamma(Q^0) & \to & Z_\Gamma(Q^1) & \to & Z_\Gamma(Q^2) & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ S(Q^0) & \to & S(Q^1) & \to & S(Q^2) & \to & \cdots \end{array}$$

in which the vertical arrows are isomorphisms $\varphi_{Q^i} : Z_\Gamma(Q^i) \to S(Q^i)$ and the diagram commutes because φ_A is a natural transformation. Then $Z^n_\Gamma(A)$ is the cohomology of the complex corresponding to the first row. On the other hand, $H^n(X, A)$ is the cohomology of the second row since $H^n(X, F) = 0$ if F is flabby and $n > 0$ (see [25], 3.15, [10], II.3.5). ∎

COROLLARY 2.3. *The long exact sequence* (2.1) *induces the long exact sequence*

$$\begin{aligned} &0 \to H^0(\Gamma, A) \to H^0(X, A) \xrightarrow{1-\sigma^*} H^0(X, A) \to H^1(\Gamma, A) \to \cdots \\ &\to H^{n-1}(X, A) \xrightarrow{1-\sigma^*} H^{n-1}(X, A) \to H^n(\Gamma, A) \to \\ &\to H^n(X, A) \xrightarrow{1-\sigma^*} H^n(X, A) \to \cdots \end{aligned} \tag{2.2}$$

where σ^* is the map on cohomology induced by the local homeomorphism $\sigma : X \to X$.

Proof. We write σ^* also for the pullback of sheaves. That is, $\sigma^*(A)$ is the pullback sheaf on X induced by A and σ. It is isomorphic to A since A is a Γ-sheaf. Conversely, given a sheaf B over X together with an isomorphism $B \simeq \sigma^*(B)$ one may endow B with the structure of a Γ-sheaf in a natural way. For the Γ-sheaf A, the long exact sequence (2.1) arises from a map $d : S(A) \to Z_\Gamma$ defined by the equation $d(f)(\gamma) = f(r(\gamma)) - \gamma f(s(\gamma))$. Since the composition $\varphi_A d : S(A) \to S(A)$ is given by the equation

$$\begin{aligned} \varphi_A d(f)(x) &= f(r(x, 1, \sigma(x))) - (x, 1, \sigma(x)) f(s(x, 1, \sigma(x))) \\ &= f(x) - (x, 1, \sigma(x)) f(\sigma(x)), \end{aligned}$$

it follows that $\varphi_A d = 1 - \sigma^*$. With this observation, we see that the long exact sequence (2.1) may be rewritten as the long exact sequence (2.2). ∎

In the case of most interest to us, A is the sheaf $\mathcal{S}$ of germs of continuous circle-valued functions on X. Since Γ is r-discrete, elements in Γ may be viewed as germs of local homeomorphisms of X. Hence there is a canonical action of Γ on $\mathcal{S}$ given by composition of germs. There is an extension of sheaves

$$0 \to \mathbb{Z} \to \mathcal{R} \to \mathcal{S} \to 0$$

where $\mathcal{R}$ is the sheaf of germs of continuous real-valued functions on X and $\mathbb{Z}$ is the constant sheaf of integers (both endowed with canonical Γ-actions). One has the long exact sequence:

$$\begin{aligned} &0 \to H^0(X, \mathbb{Z}) \to H^0(X, \mathcal{R}) \to H^0(X, \mathcal{S}) \to H^1(X, \mathbb{Z}) \\ &\to H^1(X, \mathcal{R}) \to H^1(X, \mathcal{S}) \to H^2(\Gamma, \mathbb{Z}) \to \cdots. \end{aligned} \tag{2.3}$$

Since $\mathcal{R}$ is soft, we have $H^n(X, \mathcal{R}) = 0$ for $n > 0$ and, hence, $H^n(X, \mathcal{S}) \simeq H^{n+1}(X, \mathbb{Z})$, for $n > 0$. Since one also has a short exact sequence of Γ-sheaves, there is also the long exact sequence (see [13], Definition 0.11):

$$\begin{aligned} &0 \to H^0(\Gamma, \mathbb{Z}) \to H^0(\Gamma, \mathcal{R}) \to H^0(\Gamma, \mathcal{S}) \to H^1(\Gamma, \mathbb{Z}) \\ &\to H^1(\Gamma, \mathcal{R}) \to H^1(\Gamma, \mathcal{S}) \to H^2(\Gamma, \mathbb{Z}) \to \cdots. \end{aligned} \tag{2.4}$$

By Corollary 2.3 and the fact that $\mathcal{R}$ is soft, $H^n(\Gamma, \mathcal{R}) = 0$ for $n > 1$; it follows that $H^n(\Gamma, \mathcal{S}) \simeq H^{n+1}(\Gamma, \mathbb{Z})$ for all $n > 1$. In [16], 11.3 (see also [15], 4.19) the second cohomology group $H^2(\Gamma, \mathcal{S})$ was identified with the so-called Brauer group $\mathrm{Br}(\Gamma)$ of Γ. This is the collection of strong Morita equivalence classes of Γ-bundles of elementary C^*-algebras satisfying Fell's condition. These equivalence classes form a group under tensor product that generalizes the Brauer group of finite dimensional, central simple algebras over a field. These facts together with Corollary 2.3 yield the following:

Corollary 2.4. *We have* $\mathrm{Br}(\Gamma) \simeq H^2(\Gamma, \mathcal{S}) \simeq H^3(\Gamma, \mathbb{Z})$. *Hence, in the notation of Corollary* 2.3*, one has the following exact sequence:*

$$H^2(X,\mathbb{Z}) \xrightarrow{1-\sigma^*} H^2(X,\mathbb{Z}) \to \mathrm{Br}(\Gamma) \to H^3(X,\mathbb{Z}) \xrightarrow{1-\sigma^*} H^3(X,\mathbb{Z}). \tag{2.5}$$

Note that this includes the fact that $\mathrm{Br}(\Gamma) = 0$ when X is the path space of a graph and σ is the unilateral shift ([16], Proposition 11.8).

3. CUNTZ-PIMSNER ALGEBRAS

Suppose Γ is a general r-discrete groupoid and that A is a Γ-sheaf. Then a *twist* by A over Γ is an r-discrete groupoid Σ, with $\Sigma^0 = \Gamma^0$, together with two groupoid homomorphisms, j and π, so that

$$A \underset{j}{\longrightarrow} \Sigma \underset{\pi}{\longrightarrow} \Gamma,$$

with j injective and π surjective, $\pi^{-1}(\Gamma^0) = j(A)$, and so that $\sigma j(a)\sigma^{-1} = j(\pi(\sigma)a)$ for all $\sigma \in \Sigma$ and all $a \in A_{s(\sigma)}$. Thus a twist by A over Γ is simply an extension of Γ by A. Two twists are called isomorphic in case they are isomorphic as extensions in the usual sense. The isomorphism classes of twists by A over Γ becomes an abelian group under Baer sum that is denoted $T_\Gamma(A)$.

In the special case when $A = \mathcal{S}$, $T_\Gamma(\mathcal{S})$ is also written $\mathrm{Tw}(\Gamma)$. Further, as we mentioned in the introduction, a twist Σ by $\mathcal{S}$ over Γ may be viewed as a principal circle bundle over Γ where the circle action is compatible with the groupoid actions. For the details on the theory of twists, see [14] and [15], Section 2. Note, however, that in [14], twists are restricted to extensions by $\mathcal{S}$ of *principal* groupoids. The restriction to principal groupoids is not necessary for our purposes and indeed, in general, $\Gamma(X,\sigma)$ is not principal. In Corollary 3.4 of [15], it is proved that $T_\Gamma(A)$ is naturally isomorphic to $Z^1_\Gamma(A)$ for any Γ-sheaf A. Hence, when $\Gamma = \Gamma(X,\sigma)$ and $A = \mathcal{S}$ we conclude from Theorem 2.2 that

$$H^1(X,\mathcal{S}) \simeq Z^1_\Gamma(\mathcal{S}) \simeq T_\Gamma(\mathcal{S}) \simeq \mathrm{Tw}(\Gamma).$$

Our objective in this section is to construct this isomorphism directly and then to show that the twisted groupoid C^*-algebra is a Cuntz-Pimsner algebra.

Recall that $H^1(X,\mathcal{S})$ may be identified with the group of isomorphism classes of principal circle bundles over X. We want to see how to pass between circle bundles over X to twists — i.e., certain circle bundles over Γ. To this end we shall write j for the map that sends $x \in X$ to $(x, 1, \sigma(x))$ in Γ (see Lemma 2.1); then j induces a homeomorphism from X to its image, the clopen subset $X' := \{(x, 1, \sigma(x)) \mid x \in X\}$. (This j should not be confused with the j discussed above in the general theory of twists, which will never be mentioned again.) Given a twist Λ over Γ, viewed as a principal circle bundle over Γ, we may pull the circle bundle back to X, via j, to obtain a principal circle bundle over X. In symbols, $\Lambda \mapsto j^*(\Lambda)$. We are thus led to

THEOREM 3.1. *The map* $\Lambda \to j^*(\Lambda)$ *implements an isomorphism from* $\mathrm{Tw}(\Gamma)$ *onto* $H^1(X, \mathcal{S})$ *viewed as isomorphism classes of principal circle bundles over* X.

Proof. Our comments prior to the statement of the theorem together with a moment's reflection reveal that the map $\Lambda \to j^*(\Lambda)$ induces a homomorphism from $\mathrm{Tw}(\Gamma)$ into $H^1(X, \mathcal{S})$. It remains to show that the map is bijective. We first show the surjectivity, i.e., how to construct a twist Λ_T from a principal circle bundle T so that $T \simeq j^*(\Lambda_T)$. So let the principal circle bundle T over X be given, write $p : T \to X$ for the quotient map, and for $k, l \geqslant 0$ set

$$X_{k,l} = \{(x, k-l, y) \mid \sigma^k(x) = \sigma^l(y)\} \subset \Gamma.$$

It is straightforward to verify the following assertions:

(i) $X_{k,l}$ is a clopen subset of Γ,

(ii) $X_{k,l} \subset X_{k+1,l+1}$, and

(iii) $c^{-1}(k-l) = \bigcup_{j=0}^{\infty} X_{k+j,l+j}$, where c is the position cocycle: $c(x, k-l, y) = k - l$.

Our strategy is to "extend" T to $X_{k,l}$ in such a way that the extensions to $X_{k,l}$ and to $X_{k+1,l+1}$ are compatible. This will give bundles over the disjoint sets, $c^{-1}(n)$, $n \in \mathbb{Z}$, which may then be pieced together in the obvious way. Given circle bundles T_1 and T_2 over X_1 and X_2 one may form the "product" circle bundle over $X_1 \times X_2$, $T_1 \star T_2 = T_1 \times T_2 / \sim$, where $(zt_1, t_2) \sim (t_1, zt_2)$ for $z \in \mathbb{T}$. We let $\overline{T}$ denote the conjugate circle bundle; there is a fiber preserving homeomorphism $T \to \overline{T}$, written $t \mapsto \overline{t}$, such that $\overline{zt} = \overline{z}\overline{t}$ for all $z \in \mathbb{T}$. Note that the pullback of $T \star \overline{T}$ along the diagonal is canonically isomorphic to the trivial circle bundle $X \times \mathbb{T}$. Indeed, given $t_1, t_2 \in T$ with $p(t_1) = p(t_2)$, there is a unique $z \in \mathbb{T}$ so that $t_1 = zt_2$; write $z = t_1\overline{t_2}$. The desired isomorphism is then given by $(t_1, t_2) \mapsto (p(t_1), t_1\overline{t_2})$ for all $t_1, t_2 \in T$ with $p(t_1) = p(t_2)$ (note that this is well-defined). By a slight abuse of notation let T^k denote the circle bundle $T \star \cdots \star T$ (k-factors) over X^k. Observe that there is a natural embedding $\iota_{k,l} : X_{k,l} \to X^k \times X^l$ given by the formula

$$\iota_{k,l}(x, m, y) = (x, \sigma(x), \ldots, \sigma^{k-1}(x), \sigma^{l-1}(y), \ldots, \sigma(y), y).$$

Set $T_{k,l} = \iota_{k,l}^*(T^k \star \overline{T}^l)$ (note that $T_{0,0}$ is the trivial circle bundle over $X_{0,0} = X$). One verifies that the restriction of $T_{k+1,l+1}$ to $X_{k,l}$ is isomorphic to $T_{k,l}$: if $(u_1, \ldots, u_{k+1}, \overline{v_{l+1}}, \ldots, \overline{v_1}) \in T_{k+1,l+1}$ lies in the fiber over $(x, k-l, y) \in X_{k,l}$, then $p(u_{k+1}) = \sigma^k(x) = \sigma^l(y) = p(v_{l+1})$ and the desired isomorphism is given by

$$(u_1, \ldots, u_{k+1}, \overline{v_{l+1}}, \ldots, \overline{v_1}) \mapsto u_{k+1}\overline{v_{l+1}}(u_1, \ldots, u_k, \overline{v_l}, \ldots, \overline{v_1}).$$

The desired twist Λ_T is obtained by piecing together the circle bundles $T_{k,l}$. We also denote the quotient map from Λ_T to Γ by p. The source and range maps on Λ_T are defined via p, i.e. $s(\lambda) = s(p(\lambda))$ and $r(\lambda) = r(p(\lambda))$ for all $\lambda \in \Lambda_T$. Suppose that $\lambda, \mu \in \Lambda_T$ are composable. Then there are $j, k, l \geqslant 0$ so that $\lambda \in T_{j,k}$, $\mu \in T_{k,l}$ and $x, y, z \in X$ such that $p(\lambda) = (x, j-k, y)$, $p(\mu) = (y, k-l, z)$. Given $\lambda = (t_1, \ldots, t_j, \overline{u_k}, \ldots, \overline{u_1})$ and $\mu = (v_1, \ldots, v_k, \overline{w_l}, \ldots, \overline{w_1})$, define multiplication by the formula,

$$\lambda\mu = \left(\prod_{i=1}^{k} v_i\overline{u_i}\right)(t_1, \ldots, t_j, \overline{w_l}, \ldots, \overline{w_1}).$$

(Note that $p(u_i) = p(v_i) = \sigma^{i-1}(y)$, so this formula makes sense.) It is easy to verify that multiplication is well-defined, continuous and associative. Finally, the inverse map is defined by

$$(u_1, \ldots, u_k, \overline{v_l}, \ldots, \overline{v_1})^{-1} = (v_1, \ldots, v_l, \overline{u_k}, \ldots, \overline{u_1}).$$

Thus, we see that Λ_T is a twist over Γ, and it is evident that $T = j^*(\Lambda_T)$. Hence, the map is surjective.

If the circle bundle is trivial, then any continuous section may be used to construct a continuous section of the twist (along the above lines) which is easily seen to be a groupoid homomorphism. Hence, the twist is trivial and the map is injective. ∎

Our objective now is to show how to realize $C^*(\Gamma;\Lambda)$ as a Cuntz-Pimsner algebra for each twist Λ over Γ. As we have just seen, each twist Λ over Γ comes from a unique circle bundle over X. So we begin with these. Given a circle bundle T over X, let L_T denote the space of continuous sections of the associated complex line bundle $T \times_T \mathbb{C}$ that vanish at infinity on T. We think of L_T as the space of all continuous $\mathbb{C}$-valued functions f on T such that $f(zt) = zf(t)$ for all $z \in \mathbb{T}$ and all $t \in T$ and such that $|f| \in C_0(X)$. Then, in fact, L_T has the structure of an *imprimitivity* bimodule over $C_0(X)$. The action of $C_0(X)$ is central, i.e., for $f \in C_0(X)$ and $\xi \in L_T$, $f \cdot \xi(t) = \xi \cdot f(t) := \xi(t)f(\dot{t})$; and the inner products are given by the formulae: ${}_{C_0(X)}\langle \xi, \eta\rangle(\dot{t}) = \xi(t)\overline{\eta}(t)$ and $\langle \xi, \eta\rangle_{C_0(X)}(\dot{t}) = \overline{\xi}(t)\eta(t)$. Here $\dot{t} = p(t)$. Note that by the transformation properties of ξ and η these are bona fide $C_0(X)$-valued inner products. In fact, the most general $C_0(X)$-$C_0(X)$ equivalence bimodule that fixes the spectrum X is of this form. To say the same thing differently, the isomorphism classes of these $C_0(X)$-$C_0(X)$ equivalence bimodules form a group under tensor product, a subgroup of the Picard group of X that is isomorphic to $H^1(X, \mathcal{S})$ viewed as the isomorphism classes of line bundles over X. For these things, see [21].

Next, we "twist" $\ell^2(\sigma)$ by L_T. Recall that $\ell^2(\sigma)$ is a C^*-correspondence over $C_0(X)$ and so the tensor product $\mathcal{H} := L_T \otimes_{C_0(X)} \ell^2(\sigma)$ makes sense as a right Hilbert C^*-module over $C_0(X)$ (see [24], 5.9). However, since L_T is a $C_0(X)$-$C_0(X)$ imprimitivity bimodule, the left action of $C_0(X)$ on L_T passes to one of $C_0(X)$ on $\mathcal{H}$, making $\mathcal{H}$ a C^*-correspondence over $C_0(X)$. From the definitions of L_T and $\ell^2(\sigma)$, it is clear that $\mathcal{H}$ may be viewed as the completion of the compactly supported sections in L_T with the following pre-C^*-correspondence structure:

$$\begin{aligned} &\langle \xi, \eta\rangle_{C_0(X)}(x) = \sum_{\sigma(\dot{t})=x} \overline{\xi(t)}\eta(t); \\ &\xi \cdot f(t) = \xi(t) f(\sigma(\dot{t})) \quad \text{and} \quad f \cdot \xi(t) = f(\dot{t})\xi(t). \end{aligned} \tag{3.1}$$

The claim is then that the Pimsner algebra $\mathcal{O}_{\mathcal{H}}$ is isomorphic to the twisted groupoid C^*-algebra $C^*(\Gamma;\Lambda_T)$ associated to Λ_T, where Λ_T is the twist over Γ determined by the bundle T.

To prove this claim, we need to invoke a result proved in [8]. Recall that if $\mathcal{E}$ is a C^*-correspondence over a C^*-algebra A, then an (*isometric*) *covariant representation* of $\mathcal{E}$ in a C^*-algebra B is a pair (V, π) consisting of a $\mathbb{C}$-linear map $V : \mathcal{E} \to B$ and a C^*-homomorphism $\pi : A \to B$ such that the following two conditions are satisfied:

(1) V is a bimodule map; i.e., $V(\varphi(a)\xi b) = \pi(a)V(\xi)\pi(b)$ for all $a, b \in A$ and all $\xi \in \mathcal{E}$.

(2) $V(\xi)^*V(\eta) = \pi(\langle \xi, \eta \rangle)$, for all $\xi, \eta \in \mathcal{E}$.

It can be shown easily that V is bounded and, in fact, $\|V(\xi)\| \leqslant \|\xi\|$. Furthermore, the map from $\mathcal{E} \times \widetilde{\mathcal{E}}$ to B that sends $(\xi, \widetilde{\eta})$ to $V(\xi)V(\eta)^*$ extends to a C^*-homomorphism $\pi^{(1)}$ from $K(\mathcal{E})$, identified with $\mathcal{E} \otimes_A \widetilde{\mathcal{E}}$, into B. (See [22].) The covariant representation (V, π) is said to satisfy the *Cuntz condition* or to be a *Cuntz covariant representation* in case $\pi^{(1)} \circ \varphi(a) = \pi(a)$ for all a in the ideal J in A, which is defined to be $\varphi^{-1}(K(\mathcal{E}))$. It is proved in [22] that an isometric representation (V, π) of $\mathcal{E}$ in a C^*-algebra B defines a C^*-representation of the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ in B if and only if (V, π) satisfies the Cuntz condition. The representation of $\mathcal{O}_{\mathcal{E}}$ determined by (V, π) is denoted $V \times \pi$ and is called the *integrated form* of (V, π). Conversely, every representation of the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ in a C^*-algebra B is of the form $V \times \pi$ for a (unique) isometric covariant representation of $\mathcal{E}$ in B that satisfies the Cuntz condition. A condition for the *faithfulness* of a representation $V \times \pi$ of $\mathcal{O}_{\mathcal{E}}$ into B, proved in [8], is the following; it is essential for our analysis.

LEMMA 3.2. *Suppose that (V, π) is an isometric covariant representation of $\mathcal{E}$ into a C^*-algebra B. Then $V \times \pi$ is faithful if π is faithful and there is a (strongly continuous) action $\beta : \mathbb{T} \to \operatorname{Aut}(B)$ such that $\beta_z \circ \pi = \pi$ and $\beta_z \circ V = zV$ for all $z \in \mathbb{T}$.*

Fix a principal circle bundle T over X and let $\mathcal{H} = \mathcal{H}_T$ be the C^*-correspondence over $C_0(X)$ defined above. Also, let $j : X \to \Gamma$ be defined as above by the formula $j(x) = (x, 1, \sigma(x))$. The bundles T and Λ_T are related by the formula

$$T = j^*(\Lambda_T).$$

We define the pair (V, π) by the formulae:

$$\pi(f)(\lambda) = \begin{cases} f(p(\lambda)), & \lambda \in \Lambda_T|\Gamma^0, \\ 0, & \text{otherwise}, \end{cases} \tag{3.2}$$

$f \in C_0(X)$, and

$$V(\xi)(\lambda) = \begin{cases} \xi \circ (j^*)^{-1}(\lambda), & \lambda \in \Lambda_T|X', \\ 0, & \text{otherwise}, \end{cases} \tag{3.3}$$

$\xi \in \mathcal{H}_T$. It is routine to check that (V, π) is an isometric covariant representation of $\mathcal{H}_T$ in $C^*(\Gamma; \Lambda_T)$. Thus, the assertions in the following theorem make sense.

THEOREM 3.3. *The pair (V, π) defined by equations* (3.2) *and* (3.3) *is a faithful isometric covariant representation mapping $\mathcal{H} = \mathcal{H}_T$ into $C^*(\Gamma; \Lambda_T)$ in such a way that its integrated form, $V \times \pi$, is a C^*-isomorphism mapping $\mathcal{O}_{\mathcal{H}_T}$ onto $C^*(\Gamma; \Lambda_T)$.*

Proof. As we just mentioned, it is routine to check that (V, π) is an isometric covariant representation of $\mathcal{H}$ in $C^*(\Gamma; \Lambda_T)$. Also, it is evident that the image generates $C^*(\Gamma; \Lambda_T)$. Of course, π is faithful and if $\beta : \mathbb{T} \to \operatorname{Aut}(C^*(\Gamma; \Lambda_T))$ is the automorphism group induced by the position cocycle $c : \Gamma \to \mathbb{Z}$, $c(x, m, y) = m$, then $\beta_z(f)(\lambda) = z^{c(\dot{\lambda})} f(\lambda)$, $f \in C_c(\Gamma; \Lambda_T)$, and it is clear that $\beta_z \circ \pi = \pi$, while

$\beta_z \circ V = zV$. Thus, the only thing that needs to be checked is that (V, π) satisfies the Cuntz condition.

This, however, is quite easy. It is a matter of a couple of identifications coupled with the appropriate references. First note that $K(\mathcal{H}_T) = \mathcal{H}_T \otimes_{C_0(X)} \mathcal{H}_T^*$ which, in turn, may be identified with the twisted groupoid C^*-algebra $C^*(R(\sigma), T * \overline{T}|R(\sigma))$, where $R(\sigma) = \{(x, y) \in X \times X \mid \sigma(x) = \sigma(y)\}$. This is a straightforward computation, given the representation of $\mathcal{H}_T$ in terms of functions on T that transform according to the formula: $\xi(zt) = z\xi(t)$, $t \in T$, $z \in \mathbb{T}$. (See [15] and in particular Section 3 therein, where a relation between the sheaf cohomology of X and the groupoid cohomology of $R(\sigma)$ is established.) Observe, too, that $\pi^{(1)}$ identifies $K(\mathcal{H}_T) = C^*(R(\sigma), T * \overline{T}|R(\sigma))$ with the subalgebra of $C^*(\Gamma; \Lambda_T)$ consisting of all elements that are supported on $\Lambda_T|X_{1,1}$, in the notation of the proof of Theorem 3.1. Indeed, $\iota_{1,1}(X_{1,1}) = R(\sigma)$.

Next, we claim that $\varphi(C_0(X)) \subseteq K(\mathcal{H}_T)$. This, however, is obvious from the following facts:

(i) X may be identified as the clopen subset $\Delta = \{(x, x) \in R(\sigma) \mid x \in X\}$;
(ii) $T * \overline{T}|\Delta$ is trivial (see the proof of Theorem 3.1); and
(iii) in the identification of $K(\mathcal{H}_T)$ with $C^*(R(\sigma), T * \overline{T}|R(\sigma))$,

$$\varphi(f)([t_1, t_2]) = \begin{cases} f(p(t_1)), & [t_1, t_2] \in T * \overline{T}|\Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we see that the Cuntz condition is satisfied by (V, π) simply by noting that the calculation of the previous paragraph and the identification $\iota_{1,1}$ of $X_{1,1}$ with $R(\sigma)$ allows us to identify $\pi^{(1)} \circ \varphi$ with π. ∎

4. EXAMPLES

In this section we gather together several examples that illustrate some of the theory we have developed.

Example 4.1. Let $X = \mathbb{T}^k$ be the k-dimensional torus and $\sigma : X \to X$ be a covering map given by a k by k integer matrix R with $|\det R| \geqslant 2$. Then $H^0(X, \mathbb{Z}) = \mathbb{Z}$, and for $n \geqslant 1$, $H^n(X, \mathbb{Z})$ may be identified with $\mathbb{Z}^k \wedge \cdots \wedge \mathbb{Z}^k$, n times. The map induced on cohomology, σ^n, can be identified with $R \wedge \cdots \wedge R$, n times. For $n = 0$, the wedge product is taken to be the identity map. From the exact sequence (2.2) we get

$$H^0(\Gamma, \mathbb{Z}) \simeq \mathbb{Z}$$

$$H^n(\Gamma, \mathbb{Z}) \simeq \ker(I - \underbrace{R \wedge \cdots \wedge R}_{n}) \oplus \operatorname{coker}(I - \underbrace{R \wedge \cdots \wedge R}_{n-1}), \quad n \geqslant 1.$$

Hence, by Corollary 2.4,

$$\operatorname{Br}(\Gamma) \simeq H^3(\Gamma, \mathbb{Z}) \simeq \ker(I - R \wedge R \wedge R) \oplus \operatorname{coker}(I - R \wedge R).$$

In particular, the Brauer group can be infinite for $k \geqslant 3$.

EXAMPLE 4.2. Let X be the infinite path space of the topological graph

$$\mathbb{T} \xleftarrow{s} \mathbb{T} \xrightarrow{r} \mathbb{T},$$

where s and r are the covering maps given by $x \mapsto x^p$ and $x \mapsto x^q$, respectively. Then

$$X = \{(x_1, x_2, \ldots) \in \mathbb{T}^{\mathbb{N}} \mid (x_n)^q = (x_{n+1})^p,\ n \geqslant 1\}.$$

We assume that $|p|, |q| \geqslant 2$ and $(p, q) = 1$. Then X is a solenoid,

$$X = \varprojlim (X_m, \pi_m),$$

where X_m is the space of paths of length m, and where the maps $\pi_m : X_{m+1} \to X_m$ are the projections

$$\pi_m(x_1, x_2, \ldots, x_{m+1}) = (x_1, x_2, \ldots, x_m).$$

The fact that p and q are relatively prime implies that X is connected (see [4]). It is easy to see that each X_m is homeomorphic to $\mathbb{T}$, and that the projections π_m are given by the map s. Indeed, the maps $f_m : \mathbb{T} \to X_m$,

$$f_m(x) = (s^m x, s^{m-1} r x, \ldots, s r^{m-1} x, r^m x)$$

realize the homeomorphisms, and the diagram

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{s} & \mathbb{T} \\ f_{m+1} \downarrow & & \downarrow f_m \\ X_{m+1} & \xrightarrow{\pi_m} & X_m \end{array}$$

is commutative.

Moreover, in this identification, the unilateral shift $\sigma : X \to X$ is given by

$$\begin{array}{ccccccccc} \mathbb{T} & \xleftarrow{s} & \mathbb{T} & \xleftarrow{s} & \mathbb{T} & \xleftarrow{s} & \cdots & \longleftarrow & X \\ & r \swarrow & & r \swarrow & & & & & \sigma \downarrow \\ \mathbb{T} & \xleftarrow{s} & \mathbb{T} & \xleftarrow{s} & \mathbb{T} & \xleftarrow{s} & \cdots & \longleftarrow & X. \end{array}$$

This allows us to calculate the integer cohomology of X and to identify the maps induced by σ:

$$\begin{aligned} &H^0(X, \mathbb{Z}) = \varinjlim(\mathbb{Z}, \mathrm{id}) = \mathbb{Z}, && \sigma^0 = \mathrm{id}, \\ &H^1(X, \mathbb{Z}) = \varinjlim(\mathbb{Z}, p) = \mathbb{Z}[\tfrac{1}{p}], && \sigma^1 = \frac{q}{p}, \\ &H^k(X, \mathbb{Z}) = 0, && k \geqslant 2. \end{aligned}$$

From the long exact sequence (2.2) we determine the integer cohomology of the groupoid $\Gamma = \Gamma(X, \sigma)$:

$$\begin{aligned} &H^0(\Gamma, \mathbb{Z}) = \mathbb{Z}, \\ &H^1(\Gamma, \mathbb{Z}) = \mathbb{Z} \oplus \ker\left(1 - \frac{q}{p}\right) = \mathbb{Z}, \\ &H^2(\Gamma, \mathbb{Z}) = \operatorname{coker}\left(1 - \frac{q}{p}\right) = \mathbb{Z}/(p - q)\mathbb{Z}, \\ &H^k(\Gamma, \mathbb{Z}) = 0, \quad k \geqslant 3. \end{aligned}$$

In particular, by Corollary 2.4 again,

$$\mathrm{Br}(\Gamma) \simeq H^3(\Gamma, \mathbb{Z}) = 0.$$

Example 4.3. Given a sequence of local homeomorphisms as in [15], Addendum 3

$$X_0 \xrightarrow{\sigma_0} X_1 \xrightarrow{\sigma_1} X_2 \xrightarrow{\sigma_2} \cdots,$$

take X to be the disjoint union of the spaces, $X = \coprod_k X_k$ and define $\sigma : X \to X$ in the natural way: if $x \in X_k \subset X$ set $\sigma(x) = \sigma_k(x)$. Let $\Gamma = \Gamma(X, \sigma)$; if σ_n is surjective for all n, then X_0 meets every orbit. It follows that the reduction $\Gamma|X_0$ is equivalent to Γ and therefore has the same cohomology. Further, $\Gamma|X_0$ is precisely the ultraliminary groupoid considered in [15], Addendum 3 (the equivalence relation on X_0 induced by the maps $\sigma_n \cdots \sigma_0$). We show how Corollary 2.3 allows one to recover the short exact sequence for the cohomology given in [15], Addendum 3.

A Γ-sheaf A is given by a sequence of sheaves A_k over X_k together with identifications $A_k = \sigma_k^*(A_{k+1})$ (see the proof of Corollary 2.3). Given such a Γ-sheaf A, we have $H^n(X, A) = \prod_k H^n(X_k, A_k)$ and $\sigma^* = \prod_k \sigma_k^*$ (the map induced on cohomology). By Corollary 2.3, $H^0(\Gamma, A)$ is isomorphic to the kernel of the map

$$1 - \prod_k \sigma_k^* : \prod_k H^0(X_k, A_k) \to \prod_k H^0(X_k, A_k);$$

that is, $H^0(\Gamma, A)$ is isomorphic to the subgroup consisting of all $(g_k) \in \prod_k H^0(X_k, A_k)$ for which $g_k = \sigma_k^*(g_{k+1})$. Hence,

$$H^0(\Gamma, A) = \varprojlim H^0(X_k, A_k).$$

Similarly, for $n > 0$, $H^n(\Gamma, A)$ is an extension of the cokernel of the map

$$1 - \prod_k \sigma_k^* : \prod_k H^{n-1}(X_k, A_k) \to \prod_k H^{n-1}(X_k, A_k)$$

by the kernel of the map

$$1 - \prod_k \sigma_k^* : \prod_k H^n(X_k, A_k) \to \prod_k H^n(X_k, A_k).$$

Hence, we obtain the short exact sequence

$$0 \to \varprojlim{}^1 H^{n-1}(X_k, A_k) \to H^n(\Gamma, A) \to \varprojlim H^n(X_k, A_k) \to 0.$$

Example 4.4. (Skew product construction) The following construction is adapted from [17]. Given a local homeomorphism $\sigma : X \to X$ and a continuous map $c : X \to G$, where G is a locally compact group, one constructs a new local homeomorphism $\tau : X \times G \to X \times G$ by the formula

$$\tau(x, g) = (\sigma(x), gc(x)).$$

We define a continuous one-cocycle $\widetilde{c} : \Gamma(X, \sigma) \to G$ in the following way (see Lemma 2.1). For $\gamma = (x, k - l, y) \in \Gamma$ with $\sigma^k(x) = \sigma^l(y)$, set

$$\widetilde{c}(\gamma) = c(x)c(\sigma(x)) \cdots c(\sigma^{k-1}(x))c(\sigma^{l-1}(y))^{-1} \cdots c(\sigma(y))^{-1}c(y)^{-1}$$

($\widetilde{c}$ is clearly well-defined and satisfies the cocycle property). One may construct the skew-product as defined by Renault, $\Gamma(X,\sigma)(\widetilde{c}) = \Gamma(X,\sigma) \times_{\widetilde{c}} G$ (see [23], Definition I.1.6); it is straightforward to verify that

$$\Gamma(X \times G, \tau) \simeq \Gamma(X,\sigma) \times_{\widetilde{c}} G,$$

see [17], 2.4. It follows by [23], II.5.7 that when G is abelian

$$C^*(\Gamma(X \times G, \tau)) \simeq C^*(\Gamma(X,\sigma)) \times_{\alpha_{\widetilde{c}}} \widehat{G},$$

where $\alpha = \alpha_{\widetilde{c}}$ is the action of $\widehat{G}$ on $C^*(\Gamma(X,\sigma))$ induced by the cocycle. In particular, taking $G = \mathbb{R}$, $X = \{1, 2, \dots, n\}^{\mathbb{N}}$, σ the Bernoulli shift, and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, we may define the continuous function $c : X \to \mathbb{R}$ by $c(x) = \lambda_k$ if $x_1 = k$. Then $C^*(\Gamma(X,\sigma)) \simeq \mathcal{O}_n$ and the induced action of $\mathbb{R}$ on $\mathcal{O}_n$ is given by $\alpha_t(S_k) = \mathrm{e}^{\mathrm{i}t\lambda_k} S_k$; the associated crossed product $\mathcal{O}_n \times \mathbb{R}$ is a special case of those studied by Kishimoto in [11], Section 4 (see also [12]).

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