J. OPERATOR THEORY
46(2001), 265–280

RELATIVE ANGULAR DERIVATIVES

JONATHAN E. SHAPIRO

Communicated by Norberto Salinas

ABSTRACT. We generalize the notion of the angular derivative of a holomorphic self-map b, of the unit disk, by replacing the usual difference quotient $\frac{b(z)-b(z_0)}{z-z_0}$ with a difference quotient relative to an inner function u, $\frac{1-b(z)}{1-u(z)}$. We relate properties of this generalized difference quotient to properties of the Aleksandrov measures associated with the functions b and u. Six conditions are shown to be equivalent to each other, and these are used to define the notion of a relative angular derivative. We see that this generalized derivative can be used to reproduce some known results about ordinary angular derivatives, and the generalization is shown to obey a form of the product rule.

KEYWORDS: *Angular derivative, Hardy space, Aleksandrov measure, de Branges-Rovnyak space.*

MSC (2000): 46E22, 46E30.

1. INTRODUCTION

In this paper, we will define and analyze the notion of an angular derivative of a holomorphic self-map of the unit disk relative to a nonconstant inner function.

Let b be a holomorphic self-map of the unit disk, that is, an analytic function on the unit disk $\mathbb{D}$ of the complex plane with $|b| < 1$ on $\mathbb{D}$. We will take u to be our nonconstant inner function — a holomorphic function on $\mathbb{D}$ with $|u| = 1$ almost everywhere on $\partial\mathbb{D}$. This notation will remain fixed.

Our analysis, and even our definition, of relative angular derivatives will come primarily from the viewpoint of the Aleksandrov measures μ_λ and ν_λ ($\lambda \in \partial\mathbb{D}$), which we derive from our functions b and u. These measures are defined and discussed in Section 2. Throughout this paper, we will use m to denote the usual normalized Lebesgue measure on the unit circle. We will also use the notation $\mu^{\mathrm{a.c.}}$ and μ^{s} to denote the absolutely continuous and singular parts of the measure μ ($= \mu_1$) with respect to m. For any function f on the unit disk $\mathbb{D}$, f_r will denote the function on the boundary $\partial\mathbb{D}$ such that $f_r(\mathrm{e}^{\mathrm{i}\theta}) = f(r\mathrm{e}^{\mathrm{i}\theta})$ for $r < 1$.

The relationship between Aleksandrov measures and angular derivatives has been developed by many people recently. The most direct connection comes from

THEOREM 1.1. *The function b has angular derivative at a point $z_0 \in \partial\mathbb{D}$ (where $|b(z_0)| = 1$) exactly where its corresponding Aleksandrov measure μ_λ has an atom, and $\mu_\lambda(\{z_0\}) = 1/|b'(z_0)|$.*

This theorem, which appears (somewhat hidden) in [4], Chapter 7 (see also [7], VI-7), is discussed and even given a different proof in this paper in Section 9. Indeed, Aleksandrov measures have even been used to provide improvements on angular derivative conditions in several theorems about composition operators. For example, Joel Shapiro and P. Taylor showed in [10] that if b has an angular derivative, then the corresponding composition operator C_b acting on the Hardy space H^2 is not compact. This result was generalized in papers by D. Sarason ([6]) and Shapiro and C. Sundberg ([9]), which together prove that a composition operator C_b is compact on H^2 if and only if the corresponding Aleksandrov measures μ_λ are all absolutely continuous, i.e., have not only no atoms (which is equivalent to no angular derivative for b) but no component singular with respect to Lebesgue measure. This theorem was proved in a different way by J. Cima and A. Matheson ([2]), who show that the square of the essential norm of C_b (operating on H^2) is equal to $\sup\limits_{\lambda\in\partial\mathbb{D}} \|\mu_\lambda^{\mathrm{s}}\|$. The author has also used Aleksandrov measures to generalize other theorems which give properties of composition operators and composition operator differences in terms of angular derivatives — see [8]. In the language of this paper, we will be able to restate the compactness condition for composition operators as: C_b is compact on H^2 if and only if for any $\zeta \in \partial\mathbb{D}$, the function $\overline{\zeta}b$ has no angular derivative relative to any inner function u.

In this paper we aim to develop further this type of useful generalization of angular derivatives by studying the more broad category of relative angular derivatives. We will find new perspectives from which to view the already known relationships between Aleksandrov measures and angular derivatives, and find new relationships by studying in detail the behavior of the relative angular derivative from many perspectives, beginning with the generalization of the difference quotient and using primarily the Aleksandrov measures, but also relating their properties to the Hardy space H^2, and the de Branges–Rovnyak spaces (as done by Sarason in [7]).

Section 3 contains some background material about angular derivatives. In Section 4, we generalize the difference quotient, $\frac{b(z)-b(z_0)}{z-z_0}$, which appears in the definition of an angular derivative, to $\frac{1-b}{1-u}$, which will be the primary object under investigation throughout the paper. The study of this generalized difference quotient allows us to define the notion of an angular derivative relative to an inner function. It is in this section that we present the theorem which lists six conditions, all of which will be shown to be equivalent, any one of which can be used as a definition of a relative angular derivative.

Section 5 contains an introduction to the de Branges–Rovnyak spaces, which will be useful in Section 6, when we analyze the boundary behavior of the generalized difference quotient with respect to μ^{s}. This, then, allows us to get an integral condition for the existence of a relative angular derivative which is similar in form to that in the definition of the Hardy spaces. In Section 7, we will similarly analyze the behavior of the generalized difference quotient with respect to $\mu^{\mathrm{a.c.}}$. Section 8 presents a use for our characterization of relative angular derivatives to produce an analog of the product rule for ordinary derivatives. Finally, in Section 9, we see that many known results from the theory of angular derivatives can be obtained easily by viewing an angular derivative as a special case of a relative angular derivative.

2. THE ALEKSANDROV MEASURES

For $\lambda \in \partial\mathbb{D}$, the function $\operatorname{Re}\left(\frac{\lambda+b}{\lambda-b}\right)$ is positive, and, as the real part of an analytic function, harmonic (on the disk $\mathbb{D}$). It is thus the Poisson integral of a positive measure on $\partial\mathbb{D}$, which we will call μ_λ. We have, then,

$$\operatorname{Re}\left(\frac{\lambda+b(z)}{\lambda-b(z)}\right) = \int_{\partial\mathbb{D}} P(\theta,z)\,\mathrm{d}\mu_\lambda(\mathrm{e}^{\mathrm{i}\theta}) = P\mu_\lambda(z)$$

and the Herglotz integral representation,

$$\frac{\lambda+b(z)}{\lambda-b(z)} = \int_{\partial\mathbb{D}} H(\theta,z)\,\mathrm{d}\mu_\lambda(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}\operatorname{Im}\frac{\lambda+b(0)}{\lambda-b(0)}.$$

Note that for $z \in \mathbb{D}$, the Poisson kernel, $P(\theta,z) = \frac{1-|z|^2}{|\mathrm{e}^{\mathrm{i}\theta}-z|^2}$, is the real part of the Herglotz kernel, $H(\theta,z) = \frac{\mathrm{e}^{\mathrm{i}\theta}+z}{\mathrm{e}^{\mathrm{i}\theta}-z}$. The measure μ_1 we shall simply call μ. The measure ν is similarly defined to correspond with the inner function u.

The following are some properties of the Aleksandrov measures defined above:

- All positive Borel measures on $\partial\mathbb{D}$ are associated with functions in this way.
- The absolutely continuous part of μ is given by $\frac{1-|b|^2}{|1-b|^2}$ times the normalized Lebesgue measure (on $\partial\mathbb{D}$).
- The measure μ is singular if and only if b is an inner function, i.e., $|b| = 1$ almost everywhere on $\partial\mathbb{D}$.
- For μ_λ^{s}-a.e. $\xi \in \partial\mathbb{D}$ we have $P\mu_\lambda(\xi) = \infty$ and thus $b(\xi) = \lambda$.

3. ANGULAR DERIVATIVES

The following is an overview of some of the properties of angular derivatives. Much of this material can be found in [1], Section 299. I use it primarily as presented by Sarason in [7], Chapter VI. We use this material as a starting point.

For a holomorphic function b, we can talk about its derivative, $b'(z)$ for $z \in \mathbb{D}$, or, looking at the boundary behavior, the angular derivative of the function at a point $z_0 \in \partial\mathbb{D}$.

THEOREM 3.1. *For a function b, holomorphic in $\mathbb{D}$, and a point z_0 of $\partial\mathbb{D}$, the following are equivalent:*

(i) *The function b has a nontangential limit, $b(z_0)$, at the point z_0, and the difference quotient $(b(z) - b(z_0))/(z - z_0)$ has a nontangential limit at z_0.*

(ii) *The derivative b' has a nontangential limit at z_0.*

The theorem above is true for any holomorphic b, but if we restrict ourselves to holomorphic self-maps of the disk, as we do here, and require that the function b have unit modulus at the boundary point z_0, then we say that the function b has an angular derivative in the sense of Carathéodory at the point z_0.

THEOREM 3.2. [Carathéodory] *If z_0 is a point of $\partial\mathbb{D}$ and*

$$c = \liminf_{z \to z_0} \frac{1 - |b(z)|}{1 - |z|} < \infty,$$

then b has an angular derivative in the sense of Carathéodory at z_0. The relation $b'(z_0) = c\, b(z_0)/z_0$ holds, and $\frac{1-|b(z)|}{1-|z|}$ tends to c as z tends nontangentially to z_0. The number c is positive.

It is this notion of angular derivative which we will generalize in this paper.

4. GENERALIZATIONS OF ANGULAR DERIVATIVES

Now we will examine extended notions of the angular derivative of the function b by replacing the identity function z by an arbitrary (nonconstant) inner function u in the denominator of the standard difference quotient, $\frac{b(z)-b(z_0)}{z-z_0}$. We will then examine the behavior of this generalized difference quotient, $\frac{1-b(z)}{1-u(z)}$.

The main theorem which will provide the basis for our definition of the relative angular derivative is:

THEOREM 4.1. [Main Theorem] *The following conditions are equivalent:*

(i) $\nu \ll \mu$ *and* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$;

(ii) $\frac{1-b}{1-u} k_0^u \in \mathcal{H}(b)$;

(iii) $\frac{1-b}{1-u} k_w^u \in \mathcal{H}(b)$ *for all* $w \in \mathbb{D}$;

(iv) $\int\limits_{\partial\mathbb{D}} \left| \frac{1-b_r}{1-u_r} \right| \mathrm{d}\nu$ *stays bounded as* $r \nearrow 1$;

(v) $\frac{1-b}{1-u} \in H^2$ *and* $\frac{1-b}{1-u} \in H^2(\mu^{\text{a.c.}})$;

(vi) $\frac{1-b}{1-u} \in H^2$ *and* $\frac{1-b}{1-u} \in H^2(\mu)$.

If any of the above hold, then we will say that b has an angular derivative relative to u.

Of these conditions listed above, the equivalence of (i), (ii), and (v) were previously known, and shown, in some form, in various parts of [7].

Note that both the definition of an angular derivative and Carathéodory's theorem about angular derivatives depend on the behavior of a difference quotient near only one point. It is clear from the nature of the conditions above that the notion of relative angular derivative depends on the behavior of the generalized difference quotient at more than one point. Condition (v), for example, is a condition on the boundary values m-a.e., and condition (iv) is a condition ν-a.e.. Since ν is singular with respect to m, the equivalence of these two conditions in defining a relative angular derivative is somewhat unexpected. Also note that in the definition of the angular derivative of a function b, we have a value, $b'(z_0)$, to associate with this derivative. Theorem 6.1 will provide us with the basis for defining the "value" of an angular derivative of a function b relative to a function u. In this case, we can define the value μ^{s}-a.e. on $\partial\mathbb{D}$, and that value can be taken to be $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. In Section 9, we will see that this notion of the value of a relative angular derivative is a good generalization of that for angular derivatives.

Remark 4.2. The definition of relative angular derivatives could have been expanded to include cases where the functions b and u might not meet the conditions above, but the functions $\overline{\xi}b$ (for some $\xi \in \partial\mathbb{D}$) and u do. The function b, then, "almost" has an angular derivative relative to u. In fact, Theorem 3.2 shows clearly that the condition for b to have an angular derivative in the sense of Carathéodory is not altered by the multiplication of b by a constant of unit modulus. A better generalization, perhaps, would maintain this property. We can accomplish this by altering the definition to:

> The holomorphic self-map of the unit disk b has an angular derivative relative to the inner function u if there is some $\xi \in \partial\mathbb{D}$ such that any of the six conditions above hold for the functions $\overline{\xi}b$ and u and their corresponding measures.

We will not use this modified definition, however, because of the added complication and the fact that this change does not alter the fundamental notion at all. We should, in any case, remember this method of generalizing angular derivatives as it will show up again in Section 9.

5. THE DE BRANGES–ROVNYAK SPACES

The de Branges–Rovnyak spaces are defined as the ranges of certain operators on H^2. For $\varphi \in L^\infty$ of the unit circle with $\|\varphi\|_\infty \leqslant 1$, we can define the de Branges–Rovnyak space $\mathcal{H}(\varphi)$. Since we are interested in holomorphic self-maps of the disk b, we will talk about the spaces $\mathcal{H}(b)$ for such b. The space $\mathcal{H}(b)$ is defined to be the range of the operator $(1 - T_bT_{\overline{b}})^{1/2}$, where T_b denotes the Toeplitz operator with symbol b (a function, in our case, in the unit ball of H^∞). The Toeplitz operator is the multiplication operator followed by the projection onto H^2, that is, $T_\phi f = P_+(\phi f)$ where P_+ is the projection operator from L^2 to H^2. The space $\mathcal{H}(b)$ defined this way becomes a Hilbert space, with norm $\|\cdot\|_b$ and inner product $\langle\cdot,\cdot\rangle_b$, where the inner product is defined by $\langle(1-T_bT_{\overline{b}})^{1/2}x, (1-T_bT_{\overline{b}})^{1/2}y\rangle_b = \langle x, y\rangle_{H^2}$, for $x, y \perp \ker(1 - T_bT_{\overline{b}})$.

We will not here go into full detail on the properties of these de Branges–Rovnyak spaces. What is presented is an overview of the properties which we will find useful in relation to relative angular derivatives. Most of this material, as well as a detailed study of these spaces, can be found in the works of Sarason, particularly in [7].

The space H^2 has kernel functions k_w for $w \in \mathbb{D}$, where $k_w(z) = (1-\overline{w}z)^{-1}$ are such that for f in H^2, we have $f(w) = \langle f, k_w \rangle$. Similarly, there are functions k_w^b in $\mathcal{H}(b)$ which have the property that for f in $\mathcal{H}(b)$, $f(w) = \langle f, k_w^b \rangle_b$. These are given by $k_w^b(z) = (1 - T_b T_{\overline{b}}) k_w(z) = \frac{(1-\overline{b(w)}b(z))}{1-\overline{w}z}$. Note that we can calculate the norms of these kernel functions in $\mathcal{H}(b)$:

$$\|k_w^b\|_b^2 = k_w^b(w) = \frac{1-|b(w)|^2}{1-|w|^2}.$$

The space $H^2(\mu)$ can be transformed into the space $\mathcal{H}(b)$ by an operator V_b, which we will make use of in this paper. In order to define V_b, we will consider the Cauchy integral of a complex Borel measure ρ on $\partial\mathbb{D}$, which is defined (for our purposes, for $z \in \mathbb{D}$) by

$$K_\rho f(z) = K f\rho(z) = \int\limits_{\partial\mathbb{D}} \frac{f(\mathrm{e}^{\mathrm{i}\theta})}{1-\mathrm{e}^{-\mathrm{i}\theta}z}\,\mathrm{d}\rho(\mathrm{e}^{\mathrm{i}\theta}).$$

This lets us define the operator V_b on the space $L^2(\mu)$ by

$$V_b f(z) = (1-b(z))K_\mu f(z) \quad \text{ for } z \in \mathbb{D}.$$

This operator has the properties that $V_b k_w = (1-\overline{b(w)})^{-1} k_w^b$, and also $\langle k_w, k_z \rangle_\mu = \langle V_b k_w, V_b k_z \rangle_b$. (The inner product on $H^2(\mu)$ is $\langle \cdot\,,\cdot \rangle_\mu$, defined by $\langle f, g \rangle_\mu = \int\limits_{\partial\mathbb{D}} f\overline{g}\,\mathrm{d}\mu$ for $f, g \in H^2(\mu)$.) The operator V_b, then, is an isometry of $H^2(\mu)$ onto $\mathcal{H}(b)$, since it maps the kernel functions k_w of H^2, the span of which is a dense linear manifold in $H^2(\mu)$, to (a constant times) the kernel functions of $\mathcal{H}(b)$, the span of which is a dense linear manifold in $\mathcal{H}(b)$, and it preserves norms for linear combinations of kernel functions. For more details about the transformation above, see [7], III-6,7.

The following theorem relates the existence of an angular derivative of b relative to u to the existence of a particular function in the de Branges–Rovnyak space. It can be found in [7], III-11.

THEOREM 5.1. *The following are equivalent:*
(i) $\nu \ll \mu$ *and* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$*;*
(ii) *the function* $\left(\frac{1-b}{1-u}\right) k_0^u$ *is in* $\mathcal{H}(b)$*.*

We can extend this theorem to get

THEOREM 5.2. *The two conditions in Theorem* 5.1 *are equivalent to the following third condition:*

$$\left(\frac{1-b}{1-u}\right) k_w^u \in \mathcal{H}(b) \quad \textit{for all } w \in \mathbb{D}.$$

Proof. First, notice that for $w = 0$, this third condition is part (ii) of Theorem 5.1, so it implies part (ii). Then, assuming part (i), we (imitating the proof of the Theorem 5 in [7]) consider

$$\begin{aligned} V_b\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}k_w\right) &= (1-b)K_\mu\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}k_w\right) = (1-b)K_\nu(k_w) \\ &= \left(\frac{1-b}{1-u}\right)V_u(k_w) = (1-\overline{u(w)})^{-1}\left(\frac{1-b}{1-u}\right)k_w^u. \end{aligned}$$

Since $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}k_w$ is in $L^2(\mu)$, it is mapped by V_b to an element of $\mathcal{H}(b)$, so we have what we need to prove Theorem 5.2, and thus the equivalence of parts (i), (ii) and (iii) of our main theorem. ∎

6. THE SINGULAR PART OF THE MEASURE μ

We will now examine the boundary behavior of our generalized difference quotient $\frac{1-b}{1-u}$ with respect to the singular part of the measure μ.

THEOREM 6.1. *The conditions $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$ imply that $\frac{1-b_r}{1-u_r} \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ in $L^2(\mu^{\mathrm{s}})$ as $r \nearrow 1$.*

Proof. For this, we need to make use of the following theorem by A.G. Poltoratskii in [5]:

THEOREM 6.2. *For an element $h \in \mathcal{H}(b)$, we have*

$$h_r \longrightarrow V_b^{-1} h$$

in $L^2(\mu^{\mathrm{s}})$ norm, as $r \nearrow 1$.

Assuming $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$, we may take $h = \frac{1-b}{1-u}k_0^u(1-\overline{u(0)})^{-1}$. We can see from our proof of Theorem 5.2 that $V_b(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}) = h$, and thus $h \in \mathcal{H}(b)$. Then, since $k_0^u = (1-\overline{u(0)}u)$, we have

$$h_r = \frac{1-b_r}{1-u_r}\left(\frac{1-\overline{u(0)}u_r}{1-\overline{u(0)}}\right)$$

and we use the theorem of Poltoratskii to get

$$h_r \longrightarrow V_b^{-1}(h) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$$

in $L^2(\mu^{\mathrm{s}})$ as $r \nearrow 1$. Now we see that

$$\begin{aligned}\frac{1-b_r}{1-u_r} &= h_r\left(\frac{1-\overline{u(0)}}{1-\overline{u(0)}u_r}\right) = h_r\left(1-\frac{\overline{u(0)}-\overline{u(0)}u_r}{1-\overline{u(0)}u_r}\right)\\ &= h_r - h_r\left(\frac{\overline{u(0)}(1-u_r)}{1-\overline{u(0)}u_r}\right) \longrightarrow \frac{\mathrm{d}\nu}{\mathrm{d}\mu}.\end{aligned}$$

This last convergence is seen to be true because $h_r \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, and we can now show that $h_r\left(\frac{\overline{u(0)}(1-u_r)}{1-\overline{u(0)}u_r}\right) \to 0$ in $L^2(\mu^{\mathrm{s}})$. We do this by noting that $\left|h_r\left(\frac{\overline{u(0)}(1-u_r)}{1-\overline{u(0)}u_r}\right)\right|^2$ is both uniformly integrable (with respect to μ^{s}) and tends to zero μ^{s}-a.e.. It is uniformly integrable since $\left(\frac{\overline{u(0)}(1-u_r)}{1-\overline{u(0)}u_r}\right)$ is bounded (by $\frac{2}{1-\overline{u(0)}}$) and $|h_r|^2$ converges in $L^1(\mu^{\mathrm{s}})$, and it tends to zero μ^{s}-a.e. since $h_r \to \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ and $\left(\frac{\overline{u(0)}(1-u_r)}{1-\overline{u(0)}u_r}\right) \to 0$ ν-a.e.. This proves Theorem 6.2. ∎

We now continue to prove parts of the main theorem.

THEOREM 6.3. *The conditions $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$ imply that*

$$\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right|^2 \mathrm{d}\mu^{\mathrm{s}} \longrightarrow \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)} \qquad \textit{and} \qquad \int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right| \mathrm{d}\nu \longrightarrow \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)}$$

as $r \nearrow 1$.

Proof. The first part of this theorem is true since it just expresses the fact that, for the functions $\frac{1-b_r}{1-u_r}$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, Hilbert space convergence (in $L^2(\mu^{\mathrm{s}})$ — by Theorem 6.2) implies convergence of norms (where $\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu^{\mathrm{s}})} = \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)}$).

The second part expresses the fact that, for the functions $\frac{1-b_r}{1-u_r}$ and $\frac{\mathrm{d}u}{\mathrm{d}\mu}$, the norm convergence implies weak convergence. This can give us

$$\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right| \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu \longrightarrow \int_{\partial\mathbb{D}} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^2 \mathrm{d}\mu,$$

which is equivalent to what we want. ∎

We can express this limit, which appeared in both parts of the theorem, in several ways:

$$\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)} = \left\|V_b\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)\right\|^2_b = \left\|\frac{1-b}{1-u}k_0^u\right\|^2_b |1-\overline{u(0)}|^{-2}$$

and

$$\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|^2_{L^2(\mu)} = \int_{\partial\mathbb{D}} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^2 \mathrm{d}\mu = \int_{\partial\mathbb{D}} \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\nu = \left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right\|_{L^1(\nu)}.$$

Now we will prove the converse of the second part of Theorem 6.3 above, which establishes the equivalence of conditions (iv) and (i) in the main theorem.

THEOREM 6.4. *If $\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right| \mathrm{d}\nu$ is bounded as $r \nearrow 1$, then we have $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$.*

Proof. We begin the proof by rewriting the relative difference quotient in terms of Herglotz integrals:

$$\frac{1-b}{1-u} = \left(\frac{1+u}{1-u}\right)\left(\frac{1+b}{1-b}\right)^{-1}\left(\frac{1+b}{1+u}\right),$$

where we have $\frac{1+u}{1-u} = \int_{\partial\mathbb{D}} H(\theta,z)\,\mathrm{d}\nu(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}\,\mathrm{Im}\frac{1+u(0)}{1-u(0)}$, and the similar formula for $\frac{1+b}{1-b}$.

We can write

$$\left(\frac{1+u}{1-u}\right)\left(\frac{1+b}{1-b}\right)^{-1} = \frac{H\nu}{H\mu} = \frac{H\nu^{\mathrm{s}} + H\nu^{\mathrm{a.c.}}}{H\mu}.$$

We will now show

LEMMA 6.5. *The following are true:*
(i) $\frac{H\nu^{\mathrm{s}}(z)}{H\mu(z)} \to \infty$ *as* $z \to \xi$ *nontangentially, for* ν^{s} *almost all* $\xi \in \partial\mathbb{D}$.
(ii) $\frac{H\nu^{\mathrm{a.c.}}(z)}{H\mu(z)} \to \frac{\mathrm{d}u^{\mathrm{a.c.}}}{\mathrm{d}\mu}(\xi)$ *as* $z \to \xi$ *nontangentially, for* μ^{s} *almost all* $\xi \in \partial\mathbb{D}$.

To do this, we will need a lemma by Poltoratskii in [5].

LEMMA 6.6. *For* ρ *a positive Borel measure, and* $f \in L^1(\rho)$, *the nontangential limit* $\lim\limits_{z\to\xi} \frac{Kf\rho(z)}{K\rho(z)}$ *exists for* ρ *almost every* $\xi \in \partial\mathbb{D}$ *and is equal to* $f(\xi)$ ρ^{s}*-a.e., where* ρ^{s} *is the singular part of* ρ *with respect to Lebesgue measure.*

For $K(\theta, z) = \frac{1}{1-\mathrm{e}^{-\mathrm{i}\theta}z}$, the Cauchy kernel, we easily see that

$$H(\theta, z) = \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} = 2K(\theta, z) - 1,$$

so this lemma by Poltoratskii works just as well with the Herglotz kernel in place of the Cauchy kernel. Part (ii) of Lemma 6.5 above is an almost immediate consequence of this (modified version of the) lemma by Poltoratskii, with $\rho = \mu$, and $f = \frac{\mathrm{d}\nu^{\mathrm{a.c.}}}{\mathrm{d}\mu}$. We have $\frac{\mathrm{d}\nu^{\mathrm{a.c.}}}{\mathrm{d}\mu} \in L^1(\mu)$, and

$$\frac{H\nu^{\mathrm{a.c.}}(z)}{H\mu(z)} = \frac{H\frac{\mathrm{d}\nu^{\mathrm{a.c.}}}{\mathrm{d}\mu}\mu(z)}{H\mu(z)},$$

so the result follows.

To prove part (i) of Lemma 6.5, we consider a function f defined on $\partial\mathbb{D}$ so that $f = 0$ μ-a.e., $f = 1$ ν^{s}-a.e. (as done by Poltoratskii in a similar situation in [5]). Then we consider the following nontangential limits:

$$\lim_{z\to\xi} \frac{Hf(\mu + \nu^{\mathrm{s}})(z)}{H(\mu + \nu^{\mathrm{s}})(z)} = \lim_{z\to\xi} \frac{H\nu^{\mathrm{s}}(z)}{H(\mu + \nu^{\mathrm{s}})(z)} = 1 \quad \nu^{\mathrm{s}}\text{-a.e. } \xi$$

by using the above lemma by Poltoratskii with the function f and $\rho = \mu + \nu^{\mathrm{s}}$. (Note that ν is already singular with respect to Lebesgue measure, and here we are only interested in the behavior of the above limit ν^{s}-a.e..)

We continue to examine other nontangential limits:

$$\lim_{z\to\xi} \frac{H\mu(z)}{H\nu^{\mathrm{s}}(z)} = \lim_{z\to\xi} \left(\frac{H(\mu + \nu^{\mathrm{s}})(z)}{H\nu^{\mathrm{s}}(z)} - 1 \right) = 0 \quad \nu^{\mathrm{s}}\text{-a.e. } \xi.$$

Finally, we get

$$\lim_{z\to\xi} \frac{H\nu^{\mathrm{s}}(z)}{H\mu(z)} = \infty \quad \nu^{\mathrm{s}}\text{-a.e. } \xi,$$

which is what we wanted to show.

Now we can get around to proving the theorem. To do this we assume that $\int\limits_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right| \mathrm{d}\nu$ is bounded as $r \nearrow 1$. We also assume first that $\nu \not\ll \mu$, i.e., that ν^{s} is nonzero, and we derive a contradiction. This will give us $\nu \ll \mu$. We will then show that $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$.

Assume ν^{s} is nonzero. To get our contradiction, we would like to show that the integrand, $\left|\frac{1-b_r(\xi)}{1-u_r(\xi)}\right|$, approaches ∞ for all ξ on a set of positive ν-measure.

Note that if $b_r(\xi)$ stays bounded away from 1 for all ξ on a set of positive ν-measure, then the condition above is certainly true, since $u_r(\xi) \to 1$ ν-a.e. ξ.

Otherwise, we see that the function $\frac{1+b_r(\xi)}{1+u_r(\xi)}$ is bounded away from zero, in fact, it approaches 1 ν-a.e.. We can then write

$$\begin{aligned}\frac{1-b_r(\xi)}{1-u_r(\xi)} &= \left(\frac{1+b_r(\xi)}{1+u_r(\xi)}\right)\left(\frac{1+u_r(\xi)}{1-u_r(\xi)}\right)\left(\frac{1+b_r(\xi)}{1-b_r(\xi)}\right)^{-1} \\ &= \left(\frac{1+b_r(\xi)}{1+u_r(\xi)}\right)\left(\frac{H\nu^{\mathrm{s}}(r\xi)+H\nu^{\mathrm{a.c.}}(r\xi)}{K\mu(r\xi)}\right).\end{aligned}$$

The right side $\to \infty$ as $r \nearrow 1$, since the first factor $\to 1$, and the second factor $\to \infty$, for ν^{s}-a.e. ξ. Hence if ν^{s} is not the zero-measure, we do have $\left|\frac{1-b_r(\xi)}{1-u_r(\xi)}\right| \to \infty$ for all ξ on a set of positive ν-measure. Now, by a standard measure-theory argument, we must have $\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right| \mathrm{d}\nu \to \infty$ as $r \nearrow 1$. This is the contradiction we were looking for, so we must have ν^{s} equal zero, and thus we must have $\nu \ll \mu$. This means that $\nu^{\mathrm{a.c.}} = \nu$, so we can write

$$\frac{1-b_r(\xi)}{1-u_r(\xi)} = \left(\frac{1+b_r(\xi)}{1+u_r(\xi)}\right)\left(\frac{H\nu(r\xi)}{H\mu(r\xi)}\right).$$

For ν-a.e. ξ, $H\nu(r\xi)$ and $H\mu(r\xi) \to \infty$ as $r \nearrow 1$. So for ν-a.e. ξ,

$$\lim_{r\nearrow 1} \frac{1-b_r(\xi)}{1-u_r(\xi)} = \lim_{r\nearrow 1} \frac{H\nu(r\xi)}{H\mu(r\xi)}\left(\frac{1+b_r(\xi)}{1+u_r(\xi)}\right) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi)$$

by part (ii) of Lemma 6.5, the fact that $\nu \ll \mu^{\mathrm{s}}$, and the fact that $b(r\xi)$ and $u(r\xi) \to 1$ ν-a.e..

We put everything together now to see that since $\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right| \mathrm{d}\nu$ is bounded as $r \nearrow 1$, we have $\frac{1-b}{1-u} \in L^1(\nu) = H^1(\nu)$ since ν is singular. This tells us that for $f(\mathrm{e}^{\mathrm{i}\theta})$ the boundary values of the function $\frac{1-b}{1-u}$ (defined ν-a.e.), we have $\int_{\partial\mathbb{D}} |f(\mathrm{e}^{\mathrm{i}\theta})| \mathrm{d}\nu(\mathrm{e}^{\mathrm{i}\theta}) < \infty$, and, in fact, equal to the $H^1(\nu)$ norm of $\frac{1-b}{1-u}$, or the $L^1(\nu)$ norm of f.

From the above, we see that the boundary function, $f(\mathrm{e}^{\mathrm{i}\theta})$, is, ν-a.e., $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\mathrm{e}^{\mathrm{i}\theta})$, so $\int_{\partial\mathbb{D}} \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\, \mathrm{d}\nu < \infty$. Since $\mathrm{d}\nu = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\, \mathrm{d}\mu$, we get, finally, $\int_{\partial\mathbb{D}} \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^2 \mathrm{d}\mu < \infty$, or $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$. This completes the proof of the theorem. ∎

REMARK 6.7. We might want to add the following to the list of equivalent statements in the main theorem:

$$\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right|^2 \mathrm{d}\mu^{\mathrm{s}} \text{ stays bounded as } r \nearrow 1.$$

We did, in fact, prove that if b has an angular derivative relative to u, then the above must hold. This was part of Theorem 6.3. The converse, however, is not true. We can find b and u such that the above holds, but such that we do not have $\nu \ll \mu$. Take $b(z) = z$ and $u(z) = z^2$, for example. We get $\mu = \delta_1$ and ν will have atoms at both 1 and -1. The condition, $\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right|^2 \mathrm{d}\mu^{\mathrm{s}}$ remains bounded as $r \nearrow 1$ is satisfied, since

$$\int_{\partial\mathbb{D}} \left|\frac{1-rz}{1-(rz)^2}\right|^2 \mathrm{d}\delta_1 = \left(\frac{1-r}{1-r^2}\right)^2 \to \frac{1}{4}$$

as $r \nearrow 1$, but we do not have $\nu \ll \mu$. We can, however, imitate the proofs given in this section to prove

THEOREM 6.8. *The function b has an angular derivative relative to u if and only if $\nu \ll \mu$ and $\int_{\partial\mathbb{D}} \left|\frac{1-b_r}{1-u_r}\right|^2 \mathrm{d}\mu^{\mathrm{s}}$ stays bounded as $r \nearrow 1$.*

7. THE ABSOLUTELY CONTINUOUS PART OF μ

Here we will discuss the equivalence of parts (i), (v) and (vi) in the main theorem. The equivalence of part (v) with part (i) comes from a consolidation of two separate theorems of Sarason in [7]. The reason for this is that the proof given of the equivalence of the condition $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$ and the condition $\frac{1-b}{1-u} \in H^2$ and $H^2(\mu^{\mathrm{a.c.}})$ depends on whether b is a nonextreme or extreme point in the unit ball of H^∞, i.e., whether or not the function $\log(1-|b|^2)$ is integrable (on $\partial\mathbb{D}$).

In the case b is nonextreme, i.e., $\log(1-|b|^2)$ is integrable, Sarason shows in [7], IV-8:

THEOREM 7.1. [Comparison of Measures] *For b nonextreme, the following are equivalent:*

(i) *$\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$;*

(ii) *$\frac{1-b}{1-u}$ and $\frac{a}{1-u}$ are in H^2.*

Here the function a is defined to be the outer function whose boundary values have modulus $(1-|b|^2)^{1/2}$ and which is positive at the origin. The condition, then, that $\frac{a}{1-u} \in H^2$ is equivalent to $\frac{1-b}{1-u} \in H^2(\mu^{\mathrm{a.c.}})$ (remember that $\mu^{\mathrm{a.c.}} = \frac{1-|b|^2}{|1-b|^2} m$).

Note that both of these conditions, that $\frac{a}{1-u} \in H^2$ and $\frac{1-b}{1-u} \in H^2(\mu^{\mathrm{a.c.}})$ are, in fact, equivalent to

$$\int_{\partial\mathbb{D}} \frac{1-|b|^2}{|1-u|^2}\,\mathrm{d}m < \infty,$$

since this integral condition is the same as the assertion that the boundary function of $\frac{1-b}{1-u}$ is in $L^2(\mu^{\mathrm{a.c.}})$. Also, we have the function $\frac{1-b}{1-u}$, as well as the function $\frac{a}{1-u}$ in the Nevanlinna class N^+, since they are both the quotients of outer functions. (The functions $1-b$ and $1-u$ are outer since they both have real parts which are positive everywhere in the disk — see [3], page 51). Functions in N^+ which are in L^2 are also in H^2 (see [3], page 28).

For the case where b is an extreme point of the unit ball of H^∞, i.e., $\log(1-|b|^2)$ is not integrable, Sarason has in [7], V-9:

Theorem 7.2. [Comparison of Measures] *If* b *is extreme, then the following are equivalent:*

(i) $\nu \ll \mu$ *and* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$*;*

(ii) *the function* $\frac{1-b}{1-u}$ *is in* H^2*, and the function* $\frac{1}{1-u}$ *is in* $L^2(\rho)$*.*

Here the measure ρ is $(1-|b|^2)m$, and, for b extreme (and only for b extreme), we have $H^2(\rho) = L^2(\rho)$. Part (ii) above, then, also is easily seen to be equivalent to $\frac{1-b}{1-u} \in H^2$ and $\frac{1-b}{1-u} \in H^2(\mu^{\mathrm{a.c.}})$.

When we put the extreme and the nonextreme cases together, we get part (v) of the main theorem.

We already proved earlier (Theorem 6.3) that if b has an angular derivative relative to u then $\frac{1-b}{1-u} \in L^2(\mu^{\mathrm{s}})$, and, since $H^2(\mu^{\mathrm{s}}) = L^2(\mu^{\mathrm{s}})$, we have $\frac{1-b}{1-u} \in H^2(\mu^{\mathrm{s}})$. This, together with the previous result gives us $\frac{1-b}{1-u} \in H^2(\mu)$, which then gives us part (vi) of the main theorem.

8. The Product Rule

We can use the characterization of when b has an angular derivative relative to u from Section 6 to create an analog of the regular product rule for derivatives. In this case, we will consider what happens when we have two functions, b_1 and b_2, both holomorphic self-maps of the disk, and the function $b = b_1 b_2$. Under what conditions will this b have an angular derivative relative to an inner function u? It will be shown to be sufficient to assume that both b_1 and b_2 have angular derivatives relative to u.

Theorem 8.1. *If* b_1 *and* b_2 *are holomorphic self maps of the unit disk, both with angular derivatives relative to* u*, then the function* $b = b_1 b_2$ *has an angular derivative relative to* u*.*

Proof. Under the assumptions of the theorem, we have, by condition (iv) of our main theorem (proved in the Section 6):

$$\int_{\partial\mathbb{D}} \left| \frac{1-b_{1r}}{1-u_r} \right| \mathrm{d}\nu \quad \text{and} \quad \int_{\partial\mathbb{D}} \left| \frac{1-b_{2r}}{1-u_r} \right| \mathrm{d}u$$

are both bounded as $r \nearrow 1$. We now use

$$1 - b = 1 - b_1 b_2 = (1-b_1) + (1-b_2) - (1-b_1)(1-b_2)$$

to get

$$\begin{aligned}
\int_{\partial\mathbb{D}} \left| \frac{1-b_r}{1-u_r} \right| \mathrm{d}\nu &= \int_{\partial\mathbb{D}} \left| \frac{(1-b_{1r})}{1-u_r} + \frac{(1-b_{2r})}{1-u_r} - \frac{(1-b_{1r})(1-b_{2r})}{1-u_r} \right| \mathrm{d}\nu \\
&\leqslant \int_{\partial\mathbb{D}} \left| \frac{(1-b_{1r})}{1-u_r} \right| \mathrm{d}\nu + \int_{\partial\mathbb{D}} \left| \frac{(1-b_{2r})}{1-u_r} \right| \mathrm{d}\nu \\
&\quad + \int_{\partial\mathbb{D}} \left| \frac{(1-b_{1r})(1-b_{2r})}{1-u_r} \right| \mathrm{d}\nu.
\end{aligned}$$

We know that ν-almost everywhere the integrand in the third term on the right above tends to zero as $r \nearrow 1$, and, by the same argument as used in the proof of Theorem 6.1, the integral, too, tends to zero. The first two terms stay bounded (and even approach $\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}\right\|_{L^1(\nu)}$ and $\left\|\frac{\mathrm{d}\nu}{\mathrm{d}\mu_2}\right\|_{L^1(\nu)}$ as $r \nearrow 1$), so we get from this, and part (iv) of the main theorem, b has an angular derivative relative to u. ∎

From the point of view of the measures, this theorem tells us that if two measures, μ_1 and μ_2 both have a common singular measure ν satisfying $\nu \ll \mu_1$ and $\nu \ll \mu_2$, as well as $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1} \in L^2(\mu_1)$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_2} \in L^2(\mu_2)$, then the measure μ which corresponds to the function which is the product of the two functions corresponding to μ_1 and μ_2 satisfies the conditions $\nu \ll \mu$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \in L^2(\mu)$.

In fact, because of Theorem 6.1, we get the value, ν-a.e., of the angular derivative; $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is just given by the limit as $r \nearrow 1$ of the function $\frac{1-b_r}{1-u_r} = \frac{(1-b_{1r})}{1-u_r} + \frac{(1-b_{2r})}{1-u_r} - \frac{(1-b_{1r})(1-b_{2r})}{1-u_r}$. The first term on the right has limit μ_1^{s}-a.e. of $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}$, and the second term has limit μ_2^{s}-a.e. of $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_2}$, and third term has limit zero ν-a.e.. Thus, since $\nu \ll \mu_1$ and $\nu \ll \mu_2$, we get

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(\xi) = \frac{\mathrm{d}\nu}{\mathrm{d}\mu_1}(\xi) + \frac{\mathrm{d}\nu}{\mathrm{d}\mu_2}(\xi)$$

for ν-a.e. ξ.

9. RELATIVE ANGULAR DERIVATIVES AS GENERALIZATIONS OF ANGULAR DERIVATIVES

We can now examine the special case of a relative angular derivative which we have when our inner function u is a multiple of the identity function, $u(z) = \overline{z}_0 z$ for some $z_0 \in \partial\mathbb{D}$. Since we are interested in the behavior of a holomorphic function b near the point z_0, we will let $\zeta = b(z_0)$ and analyze the angular derivative of $\overline{\zeta} b$ relative to u. Our generalized difference quotient is now

$$\frac{1-\overline{\zeta}b(z)}{1-u(z)} = \frac{1-\overline{\zeta}b(z)}{1-\overline{z}_0 z} = \frac{z_0}{\zeta}\left(\frac{\zeta - b(z)}{z_0 - z}\right), \tag{9.1}$$

which is just (z_0/ζ times) a regular difference quotient for a function b. We should thus expect that in our theorems, if we take this choice for u, we will get results that apply to angular derivatives (in the sense of Carathéodory — assuming, as we shall, that $|b(z_0)| = 1$).

For this choice of u, we can easily see that $\nu = \delta_{z_0}$, since

$$\frac{1+u(z)}{1-u(z)} = \frac{z_0+z}{z_0-z} = H(z_0, z) = \int_{\partial\mathbb{D}} H(\theta, z)\,\mathrm{d}\delta_{z_0}.$$

Thus if we are to have a function b such that $\overline{\zeta}b$ has an angular derivative relative to this u, we must have $\delta_{z_0} \ll \mu_\zeta$, that is, μ_ζ must have an atom at the point z_0. Note that the second condition, that $\frac{\mathrm{d}\nu}{d\mu_\zeta} \in L^2(\mu)$ is then automatic.

This gives us, as in [7], VI-7,

THEOREM 9.1. [Special Case] *For b a holomorphic self-map of the disk, with $\zeta = b(z_0)$, $\overline{\zeta}b$ has an angular derivative at a point z_0 of $\partial\mathbb{D}$ if and only if the corresponding measure μ_ζ has an atom at z_0.*

If $z \to z_0$ nontangentially, then the limit of the left side in equation (9.1) above must be $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_\zeta}(z_0)$, and the limit of the right side must be $z_0 b'(z_0)/\zeta$. Thus

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu_\zeta}(z_0) = z_0 b'(z_0)/\zeta.$$

Since $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_\zeta}$ is real and positive, we get

THEOREM 9.2. [Special Case] *For a holomorphic function b with an angular derivative in the sense of Carathéodory at z_0, we must have $z_0 b'(z_0)/b(z_0) = |b'(z_0)|$.*

(This result can be found in slightly different form in [1], Section 299 and is part of [7], VI-3, presented here as Theorem 3.2.)

As we have already mentioned, μ_ζ must have an atom at z_0, and since $\nu = \delta_{z_0} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu_\zeta}\mu_\zeta$ we must have $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_\zeta}(z_0) = 1/\mu_\zeta(\{z_0\})$, so this gives us

THEOREM 9.2. [Special Case] *If b has an angular derivative at a point z_0, and $\zeta = b(z_0)$, then $\mu_\zeta(\{z_0\}) = 1/|b'(z_0)|$.*

This is a result proved in a different way in [7], VI-7.

Now let us examine the case where, for some point $z_0 \in \partial\mathbb{D}$, both b and u have angular derivatives at z_0. The measures $\mu_{b(z_0)}$ and $\nu_{u(z_0)}$ will then both have atoms at z_0, with

$$\mu_{b(z_0)}(\{z_0\}) = \frac{b(z_0)}{z_0 b'(z_0)} \qquad \text{and} \qquad \nu_{u(z_0)}(\{z_0\}) = \frac{u(z_0)}{z_0 u'(z_0)}.$$

We can now use $\nu_{u(z_0)} = \frac{\mathrm{d}\nu_{u(z_0)}}{\mathrm{d}\mu_{b(z_0)}}\mu_{b(z_0)}$ to get

$$\frac{\mathrm{d}\nu_{u(z_0)}}{\mathrm{d}\mu_{b(z_0)}}(z_0) = \frac{\nu_{u(z_0)}(\{z_0\})}{\mu_{b(z_0)}(\{z_0\})} = \frac{z_0 b'(z_0) u(z_0)}{z_0 b(z_0) u'(z_0)} = \frac{b'(z_0)}{u'(z_0)}\frac{u(z_0)}{b(z_0)}.$$

Another way to get the above result is by considering

$$\begin{aligned}\frac{\mathrm{d}\nu_{u(z_0)}}{\mathrm{d}\mu_{b(z_0)}}(z_0) &= \lim_{r\nearrow 1}\frac{1-\overline{b(z_0)}b_r}{1-\overline{u(z_0)}u_r}(z_0)\\ &= \lim_{r\nearrow 1}\left(\frac{u(z_0)}{b(z_0)}\right)\left(\frac{b(z_0)-b_r(z_0)}{z_0 - rz_0}\right)\left(\frac{z_0 - rz_0}{u(z_0)-u_r(z_0)}\right)\\ &= \frac{b'(z_0)}{u'(z_0)}\frac{u(z_0)}{b(z_0)}.\end{aligned}$$

This gives us

THEOREM 9.4. [Special Case] *At any point z_0 where both b and u have angular derivatives (in the sense of Carathéodory), the value of the angular derivative of $\overline{b(z_0)}b$ relative to $\overline{u(z_0)}u$ at z_0 is equal to the quotient of the angular derivatives of b and of u at z_0 divided by the quotient of the values of b and u at z_0.*

Note that at any boundary point z_0 where u has an angular derivative, it is necessary for b to have an angular derivative in the sense of Carathéodory, too, if we are to have any $\zeta \in \partial\mathbb{D}$ such that $\overline{\zeta}b$ has an angular derivative relative to $\overline{u(z_0)}u$, and if this is the case, then we necessarily have $\zeta = b(z_0)$, and the theorem above holds.

From the Hilbert Space perspective, we get

THEOREM 9.5. [Special Case] *A holomorphic self-map of the disk b has an angular derivative at a point z_0 if and only if there is some $\zeta \in \partial\mathbb{D}$ such that the function $\frac{b(z)-\zeta}{z-z_0}$ lies in $\mathcal{H}(b)$.*

This comes as a consequence of applying part (ii) of our main theorem to the function $\overline{\zeta}b$, where $\zeta = b(z_0)$ (again, with $u(z) = \overline{z_0}z$). The theorem then tells us that b has an angular derivative at z_0 if and only if $\frac{1-\overline{\zeta}b}{1-\overline{z}_0 z} \in \mathcal{H}(b)$ (note: $k_0^u = 1$ for this u) which is the same as $\frac{b(z)-\zeta}{z-z_0} \in \mathcal{H}(b)$. We must choose $\zeta = b(z_0)$, by the way, since, for any other value of ζ, the function $\frac{b(z)-\zeta}{z-z_0}$ will not even be in H^2. This theorem is part of [7], VI-4, in which it is further proved that the above are equivalent to: Every function in $\mathcal{H}(b)$ has a nontangential limit at the point z_0.

REFERENCES

1. C. CARATHÉODORY, *Theory of Functions of a Complex Variable*, vol. 2, Chelsea Publishing Co., New York 1954.
2. J. CIMA, A. MATHESON, Essential norms of composition operators and Aleksandrov measures, *Pacific J. Math.* **179**(1997), 59–64.
3. P.L. DUREN, *Theory of H^p Spaces*, Academic Press, New York–London, 1970.
4. S.D. FISHER, *Function Theory on Planar Domains*, John Wiley & Sons, New York 1983.
5. A.G. POLTORATSKII, Boundary behavior of pseudocontinuable functions, *Algebra i Analiz* **5**(1993), 189–210.
6. D. SARASON, Composition operators as integral operators, in *Analysis and Partial Differential Equations*, 1990, pp. 545–565.
7. D. SARASON, *Sub-Hardy Hilbert Spaces in the Unit Disk*, John Wiley & Sons, Inc., New York 1994.
8. J.E. SHAPIRO, Aleksandrov measures used in essential norm inequalities for composition operators, *J. Operator Theory* **40**(1998), 133–146.
9. J.H. SHAPIRO, C. SUNDBERG, Compact composition operators on L^1, *Proc. Amer. Math. Soc.* **108**(1990), 443–449.

10. J.H. Shapiro, P.D. Taylor, Compact, nuclear, and Hilbert-Schmidt composition operators on H^2, *Indiana Univ. Math. J.* **23**(1973), 471–496.

JONATHAN E. SHAPIRO
Mathematics Department
California Polytechnic State University
San Luis Obispo, CA 93407
USA

E-mail: jshapiro@calpoly.edu

Received January 30, 1999; revised August 26, 1999.