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ADJOINING A UNIT TO AN OPERATOR ALGEBRA

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Abstract. We show that the matricial norms of a non-unital operator algebra determine those of the algebra obtained by adjoining a unit to it. As applications, we classify two-dimensional unital operator algebras and show that the algebra of bounded holomorphic functions on a strongly pseudoconvex domain has a contractive representation that is not completely contractive.

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1. INTRODUCTION

A (concrete) *operator algebra* on a Hilbert space $\mathcal{H}$ is a closed subalgebra of the algebra $\mathbb{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. An operator algebra $\mathcal{A}$ on $\mathcal{H}$ is called *unital* iff $\mathrm{id}_{\mathcal{H}} \in \mathcal{A}$. If $\mathcal{A}$ is an operator algebra on $\mathcal{H}$, then the algebra $\mathbb{M}_n(\mathcal{A})$ of $n \times n$-matrices with entries in $\mathcal{A}$ is an operator algebra on the Hilbert space $\mathbb{C}^n \otimes \mathcal{H}$. The C^*-norms on $\mathbb{B}(\mathbb{C}^n \otimes \mathcal{H})$ therefore yield canonical norms $\|\cdot\|_n$ on $\mathbb{M}_n(\mathcal{A})$ for all $n \in \mathbb{N}$. We write $\mathrm{Ball}(\mathbb{M}_n(\mathcal{A}))$ for the *open* unit ball of $\mathbb{M}_n(\mathcal{A})$.

Two operator algebras $\mathcal{A}, \mathcal{B}$ are called *completely isometric* iff there is an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that the induced maps $\varphi_n : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B})$ are isometric for all $n \in \mathbb{N}$.

A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called *completely contractive* iff φ_n is contractive for all $n \in \mathbb{N}$; a *complete quotient map* iff φ_n is a quotient map for all $n \in \mathbb{N}$; and *completely isometric* iff φ_n is isometric for all $n \in \mathbb{N}$. Finally, we define $\|\varphi\|_n := \|\varphi_n\|$ for $n \in \mathbb{N}$ and $\|\varphi\|_\infty := \sup\limits_{n\in\mathbb{N}} \|\varphi\|_n$. See [6] for this terminology.

Let $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ be a closed subalgebra with $\mathrm{id}_{\mathcal{H}} \notin \mathcal{A}$. Consider the corresponding unital operator algebra

$$\mathcal{A}^+ := \{x + \lambda \cdot \mathrm{id}_{\mathcal{H}} \mid x \in \mathcal{A}, \lambda \in \mathbb{C}\} \subset \mathbb{B}(\mathcal{H}).$$

We show that if $\varphi : \mathcal{A} \to \mathcal{B}$ is a complete isometry, then the unital extension $\varphi^+ : \mathcal{A}^+ \to \mathcal{B}^+$ is also a complete isometry. That is, the norms on $\mathbb{M}_n(\mathcal{A}^+)$ do not depend on the choice of a completely isometric representation of $\mathcal{A}$. Moreover, if $\varphi : \mathcal{A} \to \mathcal{B}$ is completely contractive or a complete quotient map, then so is $\varphi^+ : \mathcal{A}^+ \to \mathcal{B}^+$.

The uniqueness of the matricial norms on $\mathcal{A}^+$ has already been noticed by Poon and Ruan ([8]) in the special case of operator algebras with a contractive approximate identity. However, this special case is quite restrictive and does not cover the applications in Section 4 below. There we deal mainly with finite dimensional operator algebras. It is easy to verify that a finite dimensional operator algebra with a contractive approximate identity is automatically unital with $\|1\| \leqslant 1$.

The reason for the uniqueness of the matricial norms on $\mathcal{A}^+$ is that the domains $\mathfrak{D}(n) := \mathrm{Ball}(\mathbb{M}_n)$ have a large group of automorphisms. Certain automorphisms of $\mathfrak{D}(n)$ operate also on $\mathrm{Ball}(\mathcal{B})$ for any unital operator algebra $\mathcal{B}$. We show that we get all of $\mathrm{Ball}(\mathcal{A}^+)$ by applying these automorphisms to elements of $\mathrm{Ball}(\mathcal{A})$. To make the computations more transparent, we define the *positive cone* $\mathrm{Cone}(\mathcal{B})$ of a *unital* operator algebra $\mathcal{B} \subset \mathbb{B}(\mathcal{H})$ to be the set of all $x \in \mathcal{B}$ for which $\mathrm{Re}\, x := (x + x^*)/2$ is positive *and invertible*. We show that functional calculus with the rational function $\mathcal{C}(z) := (1 - z)/(1 + z)$ gives rise to a bijection between $\mathrm{Ball}(\mathcal{B})$ and $\mathrm{Cone}(\mathcal{B})$.

The last section contains several applications. Let $\mathcal{A}$ be a *commutative*, *unital* operator algebra. Then a $d-1$-contractive unital representation $\mathcal{A} \to \mathbb{M}_d$ is necessarily completely contractive. In particular, a contractive unital representation $\mathcal{A} \to \mathbb{M}_2$ is completely contractive. This generalizes a result of Agler ([1]).

If $\mathcal{A}$ is a 2-dimensional unital operator algebra, then $\mathcal{A}$ has a completely isometric representation $\mathcal{A} \to \mathbb{M}_2$.

Another simple case is $\mathcal{B} \cong \mathcal{I}^+$ with $\mathcal{I} \cdot \mathcal{I} = 0$. Then $\mathcal{B}$ is called a *unital zero algebra*. These algebras occur as quotients of less trivial operator algebras as follows. Let $\mathcal{A}$ be a commutative, unital operator algebra, $\mathcal{I} \subset \mathcal{A}$ a maximal ideal, and $\mathcal{J} \subset \mathcal{A}$ an ideal with $\mathcal{I} \cdot \mathcal{I} \subset \mathcal{J} \subset \mathcal{I}$. Then $\mathcal{A}/\mathcal{J}$ is algebraically isomorphic to $(\mathcal{I}/\mathcal{J})^+$. It is shown in [2] that quotients of unital operator algebras with the obvious matricial norms are again completely isometric to unital operator algebras. Thus $\mathcal{A}/\mathcal{J} \cong (\mathcal{I}/\mathcal{J})^+$ completely isometrically, that is, $\mathcal{A}/\mathcal{J}$ is a unital zero algebra. We compute $\mathcal{I}/\mathcal{J}$ in some cases where $\mathcal{A} = H^\infty(M)$ is the algebra of bounded holomorphic functions on a domain $M \subset \mathbb{C}^k$. If $\mathcal{I}/\mathcal{J}$ has a contractive, not completely contractive representation, then this carries over to $H^\infty(M)$. Using this we reprove and extend a result of Paulsen ([7]): The operator algebra $H^\infty(M)$ has a contractive, not completely contractive representation if M is an absolutely convex domain with $\dim M \geqslant 5$. Furthermore, such a representation exists if M is a strongly pseudoconvex domain with $\dim M \geqslant 2$.

2. THE POSITIVE CONE OF A UNITAL OPERATOR ALGEBRA

Evidently, the rational function $\mathcal{C}(z) := (1-z)/(1+z)$ maps the domain $\mathbb{C} \setminus \{-1\}$ into itself and satisfies $\mathcal{C} \circ \mathcal{C} = \mathrm{id}$ on $\mathbb{C} \setminus \{-1\}$. Let

$$\mathfrak{D} := \mathrm{Ball}(\mathbb{B}(\mathcal{H})) := \{X \in \mathbb{B}(\mathcal{H}) \mid \|X\| < 1\};$$
$$\mathfrak{D}_+ := \mathrm{Cone}(\mathbb{B}(\mathcal{H})) := \{X \in \mathbb{B}(\mathcal{H}) \mid \mathrm{Re}\, X \text{ positive and invertible}\}.$$

LEMMA 2.1. *If $X \in \mathfrak{D}$ then the spectrum of X is contained in the open disk $\{z \in \mathbb{C} \mid |z| < 1\}$. If $X \in \mathfrak{D}_+$, then the spectrum of X is contained in the right half plane $\{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$. Consequently, $\mathcal{C}(X)$ is well-defined for $X \in \mathfrak{D} \cup \mathfrak{D}_+$.*
$X \mapsto \mathcal{C}(X)$ is a bijection $\mathfrak{D} \to \mathfrak{D}_+$ with inverse $\mathcal{C}$.

Proof. It is well-known that the spectrum of $X \in \mathfrak{D}$ is contained in the open unit ball. If $\mathrm{Re}\, z \leqslant 0$, $X \in \mathfrak{D}_+$, then $X - z \in \mathfrak{D}_+$ as well. Hence if we show that all $X \in \mathfrak{D}_+$ are invertible, it follows that $X - z$ is invertible for all $X \in \mathfrak{D}_+$ and $\mathrm{Re}\, z \leqslant 0$. That is, the spectrum of X is contained in the right half plane. To invert $X \in \mathfrak{D}_+$, first conjugate X by the invertible operator $(\mathrm{Re}\, X)^{-1/2}$ to reduce to the case $\mathrm{Re}\, X = 1$. Then $X = 1 + \mathrm{i}S$ with S self-adjoint. Such an operator is evidently invertible.

It remains to prove $\mathcal{C}(\mathfrak{D}) \subset \mathfrak{D}_+$ and $\mathcal{C}(\mathfrak{D}_+) \subset \mathfrak{D}$. The computation

$$\begin{aligned}\mathrm{Re}\, \mathcal{C}(X) &= \frac{1}{2}(1+X)^{-1}((1-X)(1+X^*) + (1+X)(1-X^*))(1+X^*)^{-1}\\ &= (1+X)^{-1}(1-XX^*)(1+X^*)^{-1}\end{aligned}$$

shows that $\mathrm{Re}\, \mathcal{C}(X)$ is positive and invertible for $X \in \mathfrak{D}$. That is, $\mathcal{C}(\mathfrak{D}) \subset \mathfrak{D}_+$. Similarly,

$$1 - \mathcal{C}(X)\mathcal{C}(X)^* = 4(1+X)^{-1}\,\mathrm{Re}(X)(1+X^*)^{-1}$$

is positive and invertible for $X \in \mathfrak{D}_+$, so that $\mathcal{C}(\mathfrak{D}_+) \subset \mathfrak{D}$. ∎

THEOREM 2.2. *Let $\mathcal{A}$ be a unital operator algebra, let $n \in \mathbb{N}$, and let $X \in \mathbb{M}_n(\mathcal{A})$. The following assertions are equivalent:*

(i) *$X = \mathcal{C}(Y)$ for some $Y \in \mathrm{Ball}(\mathbb{M}_n(\mathcal{A}))$;*
(ii) *$1 + X$ is invertible in $\mathbb{M}_n(\mathcal{A})$ and $\|\mathcal{C}(X)\| < 1$;*
(iii) *$\rho_n(X) \in \mathfrak{D}_+$ for all n-contractive unital representations $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$;*
(iv) *$\rho_n(X) \in \mathfrak{D}_+$ for at least one n-isometric unital representation $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$.*

DEFINITION 2.3. The set of elements satisfying one of these equivalent conditions is called the *(positive) cone* $\mathrm{Cone}(\mathbb{M}_n(\mathcal{A}))$ of $\mathbb{M}_n(\mathcal{A})$.

Proof. Replacing $\mathcal{A}$ by $\mathbb{M}_n(\mathcal{A})$, if necessary, reduce to the case $n = 1$, that is, $X \in \mathcal{A}$. Since $\mathcal{A}$ is complete, all elements of $1 + \mathrm{Ball}(\mathcal{A})$ are invertible in $\mathcal{A}$, so that $\mathcal{C}(Y)$ is defined and lies in $\mathcal{A}$ for all $Y \in \mathrm{Ball}(\mathcal{A})$. Thus the equivalence of (i) and (ii) follows easily from $\mathcal{C} \circ \mathcal{C} = \mathrm{id}$. If ρ is a unital contractive representation and $X = \mathcal{C}(Y)$ with $Y \in \mathrm{Ball}(\mathcal{A})$, then $\rho(X) = \mathcal{C}(\rho(Y)) \in \mathfrak{D}_+$ by Lemma 2.1. Hence (i) implies (iii). (iii) trivially implies (iv). It remains to show that (iv) implies (ii).

Let $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be any isometric unital representation. Suppose that $\rho(X) \in \mathfrak{D}_+$. We will show below that $1 + X$ is invertible in $\mathcal{A}$. Taking this for granted, we get that $\mathcal{C}(X)$ is a well-defined element of $\mathcal{A}$. Furthermore, $\rho(\mathcal{C}(X)) = \mathcal{C}(\rho(X)) \in \mathfrak{D}$ by Lemma 2.1. Since ρ is isometric, $\|\mathcal{C}(X)\| < 1$ as desired.

It remains to show that if $\rho(X) \in \mathfrak{D}_+$, then $1 + X$ is invertible in $\mathcal{A}$. By Lemma 2.1, the spectrum of $\rho(X)$ is contained in the simply connected domain $\{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Thus the function $z \mapsto 1/(1+z)$ can be approximated by polynomials uniformly on the spectrum of $\rho(X)$. The inverse of $1 + \rho(X)$ therefore lies in $\rho(\mathcal{A})$ because $\rho(\mathcal{A})$ is closed. Hence $1 + X$ is invertible in $\mathcal{A}$. ∎

Consequently, the matricial norms on a *unital* operator algebra $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ can equally well be described by the collection of sets $\operatorname{Cone}(\mathbb{M}_n(\mathcal{A}))$, $n \in \mathbb{N}$.

Theorem 2.2 implies that a unital homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is n-contractive iff it maps $\operatorname{Cone}(\mathbb{M}_n\mathcal{A})$ into $\operatorname{Cone}(\mathbb{M}_n\mathcal{B})$.

3. Adjoining a unit to an operator algebra

If A is a not necessarily unital algebra, let A^+ be the algebra obtained by adjoining a unit to A. If $\rho : A \to B$ is a homomorphism of algebras, let $\rho^+ : A^+ \to B^+$ be the unital homomorphism extending ρ.

Let $\mathcal{A}$ be a not necessarily unital operator algebra. We show that the norms on $\mathbb{M}_n(\mathcal{A})$ can be extended to $\mathbb{M}_n(\mathcal{A}^+)$ in a unique way so as to obtain a unital operator algebra. The proof uses certain natural automorphisms of $\operatorname{Cone}(\mathbb{M}_n(\mathcal{A}^+))$.

The domain $\mathfrak{D}_+(n) := \operatorname{Cone}(\mathbb{M}_n)$ is one of the classical symmetric domains. If $S \in \mathbb{M}_n$ is invertible and $T \in \mathbb{M}_n$ satisfies $\operatorname{Re} T = 0$, then

$$\Phi_{S,T} : X \mapsto SXS^* + T \tag{3.1}$$

defines a bijection from $\mathfrak{D}_+(n)$ onto itself. The inverse is $\Phi_{S^{-1},T'}$ with $T' := -S^{-1}T(S^{-1})^*$. These maps $\Phi_{S,T}$ form a subgroup G of the automorphism group of $\mathfrak{D}_+(n)$. It operates transitively on $\mathfrak{D}_+(n)$ because any $X \in \mathfrak{D}_+(n)$ is of the form $\Phi_{S,T}(1)$ with $S := (\operatorname{Re} X)^{1/2}$, $T := \operatorname{Im} X$.

If $\mathcal{B}$ is a unital operator algebra, let $\Phi_{S,T}$ operate on $\operatorname{Cone}(\mathbb{M}_n(\mathcal{B}))$ by the same formula (3.1), considering $\mathbb{M}_n \subset \mathbb{M}_n(\mathcal{B})$ via the inclusion $X \mapsto X \otimes 1_{\mathcal{B}}$. Evidently, $\Phi_{S,T}$ maps $\operatorname{Cone}(\mathbb{M}_n(\mathcal{B}))$ into itself. Consequently, the map $\mathcal{C} \circ \Phi_{S,T} \circ \mathcal{C}$ is a bijection $\operatorname{Ball}(\mathbb{M}_n(\mathcal{B})) \to \operatorname{Ball}(\mathbb{M}_n(\mathcal{B}))$.

Theorem 3.1. *Let $\mathcal{B}$ be a unital operator algebra and let $\mathcal{A} \subset \mathcal{B}$ be a 1-codimensional ideal. Thus algebraically $\mathcal{B} \cong \mathcal{A}^+$.*

Then $Y \in \operatorname{Cone}(\mathbb{M}_n(\mathcal{B}))$ iff $Y = \Phi_{S,T} \circ \mathcal{C}(X)$ for some $X \in \operatorname{Ball}(\mathbb{M}_n(\mathcal{A}))$ and some $\Phi_{S,T} \in G$. Hence

$$\operatorname{Ball}(\mathbb{M}_n(\mathcal{B})) = \mathcal{C}(\operatorname{Cone}(\mathbb{M}_n\mathcal{B})) = \bigcup_{\Phi \in G} \mathcal{C} \circ \Phi \circ \mathcal{C}(\operatorname{Ball}(\mathbb{M}_n\mathcal{A})). \tag{3.2}$$

As a result, the norm on $\mathbb{M}_n(\mathcal{A})$ uniquely determines the norm on $\mathbb{M}_n(\mathcal{B})$.

Proof. Since $\mathcal{C}$ maps $\operatorname{Ball}(\mathbb{M}_n(\mathcal{B}))$ onto $\operatorname{Cone}(\mathbb{M}_n(\mathcal{B}))$ and $\Phi_{S,T}$ maps $\operatorname{Cone}(\mathbb{M}_n(\mathcal{B}))$ into itself it is clear that elements of the form $\Phi_{S,T} \circ \mathcal{C}(X)$ are in $\operatorname{Cone}(\mathbb{M}_n(\mathcal{B}))$. Conversely, let $Y \in \operatorname{Cone}(\mathbb{M}_n(\mathcal{B}))$.

Let $\pi : \mathcal{B} \to \mathbb{C}$ be the character with $\ker \pi = \mathcal{A}$, $\pi(1) = 1$. It is well-known that characters are completely contractive. Thus $\pi_n(Y) \in \operatorname{Cone}(\mathbb{M}_n) = \mathfrak{D}_+(n)$.

Since G operates transitively on $\mathfrak{D}_+(n)$ we have $\Phi_{S,T}(1) = \pi_n(Y)$ for some $\Phi_{S,T} \in G$. Put $X := \mathcal{C} \circ \Phi_{S,T}^{-1}(Y)$, then $\Phi_{S,T} \circ \mathcal{C}(X) = Y$ as desired. Theorem 2.2 yields $X \in \mathrm{Ball}(\mathbb{M}_n(\mathcal{B}))$. In addition, $X \in \mathbb{M}_n(\mathcal{A})$ because

$$\pi_n(X) = \mathcal{C} \circ \Phi_{S,T}^{-1} \circ \pi_n(Y) = \mathcal{C} \circ \Phi_{S,T}^{-1} \circ \Phi_{S,T}(1) = \mathcal{C}(1) = 0.$$

This yields the desired description of $\mathrm{Cone}(\mathbb{M}_n(\mathcal{B}))$. Equation (3.2) and the last assertion follow immediately. ∎

Corollary 3.2. *There are unique matricial norms on $\mathcal{A}^+$ for which $\mathcal{A}^+$ is a unital operator algebra with $\|1\| \leqslant 1$ and the injection $\mathcal{A} \to \mathcal{A}^+$ is completely isometric.*

Proof. Uniqueness is dealt with by Theorem 3.1. Existence is easy. Choose a completely isometric representation $\rho : \mathcal{A} \to \mathbb{B}(\mathcal{H})$. If it happens that $\mathrm{id}_{\mathcal{H}} \in \rho(\mathcal{A})$, replace ρ by the degenerate representation $\rho \oplus 0$ on $\mathcal{H} \oplus \mathbb{C}$. Thus we may assume that $\mathrm{id}_{\mathcal{H}} \notin \rho(\mathcal{A})$. Then $\rho^+(z + x) := z \cdot \mathrm{id}_{\mathcal{H}} + \rho(x)$ for $z \in \mathbb{C}$, $x \in \mathcal{A}$ defines a representation of $\mathcal{A}^+$ having the desired properties. ∎

If $\mathcal{A}$ itself is a unital operator algebra, then $\mathcal{A}^+$ is completely isometric to the orthogonal direct sum $\mathcal{A} \oplus \mathbb{C}$ because the latter is a unital operator algebra containing $\mathcal{A}$ as a 1-codimensional ideal.

Corollary 3.3. *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism between operator algebras and let $n \in \mathbb{N}$. φ is n-contractive if and only if φ^+ is n-contractive. φ is n-isometric if and only if φ^+ is n-isometric. φ is an n-quotient map if and only if φ^+ is.*

Proof. The naturality of the automorphisms $\Phi_{S,T}$ implies $\varphi_n^+ \circ \Phi_{S,T} = \Phi_{S,T} \circ \varphi_n^+$.

If φ is n-contractive, then φ_n maps $\mathrm{Ball}(\mathbb{M}_n(\mathcal{A}))$ into $\mathrm{Ball}(\mathbb{M}_n(\mathcal{B}))$. Thus $(\varphi^+)_n$ maps $\mathrm{Ball}(\mathbb{M}_n(\mathcal{A}^+))$ into $\mathrm{Ball}(\mathbb{M}_n(\mathcal{B}^+))$ by Theorem 3.1. This means that φ^+ is n-contractive. Conversely, if φ^+ is n-contractive, so is φ as the restriction of φ^+ to $\mathcal{A}$. The remaining assertions are proved similarly. ∎

4. APPLICATIONS

Let $\mathcal{A}$ be a unital operator algebra and let $\mathcal{I} \subset \mathcal{A}$ be a 1-codimensional ideal. Then $\mathcal{A} \cong \mathcal{I}^+$. Thus the study of $\mathcal{A}$ can be reduced to the study of $\mathcal{I}$. In this section, we give some applications of this idea.

Theorem 4.1. *Let $\mathcal{A}$ be a commutative unital operator algebra, $d \in \mathbb{N}$. Then any $d-1$-contractive unital homomorphism $\rho : \mathcal{A} \to \mathbb{M}_d$ is completely contractive.*

Proof. Let $\mathcal{B} := \rho(\mathcal{A})$, let $\mathcal{J}$ be a maximal ideal in $\mathcal{B}$, and let $\mathcal{I} := \rho^{-1}(\mathcal{J})$. Then $\mathcal{A} = \mathcal{I}^+$, $\mathcal{B} = \mathcal{J}^+$, and $\rho = (\rho|\mathcal{I})^+$. By Corollary 3.3, it suffices to show that $\rho|\mathcal{I}$ is completely contractive.

There is a vector $x \in \mathbb{C}^d \setminus \{0\}$ that is annihilated by all elements of $\mathcal{J}$, because $\mathcal{J}$ is a non-unital, commutative subalgebra of $\mathbb{M}_d$. The same reasoning yields a vector $y \in \mathbb{C}^d \setminus \{0\}$ annihilated by all adjoints T^* of elements $T \in \mathcal{J}$.

Thus elements of $\mathcal{J}$ can be viewed as operators from $\mathbb{C}^d \ominus x$ to $\mathbb{C}^d \ominus y$, with $\ominus$ denoting the orthogonal complement. This yields a completely isometric *linear* representation $\varphi : \mathcal{J} \to \mathbb{M}_{d-1}$. Since $\varphi \circ \rho|\mathcal{I}$ is a $d-1$-contractive linear map to $\mathbb{M}_{d-1}$, Theorem 5.1 of [6] yields that $\varphi \circ \rho|\mathcal{I}$ is completely contractive. Thus $\rho|\mathcal{I}$ is completely contractive. ∎

In particular, if $\mathcal{A}$ is a commutative unital operator algebra, then any contractive unital representation $\mathcal{A} \to \mathbb{M}_2$ is completely contractive. For certain representations of function algebras, this was observed by Agler in his proof of Lempert's theorem ([1]) and later by Salinas ([9]) and Chu ([3]).

4.1. Two-dimensional unital operator algebras. For $c \in [0,1]$, let

$$T_c := \begin{pmatrix} 0 & \sqrt{1-c^2} \\ 0 & c \end{pmatrix}.$$

Clearly, $\|T_c\| = 1$ and $T_c^2 = cT_c$. Thus the linear span $\mathcal{Q}_c$ of 1 and T_c is a unital subalgebra of $\mathbb{M}_2$.

Theorem 4.2. *Let $\mathcal{A}$ be a two-dimensional unital operator algebra. Then $\mathcal{A}$ is completely isometric to $\mathcal{Q}_c$ for a unique $c \in [0,1]$ and thus has a completely isometric unital representation by 2×2-matrices.*

Proof. Let $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ be a 2-dimensional unital operator algebra. $\mathcal{A}$ is necessarily commutative and thus contains a maximal ideal $\mathcal{I}$. Choose $T \in \mathcal{A}$ with $\|T\| = 1$. We have $T^2 = c \cdot T$ for some $c \in \mathbb{C}$. Rescaling T by some constant of modulus 1, we can achieve $c \geqslant 0$. Actually, $c \in [0,1]$ because $c = \|cT\| = \|T^2\| \leqslant \|T\|^2 = 1$. Let $x \in \mathbb{M}_n$, then $\|x \otimes T\| = \|x\| \cdot \|T\| = \|x\| \cdot \|T_c\| = \|x \otimes T_c\|$. Thus the homomorphism $\varphi : \mathcal{I} \to \mathbb{M}_2$ defined by $T \mapsto T_c$ is completely isometric. By Corollary 3.3, it follows that $\varphi^+ : \mathcal{A} \to \mathcal{Q}_c \subset \mathbb{M}_2$ is completely isometric. It is elementary to verify that c is unique, that is, the algebras $\mathcal{Q}_c$ are not isometric for different values of c. ∎

If $\rho : \mathcal{Q}_c \to \mathcal{Q}_d$ is a homomorphism, then $\|\rho\|_\infty = \|\rho\|$. This peculiarity was first observed by Holbrook ([4]) and can be established by direct computations in $\mathbb{M}_2$.

4.2. Unital zero algebras. A unital operator algebra $\mathcal{A}$ is called a *unital zero algebra* iff it is obtained by adjoining a unit to an algebra with zero multiplication. A unital operator algebra $\mathcal{A}$ is a unital zero algebra iff there is a 1-codimensional ideal $\mathcal{I} \subset \mathcal{A}$ with $\mathcal{I} \cdot \mathcal{I} = 0$. The ideal $\mathcal{I}$ is the only maximal ideal of $\mathcal{A}$ and thus uniquely determined. Any unital homomorphism $\mathcal{I}^+ \to \mathcal{J}^+$ between unital zero algebras is of the form ρ^+ for some linear map $\rho : \mathcal{I} \to \mathcal{J}$.

If $\mathbf{V} \subset \mathbb{B}(\mathcal{H})$ is an operator space, then $\mathbf{V}$ endowed with the zero multiplication is an operator algebra. The map $x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ defines a completely isometric multiplicative representation of $\mathbf{V}$ on $\mathbb{B}(\mathcal{H} \oplus \mathcal{H})$. More generally, any linear representation $\rho : \mathbf{V} \to \mathbb{B}(\mathcal{H}')$ yields a multiplicative representation $\mathbf{V} \to \mathbb{B}(\mathcal{H}' \oplus \mathcal{H}')$ and thus a unital, multiplicative representation $\widehat{\rho} : \mathbf{V}^+ \to \mathbb{B}(\mathcal{H}' \oplus \mathcal{H}')$. If $\mathbf{V}$ has badly behaved linear representations, say, contractive representations that are not completely contractive, then this carries over to $\mathbf{V}^+$ by Corollary 3.3. We can indeed prove the following strengthening of Corollary 3.3 that is only true for unital zero algebras.

Theorem 4.3. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be operator spaces, let* $\rho : \mathbf{V} \to \mathbf{W}$ *be a linear map, and let* $\rho^+ : \mathbf{V}^+ \to \mathbf{W}^+$ *be its unital extension. Then, for all* $n \in \mathbb{N} \cup \{\infty\}$,

$$\|\rho^+\|_n = \max\{1, \|\rho\|_n\}. \tag{4.1}$$

Proof. The inequality "$\geqslant$" is trivial. To prove "$\leqslant$", assume $C := \|\rho\|_n < \infty$. If $C \leqslant 1$, the assertion follows from Theorem 3.1. Thus assume $C > 1$ and let $\mu : \mathbf{V} \to \mathbf{V}$ be the map $T \mapsto CT$. Then $\rho = \rho \circ \mu^{-1} \circ \mu$, and $\|\rho \circ \mu^{-1}\|_n = 1$. Hence $\|\rho^+ \circ (\mu^{-1})^+\|_n \leqslant 1$, so that it remains to prove $\|\mu^+\|_\infty \leqslant C$.

Therefore, consider $\mathbf{V} \subset \mathbb{B}(\mathcal{H})$ and represent $\mathbf{V}^+ \subset \mathbb{B}(\mathcal{H} \oplus \mathcal{H})$ as above. Define

$$S := \begin{pmatrix} C^{1/2} & 0 \\ 0 & C^{-1/2} \end{pmatrix} \in \mathbb{B}(\mathcal{H} \oplus \mathcal{H}).$$

Then $\mu^+(T) = STS^{-1}$ for all $T \in \mathbf{V}^+$ because both sides of this equation are unital maps that coincide on $\mathbf{V}$. Thus $\|\mu^+\|_\infty \leqslant \|S\| \cdot \|S^{-1}\| = C$ as desired. ∎

Let $M \subset \mathbb{C}^k$ be a domain and let $x \in M$. Let $\mathcal{A} := H^\infty(M)$,

$$\mathcal{I} := \{f \in \mathcal{A} \mid f(x) = 0\}, \quad \mathcal{J} := \{f \in \mathcal{A} \mid f(x) = 0,\ Df(x) = 0\}.$$

We have written Df for the derivative of f. We call $\mathcal{I}/\mathcal{J} \cong \mathbf{T}^*_x M$ the *cotangent space of* M *at* x. The axiomatic description of abstract operator spaces and abstract unital operator algebras in [2] yields that $\mathcal{I}/\mathcal{J}$ is an operator space and that $\mathcal{A}/\mathcal{J}$ is a unital operator algebra. Theorem 3.1 implies $\mathcal{A}/\mathcal{J} \cong (\mathcal{I}/\mathcal{J})^+$ completely isometrically, so that $\mathcal{A}/\mathcal{J}$ is a unital zero algebra.

An element of $\mathbb{M}_n(\mathcal{I}/\mathcal{J})$ may be viewed as a linear function $\mathbf{T}_x M \to \mathbb{M}_n$. It satisfies $\|f\| \leqslant 1$ iff f is the derivative of a holomorphic function $f : M \to \mathrm{Ball}(\mathbb{M}_n)$ with $f(x) = 0$.

If M is a balanced domain (that is, $\lambda y \in M$ whenever $y \in M$, $\lambda \in \mathbb{C}$, $|\lambda| \leqslant 1$) and $x = 0$, then $\mathcal{I}/\mathcal{J}$ can be computed precisely. If M is strongly pseudoconvex instead, then $\mathcal{I}/\mathcal{J}$ can be computed approximately if x approaches the boundary. In both cases, if the dimension of M is sufficiently big, then $\mathcal{I}/\mathcal{J}$ has a contractive, but not completely contractive representation. Thus $H^\infty(M)$ has a contractive, but not completely contractive representation in the following cases: If M is balanced and $\dim M \geqslant 5$ (this is due to Paulsen ([7])); if M is a strongly pseudoconvex domain and $\dim M \geqslant 2$.

Let M be a balanced domain and let $x := 0$. Let $f : \mathbf{T}_x M \to \mathbb{M}_n$ be a linear map. We view f as a function from $M \subset \mathbf{T}_x M$ to $\mathbb{M}_n$. Evidently, if $f(M) \subset \mathrm{Ball}(\mathbb{M}_n)$ then $\|f\| \leqslant 1$. The converse is also true: If $\widetilde{f} : M \to \mathrm{Ball}(\mathbb{M}_n)$ is a holomorphic function with $\widetilde{f}(0) = 0$, $D\widetilde{f}(0) = f$, then $f(M) \subset \mathrm{Ball}(\mathbb{M}_n)$. This follows from the Schwarz inequality applied to the restrictions of $\widetilde{f}$ to disks $\mathbb{D} \cdot y$ with $y \in \partial M$. Hence we get a completely isometric embedding $\mathbf{T}^*_x M \to C_b(M)$. Thus $\mathbf{T}^*_x M$ is completely isometric to $\mathrm{MIN}(V)$, where V is the normed space whose unit ball is the polar $\check{M} \subset \mathbb{C}^n$ of M. The minimal and maximal operator space structures $\mathrm{MIN}(V)$ and $\mathrm{MAX}(V)$ are defined in [7].

Let $i : \mathrm{MIN}(V) \to \mathrm{MAX}(V)$ be the identity map and let $\alpha := \|i\|_\infty$. Consider the homomorphism

$$\varphi : \mathcal{A} \to \mathcal{A}/\mathcal{I} \cong \mathrm{MIN}(V)^+ \to \mathrm{MAX}(V)^+.$$

By Theorem 4.3, φ is contractive and $\|\varphi\|_\infty = \alpha \geqslant 1$. Hence if M has the desirable property that all contractive representations of $\mathcal{A}$ are completely contractive, then necessarily $\alpha = 1$. As Paulsen shows in [7], this fails for most normed spaces V. It fails if $\dim V \geqslant 5$ or if V is a Hilbert space with $\dim V \geqslant 2$.

This negative result of Paulsen can be extended to strongly pseudoconvex domains M (with C^2 boundary) with $d = \dim M \geqslant 2$. Close to the boundary, such a domain looks more and more like the unit ball $\mathbb{D}_d$ of the Hilbert space ℓ^2_d of dimension d. This is made more precise by Ma ([5]). Theorem 3.3 and Lemma 3.4 of [5] imply that there are holomorphic maps $f : \mathbb{D}_d \to M$ and $g : M \to \mathbb{D}_d$ such that $f(0) = x$, $g(x) = 0$, and $Dg(x) \circ Df(0) = (1 - \varepsilon)\mathrm{id}$ with $\varepsilon \to 0$ for $x \to \partial M$. It follows that $g \circ f$ converges towards the identity map on $\mathbb{D}_d$ for $x \to \partial M$.

Define the distance between two operator spaces by

$$\mathrm{dist}_\infty(\mathbf{V}, \mathbf{W}) := \log(\inf\{\|\rho\|_\infty \cdot \|\rho^{-1}\|_\infty \mid \rho : \mathbf{V} \to \mathbf{W} \text{ invertible}\}).$$

It follows that $\mathrm{dist}_\infty(\mathbf{T}^*_x M, \mathbf{T}^*_0 \mathbb{D}_d) \to 0$ for $x \to \partial M$. We have seen above that $\mathbf{T}^*_0 \mathbb{D}_d \cong \mathrm{MIN}(\ell^2_d)$. Since $\mathrm{MIN}(\ell^2_d) \neq \mathrm{MAX}(\ell^2_d)$ for $d \geqslant 2$, the operator space $\mathbf{T}^*_x M$ has a contractive representation that is not completely contractive if x is sufficiently close to ∂M. Thus $H^\infty(M)$ has a contractive representation that is not completely contractive.

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