J. OPERATOR THEORY
46(2001), 289–321

ON THE SPECTRAL THEORY OF SINGULAR DIRAC TYPE HAMILTONIANS

V. GEORGESCU and M. MĂNTOIU

Communicated by Florian-Horia Vasilescu

ABSTRACT. We introduce a class of matrix valued pseudo-differential operators that admit scalar locally conjugate operators (in the sense of E. Mourre) and we give a general method of study of singular perturbations of such operators. In particular, we develop the spectral and scattering theory for a class of hamiltonians which contains the Dirac operators with arbitrary Coulomb singularities.

KEYWORDS: *Dirac operator, singular perturbation, Mourre estimate, limiting absorption principle, wave operators.*

MSC (2000): Primary 35P05; Secondary 81Q10, 47F05.

1. INTRODUCTION AND MAIN RESULTS

1.1. Let $\mathbf{E}$ be a finite-dimensional complex Hilbert space, $B(\mathbf{E})$ the space of linear operators in $\mathbf{E}$ (such operators will often be called matrices), and let us set $\mathcal{H} = L^2(\mathbb{R}^n; \mathbf{E})$. If $\varphi : \mathbb{R}^n \to B(\mathbf{E})$ is a Borel function, then the operators $\varphi(Q)$ and $\varphi(P)$ in the Hilbert space $\mathcal{H}$ are defined by the rules $[\varphi(Q)f](x) = \varphi(x)f(x)$ and $\varphi(P) = \mathcal{F}^*\varphi(Q)\mathcal{F}$. Here $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ is the (unitary) Fourier transform. Q_j is just the operator of multiplication by the x_j variable and $P_j = -i\partial_j = -i\partial/\partial x_j$.

If $h : \mathbb{R}^n \to B(\mathbf{E})$ is a symmetric operator valued borelian function then we associate to it the self-adjoint operator in $\mathcal{H}$ defined by $H_0 = h(P)$. This is a matrix valued pseudo-differential operator with constant coefficients. Our purpose is to study self-adjoint operators H such that $(H + \mathrm{i})^{-1} - (H_0 + \mathrm{i})^{-1}$ is a compact operator. For this we use the conjugate operator method initiated by E. Mourre in [13], which is one of the most powerful techniques that are now available in the spectral analysis of self-adjoint operators.

We first isolate a class of operators H_0, the locally scalar (or Dirac type) operators, which admit conjugate operators of a simple form (see (2.7)). This class contains the usual scalar operators (i.e. those associated to real functions h;

note that $\mathbb{R} \subset B(\mathbf{E})$) and the Dirac operators, but is considerably larger. Then we consider perturbed operators H which admit conjugate operators of the same form. We show that the theory developed in Section 7.6 of [1] for the case of scalar hypoelliptic operators can be extended (in an improved version) to locally scalar operators. In other terms, the class of locally scalar operators is a natural and rich extension of the class of scalar operators. We observe that here "locally" should be interpreted as "locally in the spectrum of H_0", and not in the spatial variable x. The main results of this part of our paper are Theorems 2.17 and 2.18.

Then we go further and treat perturbations H of H_0 which have high singularities on a compact set. In rough terms, we show that if H is locally compact and if, on some neighbourhood of infinity, H differs from H_0 by a short-range plus a long-range operator, then the limiting absorption principle holds for H in a natural space, in particular H has no singular continuous spectrum. These results are summarized in Theorem 3.6 (see also Lemma 3.8).

We feel that the spectral theory of (perturbed) matrix valued pseudo-differential operators is an important and not enough studied subject. One of the main problems to which one is confronted here is the lack of a satisfactory definition of the "threshold set" $\tau(H)$ of the hamiltonian H. In any case, $\tau(H)$ should be a real set such that H has conjugate operators outside it. A possible definition of $\tau(H)$ is presented and discussed in Subsection 2.4: although it is not as general as one would like, it has the quality that outside threshold points the hamiltonian formally admits a conjugate operator which is naturally related to the asymptotic velocity observable. This point of view will be further developed in [6]; here we restrict ourselves mainly to the case of locally scalar hamiltonians. In [7], C. Gérard and F. Nier are able to show that conjugate operators exist outside a discrete real set if the function h is analytic and proper. So, although they do not explicitly describe $\tau(H)$, they show that it is a small set. By starting with our Definition 2.5 of the threshold set (which is well suited for the operators considered in [7]) one can avoid the analyticity requirement: outside this set conjugate operators exist. The advantage is that $\tau(H)$ is now explicitly defined. Of course, without analyticity conditions, $\tau(H)$ can be quite large (even in the scalar case).

In this paper the space $\mathbf{E}$ is assumed to be finite dimensional only because this suffices for the main example we have in mind, the Dirac operator. Generalizations to the infinite dimensional case, with $h(p)$ operators having purely discrete spectra, are straightforward.

1.2. The rest of this introduction is devoted to a detailed description of our results in the particular but important case of the usual Dirac operators.

Let $\alpha_0 = \beta$, $\alpha_1, \ldots, \alpha_n$ be a set of symmetric operators in $\mathbf{E}$ such that the following anticommutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk} \tag{1.1}$$

hold for all $j, k \in \{0, 1, \ldots, n\}$. We fix a strictly positive real number m and consider the free Dirac operator

$$H_0 = \alpha \cdot P + m\beta = \sum_{j=1}^{n} \alpha_j P_j + m\beta. \tag{1.2}$$

H_0 is a self-adjoint operator in the Hilbert space $\mathcal{H}$, the operator $\mathcal{F}H_0\mathcal{F}^*$ acting as multiplication by the $B(\mathbf{E})$-valued function $h(p) = \alpha \cdot p + m\beta$, $p \in \mathbb{R}^n$. We may also write $H_0 = h(P)$.

Recall that the Sobolev space $\mathcal{H}^s$ is defined for all $s \in \mathbb{R}$ by the norm $\|\langle P\rangle^s \cdot\|$, where $\|\cdot\|$ is the norm of $\mathcal{H}$ and $\langle x\rangle = (1+x^2)^{1/2}$. Then the domain and form domain of H_0 are given by $D(H_0) = \mathcal{H}^1$, $D(|H_0|^{1/2}) = \mathcal{H}^{1/2}$.

We are interested in the spectral theory of perturbed Dirac operators $H = H_0 + V$. We shall treat perturbations V which are non-local, have high local singularities, and at infinity are a mixture of short-range and long-range components. If V is a (pseudo-)differential operator, we allow it to be of the same order as H_0, so, for example, in H the matrices α_j, β could depend on x and p.

In the next definition, where we give a precise meaning to the notions of short-range and long-range behaviour at infinity, ξ is an arbitrary C^∞-function on $\mathbb{R}^n$ such that $\xi(x) = 0$ if $|x| \leqslant 1$ and $\xi(x) = 1$ if $|x| \geqslant 2$. The properties stated below are clearly independent of the choice of ξ. We denote by $\|S\|_{(1/2)}$ the norm of an operator $S : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$.

Definition 1.1. Let $T : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ be linear continuous and symmetric. We say that T is a *short-range operator* if

$$\int_1^\infty \mathrm{d}r \|\xi(Q/r)T\|_{(1/2)} < \infty.$$

We say that T is a *long-range operator* if

$$\sum_{j=1}^n \int_1^\infty \frac{\mathrm{d}r}{r} \left\{ \|\xi(Q/r)[Q_j, T]\|_{(1/2)} + \|\xi(Q/r)|Q|[P_j, T]\|_{(1/2)} \right\} < \infty.$$

T is called *small at infinity* if

$$\lim_{r\to\infty} \|\xi(Q/r)T\|_{\mathcal{H}^1 \to \mathcal{H}^{-1/2}} = 0.$$

In order to clarify the content of the next hypothesis, we make a comment now on the definition of a sum $H = H_0 + V$ as a self-adjoint operator in $\mathcal{H}$. Assume, more generally, that H_0 is a self-adjoint operator in $\mathcal{H}$ with form domain $\mathcal{G} = D(|H_0|^{1/2})$ (equipped with the graph-topology) and identify as usual $\mathcal{G} \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{G}^*$ (in the Dirac case $\mathcal{G} = \mathcal{H}^{1/2}$ and $\mathcal{G}^* = \mathcal{H}^{-1/2}$). If $V : \mathcal{G} \to \mathcal{G}^*$ is a linear, symmetric (hence continuous) operator, then $H = H_0 + V : \mathcal{G} \to \mathcal{G}^*$ is well-defined, linear and symmetric. One can associate to H an operator in $\mathcal{H}$, namely its restriction $\widehat{H}$ to $D(\widehat{H}) = \{f \in \mathcal{G} \mid (H_0 + V)f \in \mathcal{H}\}$. It is easy to prove that $\widehat{H}$ is self-adjoint if $H - z : \mathcal{G} \to \mathcal{G}^*$ is bijective for some complex number z. In this case we keep the same notation H for $\widehat{H}$ and for its extension $H : \mathcal{G} \to \mathcal{G}^*$. In the Dirac case $\widehat{H}$ is sometimes called the "distinguished" self-adjoint realization of $H_0 + V$ (see Section 4.3 in [18] and references therein). It is important to note that the form domain of H could be different from $\mathcal{G}$ in general. For a detailed discussion of these points see the beginning of Section 7.5.2 and the last part of Section 2.8 in [1].

Hypothesis 1.2. The operator $V : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ can be written as the sum $V = V_{\mathrm{S}} + V_{\mathrm{L}}$ of a short-range operator V_{S} and a long-range small at infinity operator V_{L}. The operator $H = H_0 + V$ is such that $H - z : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ is bijective for some z; we use the same notation H for the self-adjoint operator in $\mathcal{H}$ associated to it.

One more notion will be useful.

Definition 1.3. Let H be a self-adjoint operator and J a real open set. We shall say that *the spectrum of H in J is normal* if H has no singularly continuous spectrum in J and the eigenvalues of H in J have finite multiplicity and can accumulate only towards the boundary of J.

In other terms, this means that there is a locally finite subset Σ of J such that, if E is the spectral measure of H, then $E(\{\lambda\})$ is of finite rank for $\lambda \in \Sigma$ and $E(N) = 0$ if $N \subset J \setminus \Sigma$ and N has Lebesgue measure zero.

Theorem 1.4. *Assume that H verifies Hypothesis* 1.2. *Then the spectrum of H in $\mathbb{R} \setminus \{-m, +m\}$ is normal.*

The continuous parts of two operators H which differ by a short-range perturbation are unitarily equivalent. Indeed, the relative wave operators exist and are complete:

Theorem 1.5. *Let H_1, H_2 be two operators both satifying Hypothesis* 1.2. *Denote by E_k^{c} the continuous component of the spectral measure of H_k $(k = 1, 2)$. If the operator $H_1 - H_2 : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ is short-range, then the wave operators*

$$\operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH_2} \mathrm{e}^{-\mathrm{i}tH_1} E_1^{\mathrm{c}}(\mathbb{R})$$

exist and their ranges are equal to $E_2^{\mathrm{c}}(\mathbb{R})\mathcal{H}$.

The absence of the singularly continuous spectrum and the existence of the wave operators are straightforward consequences of the following strong version of *the limiting absorption principle*. We use the notation

$$\mathcal{R}(H) = \{\lambda \in \mathbb{R} \mid \lambda \neq \pm m \text{ and } \lambda \text{ is not an eigenvalue of } H\}.$$

As a consequence of Theorem 1.4, this is an open real set if H satisfies Hypothesis 1.2. The spaces $\mathcal{H}_{s,q}^{\pm 1/2}$ are weighted Sobolev spaces; their definition, in a more general context, can be found in (2.19) and (2.20).

Theorem 1.6. *Let H be as in Hypothesis* 1.2. *Then the limits*

$$\lim_{\varepsilon \searrow 0} (H - \lambda \mp \mathrm{i}\varepsilon)^{-1} \equiv R(\lambda \pm \mathrm{i}0)$$

exist in the weak topology of $B(\mathcal{H}_{1/2,1}^{-1/2}; \mathcal{H}_{-1/2,\infty}^{1/2})$, locally uniformly in $\lambda \in \mathcal{R}(H)$. In particular, the maps*

$$\lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{H}_{1/2,1}^{-1/2}; \mathcal{H}_{-1/2,\infty}^{1/2})$$

are weak continuous on $\mathcal{R}(H)$.*

We have more detailed results on the regularity of the applications $\lambda \mapsto R(\lambda \pm \mathrm{i}0)$:

Theorem 1.7. *Let H be as in Hypothesis 1.2. Choose a real number $s > 1/2$ and assume that we can decompose $V = \sum_{0 \leqslant k < s+1/2} V_k$, where $V_k : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ satisfies*

$$\|\xi(Q/r)\,\mathrm{ad}_Q^a\,\mathrm{ad}_P^b\,V_k\|_{(1/2)} \leqslant Cr^{-s-1/2+k-|b|} \tag{1.3}$$

for all multi-indices a, b with $|a| + |b| \leqslant k$. Then the maps

$$\mathcal{R}(H) \ni \lambda \to R(\lambda \pm \mathrm{i}0) \in B(\mathcal{H}^{-1/2}_{s,\infty}; \mathcal{H}^{1/2}_{-s,1})$$

are locally of Hölder-Zygmund class $\Lambda^{s-1/2}$.

We have set, for example, $\mathrm{ad}_{Q_j} T = [Q_j, T]$ and $\mathrm{ad}_Q^a = \mathrm{ad}_{Q_1}^{a_1} \cdots \mathrm{ad}_{Q_n}^{a_n}$. If V_k is the operator of multiplication by a matrix valued function then (1.3) means

$$\|\xi(Q/r)V_k^{(b)}\|_{(1/2)} \leqslant Cr^{-s-1/2+k-|b|} \tag{1.4}$$

for $|b| \leqslant k$, and this condition is satisfied if

$$\|\langle Q\rangle^{s+1/2-k+|b|}V_k^{(b)}\|_{(1/2)} < \infty.$$

The following is a quite explicit sufficient condition for (1.4) to hold:

$$\left[\int_{|x-x'|\leqslant 1} \mathrm{d}x' \|V_k^{(b)}(x')\|^n_{B(\mathbf{E})}\right]^{1/n} \leqslant C|x|^{-s-1/2+k-|b|}$$

for large x (stronger singularities are allowed locally). For the definition of the Hölder-Zygmund classes see [5]. We also send to this paper for several other abstract results which can be applied in the present context (once a conjugate operator for H is constructed) in order to get, for example, propagation properties.

1.3. All but one of the conditions we put on V in the preceding theorems are rather explicit, general and natural. The only implicit condition, namely the requirement that the operator $H_0 + V - z$ be a bijection between $\mathcal{H}^{1/2}$ and $\mathcal{H}^{-1/2}$ for some $z \in \mathbb{C}$, is mainly a restriction on the local singularities of V. We make now more detailed comments in connection with this question, because it is of some physical interest. For the same reason we assume $n = 3$. Observe that the usual hypothesis $\dim \mathbf{E} = 4$ (which amounts to asking that the representation of the relations (1.1) be irreducible) is not necessary.

For simplicity we consider now only the case when V is the operator of multiplication by a function. Let $V = V^0 + V^{\mathrm{c}}$ where $V^0 : \mathbb{R}^3 \to B(\mathbf{E})$ and $V^{\mathrm{c}} : \mathbb{R}^3 \to B(\mathbf{E})$ are symmetric matrix valued Borel functions such that:

(a) V^0 is locally of class L^3 and

$$\lim_{|a|\to\infty} \int_{|x-a|\leqslant 1} \mathrm{d}x \mid V^0(x) \mid^3_{B(\mathbf{E})} = 0; \tag{1.5}$$

(b) there is a finite set $\Gamma \subset \mathbb{R}^3$ and a number $\nu > 0$ such that

$$\|V^{\mathrm{c}}(x)\|_{B(\mathbf{E})} \leqslant \sum_{a\in\Gamma} \frac{\nu}{|x-a|}.$$

From the results of Nenciu ([14], [15]; see also [9]) it follows that if $\nu < 1$ (which corresponds to a nuclear charge < 137) then $H_0 + V - z : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ is indeed bijective for every $z \notin \mathbb{R}$. So our theorems apply to this class of potentials (of course, conditions on the behaviour at infinity of V must be added).

1.4. Some explanations are needed in order to compare our results with those in [4]. For questions related to the self-adjoint realizations of the Dirac operator we send to Section 4.3 in [18] and references therein. It is easy to show that under the above conditions (a) and (b) the operator V^0 is H_0-bounded with relative bound zero (hence $H_0 + V^0$ is self-adjoint on the domain $\mathcal{H}^1$; in particular its form domain is $\mathcal{H}^{1/2}$) and the operator V^{c} is H_0-bounded with relative bound $\leqslant 2\nu$ (2 comes from the Hardy inequality in three dimensions). So $H_0 + V$ is a well-defined symmetric operator in $\mathcal{H}$ on the domain $\mathcal{H}^1$. From the Rellich theorem we get that H is self-adjoint on $\mathcal{H}^1$ if $\nu < 1/2$ and this is optimal with respect to ν in the matrix valued case. However, the result remains true if $\nu < \sqrt{3}/2 \approx 0.87$ and $V^{\mathrm{c}}(x)$ is a scalar matrix in the neighbourhood of Γ (see [11], [12], [3] and [9]). In any case, if H is self-adjoint on $\mathcal{H}^1$, then its form domain is equal to the form domain of H_0, i.e. $D(|H|^{1/2}) = \mathcal{H}^{1/2}$. As far as we know, in other cases there are no informations concerning $D(|H|^{1/2})$. More precisely, there are no informations on the form domain of a distinguished self-adjoint extension of $H_0 + V$; the difficulty is explained at an abstract level at the end of Section 2.8 in [1]. To sum up, the results of [4] cover the cases $\nu < 1/2$ with arbitrary V^{c} and $\nu < \sqrt{3}/2$ with scalar V, while our results described above cover $\nu < 1$ with arbitrary V^{c}. Below (see Subsection 1.5) we shall, however, treat much more singular situations, including scalar potentials with arbitrary Coulomb singularities.

We mention that, in order to agree with the terminology in use in the physical literature, we should call the scalar valued potentials *electric* (or *electrostatic*) potentials, cf. Section 4.2 from [18]. Indeed, a "scalar potential" is not a scalar valued map, according to this terminology.

We shall also make a comment concerning the behaviour at infinity of the potentials. In [4], the long-range part V_{L} of the interaction is required to satisfy a supplementary condition, namely

$$\sum_{j=1}^{n} \int_1^\infty \frac{\mathrm{d}r}{r} \|\xi(Q/r)[\alpha_j\beta, V_{\mathrm{L}}]\|_{(1/2)} < \infty. \tag{1.6}$$

This is clearly not a natural assumption and it appears in [4] just because the conjugate operator they use has a complicated matrix structure. We were able to get rid of (1.6) because our conjugate operator is a scalar operator. Of course, at this level (1.6) is a rather harmless hypothesis, but using a matrix valued conjugate operator has rather awkward consequences when one studies higher order regularity properties of the boundary values of the resolvent. For example, in [16] the conditions (1.4) are replaced by

$$\sum_{j=1}^{n} \left\|\xi(Q/r)\mathrm{ad}^{a}_{\alpha_j\beta} V_k^{(b)}\right\|_{(1/2)} \leqslant Cr^{-s-1/2+k-|b|-a}$$

for $a + |b| \leqslant k$. If, for example, $b = 0$, we see that the decay of V_k must improve after taking the commutator with some $\alpha_j\beta$, a rather strange condition when V_k is not a scalar function.

1.5. We come now to the case of very singular potentials. Note that the preceding results are true even if $m = 0$, but for the next two theorems the condition $m > 0$ is needed (see Hypothesis 3.2). We first point out a result which is very general, but which contains an implicit condition. We recall that a self-adjoint operator S on $\mathcal{H}$ is called *locally compact* if $\varphi(Q)(S + \mathrm{i})^{-1}$ is a compact operator for each $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ (space of C^{∞}-functions with compact support).

Theorem 1.8. *Let H be as in Hypothesis* 1.2 *and assume that V is the operator of multiplication by a $B(\mathbf{E})$-valued function. Let $\widetilde{H}$ be a locally compact self-adjoint operator such that, for an open neighbourhood Ω of infinity, if $f \in D(H)$ and $\operatorname{supp} f \subset \Omega$ then $f \in D(\widetilde{H})$ and $\widetilde{H}f = Hf$. Then:*

(i) *The spectrum of $\widetilde{H}$ in $\mathbb{R} \setminus \{-m, m\}$ is normal.*

(ii) *If $\mathcal{R}(\widetilde{H})$ is the set of $\lambda \in \mathbb{R}$ distinct from $\pm m$ and from the eigenvalues of $\widetilde{H}$, then the limits $\lim_{\varepsilon \searrow 0}(\widetilde{H} - \lambda \mp \mathrm{i}\varepsilon)^{-1}$ exist in the weak* topology of $B(\mathcal{H}_{1/2,1}, \mathcal{H}_{-1/2,\infty})$ locally uniformly in $\lambda \in \mathcal{R}(\widetilde{H})$.*

(iii) *Let E^{c}, $\widetilde{E}^{\mathrm{c}}$ be the continuous components of the spectral measures of H and $\widetilde{H}$, respectively. Then the wave operators*

$$\operatorname*{s-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}t\widetilde{H}} \mathrm{e}^{-\mathrm{i}tH} E^{\mathrm{c}}(\mathbb{R})$$

exist and are complete, i.e. their ranges are equal to $\widetilde{E}^{\mathrm{c}}(\mathbb{R})\mathcal{H}$.

Further regularity properties of the boundary values $(\widetilde{H} - \lambda \mp \mathrm{i}0)^{-1}$ are described in the Remark 4.1.

The next result covers the case when the potential has a finite number of Coulomb singularities of arbitrary strength:

Theorem 1.9. *Let $U : \mathbb{R}^3 \to B(\mathbf{E})$ be a symmetric matrix valued borelian function and $\Gamma \subset R^3$ a finite set. Assume that:*

(i) *$U = U_{\mathrm{S}} + U_{\mathrm{L}}$, where U_{S} is short-range and U_{L} is long-range.*

(ii) *U is locally L^3 in $\mathbb{R}^3 \setminus \Gamma$ and satisfies* (1.5).

(iii) *There exists $\varepsilon > 0$ such that for any $a \in \Gamma$ one has $U(x) = u_a(|x - a|)$ if $|x - a| < \varepsilon$, where $u_a : (0, \varepsilon) \to \mathbb{R}$ is a continuous function with the property that a finite limit $\lim_{t \to 0} t u_a(t)$ exists.*

Define $H_0 + U$ on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^3; \mathbf{E})$ and choose a self-adjoint extension $\widetilde{H}$ of $H_0 + U$. Then $\widetilde{H}$ has the properties (i) *and* (ii) *of Theorem* 1.8. *If $U_{\mathrm{L}} = 0$ then the wave operators associated to the pair $(\widetilde{H}, H_0)$ exist and are complete.*

In Subsection 4.7 we shall also prove the existence of the self-adjoint extension $\widetilde{H}$ in the physically most interesting cases. For example, it suffices that the function U be bounded on the region where $\operatorname{dist}(x, \Gamma) \geqslant \varepsilon$.

1.6. We shall sketch now the content of the remaining part of this paper in which, among other things, Theorems 1.4–1.9 will be proved.

As we already said in Subsection 1.1, we shall constantly rely on the conjugate operator method, so it is convenient to begin with a short description of the main ideas of this approach. Assume, for example, that we would like to show that the spectrum of a self-adjoint operator H in the open interval J is normal (see Definition 1.3). For this, it suffices to find a second self-adjoint operator A, called *conjugate operator*, such that:

(i) The map $\tau \mapsto \mathrm{e}^{-\mathrm{i}\tau A} H \mathrm{e}^{\mathrm{i}\tau A}$ has certain regularity properties, see (2.17); in particular this should imply that the commutator $[H, \mathrm{i}A]$ is a continuous sesquilinear form on $D(H)$.

(ii) One can find a number $a > 0$ and a compact operator K such that the *Mourre estimate*

$$E(J)[H, \mathrm{i}A]E(J) \geqslant aE(J) + K$$

is valid, where E is the spectral measure of H.

Of course, if such an operator A exists, then there is a large freedom in the choice of the explicit form of A. A good choice of A is important, because it allows one to cover general and physically natural classes of hamiltonians H by imposing a minimum of restrictions. The final comment in Subsection 1.4 shows just one example backing this claim. Another purpose which a suitable choice of A may serve is the simplicity of the proofs.

For these reasons we dedicated Section 2 to the study of a class of operators $H = h(P) + V$ which allow a simple (in this case scalar) conjugate operator: this is the class of "Dirac type" hamiltonians refered to in the title of our paper. A significantly more general situation will be considered in a future work ([6]), but we give already here the definition of the threshold set for a rather arbitrary function h. The general results of Section 2 are applied to Dirac operators in the first part of Section 4. In particular, we prove Theorem 1.6 in Subsection 4.2 (and this implies Theorems 1.4 and 1.5). The computations which show that (1.3) implies the conditions of Theorem A in [5] (thus giving a proof of Theorem 1.7) are identical to those from [16] (in the hypoelliptic case) and will not be repeated here.

The situation in Theorems 1.8 and 1.9 is too singular to be treated by the methods of Section 2. Therefore we dedicate Section 3 to the study of singular perturbations in a general setting. We are given a couple of self-adjoint operators $H, \widetilde{H}$. We assume that H is as in Section 2, has a spectral gap and satisfies a weak locality condition (Hypothesis 3.2). $\widetilde{H}$ is a selfadjoint operator in $\mathcal{H}$ which coincides with H outside a compact set K; inside K, $\widetilde{H}$ may be very singular. We prove then that the difference between the resolvents of $\widetilde{H}$ and of H is a short-range operator. In particular, $\widetilde{H}$ will be of class $\mathcal{C}^{1,1}(A)$ (i.e. it is of class $\mathcal{C}^{1,1}$ in the resolvent sense) hence the abstract theory developed in [1] can be used: if $\widetilde{H}$ is locally compact then $\widetilde{H}$ will admit the same conjugate operator A as H. The results of Section 3 may be used to prove Theorem 1.8; this is done in Subsection 4.3. In Theorem 1.8 there is one implicit condition, namely that $\widetilde{H}$ be locally compact. In Subsections 4.4 and 4.5 a framework is presented in which the problem of local compactness for very singular Dirac operators is naturally settled. By combining these results with Theorem 1.8, we give a proof of Theorem 1.9 in Subsection 4.6. Finally, in Subsection 4.7 we prove a result concerning the existence of self-adjoint realizations of singular Dirac operators.

2. Locally scalar matrix valued pseudo-differential operators

2.1. In this section we shall study operators of the form $H_0 = h(P)$ where h is a continuous function defined on $\mathbb{R}^n$ and taking values in $B(\mathbf{E})$, the set of all bounded linear operators in the finite-dimensional complex Hilbert space $\mathbf{E}$. The operators $h(p)$ are always assumed to be symmetric. The spectrum $\sigma(H_0)$ is the closure of the set $\bigcup_{p\in\mathbb{R}^n} \sigma(h(p))$, where $\sigma(h(p))$ is the spectrum of the operator $h(p)$ acting in $\mathbf{E}$.

For any subset I of $\mathbb{R}$ we set

$$\Omega(I) = \{p \in \mathbb{R}^n \mid \sigma(h(p)) \cap I \neq \emptyset\}.$$

If h is scalar valued (i.e. $h(p) \in \mathbb{R} \subset B(\mathbf{E})$) then $\Omega(I)$ is just $h^{-1}(I)$. Note that $\Omega(I)$ is open if I is open and closed if I is closed. The first assertion follows from the continuity in p of the eigenvalues of the matrix $h(p)$. For the second one we argue as follows. Let $\{p_k\}$ be a sequence in $\Omega(I)$ which converges to some $p \in \mathbb{R}^n$, let λ_k be an eigenvalue of $h(p_k)$ in I, and let e_k be a vector of norm 1 in $\mathbf{E}$ such that $h(p_k)e_k = \lambda_k e_k$. Since the unit sphere in $\mathbf{E}$ is compact we may assume that $\{e_k\}$ converges to some e of norm 1. From $h(p_k) \to h(p)$ we deduce that the sequence $\{\lambda_k\}$ converges to some $\lambda \in I$ and $h(p)e = \lambda e$. So λ is an eigenvalue of $h(p)$ in I, hence $p \in \Omega(I)$.

Let us denote by E_{S} the spectral measure of a self-adjoint operator S. We use the abbreviation $E_{H_0} = E_0$ and denote by χ_V the characteristic function of the Borel set V. Since $\mathcal{F}E_0(I)\mathcal{F}^* = \mathcal{F}\chi_I(h(P))\mathcal{F}^*$ is the operator of multiplication by the matrix valued function $p \mapsto \chi_I(h(p)) = E_{h(p)}(I)$, we have

$$E_0(I) = \chi_{\Omega(I)}(P)E_0(I) = E_0(I)\chi_{\Omega(I)}(P). \tag{2.1}$$

Definition 2.1. We say that the operator H_0 is *scalar on an open real set* I if

$$H_0E_0(I) = \mu(P)E_0(I) \tag{2.2}$$

for some Borel function $\mu : \mathbb{R}^n \to \mathbb{R}$. H_0 is called *scalar at a point* $\lambda \in \mathbb{R}$ if λ has an open neighbourhood on which H_0 is scalar. The (open) set of real points at which H_0 is scalar is denoted by $\Xi(H_0)$. If $\Xi(H_0) = \mathbb{R}$ then H_0 is called *locally scalar.*

We shall also refer to the function h as being scalar on a set or at a point and we shall set $\Xi(h) = \Xi(H_0)$. H_0 (or h) is locally scalar on an open set U if $U \subset \Xi(H_0)$.

The function μ obviously depends of I. Equation (2.2) is equivalent to $h(p)E_{h(p)}(I) = \mu(p)E_{h(p)}(I)$ for almost every $p \in \mathbb{R}^n$. Since $p \in \Omega(I)$ is equivalent to $E_{h(p)}(I) \neq 0$, we see that μ may be chosen arbitrarily on $\mathbb{R}^n \setminus \Omega(I)$ but its restriction to $\Omega(I)$ is uniquely defined; we shall denote by μ_I this restriction. It is clear that *a function* μ *satisfying* (2.2) *exists if and only if for each* $p \in \Omega(I)$ *the matrix* $h(p)$ *has exactly one eigenvalue in* I*; this eigenvalue is just* $\mu(p) \equiv \mu_I(p)$*.*

The function μ_I is continuous, as follows from the continuous dependence on p of the eigenvalues of $h(p)$. Moreover, μ_I shares the smoothness properties of the function $h|\Omega(I)$. This is an immediate consequence of the formulae

$$
\begin{aligned}
E_{h(p)}(I) &= \frac{1}{2\pi \mathrm{i}} \oint_{|z-\mu_0|=r} \frac{\mathrm{d}z}{z-h(p)}, \\
\mu(p)E_{h(p)}(I) &= \frac{1}{2\pi \mathrm{i}} \oint_{|z-\mu_0|=r} \frac{z\mathrm{d}z}{z-h(p)},
\end{aligned}
\tag{2.3}
$$

where $\mu_0 = \mu(p_0)$, r is sufficiently small and p belongs to some small neighbourhood of p_0.

Remark 2.2. The main example we have in mind in this paper is the Dirac operator with strictly positive mass and this is a locally scalar operator. The Dirac operator with mass $m = 0$ and, more generally, the uniformly propagative operators are locally scalar on $\mathbb{R} \setminus \{0\}$. If $n = 1$ and h is analytic and proper (in the sense of [7]) then H_0 is locally scalar outside a discrete set which can be explicitly described.

2.2. We shall give now a formal argument to justify our choice of the conjugate operator. Let us assume that (2.2) is verified and let J be an open set whose closure is a compact subset of I. Then we formally have

$$
\begin{aligned}
E_0(J)[H_0, \mathrm{i}A]E_0(J) &= E_0(J)H_0\mathrm{i}AE_0(J) - E_0(J)\mathrm{i}AH_0E_0(J) \\
&= E_0(J)\mu(P)\mathrm{i}AE_0(J) - E_0(J)\mathrm{i}A\mu(P)E_0(J) \\
&= E_0(J)[\mu(P), \mathrm{i}A]E_0(J).
\end{aligned}
\tag{2.4}
$$

We would like to reduce ourselves to the proof of the Mourre estimate for the *scalar* operator $\mu(P)$. From (2.2) we get

$$
\varphi(H_0)E_0(I) = \varphi[\mu(P)]E_0(I) \tag{2.5}
$$

for any bounded Borel function φ on $\mathbb{R}$. Hence

$$
E_0(J) = \chi_J(H_0)E_0(I) = \chi_J[\mu(P)]E_0(I) = E_\mu(J)E_0(I) = E_0(I)E_\mu(J),
$$

where we abbreviated $E_{\mu(P)} = E_\mu$. Going back to (2.4) we see that

$$
E_0(J)[H_0, \mathrm{i}A]E_0(J) = E_0(J)E_\mu(J)[\mu(P), \mathrm{i}A]E_\mu(J)E_0(J).
$$

So, if one has a Mourre estimate for $\mu(P)$ with respect to A, one also gets one for H_0. For example, if

$$
E_\mu(J)[\mu(P), \mathrm{i}A]E_\mu(J) \geqslant E_\mu(J) \tag{2.6}
$$

then

$$
E_0(J)[H_0, \mathrm{i}A]E_0(J) \geqslant E_0(J).
$$

We construct conjugate operators for $\mu(P)$ by using an observation which goes back to Mourre ([13]). If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field and

$$A = \frac{1}{2}\{F(P) \cdot Q + Q \cdot F(P)\}, \tag{2.7}$$

then

$$[\mu(P), \mathrm{i}A] = F(P) \cdot (\nabla\mu)(P).$$

Hence, if we take $F = \eta\nabla\mu$ with η a positive function, then

$$[\mu(P), \mathrm{i}A] = \eta(P)|(\nabla\mu)(P)|^2 \geqslant 0.$$

The critical points of the function μ prevent us from getting a Mourre estimate, but if one finds an J such that $|\nabla\mu(p)| \geqslant b > 0$ whenever $\mu(p) \in J$, then we may take

$$\eta(p) = \frac{\theta[\mu(p)]}{|(\nabla\mu)(p)|^2},$$

where $\theta \in C_{\mathrm{c}}^{\infty}(I)$ such that $\theta|J = 1$ and we are done. In order to avoid unnecessary differentiability requirements on μ, we shall make later a more convenient choice for F.

2.3. The next lemma gives a simple method of checking that H_0 is locally scalar. If S is a self-adjoint operator on $\mathbf{E}$, we set

$$\delta(S) = \min\{|\lambda - \lambda'| \mid \lambda, \lambda' \in \sigma(S),\ \lambda \neq \lambda'\}.$$

Lemma 2.3. *If*

$$\inf_{p\in\mathbb{R}^n} \delta(h(p)) > 0. \tag{2.8}$$

then H_0 is locally scalar.

Proof. If (2.8) is true, the number of points of $\sigma(h(p))$ is a constant N independent of p and there are real functions $\lambda_1, \dots, \lambda_N$ on $\mathbb{R}^n$ such that

$$\lambda_1(p) < \lambda_2(p) < \cdots < \lambda_N(p), \qquad \sigma(h(p)) = \{\lambda_1(p), \dots, \lambda_N(p)\}$$

for all $p \in \mathbb{R}^n$. The functions λ_k are continuous and have the same smoothness properties as h (use a suitable analogue of formula (2.3) with μ replaced by λ_k). Now let I be an open real set with diameter strictly less than the inf in (2.8). Then we have

$$\lambda_j^{-1}(I) \cap \lambda_k^{-1}(I) = \emptyset \qquad \text{if } j \neq k. \tag{2.9}$$

Clearly we have $\Omega(I) = \bigcup_{k=1}^{N} \lambda_k^{-1}(I)$, the union being disjoint. For any $p \in \lambda_k^{-1}(I)$ one has

$$E_{h(p)}(I)h(p) = E_{h(p)}(I)\lambda_k(p).$$

Then, defining μ on $\Omega(I)$ by

$$\mu(p) = \lambda_k(p) \quad \text{if } p \in \lambda_k^{-1}(I) \tag{2.10}$$

we see that (2.2) is verified. ∎

Remark 2.4. If $\pi_k(p)$ is the spectral projection of $h(p)$ corresponding to the eigenvalue $\lambda_k(p)$, one has

$$h(p) = \sum_{k=1}^{N} \lambda_k(p)\pi_k(p). \tag{2.11}$$

Replacing μ by λ_k in (2.3), it follows that the functions $\pi_k : \mathbb{R}^n \to B(\mathbf{E})$ are continuous and have the same smoothness properties as h. One has the decomposition

$$H_0 = \sum_{k=1}^{N} \lambda_k(P)\pi_k(P).$$

If we set $\Pi_k = \pi_k(P)$, then $\{\Pi_k\}_{k=1,\dots,N}$ is a family of pairwise orthogonal projections in $\mathcal{H}$ such that $\sum_{k=1}^{N} \Pi_k = 1$. So H_0 is unitarily equivalent to a direct sum of operators $\lambda_1(P), \dots, \lambda_N(P)$ ($\lambda_k(P)$ acts in $L^2(\mathbb{R}^n)$ and has to be taken $\dim \pi_k(p)$ times; note that this dimension is independent of p). This clearly suggests choosing an operator of the form

$$A = \sum_{k=1}^{N} \Pi_k A_k \Pi_k \tag{2.12}$$

in order to get a Mourre estimate for H_0, where A_k is a conjugate operator for $\lambda_k(P)$. A suitable modification of this approach works in rather general situations (see [7]), but (2.12) is quite an intricate object and, when possible, the choice we make in Subsection 2.5 is a better one. We intend to treat by our technique the class of operators considered in [7] in a separate publication ([6]).

2.4. In this subsection we shall define the threshold set $\tau(H_0)$ of H_0. The definition is adapted to our needs in this paper (in particular it facilitates the choice of a very simple conjugate operator); in some situations a more refined choice has to be made.

From now on we assume that h is at least of class C^1. It will be convenient to use the following abbreviations: for each $1 \leqslant j \leqslant n$ and $\lambda \in \mathbb{R}$ we set

$$\nabla_j^\lambda h(p) = E_{h(p)}(\{\lambda\})(\partial_j h)(p) E_{h(p)}(\{\lambda\})$$

and we denote by $\nabla^\lambda h(p) = (\nabla_1^\lambda h(p), \dots, \nabla_n^\lambda h(p))$. If $q \in \mathbb{R}^n$ then $q\nabla^\lambda h(p) = \sum_{k=1}^{n} q_k \nabla_k^\lambda h(p)$. These operators appear naturally in the perturbation theory of linear operators, see [10] for example. We always consider them as acting in the space $E_{h(p)}(\{\lambda\})\mathbf{E}$.

Definition 2.5. Let $h : \mathbb{R}^n \to B(\mathbf{E})$ be symmetric operator valued and of class C^1 and let λ be a real number.

(i) We say that λ is a *critical value* of h if there is p with $\lambda \in \sigma(h(p))$ such that for each $q \in \mathbb{R}^n$ zero is an eigenvalue of the operator $q\nabla^\lambda h(p)$ (acting in the space $E_{h(p)}(\{\lambda\})\mathbf{E}$).

(ii) We say that λ is an *asymptotic value* of h if for each neighbourhood J of λ the set $\Omega(J)$ is unbounded.

(iii) We say that λ is a *threshold value* of h if it is either a critical or an asymptotic value of h.

The *threshold set* $\tau(h)$ of h is the set of all its threshold values. If $H_0 = h(P)$ then we also say *threshold value* of H_0 and define $\tau(H_0) = \tau(h)$.

Remark 2.6. This definition applies, in particular, to scalar functions $h : \mathbb{R}^n \to \mathbb{R}$ of class C^1, because $\mathbb{R} \subset B(\mathbf{E})$. The situation (i) will correspond then to the critical values of h in the usual sense, i.e. the numbers λ such that $\lambda = h(p)$ for some $p \in \mathbb{R}^n$ with $(\nabla h)(p) = 0$. And (ii) will mean that for each neighbourhood J of λ the set $h^{-1}(J)$ is not bounded; in other terms, there is a sequence $\{p_i\}$ such that $p_i \to \infty$ and $h(p_i) \to \lambda$ (this explains the terminology we use). Now it is easy to show that $\lambda \notin \tau(h)$ if and only if there is a compact neighbourhood J of λ such that $h^{-1}(J)$ is compact and $\nabla h(p) \neq 0$ on $h^{-1}(J)$. Hence $\tau(h)$ is closed and for each compact set J disjoint from $\tau(h)$ there exists $b > 0$ such that $|(\nabla h)(p)| \geqslant b$ if $h(p) \in J$.

One of the drawbacks of the Definition 2.5 is now clear: if h is a simply characteristic scalar polynomial then we get $\tau(h) = \mathbb{R}$ in general. Or if $|h(p)| + |\nabla h(p)| \to \infty$ when $p \to \infty$ the set $\tau(h)$ should be equal to the set of critical values of h.

The next lemma clarifies our definition of the threshold set.

Lemma 2.7. *Assume that the function h has a representation of the form (2.11), where $\lambda_k : \mathbb{R}^n \to \mathbb{R}$ and $\pi_k : \mathbb{R}^n \to B(\mathbf{E})$ are functions of class C^1 such that $\pi_k \neq 0$, $\pi_j(p)\pi_k(p) = \delta_{jk}\pi_k(p)$, $\sum_{k=1}^{N} \pi_k(p) = 1$ for all $p \in \mathbb{R}^n$. Then:*

(i) *For each real set J one has $\Omega(J) = \bigcup_k \lambda_k^{-1}(J)$. In particular, $\Omega(J)$ is bounded if and only if for each k the set $\lambda_k^{-1}(J)$ is bounded.*

(ii) *For each $\lambda \in \mathbb{R}$ denote by $\Delta(\lambda, p)$ the set of $j \in \{1, \dots, N\}$ such that $\lambda_j(p) = \lambda$. Then $E_{h(p)}(\{\lambda\}) = \sum_{j\in\Delta(\lambda,p)} \pi_j(p)$ (a sum over an empty set being equal to zero) and*

$$\nabla^\lambda h(p) = \sum_{k\in\Delta(\lambda,p)} \nabla\lambda_k(p) \cdot \pi_k(p).$$

Proof. Assertion (i) is easy, we prove only (ii). Clearly $\nabla h = \sum_k (\pi_k \nabla\lambda_k + \lambda_k \nabla\pi_k)$. If we set $\Pi_p^\lambda = E_{h(p)}(\{\lambda\})$, then

$$\nabla^\lambda h(p) = \sum_{k=1}^{N} \Pi_p^\lambda \pi_k(p) \Pi_p^\lambda \nabla\lambda_k(p) + \sum_{k=1}^{N} \lambda_k(p) \Pi_p^\lambda \nabla\pi_k(p) \Pi_p^\lambda.$$

Since the projections $\pi_j(p)$ are pairwise orthogonal, one has $\Pi_p\pi_k(p)\Pi_p = \pi_k(p)$ if $k \in \Delta(\lambda, p)$ and $= 0$ otherwise. So it suffices to prove that the second sum above is equal to zero. We assume $\lambda \in \sigma(h(p))$, otherwise there is nothing to prove.

Let $r > 0$ be such that the distance between two consecutive eigenvalues of $h(p)$ is $> 2r$. If $\nu \in \sigma(h(p))$ we abbreviate $\Delta(\nu) = \Delta(\nu, p)$, so the family $\{\Delta(\nu) \mid \nu \in \sigma(h(p))\}$ is a partition of the set $\{1, \ldots, N\}$. We also denote by $I_\nu = [\nu - r/2, \nu + r/2]$. Choose $\varepsilon > 0$ such that $|\lambda_k(q) - \lambda_k(p)| < r/2$ for all k if $|q-p| < \varepsilon$. So if $|q-p| < \varepsilon$ then $\sigma(h(q)) \cap I_\nu = \{\lambda_k(q) \mid k \in \Delta(\nu)\}$. Clearly then $E_{h(q)}(I_\nu) = \sum_{k\in\Delta(\nu)} \pi_k(q)$; in particular this is a C^1-function of q in the ball $|q-p| < \varepsilon$. We set $\Pi_q^\nu = E_{h(q)}(I_\nu)$ and note that for $q = p$ and $\nu = \lambda$ we get the same operator as before.

Assume $|q-p| < \varepsilon$, differentiate with respect to q the relation $(\Pi_q^\lambda)^2 = \Pi_q^\lambda$, and multiply to the left the result by Π_q^λ; we get $\Pi_q^\lambda \cdot \nabla\Pi_q^\lambda \cdot \Pi_q^\lambda = 0$. If $\nu \neq \lambda$ we have $\Pi_q^\lambda\Pi_q^\nu = 0$. Differentiating this with respect to q and multiplying the result to the right by Π_q^λ we get $\Pi_q^\lambda \cdot \nabla\Pi_q^\nu \cdot \Pi_q^\lambda = 0$.

Finally, we are able to show that the second sum in the expression we have obtained above for $\nabla^\lambda h(p)$ is equal to zero. Indeed,

$$\sum_{k=1}^N \lambda_k(p)\Pi_p^\lambda\nabla\pi_k(p)\Pi_p^\lambda = \sum_{\nu\in\sigma(h(p))} \nu\Pi_p^\lambda\Bigg[\sum_{k\in\Delta(\nu)} \nabla\pi_k(p)\Bigg]\Pi_p^\lambda.$$

We would like to commute the sum and the derivative, but it is not clear whether we are allowed to do it because $\Delta(\nu) = \Delta(\nu, p)$ depends on p. However, for each q in the ball $|q-p| < \varepsilon$ and each $\nu \in \sigma(h(p))$ we have

$$0 = \Pi_q^\lambda \cdot \nabla\Pi_q^\nu \cdot \Pi_q^\lambda = \Pi_q^\lambda\Bigg[\sum_{k\in\Delta(\nu)} \nabla\pi_k(q)\Bigg]\Pi_q^\lambda$$

and here we may take $q = p$. This finishes the proof. ∎

For the class of locally scalar operators we have the following description of the threshold set.

Proposition 2.8. *Assume that H_0 is scalar on the open real set I and let μ be as in* (2.2)*. Then a real number $\lambda \in I$ is not a threshold value of H_0 if and only if λ has a compact neighbourhood J in I such that the set $\mu_I^{-1}(J) = \mu^{-1}(J) \cap \Omega(I)$ is compact and $\nabla\mu(p) \neq 0$ on it. In particular, $\tau(H_0) \cap \Xi(H_0)$ is a closed subset of $\Xi(H_0)$. If h is as in Lemma* 2.7*, in particular if* (2.8) *is satisfied, then $\tau(H_0) = \bigcup_{k=1}^N \tau(\lambda_k)$.*

Proof. We begin by noting that $\lambda \notin \tau(h)$ if and only if there is a neighbourhood J of λ such that $\Omega(J)$ is a bounded set and for each p such that $\lambda \in \sigma(h(p))$ there is q such that $q\nabla^\lambda h(p)$ has not zero as eigenvalue in the space $E_{h(p)}(\{\lambda\})\mathbf{E}$. We may assume that $\Omega(J)$ is compact because if J is closed then $\Omega(J)$ is also closed.

If the conditions of Definition 2.1 are satisfied then $\Omega(J) = \mu^{-1}(J) \cap \Omega(I)$ for $J \subset I$. So the proof of the first part of the proposition will be finished once we have shown that

$$\nabla^\lambda h(p) = \nabla\mu(p) \cdot E_{h(p)}(\{\lambda\}) \tag{2.13}$$

for all $\lambda \in I$, $p \in \mathbb{R}^n$. If $p \notin \Omega(I)$ then $h(p)$ has no eigenvalue in I and if $p \in \Omega(I)$ then $h(p)$ has just one eigenvalue in I, namely $\mu(p)$. Hence we may assume $p \in \Omega(I)$ and $\lambda = \mu(p)$, otherwise both sides above are zero. In this case we also have $E_{h(p)}(\{\lambda\}) = E_{h(p)}(I) \equiv \pi(p)$. Then π is a C^1-function on $\Omega(I)$ and we have $(h-\mu)\pi = 0$ on $\Omega(I)$. So we get $(\nabla h - \nabla\mu)\pi + (h-\mu)\nabla\pi = 0$ on $\Omega(I)$. Multiplying to the left and to the right by π, which commutes with $h - \mu$, and using $\pi \cdot \nabla\pi \cdot \pi = 0$, we get (2.13).

Now let us prove the last assertion of the proposition. From (i) of Lemma 2.7 it follows that λ is an asymptotic value of h if and only if there is k such that $\lambda_k^{-1}(J)$ is unbounded for each neighbourhood J of λ, i.e. if and only if there is k such that λ is an asymptotic value of λ_k. On the other hand, from (ii) of Lemma 2.7 it follows that λ is a critical value of h if and only if there is p such that $\Delta(\lambda, p) \neq \emptyset$ and such that for each q the operator $\sum_{k\in\Delta(\lambda,p)} q\nabla\lambda_k(p)\cdot\pi_k(p)$ has zero as eigenvalue in the space $E_{h(p)}(\{\lambda\})\mathbf{E}$. This last fact is clearly equivalent to $q\nabla\lambda_k(p) = 0$ for some k depending on q. Since $\mathbb{R}^n$ cannot be the union of n subspaces of dimension $n-1$ we see that there must exist k such that $q\nabla\lambda_k(p) = 0$ for all q. Hence λ is a critical value of h if and only if there is k such that $\lambda_k(p) = \lambda$ and $\nabla\lambda_k(p) = 0$, i.e. λ is a critical value of λ_k. Finally, we see that the equality $\tau(H_0) = \bigcup_{k=1}^{N} \tau(\lambda_k)$ holds under the hypotheses of Lemma 2.7. ∎

2.5. We are finally in a position to define conjugate operators for H_0 at scalar non-threshold points. Let $\lambda \notin \tau(H_0)$ and let I be an open neighbourhood of λ on which H is scalar. Let μ be such that (2.2) holds and remark that it is a C^1-function on $\Omega(I)$. Replacing, if needed, I by a slightly smaller set, we may assume that μ has been extended to a C^1-function on $\mathbb{R}^n$ (in order to be able to define $\mu(P)$ and $(\nabla\mu)(P)$; but note that the values taken by μ on $\mathbb{R}^n \setminus \Omega(I)$ will be irrelevant). Now let J be a compact neighbourhood of λ included into $I \setminus \tau(H_0)$. Then $\mu_I^{-1}(J)$ is a compact subset of the open set $\Omega(I)$ and there is a number $b > 0$ such that $|(\nabla\mu)(p)| \geqslant b$ if $p \in \Omega(I)$ and $\mu(p) \in J$. Hence there is a C^∞-function $F : \mathbb{R}^n \to \mathbb{R}^n$ with compact support contained in $\Omega(I)$ such that

$$F(p) \cdot (\nabla\mu)(p) \geqslant 1 \qquad \text{if } p \in \Omega(I) \cap \mu^{-1}(J)$$

(see page 337 in [1]). The operator A given by (2.7) is self-adjoint and (2.6) is verified.

DEFINITION 2.9. The (unique) self-adjoint realization A of an expression of the form (2.7) with $F : \mathbb{R}^n \to \mathbb{R}^n$ a C^∞-function with compact support is called *a standard (conjugate) operator*.

THEOREM 2.10. *Assume that the symmetric matrix valued function h is of class C^1 and let $\lambda \in \mathbb{R} \setminus \tau(H_0)$ such that H_0 is scalar at λ. Then there is a neighbourhood J of λ and there is a standard conjugate operator A such that*

$$E_0(J)[H_0, \mathrm{i}A]E_0(J) \geqslant E_0(J).$$

REMARK 2.11. We discuss here a technical point concerning the interpretation of the commutator $[H_0, \mathrm{i}A]$. In the sense of sesquilinear forms on $C_{\mathrm{c}}^\infty(\mathbb{R}^n; \mathbf{E})$ we clearly have $[H_0, \mathrm{i}A] = F(P) \cdot (\nabla h)(P)$, so the form $[H_0, \mathrm{i}A]$ extends to a bounded operator on $\mathcal{H}$ for which we shall keep the same notation. Moreover, the unitary group generated by A has the property $\mathrm{e}^{\mathrm{i}tA}D(H_0) \subset D(H_0)$ for any $t \in \mathbb{R}$ (see Lemma 7.5.6 and the proof of Proposition 7.6.3 (a) in [1]).

Proof of Theorem. 2.10. By taking into account the preceding considerations, we are left with proving the formal steps in the calculation (2.4). So, we have to show that the two bounded operators

$$E_0(J)[H_0, \mathrm{i}A]E_0(J) = E_0(J)F(P) \cdot (\nabla h)(P)E_0(J) \tag{2.14}$$

and

$$E_0(J)[\mu(P), \mathrm{i}A]E_0(J) = E_0(J)F(P) \cdot (\nabla \mu)(P)E_0(J) \tag{2.15}$$

are equal. For this it suffices to prove that for all $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^n; \mathbf{E})$ and for all $\varphi \in C_{\mathrm{c}}^\infty(I)$ one has

$$\langle \varphi(H_0)f, [H_0, \mathrm{i}A]\varphi(H_0)f\rangle = \langle \varphi(H_0)f, [\mu(P), \mathrm{i}A]\varphi(H_0)f\rangle. \tag{2.16}$$

From Remark 2.11 (or from the explicit form of $\varphi(H_0)$) it follows that $\varphi(H_0)D(A) \subset D(A)$. On the other hand, from (2.5) we get $\varphi(H_0) = \varphi[\mu(P)]E_0(I)$, hence $\varphi(H_0)\mathcal{H} \subset D[\mu(P)] \cap D(H_0)$. So the commutators in (2.16) may be developped ($f \in D(A)$ would be sufficient) and the computation in (2.4) is justified. ∎

REMARK 2.12. If (2.8) is satisfied then the following proof of Theorem 2.10 is simpler. Instead of proving that the left-hand sides of (2.14) and (2.15) are equal, let us do this for the right-hand sides, by using Remark 2.4. In fact, one must show that

$$\chi_J[h(p)]\,\{F(p) \cdot (\nabla h)(p) - F(p) \cdot (\nabla \mu)(p)\}\,\chi_J[h(p)] = 0 \qquad \text{for all } p \in \mathbb{R}^n.$$

But, p being fixed, there is a unique $k_0 \in \{1, \ldots, N\}$ such that $\chi_I[h(p)] = \pi_{k_0}(p)$. Hence, by also using (2.11), we need to show

$$\sum_{l=1}^n F_l(p)\pi_{k_0}(p)\left\{\partial_l\left[\sum_{k=1}^n \lambda_k(p)\pi_k(p)\right] - (\partial_l\mu)(p)\right\}\pi_{k_0}(p) = 0.$$

This follows immediatly by using the obvious relations $\pi_{k_0}(p)\pi_k(p) = \delta_{k_0k}\pi_{k_0}(p)$ and $\pi_{k_0}(p)(\partial_l\pi_k)(p)\pi_{k_0}(p) = 0$.

2.6. Our purpose is to treat perturbations $H = H_0 + V$ and for this we shall summarize some general facts. It will be convenient to stay at an abstract level. Proofs of these facts may be found in Section 7.5. of [1].

Let $\mathcal{G}, \mathcal{H}$ be Hilbert spaces such that $\mathcal{G} \subset \mathcal{H}$ continuously and densely. By taking into account $\mathcal{G}^*$, the adjoint space (i.e. topological anti-dual) of $\mathcal{G}$, we get a standard triplet $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$. Moreover, we assume that a self-adjoint operator A is given in $\mathcal{H}$ and that $\mathrm{e}^{\mathrm{i}\tau A}\mathcal{G} \subset \mathcal{G}$ for every $\tau \in \mathbb{R}$. Then

$$D(A;\mathcal{G}) = \{f \in \mathcal{G} \cap D(A) \mid Af \in \mathcal{G}\} = \{f \in \mathcal{G} \mid \lim_{\tau\to 0}(\mathrm{e}^{\mathrm{i}\tau A}f - f)\tau^{-1}\text{exists in } \mathcal{G}\}$$

is dense in $\mathcal{G}$.

Assume now that H_0, H are symmetric operators from $\mathcal{G}$ to $\mathcal{G}^*$ such that $H_0 + \mathrm{i}$ and $H + \mathrm{i}$ are isomorphisms from $\mathcal{G}$ to $\mathcal{G}^*$. We shall assume that H_0 and H are *A-regular*, more precisely

$$\int_0^1 \frac{\mathrm{d}\tau}{\tau^2} \left\|\mathrm{e}^{\mathrm{i}\tau A}H\mathrm{e}^{-\mathrm{i}\tau A} + \mathrm{e}^{-\mathrm{i}\tau A}H\mathrm{e}^{\mathrm{i}\tau A} - 2H\right\|_{\mathcal{G}\to\mathcal{G}^*} < \infty \tag{2.17}$$

and similarly for H_0. Under these assumptions one can show that the commutators $[H_0, \mathrm{i}A]$ and $[H, \mathrm{i}A]$, which are welldefined as sesquilinear forms on $D(A;\mathcal{G})$, extend to continuous, symmetric operators from $\mathcal{G}$ to $\mathcal{G}^*$. By Lemma 7.5.3 in [1], one can associate to H_0 and H self-adjoint operators $\widehat{H}_0$ and $\widehat{H}$ in $\mathcal{H}$. For example, $\widehat{H}$ has domain $D(\widehat{H}) = (H+\mathrm{i})^{-1}\mathcal{H}$ and $\widehat{H} = H|D(\widehat{H})$. Furthermore, if $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R})$, the bounded operator $\varphi(\widehat{H})$ in $\mathcal{H}$ extends to a continuous operator from $\mathcal{G}^*$ to $\mathcal{G}$ (similarly for $\widehat{H}_0$). From now on we shall set $\widehat{H}_0 = H_0$ and $\widehat{H} = H$. Let us recall that if one of the operators H_0 or H is semibounded, its form domain will coincide with $\mathcal{G}$. But this is not necessarily true in general. In fact, for the Dirac case which will be treated in Section 4 we shall take *by definition* $\mathcal{G}$ to be the form-domain of “the free operator” H_0, but the form domain of its perturbation H could differ from $\mathcal{G}$ in some important instances.

Now let I be an open, real set such that A is *locally conjugate* to H_0 on I. This means that for each $\lambda \in I$ there are a compact neighbourhood J of λ, a number $a > 0$ and a compact operator K in H such that the following Mourre estimate holds:

$$E_0(J)[H_0, \mathrm{i}A]E_0(J) \geqslant aE_0(J) + K.$$

Under the above hypothesis, the operator H_0 has nice spectral properties in I. By imposing a certain compactness assumption, the same will be true for H. Indeed, one has (see Proposition 7.5.6 in [1]):

Proposition 2.13. *Besides the preceding hypotheses, let us assume that the difference $(H+\mathrm{i})^{-1} - (H_0+\mathrm{i})^{-1}$ is a compact operator in $\mathcal{H}$. Then the spectrum of H in I is normal.*

2.7. Our next purpose is to prove a version of the so-called *limiting absorption principle* in a framework which is suited to perturbations of matrix valued hamiltonians, i.e. the Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R}^n;\mathbf{E})$ and $H_0 = h(P)$. Then $\mathcal{G}$ will be the form domain of H_0, i.e. $\mathcal{G} = D(|H_0|^{1/2})$ (equipped with the graph topology). Explicitly, $\mathcal{F}|H_0|^{1/2}\mathcal{F}^*$ is the operator of multiplication by the matrix valued function $p \mapsto |h(p)|^{1/2} = [h(p)^2]^{1/4}$. In the particular case of the Dirac operator, $\mathcal{G}$

is just the Sobolev space $\mathcal{H}^{1/2}(\mathbb{R}^n;\mathbf{E})$, but in general it might not be a standard space of distributions. For this reason and in order not to impose too many constraints on h, besides $\mathcal{G}$ we shall use a generalization of the Sobolev spaces allowing some anisotropy in P. This will also lead to a more transparent form of the limiting absorption principle. A detailed study of these spaces may be found in Subsection 10.1 from [8].

Let us denote by $\mathcal{S}=\mathcal{S}(\mathbb{R}^n;\mathbf{E})$ the space of Schwartz test functions on $\mathbb{R}^n$ taking values in $\mathbf{E}$ and by $\mathcal{S}^*$ its adjoint space. By considering the usual topologies on $\mathcal{S}$ and $\mathcal{S}^*$ and by identifying $\mathcal{H}=L^2(\mathbb{R}^n;\mathbf{E})$ with its adjoint space one gets the following continuous, dense embeddings of topological vector spaces: $\mathcal{S}\subset\mathcal{H}\subset\mathcal{S}^*$. We recall that the Fourier transform is an isomorphism both in $\mathcal{S}$ and in $\mathcal{S}^*$.

We shall consider continuous functions $\omega:\mathbb{R}^n\to(0,\infty)$ such that

$$\omega(p+p')\leqslant C\omega(p)\langle p'\rangle^N \tag{2.18}$$

for some constants $C,N>0$ and all $p,p'\in\mathbb{R}^n$. Then $\omega(P)=\mathcal{F}^*\omega(Q)\mathcal{F}$ is a well-defined continuous injective operator $\omega(P):\mathcal{S}\to\mathcal{H}$. So we may define the Hilbert space $\mathcal{H}^\omega$ as the completion of $\mathcal{S}$ with respect to the norm $\|f\|_{\mathcal{H}^\omega}=\|\omega(P)f\|$.

We shall slightly modify the terminology of Hörmander and call *weight* a function ω with the preceding properties. According to the remark after Theorem 10.1.5 in [8], there is a temperate weight function ω' (in the sense of Definition 10.1.1 in [8]) such that $\omega\leqslant\omega'\leqslant C\omega$ for a constant $C<\infty$. Hence the norms defined by ω and ω' are equivalent and they define the same topological vector space $\mathcal{H}^\omega$. In particular, from Theorem 10.1.7 we get that $\mathcal{H}^\omega$ coincides with the set of distributions $f\in\mathcal{S}^*$ such that the Fourier transform $\widehat{f}$ is a function and

$$\|f\|_{\mathcal{H}^\omega}=\left[\int_{\mathbb{R}^n}\mathrm{d}p|\omega(p)\widehat{f}(p)|^2\right]^{1/2}<\infty.$$

We have $\mathcal{S}\subset\mathcal{H}^\omega\subset\mathcal{S}^*$ continuously and densely, so the adjoint space $(\mathcal{H}^\omega)^*$ can be realized such that $\mathcal{S}\subset(\mathcal{H}^\omega)^*\subset\mathcal{S}^*$. The function $1/\omega$ clearly is a weight and Theorem 10.1.14 from [8] gives a canonical identification $(\mathcal{H}^\omega)^*=\mathcal{H}^{1/\omega}$ as topological vector spaces.

The class of weight functions is clearly stable under multiplication and under taking arbitrary real powers. So, if ω_0,ω_1 are weights and $0<\theta<1$ is a real number then $\omega_\theta=\omega_0^{1-\theta}\omega_1^\theta$ is again a weight. It follows that the class of spaces $\mathcal{H}^\omega$ is stable under complex interpolation and one has $[\mathcal{H}^{\omega_0},\mathcal{H}^{\omega_1}]_\theta=\mathcal{H}^{\omega_\theta}$. In particular, if ω,ω_0,ω_1 are weights and T is a linear operator such that $T:\mathcal{H}^\omega\to\mathcal{H}^{\omega_0}$ continuosly and $T:\mathcal{H}^\omega\to\mathcal{H}^{\omega_1}$ compactly, then $T:\mathcal{H}^\omega\to\mathcal{H}^{\omega_\theta}$ is a compact operator for each $0<\theta\leqslant 1$.

From (2.18) and the relation $\mathrm{e}^{-\mathrm{i}p\cdot Q}\omega(P)\mathrm{e}^{\mathrm{i}p\cdot Q}=\omega(P+p)$ it follows easily that $\{\mathrm{e}^{\mathrm{i}p\cdot Q}\}_{p\in\mathbb{R}^n}$ induces a strongly continuous polynomially bounded n-parameter group of bounded operators in $\mathcal{H}^\omega$. In particular, if $\varphi\in\mathcal{S}(\mathbb{R}^n)$, then $\varphi(Q)$ is a bounded linear operator in $\mathcal{H}^\omega$ and if $0<\varepsilon<1$ then the family of operators $\varphi(\varepsilon Q)$ is uniformly bounded in $B(\mathcal{H}^\omega)$. Let $\mathcal{H}^\omega_{s,q}$ ($s\in\mathbb{R}$, $1\leqslant q\leqslant\infty$) be the Besov scale associated to this group in $\mathcal{H}^\omega$ (cf. the beginning of Section 4.1 in [1]). We shall briefly describe the main properties of these spaces (see Theorems 3.6.14 and 3.4.3 in [1]).

Choose $\theta, \widetilde{\theta} \in C_c^\infty(\mathbb{R}^n)$ such that $\theta(x) > 0$ if $1 < |x| < 2$ and $\theta(x) = 0$ otherwise, and $\widetilde{\theta}(x) = 1$ if $|x| \leqslant 1$. For any $s \in \mathbb{R}$ and $q \in [1, \infty]$ we set

$$\mathcal{H}_{s,q}^\omega = \{f \in \mathcal{S}^* \mid \|f\|_{\mathcal{H}_{s,q}^\omega} < \infty\}, \tag{2.19}$$

where

$$\|f\|_{\mathcal{H}_{s,q}^\omega} = \|\widetilde{\theta}(Q)f\|_{\mathcal{H}^\omega} + \left[\int_1^\infty \frac{\mathrm{d}r}{r} \|r^s\theta(Q/r)f\|_{\mathcal{H}^\omega}^q\right]^{1/q}. \tag{2.20}$$

If $q = \infty$, the last term in (2.20) should be interpreted as $\sup_{r\geqslant 1} \|r^s\theta(Q/r)f\|_{\mathcal{H}^\omega}$. If $q = 2$, the space $\mathcal{H}_{s,2}^\omega$ is usually denoted by $\mathcal{H}_s^\omega$ and an equivalent norm on it is given by $\|\langle Q\rangle^s f\|_{\mathcal{H}^\omega}$.

$\mathcal{H}_{s,q}^\omega$ are Banach spaces such that $\mathcal{S} \subset \mathcal{H}_{s,q}^\omega \subset \mathcal{S}^*$ continuously. The first embedding is dense if and only if $q \neq \infty$; we shall denote by $\breve{\mathcal{H}}_{s,\infty}^\omega$ the closure of $\mathcal{S}$ in $\mathcal{H}_{s,\infty}^\omega$. The second embedding allows us to compare the spaces $\mathcal{H}_{s,q}^\omega$. Let us write $(s_1, q_1) \leqslant (s_2, q_2)$ if and only if $s_1 > s_2$ or $s_1 = s_2$ and $q_1 \leqslant q_2$. Then we have $\mathcal{H}_{s_1,q_1}^\omega \subset \mathcal{H}_{s_2,q_2}^\omega$ if and only if $(s_1, q_1) \leqslant (s_2, q_2)$.

Since $\mathcal{S}$ is dense in $\mathcal{H}_{s,q}^\omega$ if $q < \infty$, the dual space $(\mathcal{H}_{s,q}^\omega)^*$ is a subspace of $\mathcal{S}^*$; similarly for $(\breve{\mathcal{H}}_{s,\infty}^\omega)^*$. These spaces are explicitly given by

$$\begin{aligned} (\mathcal{H}_{s,q}^\omega)^* &= \mathcal{H}_{-s,q'}^{1/\omega} \qquad \text{if } 1 \leqslant q < \infty, \\ (\breve{\mathcal{H}}_{s,\infty}^\omega)^* &= \mathcal{H}_{-s,1}^{1/\omega}, \end{aligned}$$

where q' is given by $1/q + 1/q' = 1$.

Concerning the dependence on ω of the spaces $\mathcal{H}_{s,q}^\omega$, one has $\mathcal{H}_{s,q}^{\omega_1} \subset \mathcal{H}_{s,q}^{\omega_2}$ for some (hence for all) s, q if and only if $\omega_2 \leqslant C\omega_1$ (see Theorem 10.1.8 in [8]).

Now assume that $\omega_2(p)/\omega_1(p) \to 0$ as $p \to \infty$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. We shall prove that $\varphi(Q) : \mathcal{H}^{\omega_1} \to \mathcal{H}^{\omega_2}$ *is a compact operator*. Let $\omega = \omega_1^2$ and $w = \omega_2^2\omega_1^{-2}$. Then $\varphi(Q) : \mathcal{H} \to \mathcal{H}^w$ is compact (because $w(P)\varphi(Q)$ is a compact operator in $\mathcal{H}$) and $\varphi(Q) : \mathcal{H}^\omega \to \mathcal{H}^\omega$ is bounded. Hence for $0 < \theta < 1$

$$\varphi(Q) : [\mathcal{H}, \mathcal{H}^\omega]_\theta = \mathcal{H}^{\omega^\theta} \to [\mathcal{H}^w, \mathcal{H}^\omega]_\theta = \mathcal{H}^{w^{1-\theta}\omega^\theta}$$

is compact. The result follows by taking $\theta = 1/2$.

The particular case of the usual Sobolev spaces is important and we shall introduce a special notation for it: if $\omega(p) = \langle p\rangle^t$ for some $t \in \mathbb{R}$ (recall that $\langle p\rangle = (1 + |p|^2)^{1/2}$), then we set $\mathcal{H}_{s,q}^\omega = \mathcal{H}_{s,q}^t$. If $t = 0$ then the upper index is simply dropped. Notice that, by our rules of identification, if $\omega \geqslant c > 0$ for some constant c, then

$$\mathcal{S} \subset \mathcal{H}_{s,q}^\omega \subset \mathcal{H}_{s,q} \subset \mathcal{H}_{s,q}^{1/\omega} \subset \mathcal{S}^*.$$

2.8. The following technical result will be important later on.

Theorem 2.14. *Let ω_1, ω_2 be continuous, positive functions on $\mathbb{R}^n$ such that $\omega_i(p+p') \leqslant C\omega_i(p)\langle p'\rangle^N$ for some constants $C, N > 0$ and all $p, p' \in \mathbb{R}^n$. Let $S : \mathcal{H}^{\omega_1} \to \mathcal{H}^{\omega_2}$ be a continuous, linear map such that*

$$\int_1^\infty \mathrm{d}r \big\{ \|\theta(Q/r)S\|_{\mathcal{H}^{\omega_1}\to\mathcal{H}^{\omega_2}} + \|S\theta(Q/r)\|_{\mathcal{H}^{\omega_1}\to\mathcal{H}^{\omega_2}} \big\} < \infty$$

for some $\theta \in C^\infty(\mathbb{R}^n)$ with $\theta(x) > 0$ if $1 < |x| < 2$ and $\theta(x) = 0$ otherwise.

Then $S \in B\big(\breve{\mathcal{H}}^{\omega_1}_{-1/2,\infty}, \mathcal{H}^{\omega_2}_{1/2,1}\big)$.

Remark 2.15. (i) We have not stated the theorem in its full generality. The result remains true (with the same proof) if $\mathcal{H}^{\omega_1}$, $\mathcal{H}^{\omega_2}$ are replaced by reflexive Banach spaces equipped with polynomially bounded, strongly continuous representations of $\mathbb{R}^n$ (then Theorem 3.6.14 from [1] can be used).

(ii) If S is a local operator (i.e supp $Sf \subset$ supp f for each $f \in \mathcal{H}^{\omega_1}$), then the reciprocal assertion is also true, cf. the proof of Theorem 7.6.10 in [1].

Proof of Theorem 2.14. Since $\mathcal{S}$, hence $C_c^\infty(\mathbb{R}^n; \mathbf{E})$, is dense in $\breve{\mathcal{H}}^{\omega_1}_{-1/2,\infty}$, it suffices to show that there is $C > 0$ such that

$$\|Sf\|_{\mathcal{H}^{\omega_2}_{1/2,1}} \leqslant C\|f\|_{\mathcal{H}^{\omega_1}_{-1/2,\infty}} \tag{2.21}$$

for all $f \in C_c^\infty(\mathbb{R}^n; \mathbf{E})$. We can further simplify later arguments by observing that it is enough to prove this under the assumption $f(x) = 0$ if $|x| < 1$. Indeed, choose $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\psi(x) = 1$ if $|x| \leqslant 1$; then

$$\|Sf\|_{\mathcal{H}^{\omega_2}_{1/2,1}} \leqslant \|S[1-\psi(Q)]f\|_{\mathcal{H}^{\omega_2}_{1/2,1}} + \|S\psi(Q)f\|_{\mathcal{H}^{\omega_2}_{1/2,1}}.$$

If (2.21) holds if $f(x) = 0$ in $\{x \mid |x| \leqslant 1\}$, then the first term in the right-hand side above is dominated by $C\|[1-\psi(Q)]f\|_{\mathcal{H}^{\omega_1}_{-1/2,\infty}} \leqslant C_1\|f\|_{\mathcal{H}^{\omega_1}_{-1/2,\infty}}$. On the other hand

$$\begin{aligned}\|S\psi(Q)f\|_{\mathcal{H}^{\omega_2}_{1/2,1}} &= \|\widetilde{\theta}(Q)S\psi(Q)f\|_{\mathcal{H}^{\omega_2}} + \int_1^\infty \frac{\mathrm{d}r}{\sqrt{r}}\|\theta(Q/r)S\psi(Q)f\|_{\mathcal{H}^{\omega_2}} \\ &\leqslant C_2\|\psi(Q)f\|_{\mathcal{H}^{\omega_1}} \leqslant C_3\|f\|_{\mathcal{H}^{\omega_1}_{-1/2,\infty}}\end{aligned}$$

for some $C_2, C_3 \in \mathbb{R}$.

It remains to prove (2.21) for $f \in C_c^\infty(\mathbb{R}^n; \mathbf{E})$ and such that $f(x) = 0$ if $|x| \leqslant 1$. We do this with the help of a Littlewood-Paley type decomposition of f. Let $\xi \in C^\infty(\mathbb{R}^n)$ with $\xi(x) = 0$ if $|x| \leqslant 1$ and $\xi(x) = 1$ if $|x| \geqslant 2$ and set $\eta(x) = x \cdot (\nabla\xi)(x)$. Then $\eta \in C^\infty(\mathbb{R}^n)$ and has support in $\{x \mid 1 \leqslant |x| \leqslant 2\}$. Furthermore, ξ may be chosen such that $\eta = \theta^2$ for some $\theta \in C^\infty(\mathbb{R}^n)$ with $\theta(x) > 0$ if $1 < |x| < 2$. For $0 < a < b$ we then have

$$\xi(bx) - \xi(ax) = \int_a^b \frac{\mathrm{d}\varepsilon}{\varepsilon}\eta(\varepsilon x)$$

for all $x \in \mathbb{R}^n$, hence

$$\xi(bQ)f - \xi(aQ)f = \int_a^b \frac{\mathrm{d}\varepsilon}{\varepsilon}\eta(\varepsilon Q)f.$$

Note that the map $\varepsilon \mapsto \eta(\varepsilon Q)f \in \mathcal{H}^{\omega_1}$ is of class C^∞ on $(0,\infty)$ and is equal to zero for ε small or large (by the conditions on the support of f), so the integral exists strongly in $\mathcal{H}^{\omega_1}$. Also $\xi(aQ)f = 0$ if a is small and $\xi(bQ)f = f$ if $b > 2$. So we have

$$f = \int_0^2 \frac{\mathrm{d}\varepsilon}{\varepsilon}\eta(\varepsilon Q)f$$

in $\mathcal{H}^{\omega_1}$. The operator $S : \mathcal{H}^{\omega_1} \to \mathcal{H}^{\omega_2}$ being linear and continuous, we get

$$Sf = \int_0^2 \frac{\mathrm{d}\varepsilon}{\varepsilon} S\eta(\varepsilon Q)f = \int_{1/2}^\infty \frac{\mathrm{d}r}{r} S\theta^2(Q/r)f.$$

We have to estimate the norm

$$\|Sf\|_{\mathcal{H}^{\omega_2}_{1/2,1}} = \|\widetilde{\theta}(Q)Sf\|_{\mathcal{H}^{\omega_2}} + \int_1^\infty \frac{\mathrm{d}r}{\sqrt{r}}\|\theta(Q/r)Sf\|_{\mathcal{H}^{\omega_2}}.$$

For both terms we use the preceding representation of Sf. We have:

$$\begin{aligned}
\|\widetilde{\theta}(Q)Sf\|_{\mathcal{H}^{\omega_2}} \leqslant C_1\|Sf\|_{\mathcal{H}^{\omega_2}} &\leqslant C_1 \int_{1/2}^\infty \frac{\mathrm{d}r}{r}\|S\theta(Q/r)\|_{\mathcal{H}^{\omega_1}\to\mathcal{H}^{\omega_2}} \cdot \|\theta(Q/r)f\|_{\mathcal{H}^{\omega_1}} \\
&\leqslant C_1 \int_{1/2}^\infty \frac{\mathrm{d}r}{\sqrt{r}}\|S\theta(Q/r)\|_{\mathcal{H}^{\omega_1}\to\mathcal{H}^{\omega_2}} \cdot \sup_{|r|\geqslant 1/2}\|r^{-1/2}\theta(Q/r)f\|_{\mathcal{H}^{\omega_1}} \\
&\leqslant C_2\|f\|_{\mathcal{H}^{\omega_1}_{-1/2,\infty}}.
\end{aligned}$$

Then for the second term:

$$\begin{aligned}
&\int_1^\infty \frac{\mathrm{d}r}{\sqrt{r}}\|\theta(Q/r)Sf\|_{\mathcal{H}^{\omega_2}} \\
&\qquad \leqslant \int_1^\infty \frac{\mathrm{d}r}{\sqrt{r}} \int_{1/2}^\infty \frac{\mathrm{d}s}{\sqrt{s}}\|\theta(Q/r)S\theta(Q/s)\|_{\mathcal{H}^{\omega_1}\to\mathcal{H}^{\omega_2}} \cdot \sup_{|s|\geqslant 1/2}\|s^{-1/2}\theta(Q/s)f\|_{\mathcal{H}^{\omega_1}}.
\end{aligned}$$

So it suffices to show that

$$\int_1^\infty\int_1^\infty \frac{\mathrm{d}r\mathrm{d}s}{\sqrt{rs}}\|\theta(Q/r)S\theta(Q/s)\|_{\mathcal{H}^{\omega_1}\to\mathcal{H}^{\omega_2}} < \infty.$$

We consider separately the contributions of the regions $s < r$ and $s > r$. For example

$$\iint\limits_{1 \leqslant s < r < \infty} \frac{\mathrm{d}r\mathrm{d}s}{\sqrt{rs}} \|\theta(Q/r)S\theta(Q/s)\|_{\mathcal{H}^{\omega_1} \to \mathcal{H}^{\omega_2}}$$

$$\leqslant C_3 \int_1^\infty \frac{\mathrm{d}r}{\sqrt{r}} \int_1^r \frac{\mathrm{d}s}{\sqrt{s}} \|\theta(Q/r)S\|_{\mathcal{H}^{\omega_1} \to \mathcal{H}^{\omega_2}}$$

$$= C_3/2 \int_1^\infty \frac{\mathrm{d}r}{\sqrt{r}} (\sqrt{r} - 1) \|\theta(Q/r)S\|_{\mathcal{H}^{\omega_1} \to \mathcal{H}^{\omega_2}} < \infty.$$

We used the obvious estimate $\|\theta(Q/s)\|_{B(\mathcal{H}^{\omega_1})} \leqslant C_3$, with C_3 independent of s. This finishes the proof. ∎

2.9. Let us now introduce two classes of bounded operators : $\mathcal{H}^\omega \to \mathcal{H}^{1/\omega}$, defined in terms of the asymptotic behaviour with respect to Q. Our purpose is to generalize Definition 1.1. We recall that ξ is an arbitrary C^∞-function on $\mathbb{R}^n$ such that $\xi(x) = 0$ if $|x| \leqslant 1$ and $\xi(x) = 1$ if $|x| \geqslant 2$.

Definition 2.16. (i) We say that $T \in B(\mathcal{H}^\omega, \mathcal{H}^{1/\omega})$ is a *short-range operator with respect to* ω if T is symmetric and

$$\int_1^\infty \mathrm{d}r \|\xi(Q/r)T\|_{\mathcal{H}^\omega \to \mathcal{H}^{1/\omega}} < \infty.$$

(ii) We say that $T \in B(\mathcal{H}^\omega, \mathcal{H}^{1/\omega})$ is a *long-range operator with respect to* ω if T is symmetric and

$$\sum_{j=1}^n \int_1^\infty \frac{\mathrm{d}r}{r} \{\|\xi(Q/r)[Q_j, T]\|_{\mathcal{H}^\omega \to \mathcal{H}^{1/\omega}} + \|\xi(Q/r)|Q|[P_j, T]\|_{\mathcal{H}^\omega \to \mathcal{H}^{1/\omega}}\} < \infty.$$

A simple argument shows that one gets the same class of short-range operators if the function ξ in (i) is replaced by a function $\theta \in C_c^\infty(\mathbb{R}^n)$ which satisfies $\theta(x) > 0$ if $a < |x| < b$ (for some $0 < a < b < \infty$) and $\theta(x) = 0$ otherwise (see Remark 7.6.9 in [1]).

If $\omega(p) \to \infty$ when $|p| \to \infty$ it is useful to introduce the following generalization of the notion of smallness at infinity considered in Definition 1.1. We say that an operator T is *small at infinity with respect to* ω if $T : \mathcal{H}^\omega \to \mathcal{H}^{1/\omega}$ is symmetric and satisfies one of the equivalent conditions:

(1) $\|\xi(Q/r)T\|_{\mathcal{H}^\omega \to \mathcal{H}^{1/\omega^2}} \longrightarrow 0$ as $r \to \infty$;
(2) $\|\xi(Q/r)T\|_{\mathcal{H}^{\omega^2} \to \mathcal{H}^{1/\omega}} \longrightarrow 0$ as $r \to \infty$;
(3) $T : \mathcal{H}^\omega \to \mathcal{H}^{1/\omega^2}$ is compact;
(4) $T : \mathcal{H}^{\omega^2} \to \mathcal{H}^{1/\omega}$ is compact.

Note that the power two which appears above plays no special role: we can replace ω^2 by ω^s with $s > 1$ arbitrary and get an equivalent definition. For example, the fact that (3) implies the compacity of $T : \mathcal{H}^\omega \to \mathcal{H}^{1/\omega^s}$ for $1 < s \leqslant 2$ follows by interpolation, see Subsection 2.7.

Let us prove the equivalence of the preceding four assertions. We set $\varepsilon = 1/r$ and assume $\varepsilon \in (0,1)$. If w is an arbitrary weight then $\|\xi(\varepsilon Q)\|_{B(\mathcal{H}^w)} \leqslant C$ and $\xi(\varepsilon Q)f = 0$ if f has compact support and ε is small, so $\xi(\varepsilon Q) \to 0$ strongly in $B(\mathcal{H}^w)$. Hence (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2). One has (3) $\Leftrightarrow$ (4) by the symmetry of T. It remains to prove (1) $\Rightarrow$ (3), for example. But $1 - \xi \in C_{\mathrm{c}}^\infty(\mathbb{R}^n)$, hence $1 - \xi(\varepsilon Q) : \mathcal{H}^{1/\omega} \to \mathcal{H}^{1/\omega^2}$ is compact, see Subsection 2.7. So $(1 \cdot \xi(\varepsilon Q))T$ are compact operators $\mathcal{H}^\omega \to \mathcal{H}^{1/\omega^2}$ and they converge in norm to T in $B(\mathcal{H}^\omega, \mathcal{H}^{1/\omega^2})$.

We observe that *a short-range operator is small at infinity*. Indeed, we clearly have $\|\xi(Q/r)T\|_{\mathcal{H}^\omega \to \mathcal{H}^{1/\omega}} \to 0$, which is more than needed. In particular, if T is short-range then $T : \mathcal{H}^{\omega^2} \to \mathcal{H}^{1/\omega}$ is compact.

2.10. We shall asume now that there is a continuous function $\omega : \mathbb{R}^n \to [1, \infty)$ verifying (2.18) such that for a finite constant C independent on $p \in \mathbb{R}^n$ one has

$$\omega(p)^2 1_{\mathbf{E}} \leqslant C[1_{\mathbf{E}} + |h(p)|]. \tag{2.22}$$

In other terms, if $\lambda_0(p)$ is the nearest to zero eigenvalue of $h(p)$ then $\omega(p)^2 \leqslant C(1 + |\lambda_0(p)|)$. This implies that $\mathcal{G}$ is embedded continuously in $\mathcal{H}^\omega$, so we have the following scale of spaces:

$$\mathcal{G} \subset \mathcal{H}^\omega \subset \mathcal{H} \subset \mathcal{H}^{1/\omega} \subset \mathcal{G}^*.$$

One may always choose $\omega = 1$ (and in this case $\mathcal{H}^\omega = \mathcal{H} = \mathcal{H}^{1/\omega}$), but the assertions of Theorem 2.17 become stronger if ω is larger. Since $\mathcal{H}^\omega \subset \breve{\mathcal{H}}^\omega_{-1/2,\infty}$ and $\mathcal{H}^{1/\omega}_{1/2,1} \subset \mathcal{H}^{1/\omega}$, one has

$$B(\mathcal{G}^*, \mathcal{G}) \subset B\big(\mathcal{H}^{1/\omega}_{1/2,1}, \mathcal{H}^\omega_{-1/2,\infty}\big).$$

Under suitable hypotheses the resolvent $(H-z)^{-1}$, which does not satisfy a uniform estimate in $B(\mathcal{G}^*, \mathcal{G})$ (or in $B(\mathcal{H})$) when z approaches the spectrum of H, will have boundary values in $B\big(\mathcal{H}^{1/\omega}_{1/2,1}, \mathcal{H}^\omega_{-1/2,\infty}\big)$.

The next theorem is a consequence of the existence of standard conjugate operators, see Theorem 2.10 above, and of Propositions 7.5.6 and 7.5.7 and Theorem 7.5.8 from [1]. The details of the proof are similar to those of Theorem 7.6.8 ([1]) and will not be repeated here. Note, however, that even if h is a scalar function, the next result is much stronger than that of Theorem 7.6.8. For example, the hypoellipticity type conditions imposed on h at page 343 in [1] are no longer needed.

Theorem 2.17. *Assume that h is of class C^2. Let $\omega \geqslant 1$ be a weight function satisfying* (2.22). *Denote by $\mathcal{G}$ the form domain of $H_0 = h(P)$ and let $H : \mathcal{G} \to \mathcal{G}^*$ be a symmetric operator such that $(H + \mathrm{i})\mathcal{G} = \mathcal{G}^*$ and $(H + \mathrm{i})^{-1} - (H_0 + \mathrm{i})^{-1}$ is a compact operator in $\mathcal{H}$. Assume that $H - H_0 = V_{\mathrm{S}} + V_{\mathrm{L}}$, where V_{S} is short-range with respect to ω and V_{L} is long-range with respect to ω. Then:*

(i) *Let $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm\Im z > 0\}$. The holomorphic maps*

$$\mathbb{C}_\pm \ni z \mapsto (H - z)^{-1} \in B\big(\mathcal{H}^{1/\omega}_{1/2,1}, \mathcal{H}^\omega_{-1/2,\infty}\big)$$

extend to weak continuous maps on* $[\mathbb{C}_\pm \cup \Xi(H_0)] \setminus [\tau(H_0) \cup \sigma_p(H)]$, *where* $\sigma_p(H)$ *is the point spectrum of* H.

(ii) *The spectrum of* H *in* $\Xi(H_0) \setminus \tau(H_0)$ *is normal.*

We mention that h need not be C^2, the local Besov class $B_\infty^{1,1}$ suffices.

The hypothesis concerning the compactness of $(H+\mathrm{i})^{-1} - (H_0+\mathrm{i})^{-1}$ can easily be checked in general. For example, *assume that* $\omega(p) \to \infty$ *as* $|p| \to \infty$*; then this condition is satisfied if* V_{L} *is small at infinity with respect to* ω. Indeed, one has in $B(\mathcal{G}^*, \mathcal{G})$

$$(H_0+\mathrm{i})^{-1} - (H+\mathrm{i})^{-1} = (H+\mathrm{i})^{-1}(V_{\mathrm{S}}+V_{\mathrm{L}})(H_0+\mathrm{i})^{-1}$$

so the left-hand side is compact on $\mathcal{H}$ if $V_{\mathrm{S}}+V_{\mathrm{L}}$ is a compact operator $D(H_0) \to \mathcal{G}^*$ (we avoid the use of $D(H)$ because we do not have any information on it). We have $\mathcal{H}^{1/\omega} \subset \mathcal{G}^*$ and from (2.22) it also follows that $D(H_0) \subset \mathcal{H}^{\omega^2}$, hence it suffices that $V_{\mathrm{S}}+V_{\mathrm{L}}$ be a compact operator $\mathcal{H}^{\omega^2} \to \mathcal{H}^{1/\omega}$. The short-range part always has this property (see Subsection 2.9) and the long-range part has it by hypothesis.

As a consequence of Theorems 2.17 and 2.14 we get a result on the existence and completeness of the relative wave operators (use Theorem 7.5.5 from [1]). This is an extension of Theorem 7.6.11 ([1]): indeed, we have eliminated the unnatural condition (ii) from that theorem (without imposing a locality condition).

We denote by E_T^{c} the continuous part of the spectral measure of the self-adjoint operator T.

Theorem 2.18. *Let* H_1 *and* H_2 *be two self-adjoint operators of the same form as* H *in Theorem* 2.17 *(corresponding to the same* H_0*). Assume that the operator* $H_1 - H_2 : \mathcal{G} \to \mathcal{G}^*$ *extends to a bounded operator* $: \mathcal{H}^\omega \to \mathcal{H}^{1/\omega}$ *which is short-range with respect to* ω. *Then the limits*

$$\operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_2}\mathrm{e}^{-\mathrm{i}tH_1} E_{H_1}^{\mathrm{c}}[\Xi(H_0) \setminus \tau(H_0)]$$

exist and their ranges are equal to $E_{H_2}^{\mathrm{c}}[\Xi(H_0) \setminus \tau(H_0)]\mathcal{H}$.

3. LOCALLY SINGULAR PERTURBATIONS

3.1. In this paragraph we shall show that if two self-adjoint operators H and $\widetilde{H}$ acting in $\mathcal{H} = L^2(\mathbb{R}^n; \mathbf{E})$ coincide in some neighbourhood of infinity and if one of them has a certain regularity property with respect to the position observable Q, then the difference of their resolvents is short-range. We first need a "resolvent identity" valid under general assumptions on the domains

Lemma 3.1. *Let* $H, \widetilde{H}$ *be self-adjoint operators and* S *a bounded operator such that* $SD(H) \subset D(H) \cap D(\widetilde{H})$. *For* $z \notin \sigma(H) \cup \sigma(\widetilde{H})$ *set* $R = (H-z)^{-1}$ *and* $\widetilde{R} = (\widetilde{H}-z)^{-1}$. *Then*

$$(\widetilde{R} - R)S = (\widetilde{R} - R)[S, H]R + \widetilde{R}(H - \widetilde{H})SR. \tag{3.1}$$

Proof. Assume first that $\widetilde{H}$ is a bounded operator. By using $RS = R[S,H]R + SR$, we get

$$(\widetilde{R} - R)S = \widetilde{R}(H - \widetilde{H})RS = \widetilde{R}(H - \widetilde{H})R[S,H]R + \widetilde{R}(H - \widetilde{H})SR,$$

hence we have (3.1).

For the general case, choose a sequence of bounded self-adjoint operators $\widetilde{H}_n$ such that $\sigma(\widetilde{H}_n) \subset \sigma(\widetilde{H})$ and $\widetilde{H}_n f \to \widetilde{H} f$ if $f \in D(\widetilde{H})$. Then setting $\widetilde{R}_n = (\widetilde{H}_n - z)^{-1}$, we have $\widetilde{R}_n \to \widetilde{R}$ strongly and for each n:

$$(\widetilde{R}_n - R)S = (\widetilde{R}_n - R)[S,H]R + \widetilde{R}_n(H - \widetilde{H}_n)SR.$$

Since $[S,H]R$ is a bounded operator and SR is also bounded, with range in $D(\widetilde{H})$, we get (3.1) by letting $n \to \infty$ in the last formula. ∎

We isolate the properties of the operator H which are needed for our purposes in

Hypothesis 3.2. H is a self-adjoint operator in $\mathcal{H} = L^2(\mathbb{R}^n; \mathbf{E})$ such that
(i) $\varphi(Q)D(H) \subset D(H)$ for all $\varphi \in C_c^\infty(\mathbb{R}^n)$;
(ii) H has a spectral gap;
(iii) for all $\theta \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ one has

$$\int_1^\infty \mathrm{d}r \big\{ \|[\theta(Q/r), H]\|^2_{D(H)\to\mathcal{H}} + \|[\theta(Q/r), [\theta(Q/r), H]]\|_{D(H)\to\mathcal{H}} \big\} < \infty.$$

Remark 3.3. (a) Condition (i) is fulfilled, for instance, if $H = h(P) + V$, where V is a multiplication operator which is small with respect to $h(P)$ in the operator sense and $h : \mathbb{R}^n \to B(\mathbf{E})$ is of class C^m for some $m \geqslant 2$, satisfies $(\partial^\alpha h)(p)^2 \leqslant C_\alpha(1 + h(p)^2)$ for $|\alpha| < m$, and $\partial^\alpha h$ is bounded for $|\alpha| = m$. So h could be a matrix valued hypoelliptic polynomial.

(b) Condition (iii) is correctly formulated due to (i). It is obviously implied by $H \in C^2(Q; D(H), \mathcal{H})$, which means (by definition) that for all $j, k \in \{1, \ldots, n\}$ the commutators $[Q_j, H]$ and $[Q_j, [Q_k, H]]$ are in $B(D(H), \mathcal{H})$. In fact, in this case the integrand is $\mathrm{O}(r^{-2})$.

If Ω is an open subset of $\mathbb{R}^n$, we define H_Ω as the restriction of H to the subspace $D(H_\Omega) = \{f \in D(H) \mid \operatorname{supp} f \subset \Omega\}$. Note that for $\varphi \in C_c^\infty(\Omega)$ we shall have $\varphi(Q)D(H) \subset D(H_\Omega)$. Observe also that the next lemma is valid without the spectral gap assumption (ii) from Hypothesis 3.2.

Lemma 3.4. *Assume that H satisfies Hypothesis* 3.2 *and let $\widetilde{H}$ be a self-adjoint operator in $\mathcal{H}$ such that $H_\Omega \subset \widetilde{H}$ for some neighbourhood of infinity Ω in $\mathbb{R}^n$. Then for each $z \notin \sigma(H) \cup \sigma(\widetilde{H})$ the operator $\widetilde{R} - R = (\widetilde{H} - z)^{-1} - (H - z)^{-1}$ is short-range in the following sense (see the remark after the Definition* 2.16*): for any $\theta \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$*

$$\int_1^\infty \mathrm{d}r \|\theta(Q/r)(\widetilde{R} - R)\| < \infty. \tag{3.2}$$

Proof. Replacing z by $\bar{z}$ and taking adjoints, we see that it is enough to consider the expression $\|(\widetilde{R}-R)\theta(Q/r)\|$. Moreover, it is clear that we can reduce ourselves to the case when θ is the square of a function with similar properties, so it suffices to estimate $(\widetilde{R}-R)\theta^2(\varepsilon Q)$ for $\varepsilon>0$ small.

In Lemma 3.1 we take $S=\theta(\varepsilon Q)$ and, by using $H\theta(\varepsilon Q)f=\widetilde{H}\theta(\varepsilon Q)f$ if ε is small and $f\in D(H)$, we get

$$(\widetilde{R}-R)\theta(\varepsilon Q)=(\widetilde{R}-R)[\theta(\varepsilon Q),H]R. \tag{3.3}$$

We multiply this to the right by $\theta(\varepsilon Q)$ and use

$$R\theta(\varepsilon Q)=(\widetilde{R}-R)[\theta(\varepsilon Q),H]R+\theta(\varepsilon Q)R$$

to get

$$\begin{aligned}(\widetilde{R}-R)\theta^2(\varepsilon Q)&=(\widetilde{R}-R)[\theta(\varepsilon Q),H]R[\theta(\varepsilon Q),H]R\\&\quad+(\widetilde{R}-R)[[\theta(\varepsilon Q),H],\theta(\varepsilon Q)]R+(\widetilde{R}-R)\theta(\varepsilon Q)[\theta(\varepsilon Q),H]R.\end{aligned}$$

Finally, using once again (3.3), we obtain

$$(\widetilde{R}-R)\theta^2(\varepsilon Q)=2(\widetilde{R}-R)[\theta(\varepsilon Q),H]R[\theta(\varepsilon Q),H]R-(\widetilde{R}-R)[\theta(\varepsilon Q),[\theta(\varepsilon Q),H]]R$$

for all small ε. This clearly implies the assertion of the lemma. ∎

Remark 3.5. The relation (3.2) suffices for Theorem 3.6, where the A-regularity of $\widetilde{H}$ is deduced from the corresponding property of H. But the idea of the preceding proof can be used in order to prove other useful estimates. For example, if $H\in C^\infty(Q;D(H),\mathcal{H})$ (i.e. all the succesive commutators $\mathrm{ad}_Q^a(H)$, $a\in\mathbb{N}^n$, are in $B(D(H),\mathcal{H})$), then $\widetilde{R}-R$ will decay rapidly at infinity: $\langle Q\rangle^{m_1}(\widetilde{R}-R)\langle Q\rangle^{m_2}$ is bounded for all $m_1,m_2\in\mathbb{N}$. In fact, this stronger conclusion will effectively hold for Dirac operators with potentials which are multiplication operators by matrix valued functions.

3.2. We recall that a self-adjoint realization A of an expression of the form $\frac{1}{2}\{F(P)\cdot Q+Q\cdot F(P)\}$ with $F:\mathbb{R}^n\to\mathbb{R}^n$ a C^∞-function with compact support was called *a standard (conjugate) operator.*

Theorem 3.6. *Let H be a self-adjoint operator in $\mathcal{H}$ satisfying Hypothesis* 3.2. *Assume that there is a standard operator A such that H is A-regular (see* (2.17)) *and A is locally conjugate to H on an open real set J. Let $\widetilde{H}$ be a second self-adjoint operator in $\mathcal{H}$ such that $H_\Omega\subset\widetilde{H}$ for some neighbourhood of infinity Ω and such that $(\widetilde{H}+\mathrm{i})^{-1}-(H+\mathrm{i})^{-1}$ is a compact operator. Then:*

(i) *The spectrum of $\widetilde{H}$ in J is normal.*

(ii) *Let E^{c}, $\widetilde{E}^{\mathrm{c}}$ the continuous components of the spectral measures of H and $\widetilde{H}$ respectively. Then the wave operators*

$$\operatorname*{s-lim}_{t\to\pm\infty}\mathrm{e}^{\mathrm{i}t\widetilde{H}}\mathrm{e}^{-\mathrm{i}tH}E^{\mathrm{c}}(J)$$

exist and their ranges are equal to $\widetilde{E}^{\mathrm{c}}(J)\mathcal{H}$.

(iii) *Let* $\widetilde{J}_0 = J \setminus \sigma_p(\widetilde{H})$. *Then the limits* $\lim_{\varepsilon \searrow 0}(H - \lambda \mp \mathrm{i}\varepsilon)^{-1}$ *exist in the weak* topology of* $B\left(\mathcal{H}_{1/2,1}, \mathcal{H}_{-1/2,\infty}\right)$, *locally uniformly in* $\lambda \in \widetilde{J}_0$.

Proof. Since the operators H and $\widetilde{H}$ have the same essential spectrum and H has a spectral gap, they have a common spectral gap, so there is $z \in \mathbb{R} \setminus \sigma(H) \cup \sigma(\widetilde{H})$. We set $R = (H - z)^{-1}$ and $\widetilde{R} = (\widetilde{H} - z)^{-1}$, so $R, \widetilde{R}$ are bounded, self-adjoint operators and $\widetilde{R} - R$ is compact and short-range in the sense described in Lemma 3.4. If we apply Theorem 7.5.8 from [1] with $\mathcal{G} = \mathcal{H}$ and $\Lambda = \langle Q \rangle$ we get that $\widetilde{R} - R$ is A-regular in the following sense:

$$\int_0^1 \frac{\mathrm{d}\tau}{\tau^2} \left\| \mathrm{e}^{\mathrm{i}\tau A}(\widetilde{R} - R)\mathrm{e}^{-\mathrm{i}\tau A} + \mathrm{e}^{-\mathrm{i}\tau A}(\widetilde{R} - R)\mathrm{e}^{\mathrm{i}\tau A} - 2(\widetilde{R} - R) \right\| < \infty.$$

This immediatly implies that $\widetilde{H}$ is of class $\mathcal{C}^{1,1}$ with respect to A (see Definition 6.2.2 in [1]). So we may apply Theorem 7.2.9 from [1] and deduce that A is locally conjugate to $\widetilde{H}$ on J.

Now (i) and (iii) follow from Theorems 7.4.1 and 7.4.2 from [1] (remark that the domain of $\langle Q \rangle$ is contained in $D(A)$). Finally, (ii) is an immediate consequence of Theorem 7.4.3 from [1] because of Theorem 2.14 (set $\omega_1 = \omega_2 = 1$ and $S = \widetilde{R} - R$) and of Lemma 3.4. ∎

3.3. There is one non-explicit assumption in Theorem 3.6, namely the condition that $(\widetilde{H} + \mathrm{i})^{-1} - (H + \mathrm{i})^{-1}$ be a compact operator. We shall check this property with the help of the next lemma.

We first recall the notion of local compactness.

Definition 3.7. Let S be a closed operator in $\mathcal{H}$ and $D(S)$ its domain equipped with the graph-norm. We say that S is *locally compact* if $\theta(Q) : D(S) \to \mathcal{H}$ is a compact operator for each $\theta \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$.

If S is self-adjoint, this may be reformulated in the following way: for each $\theta \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ the operator $\theta(Q)(S + \mathrm{i})^{-1}$ is compact in $\mathcal{H}$. Local compactness of S must be thought as a local (in Q-space) regularity condition on the domain of S (recall the Riesz-Kolmogorov compacity criterion). For exemple, if there is $s > 0$ such that $D(S) \subset \mathcal{H}^s_{\mathrm{loc}}$, then S is locally compact.

Lemma 3.8. *Let H be a locally compact self-adjoint operator such that, for each $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$, $\varphi(Q)D(H) \subset D(H)$. Let $\widetilde{H}$ be a self-adjoint operator in $\mathcal{H}$ such that $H_\Omega \subset \widetilde{H}$ for some neighbourhood Ω of infinity. Assume that $[\theta(Q), H](H + \mathrm{i})^{-1}$ is a compact operator if $\theta \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$. Then $(\widetilde{H} + \mathrm{i})^{-1} - (H + \mathrm{i})^{-1}$ is compact if and only if $\widetilde{H}$ is locally compact.*

Proof. If $\xi \in C^{\infty}(\mathbb{R}^n)$, supp $\xi \subset \Omega$ and $\xi(x) = 1$ on a neighbourhood of infinity, then $\xi(Q)D(H) \subset D(H_\Omega) \subset D(\widetilde{H})$, so we may use Lemma 3.1 and get

$$(\widetilde{R} - R)\xi(Q) = (\widetilde{R} - R)[\xi(Q), H]R.$$

Here $R = (H + \mathrm{i})^{-1}$ and $\widetilde{R} = (\widetilde{H} + \mathrm{i})^{-1}$. This clearly implies the assertion of the lemma. ∎

In applications, it will be usually easy to check that H is locally compact and $[\theta(Q), H](H+\mathrm{i})^{-1}$ compact. In consequence, we will be left with proving the local compactness of $\widetilde{H}$.

4. DIRAC OPERATORS WITH HIGHLY SINGULAR POTENTIALS

4.1. Let $H_0 = h(P)$ be the free Dirac operator defined in (1.2). One clearly has $H_0^2 = P^2 + m^2$, so $|H_0| = (H_0^2)^{1/2} = (P^2+m^2)^{1/2}$. In particular this shows that the domain of H_0 is the Sobolev space $\mathcal{H}^1$ and its form domain is $\mathcal{H}^{1/2}$. Then the spectral projections of H_0 associated with the positive and negative half axis are given by

$$\Pi_\pm = \frac{1}{2}\left[1 \pm \operatorname{sign}(H_0)\right] = \frac{1}{2}\left[1 \pm \frac{H_0}{|H_0|}\right] = \frac{1}{2}\left[1 \pm \frac{\alpha \cdot P + m\beta}{\sqrt{P^2+m^2}}\right].$$

These are orthogonal projections in $\mathcal{H}$ satisfying $\Pi_+\Pi_- = 0, \Pi_+ + \Pi_- = 1$ and $H_0 = |H_0|(\Pi_+ - \Pi_-)$. In particular $\sigma(H_0) = (-\infty, -m] \cup [m, \infty)$.

All these operators are functions of P. Define $\mu_m : \mathbb{R}^n \to \mathbb{R}$ by $\mu_m(p) = (p^2+m^2)^{1/2}$ and $\pi_\pm : \mathbb{R}^n \to B(\mathbf{E})$ by

$$\pi_\pm(p) = \frac{1}{2}\left[1 \pm \frac{h(p)}{|h(p)|}\right] = \frac{1}{2}\left[1 \pm \frac{\alpha \cdot p + m\beta}{\sqrt{p^2+m^2}}\right].$$

Then $\pi_\pm(p)$ are orthogonal projections in $\mathbf{E}$ such that $\pi_+(p)\pi_-(p) = 0$, $\pi_+(p) + \pi_-(p) = 1$, hence the identity $h(p) = \mu_m(p)(\pi_+(p) - \pi_-(p))$ gives the spectral decomposition of the operator $h(p)$ on $\mathbf{E}$. Clearly $|H_0| = \mu_m(P)$, $\Pi_\pm = \pi_\pm(P)$.

We have $\sigma(h(p)) = \{-\mu_m(p), \mu_m(p)\}$, so $\delta[h(p)] = 2\mu_m(p) \geqslant 2m > 0$, hence (2.8) is satisfied. It follows that *the free Dirac operator H_0 is locally scalar*. More precisely, for any open set I strictly contained in $(-m,\infty)$ one must define $\mu : \Omega(I) \to \mathbb{R}$ by $\mu(p) = \mu_m(p) = (p^2+m^2)^{1/2}$ in order to have (2.2). It is convenient to choose I strictly greater than $[m,\infty)$; in this case we have $\Omega(I) = \mathbb{R}^3$. But in any case, it is natural to define $\mu(p)$ by the formula above even for $p \in \mathbb{R}^3 \setminus \Omega(I)$ when this set is not void. There are analogous statements for I strictly contained in $(-\infty, m)$; just set $\mu = -\mu_m$. Obviously, the threshold set for the Dirac operator is $\tau(H_0) = \{-m, m\}$.

We notice that in [4] the choice for the conjugate operator was $\widetilde{A} = \Pi_+ A \Pi_+ + \Pi_- A \Pi_-$, where A is a standard operator. This is unnecessarily complicated and, even for the more restricted class of potentials considered in [4], it leads to some limitations (see the discussion in Subsection 1.4).

4.2. *Proof of Theorem* 1.6. Since we know that H_0 is locally scalar and $\tau(H_0) = \{\pm m\}$, we shall apply Theorem 2.17 with $\omega(p) = \langle p \rangle^{1/2}$. We are left only with proving that $(H+\mathrm{i})^{-1} - (H_0+\mathrm{i})^{-1}$ is a compact operator in $\mathcal{H}$. But this has been done in a more general context in Subsection 2.10. ∎

4.3. *Proof of Theorem* 1.8. We shall use Theorem 3.6 and Lemma 3.8. Let us first show that H satisfies Hypothesis 3.2. Since $(H+\mathrm{i})^{-1}-(H_0+\mathrm{i})^{-1}$ is a compact operator in $\mathcal{H}$ and $(-m,m)\cap\sigma(H_0)=\emptyset$, obviously H has a spectral gap. One has $[\varphi(Q),H]=[\varphi(Q),H_0]=\mathrm{i}\alpha\cdot(\nabla\varphi)(Q)$; the higher order commutators of H with multiplication operators are all zero. This gives immediatly the invariance of $D(H)$ under $\varphi(Q)$ for any $\varphi\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^3)$ (recall that $D(H)=\{f\in\mathcal{H}^{1/2}\mid(H_0+V)f\in\mathcal{H}\}$), as well as the validity of condition (iii) in Hypothesis 3.2. Now, for some standard operator A, we have shown that H is A-regular and A is locally conjugate to H on $\mathbb{R}\setminus\{\pm m\}$. The compacity of $(\widetilde{H}+\mathrm{i})^{-1}-(H+\mathrm{i})^{-1}$ follows from Lemma 3.8, since H is obviously locally compact and $[\theta(Q),H](H+\mathrm{i})^{-1}=\mathrm{i}\alpha\cdot(\nabla\theta)(Q)(H+\mathrm{i})^{-1}$ is a compact operator for any $\theta\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^3)$. ∎

Remark 4.1. Let us place ourselves in the framework of Theorem 1.8, but assume that the function V is more regular, namely that there exists $s>1/2$ such that $V=\sum_{0\leqslant k<s+1/2}V_k$ with V_k functions satisfying (1.4). Then the boundary values $(\widetilde{H}-\lambda\mp\mathrm{i}0)^{-1}$ have regularity properties (as functions of $\lambda\in\mathcal{R}(\widetilde{H})$) similar to those expressed in Theorem 1.7, namely $\lambda\to(\widetilde{H}-\lambda\mp\mathrm{i}0)^{-1}\in B(\mathcal{H}_{s,\infty},\mathcal{H}_{-s,1})$ is locally of Hölder-Zygmund class $\Lambda^{s-1/2}$. This follows from the fact that under the hypothesis of Theorem 1.8 the operator $\widetilde{R}-R$ is rapidly decaying at infinity (see Remark 3.5), so that for any $s\geqslant 0$, $\widetilde{R}\in\mathcal{C}^s(A)$ if $R\in\mathcal{C}^s(A)$ and A is standard.

4.4. In this paragraph we shall make some considerations concerning the local compactness of some self-adjoint realizations of H_0+U, where $H_0=\alpha\cdot P+m\beta$ and U is the operator of multiplication by some singular function. We restrict ourselves to the three-dimensional case, but generalizations to any dimension or to more complicated "free hamiltonians" H_0 are straightforward.

Let $U:\mathbb{R}^3\to B(\mathbf{E})$ be a symmetric matrix valued Borel application such that $U\mathcal{H}^1_{\mathrm{loc}}\subset\mathcal{H}_{\mathrm{loc}}$. This is satisfied if U is locally in the weak Lebesgue space L^3_{weak} (use Theorem II.3.6 from [17]). In particular, one may take

$$U(x)=U^0(x)+\sum_{a\in\Gamma}\frac{u_a(x)}{|x-a|},\tag{4.1}$$

where $U^0\in L^3_{\mathrm{loc}}(\mathbb{R}^3;B(\mathbf{E}))$, $\Gamma\subset\mathbb{R}^3$ is a finite set and $u_a\in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^3;B(\mathbf{E}))$.

Due to the assumption $U\mathcal{H}^1_{\mathrm{loc}}\subset\mathcal{H}_{\mathrm{loc}}$ the operator H_0+U is a well-defined map $\mathcal{H}_{\mathrm{loc}}\to\mathcal{H}^{-1}_{\mathrm{loc}}$ in a distributional sense. We shall consider the operator H_0+U initially defined on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^3;\mathbf{E})$ and we shall associate to it:

(1) *The minimal operator* $H_{\min}=$ closure of H_0+U, with domain $\mathcal{D}_{\min}$,
(2) *The maximal operator* $H_{\max}=H^*_{\min}$, with domain $\mathcal{D}_{\max}$.

So H_0+U is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^3,\mathbf{E})$ if and only if $H_{\min}=H_{\max}$. We remark that $\mathcal{D}_{\min}$ contains $\mathcal{H}^1_{\mathrm{c}}$ (approximation by mollification). One has $\mathcal{D}_{\max}=\{f\in\mathcal{H}\mid H_0f+Uf\in\mathcal{H}\}$ and $H_{\max}f=H_0f+Uf$ for any $f\in\mathcal{D}_{\max}$. We equip $\mathcal{D}_{\max}$ with the graph-norm $(\|f\|^2+\|(H_0+U)f\|^2)^{1/2}$.

For any open subset $\mathcal{O}$ of $\mathbb{R}^3$ and any vector subspace $\mathcal{K}$ of $\mathcal{H}$ we set $\mathcal{K}_{\mathrm{c}}(\mathcal{O})=\{f\in\mathcal{K}\mid\operatorname{supp}f$ is a compact subset of $\mathcal{O}\}$. We abbreviate $\mathcal{K}_{\mathrm{c}}=\mathcal{K}_{\mathrm{c}}(\mathbb{R}^3)$.

Definition 4.2 U is *regular* on the open set $\mathcal{O} \subset \mathbb{R}^3$ if $\mathcal{D}_{\max,c}(\mathcal{O}) = \mathcal{H}^1_c(\mathcal{O})$. U is *locally regular* on the open set $\Omega \subset \mathbb{R}^3$ if each $x \in \Omega$ has an open neighbourhood $\mathcal{O}$ such that U is regular on $\mathcal{O}$.

More explicitly, U is regular on $\mathcal{O}$ if each $f \in \mathcal{H}$, whose support is a compact subset of $\mathcal{O}$ and has the property $(H_0 + U)f \in \mathcal{H}$, belongs to $\mathcal{H}^1$. By using a partition of unity, one easily shows that if U is locally regular on Ω it is regular on any open set whose closure is a compact subset of Ω. The following result can be proved by standard methods, so we skip the details.

Lemma 4.3. *For each open set $\Omega \subset \mathbb{R}^3$ consider the assertions:*
(i) $U \in L^3_{\mathrm{loc}}(\Omega; B(\mathbf{E}))$*;*
(ii) *for each $x \in \Omega$, there is a neighbourhood $\mathcal{O}$ of x and there are numbers $0 \leqslant a < 1$, $b > 0$ such that for all $f \in C^\infty_c(\mathcal{O}; \mathbf{E})$*

$$\|Uf\| \leqslant a\|H_0 f\| + b\|f\|;$$

(iii) *U is locally regular on $\mathcal{O}$.*
Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (ii). *If $\Omega = \mathbb{R}^3$ then* (iii) $\Rightarrow H_{\min} = H_{\max}$.

Obviously, local regularity of U on $\mathbb{R}^3$ implies that $H_{\max}$ is locally compact. But a more refined notion is useful:

Definition 4.4. U is *quasi-regular* on the open set $\mathcal{O} \subset \mathbb{R}^3$ if the unit ball of $\mathcal{D}_{\max,c}(\mathcal{O})$ is a relatively compact subset of $\mathcal{H}$. U is *locally quasi-regular* on the open set $\Omega \subset \mathbb{R}^3$ if each $x \in \Omega$ has an open neighbourhood $\mathcal{O}$ such that U is quasi-regular on $\mathcal{O}$.

Of course,"locally regular" implies "locally quasi-regular". Once again, local quasi-regularity on Ω implies quasi-regularity on every open, relatively compact subset of Ω. The important, although easy, fact is:

Lemma 4.5. *If U is locally quasi-regular on $\mathbb{R}^3$, then each self-adjoint extension of $H_{\min}$ is locally compact.*

Proof. Since any closed restriction of a closed, locally compact operator is locally compact, we only need to show that $H_{\max}$ is locally compact. But this is obvious, since for any $\theta \in C^\infty_c(\mathbb{R}^3)U$ is quasi-regular on an open set containing $\operatorname{supp}\theta$. ∎

Remark 4.6. We want to stress the point which makes the preceding considerations useful: for each open set $\mathcal{O} \subset \mathbb{R}^3$ the space $\mathcal{D}_{\max,c}(\mathcal{O})$ depends only on the restriction of U to $\mathcal{O}$. If we replace $\mathcal{D}_{\max}$ by the domain of a self-adjoint extension of $H_{\min}$ defined by some kind of non-local boundary conditions, this property might fail. In the next paragraph we shall use quasi-regularity to treat potentials having several local singularities such that each singularity can be controlled separately.

4.5. We apply now Subsection 4.4 to a potential U which has essentially the form (4.1), with some additional conditions on the function u_a. We shall use the results of [20], from which we extract the following:

LEMMA 4.7. *Let* $H = H_0 + W(Q) + \chi_\varepsilon(Q)u(|Q|)|Q|^{-1}$, *where* χ_ε *is the characteristic function of the ball of radious* ε *centered in the origin,* u *is a continuous function which has a finite limit in* 0 *and* W *is a bounded Borel function:* $\mathbb{R}^3 \to B(\mathbf{E})$. *Then the operator* H *on* $C_c^\infty(\mathbb{R}^3; \mathbf{E})$ *has finite defect indices and for any self-adjoint extension* H_1 *of* H, *the operator* $(H_1 + \mathrm{i})^{-1} - (H_0 + W + \mathrm{i})^{-1}$ *is Hilbert-Schmidt.*

The results of Xia are more precise, but they do not cover the multi-center potentials. But one may exploit Lemma 4.7 and the notion of quasi-regularity to prove

PROPOSITION 4.8. *Let* $\Gamma \subset \mathbb{R}^3$ *be a finite set and assume that for each* $a \in \Gamma$ *there exists* $\varepsilon > 0$ *such that if* $|x-a| < \varepsilon$ *one has* $U(x) = u_a(|x-a|)|x-a|^{-1} + v_a(x)$, *where* v_a *is a bounded matrix valued function and* $u_a : (0, \varepsilon) \to \mathbb{R}$ *is a continuous function such that* $u_a(0^+) = \lim_{r \searrow 0} u_a(r)$ *exists. Moreover, assume that* U *is locally regular on* $\mathbb{R}^3 \setminus \Gamma$. *Then any self-adjoint extension* $\widetilde{H}$ *of* $H_{\min}$ *is locally compact.*

Proof. By Lemma 4.5, it will be enough to show that U is quasi-regular on $B(a;\varepsilon) = \{x \in \mathbb{R}^3 \,\big|\, |x-a| < \varepsilon\}$. By a translation, we reduce ourselves to the case $a = 0$. Since quasi-regularity on $B(a;\varepsilon)$ depends only on $U|B(a;\varepsilon)$ (cf. the definition of $\mathcal{D}_{\max}$), we only need to show that the potential considered in Lemma 4.7 is quasi-regular on $B(a;\varepsilon)$. But $H_0 + W$ is locally compact, hence H_1 is also locally compact, by Lemma 4.7. It remains to use the fact that the defect indices are finite. ∎

REMARK 4.9. For some special classes of matrix valued potentials one might be able to construct self-adjoint realizations with domain contained in a local Sobolev space $\mathcal{H}^s_{\mathrm{loc}}$ with $s > 0$. In this case, the local compactness is obvious and our method works. For example, the self-adjoint realizations constructed in [19] are covered by our formalism.

4.6. *Proof of Theorem* 1.9. From conditions (i) and (ii) of Theorem 1.9 and by using Lemma 4.3 and Proposition 4.8 we conclude that $\widetilde{H}$ is locally compact.

Let us fix an open neighbourhood Ω of infinity in $\mathbb{R}^n$ whose closure does not intersect Γ. Let θ be a C^∞-function with support disjoint of Γ, equal to 1 on Ω and set $V = \theta U$. Then $H = H_0 + V$ satisfies Hypothesis 1.2. Since the maximal operator $H_{\max}$ described in Subsection 4.4 is local and V and U coincide on Ω, it is easy to see that if $\operatorname{supp} f \subset \Omega$ then $f \in D(H)$ if and only if $f \in D(\widetilde{H})$ and $Hf = \widetilde{H}f$. Therefore one may apply Theorem 1.8 to get the result. ∎

4.7. We shall prove here that $H_{\min}$ has self-adjoint extensions if U is the sum of an electrostatic potential, a scalar one, and an arbitrary bounded potential. For another result of this type, see [2].

Proposition 4.10. *Let* $U : \mathbb{R}^3 \to B(\mathbf{E})$ *be a locally square integrable symmetric operator valued function. Denote by* $H_{\min}$ *the closure of the symmetric operator* $H_0 + U$ *with domain* $C_c^\infty(\mathbb{R}^3; \mathbf{E})$. *Assume that* $U = U_1 + \beta U_2 + U_3$ *where* U_1, U_2 *are real valued functions and* U_3 *is a bounded matrix valued function. Then* $H_{\min}$ *has self-adjoint extensions.*

Proof. Since any representation of the relations (1.1) is unitarily equivalent to a direct sum of Majorana representations (cf. page 36 in [18]), we may assume that $\mathbf{E} = \mathbb{C}^4$ (equipped with the usual conjugation operation) and that the matrices α_k are real if $1 \leqslant k \leqslant 3$ while $\alpha_0 = \beta$ is purely imaginary: $\overline{\alpha}_k = \alpha_k$, $\overline{\beta} = -\beta$. If we set $\gamma = \mathrm{i}\alpha_0\alpha_1\alpha_2\alpha_3$ then $\overline{\gamma} = \gamma$, $\gamma^* = -\gamma$, $\gamma^2 = -1$. Moreover, we have $\gamma\alpha_k = -\alpha_k\gamma$ for $0 \leqslant k \leqslant 3$. Now we define $J : \mathcal{H} \to \mathcal{H}$ by $(Jf)(x) = \gamma\overline{f(x)}$. Clearly, J is antilinear, $J^2 = -1$ and $\langle Jf, g\rangle = -\langle Jg, f\rangle$ for all f, g. Then we have $\alpha \cdot PJg = J\alpha \cdot Pg$ and $(U_1 + \beta U_2)Jg = J(U_1 + \beta U_2)g$ if $g \in C_c^\infty$. Denote by S the symmetric operator $\alpha \cdot P + U_1 + \beta U_2$ with domain $C_c^\infty(\mathbb{R}^3; \mathbf{E})$. If $f \in D(S^*)$ and $g \in C_c^\infty(\mathbb{R}^3; \mathbf{E})$ we have

$$\langle Jf, Sg\rangle = -\langle JSg, f\rangle = -\langle SJg, f\rangle = -\langle Jg, S^* f\rangle = \langle JS^* f, g\rangle,$$

hence $JS^* \subset S^* J$. Finally, we may use the von Neumann criterion for the equality of the defect indices. More precisely, if $S^* f = \pm\mathrm{i}f$, then clearly $S^* Jf = \mp\mathrm{i}Jf$, hence J is a bijective map of $\ker(S^* - \mathrm{i})$ onto $\ker(S^* + \mathrm{i})$. Since $H_{\min}$ differs from the closure of S by a bounded operator, the assertion of the proposition is proved. ∎

We mention several extensions of the preceding proposition which can be proved by the same argument. First, it suffices that U be locally square integrable outside a closed set of measure zero M: it suffices to replace $C_c^\infty(\mathbb{R}^3; \mathbf{E})$ by $C_c^\infty(\mathbb{R}^3 \setminus M; \mathbf{E})$. Then, β can be replaced by any matrix δ which has, in the preceding Majorana representation, the property $\delta\gamma = \gamma\overline{\delta}$ (for example $\delta = \gamma_5$, cf. page 37 in [18]). Finally, it suffices that the operator bound of U_3 with respect to $H_0 + U_1 + \beta U_2$ be strictly less than 1.

Acknowledgements. Part of this work was done while the first author was visiting the Department of Physics of the University of Geneva. Both authors are indebted to the Swiss National Science Foundation for financial support, and V. Georgescu thanks Werner Amrein for his kind hospitality.

REFERENCES

1. W. Amrein, A. Boutet de Monvel, V. Georgescu, *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser Verlag, Basel 1996.
2. M. Arai, On essential self-adjointness, distinguished selfadjoint extensions and essential spectrum of Dirac operators with matrix valued potentials, *Publ. Res. Inst. Math. Sci.* **19**(1983), 33–57.
3. M.M. Arai, O. Yamada, Essential self-adjointness and invariance of the essential spectrum for Dirac operators, *Publ. Res. Inst. Math. Sci.* **18**(1982), 973–985.

4. A. Boutet de Monvel, D. Manda, R. Purice, Limiting absorption for the Dirac operator, *Ann. Inst. H. Poincaré Phys. Théor.* **58**(1993), 413–431.
5. A. Boutet de Monvel, V. Georgescu, J. Sahbani, Boundary values of regular resolvent families, *Helv. Phys. Acta* **71**(1998), 518–553.
6. V. Georgescu, M. Măntoiu, Asymptotic velocity and conjugate operators for matrix valued hamiltonians, in preparation.
7. C. Gérard, F. Nier, The Mourre theory for analytically fibered operators, *J. Funct. Anal.* **152**(1998), 202–219.
8. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. II, Springer, Berlin 1983.
9. M. Klaus, Dirac operators with several Coulomb singularities, *Helv. Phys. Acta* **53**(1980), 463–482.
10. T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin 1966.
11. J. Landgren, P. Rejtö, An aplication of the maximum principle to the study of essential self-adjointness of dirac operators. I, *J. Math. Phys.* **l20**(1979), 2204–2211.
12. J. Landgren, P. Rejtö, M. Klaus, An aplication of the maximum principle to the study of essential self-adjointness of Dirac operators. II, *J. Math. Phys.* **21**(1980), 1210–1217.
13. E. Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, *Comm. Math. Phys.* **78**(1981), 391–408.
14. G. Nenciu, Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms, *Comm. Math. Phys.* **48**(1976), 235–247.
15. G. Nenciu, Distinguished self-adjoint extensions for Dirac operators with potentials dominated by multicenter Coulomb potentials, *Helv. Phys. Acta* **50**(1977), 1–3.
16. J. Sahbani, Propagation theorems for some classes of pseudo-differential operators, *J. Math. Anal. Appl.* **211**(1997), 481–497.
17. R.S. Strichartz, Multipliers on fractional Sobolev spaces, *J. Math. Mech.* **16**(1967), 1031–1060.
18. B. Thaller, *The Dirac Equation*, Texts and Monographs in Physics, Springer-Verlag, Berlin–Heidelberg 1992.
19. V. Vogelsang, Self-adjoint extension of Dirac operators for non spherically symmetric potentials in Coulomb scattering, *Integral Equations Operator Theory* **10**(1987), 841–858.
20. J. Xia, On the contribution to the Coulomb singularity of arbitrary charge to the Dirac hamiltonian, *Trans. Amer. Math.Soc.* **351**(1999), 1989–2023.

V. GEORGESCU
Département de Mathématiques
Université de Cergy-Pontoise
2 avenue Adolphe Chauvin
95302 Cergy-Pontoise Cedex
FRANCE

E-mail: vlad@oara.pst.u-cergy.fr

M. MĂNTOIU
Institute of Mathematics
of the Romanian Academy
P.O. Box 1-764
RO-70700 Bucharest
ROMANIA

Current address:

Département de Physique Théorique
Université de Genève
24 Quai Ernest Ansermet
CH–1211 Genève 4
SUISSE
E-mail: marius.mantoiu@physics.unige.ch

Received February 3, 1999.