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TENSOR PRODUCTS AND THE SEMI-BROWDER JOINT SPECTRA

ENRICO BOASSO

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Abstract. Given two complex Banach spaces X_1 and X_2, a tensor product of X_1 and X_2, $X_1 \widetilde{\otimes} X_2$, in the sense of J. Eschmeier ([5]), and two finite tuples of commuting operators, $S = (S_1, \dots, S_n)$ and $T = (T_1, \dots, T_m)$, defined on X_1 and X_2 respectively, we consider the $(n+m)$-tuple of operators defined on $X_1 \widetilde{\otimes} X_2$, $(S \otimes I, I \otimes T) = (S_1 \otimes I, \dots, S_n \otimes I, I \otimes T_1, \dots, I \otimes T_m)$, and we give a description of the semi-Browder joint spectra introduced by V. Kordula, V. Müller and V. Rakočević in [7] and of the split semi-Browder joint spectra (see Section 3) of the $(n+m)$-tuple $(S \otimes I, I \otimes T)$, in terms of the corresponding joint spectra of S and T. This result is in some sense a generalization of a formula obtained for other various Browder spectra in Hilbert spaces and for tensor products of operators and for tuples of the form $(S \otimes I, I \otimes T)$. In addition, we also describe all the mentioned joint spectra for a tuple of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5].

Keywords: *Semi-Fredholm, semi-Browder and split joint spectra.*

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1. INTRODUCTION

Given a complex Banach space X, V. Kordula, V. Müller and V. Rakočević extended in [7] the notion of upper and lower semi-Browder spectrum of an operator to n-tuples of commuting operators, and they proved the main spectral properties for this joint spectra, i.e., the compactness, nonemptiness, the projection property and the spectral mapping property.

On the other hand, there are many other joint Browder spectra, for example, we may consider the one introduced by R.E. Curto and A.T. Dash in [2], σ_{b}, and the joint Browder spectra defined by A.T. Dash in [3], σ_{b}^1, σ_{b}^2 and σ_{b}^T. By the observation which follows Definition 4 in [3] and the Example in [7], we have that the Browder spectra of V. Kordula, V. Müller and V. Rakočević, $\sigma_{\mathcal{B}_+}$ and $\sigma_{\mathcal{B}_-}$, differ, in general, from the other mentioned joint Browder spectra. However, if we consider two complex Hilbert spaces H_1 and H_2, and S and T two operators defined on H_1 and H_2 respectively, by [2] and by [3;7] we have that the joint Browder spectra σ_{b}, σ_{b}^1, σ_{b}^2 and σ_{b}^T of the tuple of operators $(S\otimes I, I\otimes T)$ defined on $H_1\overline{\otimes}H_2$, coincide with the set

$$\sigma_{\mathrm{b}}(S)\times\sigma(T)\cup\sigma(S)\times\sigma_{\mathrm{b}}(T),$$

where σ and σ_{b} denote, respectively, the usual and the Browder spectrum of an operator.

Moreover, if $S=(S_1,\ldots,S_n)$, respectively $T=(T_1,\ldots,T_m)$, is an n-tuple, respectively an m-tuple, of commuting operators defined on the Hilbert space H_1, respectively H_2, R.E. Curto and A.T. Dash computed in [2] the Browder spectum of the $(n+m)$-tuple $(S\otimes I, I\otimes T)=(S_1\otimes I,\ldots,S_n\otimes I, I\otimes T_1,\ldots,I\otimes T_m)$, and they obtained the formula

$$\sigma_{\mathrm{b}}(S\otimes I, I\otimes T)=\sigma_{\mathrm{b}}(S)\times\sigma_{\mathrm{T}}(T)\cup\sigma_{\mathrm{T}}(T)\times\sigma_{\mathrm{b}}(T),$$

where σ_{T} denotes the Taylor joint spectrum (see [9]).

In this article we give in some sense a generalization of the above formulas for commutative tuples of Banach spaces operators and for the semi-Browder joint spectra. Indeed, we consider two complex Banach spaces, X_1 and X_2, a tensor product between X_1 and X_2 in the sense of J. Eschmeier ([5]) $X_1\widetilde{\otimes}X_2$, S and T, two commuting tuples of Banach space operators defined on X_1 and X_2 respectively, and we describe the semi-Browder joint spectra introduced in [7], $\sigma_{\mathcal{B}_+}$ and $\sigma_{\mathcal{B}_-}$, and the split semi-Browder joint spectra $\mathrm{sp}_{\mathcal{B}_+}$ and $\mathrm{sp}_{\mathcal{B}_-}$ (see Section 3) of the tuple $(S\otimes I, I\otimes T)$, in terms of the corresponding semi-Browder joint spectra and of the defect and the approximate point spectra of S and T. The results that we have obtained extend in same way the above formulas, see Section 5. Furthermore, since for our objective we need to know the Fredholm joint spectra of J.J. Buoni, R. Harte and T. Wickstead of $(S\otimes I, I\otimes T)$ ([1]) and its split versions ([4]) we also describe in Section 4 these joint spectra.

In addition, by similar arguments we describe in Section 6 all the mentioned joint spectra for a tuple of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5].

However, in order to give our descriptions, we need to introduce the split semi-Browder joint spectra of a tuple of commuting Banach space operators, and to prove their main spectral properties (see Section 3).

The article is organized as follows. In Section 2 we recall several definitions and results which we need for our work. In Section 3 we introduce the split semi-Browder joint spectra and prove their main spectral properties. In Section 4 we compute the semi-Fredholm joint spectra of $(S\otimes I, I\otimes T)$. In Section 5 we compute the semi-Browder joint spectra of $(S\otimes I, I\otimes T)$, and in Section 6, the semi-Fredholm and the semi-Browder joint spectra of a tuple of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5].

2. PRELIMINARIES

Let us begin our work by recalling the definitions of the lower semi-Fredholm and of the lower semi-Browder joint spectra of a finite tuple of operators defined on a complex Banach space; for a complete exposition see [1] and [7].

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators defined on a Banach space X, and for $k \in \mathbb{N}$ define $M_k(T) = R(T_1^k) + \cdots + R(T_n^k)$. Clearly $X \supseteq M_1(T) \supseteq M_2(T) \supseteq \cdots \supseteq M_k(T) \supseteq \cdots$. Let us set $R^\infty(T) = \bigcap_{k=1}^{\infty} M_k(T)$. We may now recall the definition of the lower semi-Browder joint spectrum (see [7]).

We say that $T = (T_1, \ldots, T_n)$ is *lower semi-Browder* if $\operatorname{codim} R^\infty(T) < \infty$. The set of all lower semi-Browder n-tuples is denoted by $\mathcal{B}_-^{(n)}(X)$, and the *lower semi-Browder spectrum* is the set

$$\sigma_{\mathcal{B}_-}(T) = \{\lambda \in \mathbb{C}^n : \ T - \lambda \notin \mathcal{B}_-^{(n)}(X)\},$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $T - \lambda = (T_1 - \lambda_1 I, \ldots, T_n - \lambda_n I)$.

As usual (see [1]), we say that $T = (T_1, \ldots, T_n)$ is *lower semi-Fredholm*, i.e., $T \in \Phi_-^{(n)}(X)$, if

$$\operatorname{codim} M_1(T) = \operatorname{codim} (R(T_1) + \cdots + R(T_n)) < \infty,$$

equivalently, if the operator $\widehat{T} : X^n \to X$ defined by $\widehat{T}(x_1, \ldots, x_n) = T_1(x_1) + \cdots + T_n(x_n)$ is lower semi-Fredholm, i.e., $R(\widehat{T})$ is closed and has finite codimension. The *lower semi-Fredholm spectrum* is the set

$$\sigma_{\Phi_-}(T) = \{\lambda \in \mathbb{C}^n : \ T - \lambda \notin \Phi_-^{(n)}(X)\}.$$

An easy calculation shows that

$$\sigma_{\Phi_-}(T) \subseteq \sigma_{\mathcal{B}_-}(T) \subseteq \sigma_\delta(T),$$

where $\sigma_\delta(T)$ is the defect spectrum of T, i.e.,

$$\sigma_\delta(T) = \{\lambda \in \mathbb{C}^n : \ \operatorname{codim} M_1(T - \lambda) \neq 0\}.$$

Moreover, it is easy to see that the lower semi-Browder spectrum may be decomposed as the disjoint union of two sets,

$$\sigma_{\mathcal{B}_-}(T) = \sigma_{\Phi_-}(T) \cup \mathcal{A}(T),$$

where

$$\mathcal{A}(T) = \{\lambda \in \mathbb{C}^n : \forall k \in \mathbb{N},\ 1 \leqslant \operatorname{codim} M_k(T - \lambda) < \infty, \operatorname{codim} M_k(T - \lambda) \underset{k \to \infty}{\longrightarrow} \infty\}.$$

Now, we recall the definition of the upper semi-Fredholm and the upper semi-Browder joint spectra; as above, for a complete exposition see [1] and [7].

If T is an n-tuple of commuting operators defined on a Banach space X, then T is said *upper semi-Fredholm*, i.e., $T \in \Phi_+^{(n)}(X)$, if the map $\widetilde{T} : X \to X^n$ defined by $\widetilde{T}(x) = (T_1(x), \ldots, T_n(x))$ is upper semi-Fredholm; equivalently, if $\widetilde{T}$ has finite dimensional null space and closed range. Moreover, T is said *upper semi-Browder*, i.e., $T \in \mathcal{B}_+^{(n)}(X)$, if $T \in \Phi_+^{(n)}(X)$ and $\dim N^\infty(T) < \infty$, where

$$N^\infty(T) = \bigcup_{k\in\mathbb{N}} [N(T_1^k) \cap \cdots \cap N(T_n^k)].$$

As above, the *upper semi-Fredholm spectrum* is the set

$$\sigma_{\Phi_+}(T) = \{\lambda \in \mathbb{C}^n : T - \lambda \notin \Phi_+^{(n)}(X)\},$$

and the *upper semi-Browder spectrum* is the set

$$\sigma_{\mathcal{B}_+}(T) = \{\lambda \in \mathbb{C}^n : T - \lambda \notin \mathcal{B}_+^{(n)}(X)\}.$$

In addition, it is easy to see that

$$\sigma_{\Phi_+}(T) \subseteq \sigma_{\mathcal{B}_+}(T) \subseteq \sigma_\pi(T),$$

where $\sigma_\pi(T)$ denotes the approximate point spectrum of T,

$$\sigma_\pi(T) = \{\lambda \in \mathbb{C}^n : N(\widetilde{T-\lambda}) \neq 0 \text{ or } R(\widetilde{T-\lambda}) \text{ is not closed}\}.$$

Moreover, it is easy to see that the upper semi-Browder spectrum may be decomposed as the disjoint union of two sets,

$$\sigma_{\mathcal{B}_+}(T) = \sigma_{\Phi_+}(T) \cup \mathcal{D}(T),$$

where $\mathcal{D}(T) = \{\lambda \in \mathbb{C}^n : \forall k \in \mathbb{N},\ 1 \leqslant \dim N_k(\widetilde{T-\lambda}) < \infty,\ R(\widetilde{T-\lambda})$ is closed, and $\dim N_k(\widetilde{T-\lambda}) \underset{k\to\infty}{\longrightarrow} \infty\}$, where $N_k(\widetilde{T-\lambda}) = N(\widetilde{(T-\lambda)^k})$ and $(T-\lambda)^k = ((T_1-\lambda_1)^k, \ldots, (T_n-\lambda_n)^k)$.

Let us recall that the semi-Fredholm and the semi-Browder joint spectra are compact nonempty subsets of $\mathbb{C}^n$, which also satisfy the projection property and the analytic spectral mapping theorem for tuples of holomorphic functions defined on a neighborhood of the Taylor joint spectrum ([9]) (see [4] and [7]).

On the other hand, in order to prove our main results, we have to recall the axiomatic tensor product between Banach spaces introduced by J. Eschmeier in [5]. This notion will be central in this work. For a complete exposition see [5]. We proceed as follows.

A pair $\langle X, \widetilde{X}\rangle$ of Banach spaces will be called a *dual pairing*, if

$$\text{(i) } \widetilde{X} = X' \quad \text{or} \quad \text{(ii) } X = \widetilde{X}'.$$

In both cases, the canonical bilinear mapping is denoted by

$$X \times \widetilde{X} \to \mathbb{C}, \quad (x, u) \mapsto \langle x, u\rangle.$$

If $\langle X, \widetilde{X}\rangle$ is a dual pairing, we consider the subalgebra $\mathcal{L}(X)$ of $\mathrm{L}(X)$ consisting of all operators $T \in \mathrm{L}(X)$ for which there is an operator $T' \in \mathrm{L}(\widetilde{X})$ with

$$\langle Tx, u\rangle = \langle x, T'u\rangle,$$

for all $x \in X$ and $u \in \widetilde{X}$. It is clear that if the dual pairing is $\langle X, X'\rangle$, then $\mathcal{L}(X) = \mathrm{L}(X)$, and that if the dual pairing is $\langle X', X\rangle$, then $\mathcal{L}(X) = \{T^* : T \in \mathrm{L}(\widetilde{X})\}$. In particular, each operator of the form

$$f_{y,v} : X \to X, \quad x \mapsto \langle x, v\rangle y,$$

is contained in $\mathcal{L}(X)$, where $y \in X$ and $v \in \widetilde{X}$.

We now recall the definition of the tensor product given by J. Eschmeier in [5].

Given two dual pairings $\langle X, \widetilde{X}\rangle$ and $\langle Y, \widetilde{Y}\rangle$, a tensor product of the Banach spaces X and Y relative to the dual pairings $\langle X, \widetilde{X}\rangle$ and $\langle Y, \widetilde{Y}\rangle$, is a Banach space Z together with continuous bilinear mappings

$$\begin{array}{ll} X \times Y \to Z, & (x, y) \mapsto x \otimes y; \\ \mathcal{L}(X) \times \mathcal{L}(Y) \to \mathrm{L}(Z), & (T, S) \mapsto T \otimes S, \end{array}$$

which satisfy the following conditions:

(T1) $\|x \otimes y\| = \|x\| \, \|y\|$;
(T2) $T \otimes S(x \otimes y) = (Tx) \otimes (Sy)$;
(T3) $(T_1 \otimes S_1) \circ (T_2 \otimes S_2) = (T_1T_2) \otimes (S_1S_2)$, $I \otimes I = I$;
(T4) $\mathrm{Im}\,(f_{x,u} \otimes I) \subseteq \{x \otimes y : y \in Y\}$, $\mathrm{Im}\,(f_{y,v} \otimes I) \subseteq \{x \otimes y : x \in X\}$.

In this work, as in [5], instead of Z we shall often write $X \widetilde{\otimes} Y$. In addition, as in [5], we shall have two applications of this definition of tensor product. First of all, the completion $X \widetilde{\otimes}_\alpha Y$ of the algebraic tensor product of Banach spaces X and Y with respect to a quasi-uniform crossnorm α (see [6]) and an operator ideal between Banach spaces (see [5] and Section 6).

In Section 4 and 5, given two complex Banach spaces X_1 and X_2, and two tuples of Banach spaces operators, S and T, defined on X_1 and X_2 respectively, we shall describe the semi-Fredholm and the semi-Browder joint spectra of the tuple $(S \otimes I, I \otimes T)$, whose operators, $S_i \otimes I$ and $I \otimes T_j$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$, are defined on $X_1 \widetilde{\otimes} X_2$, a tensor product of X_1 and X_2 relative to $\langle X_1, X_1'\rangle$ and $\langle X_2, X_2'\rangle$. However, in the following section, we first introduce the split semi-Browder joint spectra, which will be necessary for our description.

3. THE SPLIT SEMI-BROWDER JOINT SPECTRA

In this section we introduce the upper and lower split semi-Browder joint spectra. We also prove their main spectral properties.

Let us consider, as in Section 2, a complex Banach space X and $T = (T_1, \ldots, T_n)$ a commuting tuple of operators defined on X. We say that T is *lower split semi-Browder* if $R^\infty(T)$ has finite codimension and $N(\widehat{T})$ has a direct complement in X^n, where $\widehat{T} : X^n \to X$ is the map considered in Section 2. We denote by $\mathcal{SB}_-^{(n)}(X)$ the set of all lower split semi-Browder n-tuples, and the *lower split semi-Browder spectrum* is the set

$$\mathrm{sp}_{\mathcal{B}_-}(T) = \{\lambda \in \mathbb{C}^n : T - \lambda \notin \mathcal{S}B_-^{(n)}(X)\}.$$

It is clear that

$$\mathrm{sp}_{\mathcal{B}_-}(T) = \sigma_{\mathcal{B}_-}(T) \cup \mathcal{C}_-(T),$$

where $\mathcal{C}_-(T) = \{\lambda \in \mathbb{C}^n : N(\widehat{T - \lambda})$ has not a direct complement in $X^n\}$. In particular, $\mathrm{sp}_{\mathcal{B}_-}(T)$ is a nonempty set.

On the other hand, if we consider the split defect spectrum and the essential split defect spectrum of T introduced in [4], $\mathrm{sp}_{\delta}(T)$ and $\mathrm{sp}_{\delta\mathrm{e}}(T)$ respectively, sets that by 2.7 from [4] may be presented as

$$\mathrm{sp}_{\delta}(T)=\sigma_{\delta}(T)\cup\mathcal{C}_{-}(T),\quad \mathrm{sp}_{\delta\mathrm{e}}(T)=\sigma_{\Phi_{-}}(T)\cup\mathcal{C}_{-}(T),$$

then we have that

$$\mathrm{sp}_{\delta\mathrm{e}}(T)\subseteq\mathrm{sp}_{\mathcal{B}_{-}}(T)\subseteq\mathrm{sp}_{\delta}(T).$$

In addition, if we consider the set $\widetilde{\mathcal{A}}(T)=\{\lambda\in\mathbb{C}^n : \lambda\notin\mathrm{sp}_{\delta\mathrm{e}}(T),\ \forall k\in\mathbb{N},\ 1\leqslant \operatorname{codim} M_k(T-\lambda)<\infty,\ \operatorname{codim} M_k(T-\lambda)\underset{k\to\infty}{\longrightarrow}\infty\}$, then it is clear that

$$\widetilde{\mathcal{A}}(T)\subseteq\mathcal{A}(T)\subseteq\sigma_{\mathcal{B}_{-}}(T)\subseteq\mathrm{sp}_{\mathcal{B}_{-}}(T).$$

In particular

$$\mathrm{sp}_{\delta\mathrm{e}}(T)\cup\widetilde{\mathcal{A}}(T)\subseteq\mathrm{sp}_{\mathcal{B}_{-}}(T).$$

On the other hand, let us consider $\lambda\in\mathrm{sp}_{\mathcal{B}_{-}}(T)$, and let us decompose the lower split semi-Browder spectrum of T as

$$\mathrm{sp}_{\mathcal{B}_{-}}(T)=\sigma_{\mathcal{B}_{-}}(T)\cup\mathcal{C}_{-}(T)=\sigma_{\Phi_{-}}(T)\cup\mathcal{A}_{-}(T)\cup\mathcal{C}_{-}(T).$$

Now, if $\lambda\in\sigma_{\Phi_{-}}(T)\cup\mathcal{C}_{-}(T)$, then $\lambda\in\mathrm{sp}_{\delta\mathrm{e}}(T)$. Moreover, if $\lambda\in\mathcal{A}(T)\setminus(\sigma_{\Phi_{-}}(T)\cup\mathcal{C}_{-}(T))$, then $\lambda\in\mathcal{A}(T)\setminus\mathrm{sp}_{\delta\mathrm{e}}(T)=\widetilde{\mathcal{A}}(T)$. Thus, we have that

$$\mathrm{sp}_{\mathcal{B}_{-}}(T)=\mathrm{sp}_{\delta\mathrm{e}}(T)\cup\widetilde{\mathcal{A}}(T).$$

We now introduce the upper split semi-Browder spectrum.

If X and $T=(T_1,\ldots,T_n)$ are as above, then we say that T is *upper split semi-Browder* if it is upper semi-Browder and $R(\widetilde{T})$ has a direct complement in X^n, where $\widetilde{T}:X\to X^n$ is the map considered in Section 2. We denote by $\mathcal{SB}_{+}^{(n)}(X)$ the set of all upper split semi-Browder n-tuples, and the *upper split semi-Browder spectrum* is the set

$$\mathrm{sp}_{\mathcal{B}_{+}}(T)=\{\lambda\in\mathbb{C}^n : T-\lambda\notin\mathcal{S}B_{+}^{(n)}(X)\}.$$

It is clear that

$$\mathrm{sp}_{\mathcal{B}_{+}}(T)=\sigma_{\mathcal{B}_{+}}(T)\cup\mathcal{C}_{+}(T),$$

where $\mathcal{C}_{+}(T)=\{\lambda\in\mathbb{C}^n : R(\widetilde{T-\lambda})$ has not a direct complement in $X^n\}$. In particular, $\mathrm{sp}_{\mathcal{B}_{+}}(T)$ is a nonempty set.

On the other hand, if we consider the split approximate point spectrum and the essential split approximate point spectrum of T (see [4]), $\mathrm{sp}_{\pi}(T)$ and $\mathrm{sp}_{\pi\mathrm{e}}(T)$ respectively, i.e., the sets

$$\mathrm{sp}_{\pi}(T)=\sigma_{\pi}(T)\cup\mathcal{C}_{+}(T),\quad \mathrm{sp}_{\pi\mathrm{e}}(T)=\sigma_{\Phi_{+}}(T)\cup\mathcal{C}_{+}(T),$$

then we have that

$$\mathrm{sp}_{\pi\mathrm{e}}(T)\subseteq\mathrm{sp}_{\mathcal{B}_{+}}(T)\subseteq\mathrm{sp}_{\pi}(T).$$

In addition, if we consider the set $\widetilde{\mathcal{D}}(T)=\{\lambda\in\mathbb{C}^n : \lambda\notin\mathrm{sp}_{\pi e}(T),\ \forall k\in\mathbb{N},\ 1\leqslant \dim N_k(\widetilde{T-\lambda})<\infty,\ \dim N_k(\widetilde{T-\lambda})\underset{k\to\infty}{\longrightarrow}\infty\}$, then it is clear that

$$\widetilde{\mathcal{D}}(T)\subseteq\mathcal{D}(T)\subseteq\sigma_{\mathcal{B}_{+}}(T)\subseteq\mathrm{sp}_{\mathcal{B}_{+}}(T).$$

In particular

$$\mathrm{sp}_{\pi\mathrm{e}}(T) \cup \widetilde{\mathcal{D}}(T) \subseteq \mathrm{sp}_{\mathcal{B}_+}(T).$$

On the other hand, let us consider $\lambda \in \mathrm{sp}_{\mathcal{B}_+}(T)$, and let us decompose the upper split semi-Browder spectrum of T as

$$\mathrm{sp}_{\mathcal{B}_+}(T) = \sigma_{\mathcal{B}_+}(T) \cup \mathcal{C}_+(T) = \sigma_{\Phi_+}(T) \cup \mathcal{D}_+(T) \cup \mathcal{C}_+(T).$$

Now, if $\lambda \in \sigma_{\Phi_+}(T) \cup \mathcal{C}_+(T)$, then $\lambda \in \mathrm{sp}_{\pi\mathrm{e}}(T)$. Moreover, if $\lambda \in \mathcal{D}(T) \setminus (\sigma_{\Phi_+}(T) \cup \mathcal{C}_+(T))$, then $\lambda \in \mathcal{D}(T) \setminus \mathrm{sp}_{\pi\mathrm{e}}(T) = \widetilde{\mathcal{D}}(T)$. Thus, we have that

$$\mathrm{sp}_{\mathcal{B}_+}(T) = \mathrm{sp}_{\pi\mathrm{e}}(T) \cup \widetilde{\mathcal{D}}(T).$$

We now see that the sets that we have introduced satisfy the main spectral properties.

PROPOSITION 3.1. *Let X be a complex Banach space and $T = (T_1, \ldots, T_n)$ a commuting tuple of bounded linear operators defined on X. Then the sets $\mathrm{sp}_{\mathcal{B}_-}(T)$ and $\mathrm{sp}_{\mathcal{B}_+}(T)$ are compact subsets of $\mathbb{C}^n$.*

Proof. Since $\mathrm{sp}_{\mathcal{B}_-}(T) = \mathrm{sp}_{\delta\mathrm{e}}(T) \cup \widetilde{\mathcal{A}}(T) \subseteq \mathrm{sp}_{\delta\mathrm{e}}(T) \cup \sigma_{\mathcal{B}_-}(T)$, we have that $\mathrm{sp}_{\mathcal{B}_-}(T)$ is a bounded subset of $\mathbb{C}^n$.

On the other hand, let us consider a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq \mathrm{sp}_{\mathcal{B}_-}(T)$, and $\lambda \in \mathbb{C}^n$ such that $\lambda_n \underset{n\to\infty}{\longrightarrow} \lambda$. If there exists a subsequence $(\lambda_{n_k})_{k\in\mathbb{N}} \subseteq \mathrm{sp}_{\delta\mathrm{e}}(T)$, then $\lambda \in \mathrm{sp}_{\delta\mathrm{e}}(T) \subseteq \mathrm{sp}_{\mathcal{B}_-}(T)$. Thus, we may suppose that there is $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geqslant n_0$, $\lambda_n \in \widetilde{\mathcal{A}}(T)$. Moreover, we may also suppose that $\lambda \notin \mathrm{sp}_{\delta\mathrm{e}}(T)$. In particular, there is an open neighborhood of λ, U, such that $U \cap \mathrm{sp}_{\delta\mathrm{e}}(T) = \emptyset$, and there is $n_1 \in \mathbb{N}$ such that $\lambda_n \in U$, for all $n \geqslant n_1$.

However, since for all $n \geqslant n_0$, $\lambda_n \in \widetilde{\mathcal{A}}(T) \subseteq \mathcal{A}(T) \subseteq \sigma_{\mathcal{B}_-}(T)$, then $\lambda \in \sigma_{\mathcal{B}_-}(T)$. But $\lambda \notin \sigma_{\Phi_-}(T)$, for $\sigma_{\Phi_-}(T) \subseteq \mathrm{sp}_{\delta\mathrm{e}}(T)$. Then, $\lambda \in \mathcal{A}(T) \setminus \mathrm{sp}_{\delta\mathrm{e}}(T) = \widetilde{\mathcal{A}}(T) \subseteq \mathrm{sp}_{\mathcal{B}_-}(T)$.

By means of a similar argument, it is possible to see that $\mathrm{sp}_{\mathcal{B}_+}(T)$ is a compact subset of $\mathbb{C}^n$. ∎

PROPOSITION 3.2. *Let X be a complex Banach space and $T = (T_1, \ldots, T_n, T_{n+1})$ a commuting tuple of bounded linear operators defined on X. If $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ denotes the projection onto the first n-coordinate, then we have that:*

(i) $\pi(\mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n, T_{n+1})) = \mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n)$;

(ii) $\pi(\mathrm{sp}_{\mathcal{B}_+}(T_1, \ldots, T_n, T_{n+1})) = \mathrm{sp}_{\mathcal{B}_+}(T_1, \ldots, T_n)$.

Proof. By 7 from [7] we know that $\pi(\sigma_{\mathcal{B}_-}(T_1, \ldots, T_n, T_{n+1})) = \sigma_{\mathcal{B}_-}(T_1, \ldots, T_n) \subseteq \mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n)$. Moreover, since $\mathcal{C}_-(T_1, \ldots, T_n, T_{n+1}) \subseteq \mathrm{sp}_{\delta\mathrm{e}}(T_1, \ldots, T_n, T_{n+1})$, by 2.6 from [4] we have that $\pi(\mathcal{C}_-(T_1, \ldots, T_n, T_{n+1})) \subseteq \pi(\mathrm{sp}_{\delta\mathrm{e}}(T_1, \ldots, T_n, T_{n+1})) = \mathrm{sp}_{\delta\mathrm{e}}(T_1, \ldots, T_n) \subseteq \mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n)$. Thus, we have that

$$\pi(\mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n, T_{n+1})) \subseteq \mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n).$$

On the other hand, by 7 from [7] we also have that

$$\sigma_{\mathcal{B}_-}(T_1, \ldots, T_n) = \pi(\sigma_{\mathcal{B}_-}(T_1, \ldots, T_n, T_{n+1})) \subseteq \pi(\mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n, T_{n+1})).$$

Furthermore, since $\mathcal{C}_-(T_1,\ldots,T_n)\subseteq \mathrm{sp}_{\delta\mathrm{e}}(T_1,\ldots,T_n)$, by 2.6 from [4] we also have that

$$\begin{aligned}\mathcal{C}_-(T_1,\ldots,T_n)&\subseteq \mathrm{sp}_{\delta\mathrm{e}}(T_1,\ldots,T_n)=\pi(\mathrm{sp}_{\delta\mathrm{e}}(T_1,\ldots,T_n,T_{n+1}))\\ &\subseteq \pi(\mathrm{sp}_{\mathcal{B}_-}(T_1,\ldots,T_n,T_{n+1})).\end{aligned}$$

Thus,

$$\mathrm{sp}_{\mathcal{B}_-}(T_1,\ldots,T_n)\subseteq \pi(\mathrm{sp}_{\mathcal{B}_-}(T_1,\ldots,T_n,T_{n+1})),$$

i.e., we have proved the first statement of the proposition.

By means of a similar argument it is possible to see the second statement. ∎

In the following proposition we shall see that the split semi-Browder joint spectra satisfy the analytic spectral mapping theorem.

Proposition 3.3. *Let X be a complex Banach space and $T=(T_1,\ldots,T_n)$ a commuting tuple of bounded linear operators defined on X. Then, if $f\in\mathcal{O}(\mathrm{sp}(T))^m$, we have that:*

(i) $f(\mathrm{sp}_{\mathcal{B}_-}(T_1,\ldots,T_n))=\mathrm{sp}_{\mathcal{B}_-}(f(T_1,\ldots,T_n))$;

(ii) $f(\mathrm{sp}_{\mathcal{B}_+}(T_1,\ldots,T_n))=\mathrm{sp}_{\mathcal{B}_+}(f(T_1,\ldots,T_n))$,

where $\mathrm{sp}(T)$ *denotes the split spectrum of T.*

Proof. By 2.6 from [4], the split sectrum of T, $\mathrm{sp}(T)$, satisfies the analytic spectral mapping theorem, i.e., there is an algebra morphism

$$\Phi:\mathcal{O}(\mathrm{sp}(T))\to \mathrm{L}(X),\quad f\mapsto f(T),$$

such that $\mathbb{1}(T)=I$, $z_i(T)=T_i$, $1\leqslant i\leqslant n$, where z_i denotes the projection of $\mathbb{C}^n$ onto the i-th coordinate, and such that the equality $\mathrm{sp}(f(T))=f(\mathrm{sp}(T))$, holds for all $f\in\mathcal{O}(\mathrm{sp}(T))^m$.

Now, as in [4], let us consider the algebra

$$A=\overline{\Phi(\mathcal{O}(\mathrm{sp}(T)))}\subseteq \mathrm{L}(X).$$

Then, we have that the split spectrum is a spectral system on A, in the sense of 1 from [4].

In order to show this claim, since the split spectrum is a compact set which also satisfies the projection property (2.6 from [4]), we have only to see that if $a=(a_1,\ldots,a_n)$ is a tuple of commuting operators such that $a_i\in A$, then $\mathrm{sp}(a)\subseteq \sigma^A_{\mathrm{joint}(a)}$.

In fact, if $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathrm{sp}(a)\setminus\sigma^A_{\mathrm{joint}(a)}$, there are $B_1,\ldots,B_n\in A$ such that $\sum\limits_{i=1}^n B_i(a_i-\lambda_i I)=I$, where I denotes the identity map of X. In particular,

$$\sum_{i=1}^n L_{B_i}(L_{a_i}-\lambda_i I_{\mathrm{L}(X)})=I_{\mathrm{L}(X)}.$$

Then $\lambda\notin\sigma(L_a)$, the Taylor joint spectrum of the tuple of left multiplication, $L_a=(L_{a_1},\ldots,L_{a_n})$, defined on $\mathrm{L}(X)$. However, by 2.5 from [4], $\lambda\notin\mathrm{sp}(a)$, which is impossible by our assumption.

Now, since $\mathrm{sp}_{\mathcal{B}_-}(T_1,\ldots,T_n)$ and $\mathrm{sp}_{\mathcal{B}_+}(T_1,\ldots,T_n)$ are contained in $\mathrm{sp}(T)$, by Propositions 3.1 and 3.2, $\mathrm{sp}_{\mathcal{B}_-}(T_1,\ldots,T_n)$ and $\mathrm{sp}_{\mathcal{B}_+}(T_1,\ldots,T_n)$ are spectral systems on A contained in $\mathrm{sp}(T)$. Then, by 1.2 and 1.3 from [4], since the split spectrum is a spectral system on A which satisfy the analytic spectral mapping theorem, $\mathrm{sp}_{\mathcal{B}_-}(T_1, \ldots, T_n)$ and $\mathrm{sp}_{\mathcal{B}_+}(T_1, \ldots, T_n)$ also satisfies the analytic spectral mapping theorem defined on $\mathcal{O}(\mathrm{sp}(T))$. ∎

In the following section we give a description of the semi-Fredholm joint spectra of the system $(S \otimes I, I \otimes T)$, which will be a central step for one of the main theorems of the present article.

4. THE SEMI-FREDHOLM JOINT SPECTRA

In this section we consider two complex Banach spaces X_1 and X_2, two tuples of bounded linear operators defined on X_1 and X_2, $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_n)$ respectively, and we describe the semi-Fredholm joint spectra of the $(n+m)$-tuple $(S \otimes I, I \otimes T)$ defined on $X_1 \tilde{\otimes} X_2$, a tensor product between X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$, where $(S \otimes I, I \otimes T) = (S_1 \otimes I, \ldots, S_n \otimes I, I \otimes T_1, \ldots, I \otimes T_m)$.

We recall that if K_1, K_2 and K are the Koszul complexes associated to the tuples S, T and $(S \otimes I, I \otimes T)$ respectively (see [9]), i.e., $K_1 = (X_1 \otimes \wedge \mathbb{C}^n, d_1)$, $K_2 = (X_2 \otimes \wedge \mathbb{C}^m, d_2)$ and $K = (X_1 \tilde{\otimes} X_2 \otimes \wedge \mathbb{C}^{n+m}, d_{12})$, then, by 3 from [5] we have that K is isomorphic to the total complex of the double complex obtained from the tensor product of the complexes K_1 and K_2; we denote this total complex by $K_1 \tilde{\otimes} K_2$. Moreover, if we consider the differential spaces associated to K_1, K_2, K, and $K_1 \tilde{\otimes} K_2$, which we denote, by $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}$, and $\mathcal{K}_1 \tilde{\otimes} \mathcal{K}_2$ respectively, then we have that $\mathcal{K} \cong \mathcal{K}_1 \tilde{\otimes} \mathcal{K}_2$; and if the boundary of these differential spaces are, $\partial_1, \partial_2, \partial_{12}$, and ∂ respectively, then we have that $\partial = \partial_1 \otimes I + \eta \otimes \partial_2$, where η is the map $\eta : \mathcal{K}_2 \to \mathcal{K}_2$, $\eta | X_2 \otimes \wedge^m \mathbb{C} = (-1)^m I$ (for a complete exposition see 3 from [5]).

In the following proposition we describe the defect, the approximate point spectrum, and the split version of these spectra for the tuple $(S \otimes I, I \otimes T)$. This result is necessary for our description of the semi-Fredholm joint spectra.

PROPOSITION 4.1. *Let X_1 and X_2 be two complex Banach spaces, and $X_1 \tilde{\otimes} X_2$ a tensor product of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$. Let us consider two tuples of commuting operators defined on X_1 and X_2, S and T respectively. Then, for the tuple $(S \otimes I, I \otimes T)$, defined on $X_1 \tilde{\otimes} X_2$, we have that:*

(i) $\sigma_\delta(S) \times \sigma_\delta(T) \subseteq \sigma_\delta(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_\delta(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_\delta(S) \times \mathrm{sp}_\delta(T)$;

(ii) $\sigma_\pi(S) \times \sigma_\pi(T) \subseteq \sigma_\pi(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_\pi(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_\pi(S) \times \mathrm{sp}_\pi(T)$.

In addition, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. Let us consider $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}^m$ and the Koszul complexes associated to $S - \lambda$, $T - \mu$ and $(S \otimes I, I \otimes T) - (\lambda, \mu) = ((S - \lambda) \otimes I, I \otimes (T - \mu))$, which we denote by K_1, K_2 and K. By the previous observation we have that $K \cong K_1 \tilde{\otimes} K_2$. Moreover, if we consider the differential spaces associated to these complexes, $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}$, then we have that $\mathcal{K} \cong \mathcal{K}_1 \tilde{\otimes} \mathcal{K}_2$.

Now, we may apply 2.2 from [5] to the differential spaces $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{K}_1 \tilde{\otimes} \mathcal{K}_2$. However, by the definition of the map φ in 2.2 from [5], the grading of the differential spaces $\mathcal{K}_1, \mathcal{K}_2$, and $\mathcal{K}_1 \tilde{\otimes} \mathcal{K}_2$, and of the isomorphism $\mathcal{K} \cong \mathcal{K}_1 \tilde{\otimes} \mathcal{K}_2$, we have the left hand side inclusion of the first statement.

The middle inclusion is clear.

Let us now suppose that $(\lambda,\mu)\notin \mathrm{sp}_\delta(S)\times \mathrm{sp}_\delta(T)$. Then, either $\lambda\notin \mathrm{sp}_\delta(S)$ or $\mu\notin \mathrm{sp}_\delta(T)$. We shall see that if $\lambda\notin \mathrm{sp}_\delta(S)$, then $(\lambda,\mu)\notin \mathrm{sp}_\delta(S\otimes I, I\otimes T)$. By means of a similar argument it is possible to see that if $\mu\notin \mathrm{sp}_\delta(T)$ then $(\lambda,\mu)\notin \mathrm{sp}_\delta(S\otimes I, I\otimes T)$.

Now, if $\lambda\notin \mathrm{sp}_\delta(S)$, there is a bounded linear operator $h\colon X_1\to X_1\otimes\wedge^n\mathbb{C}$ such that

$$d_{11}\circ h = I,$$

where $d_{11}\colon X_1\otimes\wedge\mathbb{C}^n\to X_1$ is the chain map of the Koszul complex K_1 at level $p=1$.

Let us consider the map

$$H\colon X_1\tilde{\otimes}X_2\to X_1\otimes\wedge\mathbb{C}^n\tilde{\otimes}X_2,\quad H=h\otimes I.$$

Then, by the properties of the tensor product introduced in [5], H is a well defined map which satisfies

$$d_1\circ H = d_{11}\circ h\otimes I = I\otimes I = I,$$

where d_1 is the chain map of the complex $K_1\tilde{\otimes}K_2$ at level $p=1$. Since $K\cong K_1\tilde{\otimes}K_2$, we have that $(\lambda,\mu)\notin \mathrm{sp}_\delta(S\otimes I, I\otimes T)$.

The second statement may be proved by means of a similar argument. ∎

In the following proposition we state our description of the semi-Fredholm joint spectra.

Proposition 4.2. *Let X_1 and X_2 be two complex Banach spaces, and $X_1\tilde{\otimes}X_2$ a tensor product of X_1 and X_2 relative to $\langle X_1,X_1'\rangle$ and $\langle X_2,X_2'\rangle$. Let us consider two tuples of commuting operators defined on X_1 and X_2, S and T respectively. Then, for the tuple $(S\otimes I, I\otimes T)$, defined on $X_1\tilde{\otimes}X_2$, we have that:*

(i) $\sigma_{\Phi_-}(S)\times\sigma_\delta(T)\cup\sigma_\delta(S)\times\sigma_{\Phi_-}(T)\subseteq\sigma_{\Phi_-}(S\otimes I, I\otimes T)\subseteq \mathrm{sp}_{\delta\mathrm{e}}(S\otimes I, I\otimes T)\subseteq \mathrm{sp}_{\delta\mathrm{e}}(S)\times\mathrm{sp}_\delta(T)\cup\mathrm{sp}_\delta(S)\times\mathrm{sp}_{\delta\mathrm{e}}(T)$;

(ii) $\sigma_{\Phi_+}(S)\times\sigma_\pi(T)\cup\sigma_\pi(S)\times\sigma_{\Phi_+}(T)\subseteq\sigma_{\Phi_+}(S\otimes I, I\otimes T)\subseteq \mathrm{sp}_{\pi\mathrm{e}}(S\otimes I, I\otimes T)\subseteq \mathrm{sp}_{\pi\mathrm{e}}(S)\times\mathrm{sp}_\pi(T)\cup\mathrm{sp}_\pi(S)\times\mathrm{sp}_{\pi\mathrm{e}}(T)$.

In addition, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. First of all, let us observe that we use the same notations of Proposition 4.1.

With regard to the first statement, in order to prove the left hand side inclusion, it is enough to adapt for this case the argument that we have developed in Proposition 4.1 for the corresponding inclusion.

The middle inclusion is clear.

Let us denote by E the set $E=\mathrm{sp}_{\delta\mathrm{e}}(S)\times\mathrm{sp}_\delta(T)\cup\mathrm{sp}_\delta(S)\times\mathrm{sp}_{\delta\mathrm{e}}(T)$, and let us consider $(\lambda,\mu)\in\mathrm{sp}_{\delta\mathrm{e}}(S\otimes I, I\otimes T)\setminus E$. Then, by Proposition 4.1, since $(\lambda,\mu)\in\mathrm{sp}_\delta(S\otimes I, I\otimes T)\subseteq\mathrm{sp}_\delta(S)\times\mathrm{sp}_\delta(T)$, we have that $\lambda\in\mathrm{sp}_\delta(S)\setminus\mathrm{sp}_{\delta\mathrm{e}}(S)$ and $\mu\in\mathrm{sp}_\delta(T)\setminus\mathrm{sp}_{\delta\mathrm{e}}(T)$. In particular, there are two linear bounded maps $h\colon X_1\to X_1\otimes\wedge^1\mathbb{C}^n$, $g\colon X_2\to X_2\otimes\wedge^1\mathbb{C}^m$, and two compact operators $k_1\colon X_1\to X_1$ and $k_2\colon X_2\to X_2$ such that

$$d_{11}\circ h = I-k_1,\quad d_{21}\circ g = I-k_2,$$

where d_{21} is the boundary map of the complex K_2 at level $p = 1$. Moreover, by an argument similar to 2.7 from [4] or 2.1 from [5], the maps k_i, $i = 1, 2$, may be chosen as finite rank projectors.

In addition, by the properties of the tensor product introduced in [5], we may consider the well defined map

$$H : X_1 \widetilde{\otimes} X_2 \to (K_1 \widetilde{\otimes} K_2)_1, \quad H = (h \otimes I, I \otimes g).$$

Now, an easy calculation shows that $d_1 \circ H = I - k_1 \otimes k_2$, where d_1 denotes the chain map of the complex $K_1 \widetilde{\otimes} K_2$ at level $p = 1$. However, it is not difficult to see, using in particular 1.1 from [5], that $k_1 \otimes k_2$ is a finite rank projector whose range coincide with $R(k_1) \otimes R(k_2)$. In particular, $k_1 \otimes k_2$ is a compact operator. Thus, since $K \cong K_1 \widetilde{\otimes} K_2$, $(\lambda, \mu) \notin \mathrm{sp}_{\delta\mathrm{e}}(S \otimes I, I \otimes T)$, which is impossible by our assumptions.

By means of a similar argument it is possible to prove the second statement. ∎

5. THE SEMI-BROWDER JOINT SPECTRA

In this section we give our description of the semi-Browder joint spectra of the tuple $(S \otimes I, I \otimes T)$. The following theorem is one of the main results of the present article.

THEOREM 5.1. *Let X_1 and X_2 be two complex Banach spaces, and $X_1 \widetilde{\otimes} X_2$ a tensor product of X_1 and X_2 relative to $\langle X_1, X_1' \rangle$ and $\langle X_2, X_2' \rangle$. Let us consider two tuples of commuting operators defined on X_1 and X_2, S and T respectively. Then for the tuple $(S \otimes I, I \otimes T)$, defined on $X_1 \widetilde{\otimes} X_2$, we have that:*

(i) $\sigma_{\mathcal{B}_-}(S) \times \sigma_\delta(T) \cup \sigma_\delta(S) \times \sigma_{\mathcal{B}_-}(T) \subseteq \sigma_{\mathcal{B}_-}(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_{\mathcal{B}_-}(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_{\mathcal{B}_-}(S) \times \mathrm{sp}_\delta(T) \cup \mathrm{sp}_\delta(S) \times \mathrm{sp}_{\mathcal{B}_-}(T)$;

(ii) $\sigma_{\mathcal{B}_+}(S) \times \sigma_\pi(T) \cup \sigma_\pi(S) \times \sigma_{\mathcal{B}_+}(T) \subseteq \sigma_{\mathcal{B}_+}(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_{\mathcal{B}_+}(S \otimes I, I \otimes T) \subseteq \mathrm{sp}_{\mathcal{B}_+}(S) \times \mathrm{sp}_\pi(T) \cup \mathrm{sp}_\pi(S) \times \mathrm{sp}_{\mathcal{B}_+}(T)$.

In addition, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. First of all, as in Proposition 4.2, we use the notations of Proposition 4.1.

Let us consider $(\lambda, \mu) \in \sigma_{\mathcal{B}_-}(S) \times \sigma_\delta(T)$. If $\lambda \in \sigma_{\Phi_-}(S)$, then, by Proposition 4.2, $(\lambda, \mu) \in \sigma_{\Phi_-}(S) \times \sigma_\delta(T) \subseteq \sigma_{\Phi_-}(S \otimes I, I \otimes T) \subseteq \sigma_{\mathcal{B}_-}(S \otimes I, I \otimes T)$.

Now, if $\lambda \in \mathcal{A}(S)$, since $\mu \in \sigma_\delta(T)$, by the definition of the map φ in 2.2 from [5], the grading of the complex K_1, K_2, and $K_1 \widetilde{\otimes} K_2$, and by the isomorphism $K \cong K_1 \widetilde{\otimes} K_2$, we have that $\dim H_0(K) = \dim H_0(K_1 \widetilde{\otimes} K_2) \geqslant \dim H_0(K_1) \times \dim H_0(K_2) \geqslant 1$. In particular, $(\lambda, \mu) \in \sigma_\delta(S \otimes I, I \otimes T)$.

Moreover, if $\dim H_0(K) = \infty$, then $(\lambda, \mu) \in \sigma_{\Phi_-}(S \otimes I, I \otimes T) \subseteq \sigma_{\mathcal{B}_-}(S \otimes I, I \otimes T)$.

On the other hand, if we suppose that $(\lambda, \mu) \notin \sigma_{\Phi_-}(S \otimes I, I \otimes T)$. Then we consider the tuples of operators $(S - \lambda)^l = ((S_1 - \lambda_1)^l, \ldots, (S_n - \lambda_n)^l)$ and $(T - \mu)^l = ((T_1 - \mu_1)^l, \ldots, (T_m - \mu_m)^l)$, and we denote by K_1^l and K_2^l the Koszul complexes associated to the tuples $(S - \lambda)^l$ and $(T - \mu)^l$, respectively. Moreover, if we denote by K^l the Koszul complex associated to the tuple $((S - \lambda)^l \otimes I, I \otimes (T - \mu)^l)$, as above, K^l is isomorphic to the total complex of the double complex of the tensor product of K_1^l and K_2^l, i.e., $K^l \cong K_1^l \widetilde{\otimes} K_2^l$.

In addition, as we have seen for the complexes K_1, K_2, K, and $K_1 \widetilde{\otimes} K_2$, we have that $\dim H_0(K^l) = \dim H_0(K_1^l \widetilde{\otimes} K_2^l) \geqslant \dim H_0(K_1^l) \times \dim H_0(K_2^l)$. Now, since $\mu \in \sigma_\delta(T)$, by the analytic spectral mapping theorem for the defect spectrum (see 2.1 from [4]) we have that $\dim H_0(K_2^l) \neq 0$. In addition, since $\dim H_0(K_1^l) = \operatorname{codim} M_l(S-\lambda)$, and since $\lambda \in \mathcal{A}(S)$, then $\dim H_0(K^l) \underset{l\to\infty}{\longrightarrow} \infty$. However, $\dim H_0(K^l) = \operatorname{codim} M_l((S-\lambda)\otimes I, I\otimes(T-\mu))$. In particular, $(\lambda,\mu) \in \mathcal{A}(S\otimes I, I\otimes T) \subseteq \sigma_{\mathcal{B}_-}(S\otimes I, I\otimes T)$.

By means of a similar argument it is possible to see that $\sigma_\delta(S) \times \sigma_{\mathcal{B}_-}(T) \subseteq \sigma_{\mathcal{B}_-}(S\otimes I, I\otimes T)$.

The middle inclusion is clear.

In order to see the right hand inclusion, let us consider $(\lambda,\mu) \in \operatorname{sp}_{\mathcal{B}_-}(S\otimes I, I\otimes T)$. If $(\lambda,\mu) \in \operatorname{sp}_{\delta \mathrm{e}}(S\otimes I, I\otimes T)$, then by Proposition 4.2, $(\lambda,\mu) \in \operatorname{sp}_{\delta e}(S) \times \operatorname{sp}_\delta(T) \cup \operatorname{sp}_\delta(S) \times \operatorname{sp}_{\delta\mathrm{e}}(T) \subseteq \operatorname{sp}_{\mathcal{B}_-}(S) \times \operatorname{sp}_\delta(T) \cup \operatorname{sp}_\delta(S) \times \operatorname{sp}_{\mathcal{B}_-}(T)$.

On the other hand, if $(\lambda,\mu) \in \widetilde{\mathcal{A}}(S\otimes I, I\otimes T)$, since by Proposition 4.1 $\operatorname{sp}_{\mathcal{B}_-}(S\otimes I, I\otimes T) \subseteq \operatorname{sp}_\delta(S\otimes I, I\otimes T) \subseteq \operatorname{sp}_\delta(S)\times\operatorname{sp}_\delta(T)$, if $(\lambda,\mu) \notin (\operatorname{sp}_{\mathcal{B}_-}(S) \times \operatorname{sp}_\delta(T) \cup \operatorname{sp}_\delta(S) \times \operatorname{sp}_{\mathcal{B}_-}(T))$, then $\lambda \notin \operatorname{sp}_{\mathcal{B}_-}(S)$ and $\mu \notin \operatorname{sp}_{\mathcal{B}_-}(T)$. In particular, $\lambda \notin \operatorname{sp}_{\delta\mathrm{e}}(S)$ and $\mu \notin \operatorname{sp}_{\delta\mathrm{e}}(T)$, and there is $l \in \mathbb{N}$ such that for all $r \geqslant l$, $\dim H_0(K_1^r) = \dim H_0(K_1^l)$ and $\dim H_0(K_2^r) = \dim H_0(K_2^l)$.

In addition, by the analytic spectral mapping theorem of the essential split defect spectrum (2.6 from [4]) the complex K_1^r and K_2^r are Fredholm split for all $r \in \mathbb{N}$ at level $p = 0$. In particular, for all $r \in \mathbb{N}$ there are bounded linear maps $h_r \colon X_1 \to X_1 \otimes \wedge^1\mathbb{C}^n$ and $g_r \colon X_2 \to X_2 \otimes \wedge^1\mathbb{C}^m$, and finite rank projectors (see Proposition 4.2), $k_{1r} \colon X_1 \to X_1$ and $k_{2r} \colon X_2 \to X_2$, such that

$$d_{11}^r \circ h_r = I - k_{1r}, \quad d_{21}^r \circ g_r = I - k_{2r},$$

where d_{11}^r and d_{21}^r are the chain maps of the complex K_1^r and K_2^r at level $p = 1$, respectively.

Moreover, since the complexes K_1^r and K_2^r are Fredholm split at level $p = 0$, by 2.7 from [4] the complexes K_1^r and K_2^r are Fredholm at level $p = 0$ and $N(d_{11}^r)$ and $N(d_{21}^r)$ have direct complements in $X_1 \otimes \wedge^1\mathbb{C}^n$ and $X_2 \otimes \wedge^1\mathbb{C}^m$ respectively. Now, by an argument similar to 2.7 from [4] or 2.1 from [5], we have that the maps h_r, g_r, k_{1r} and k_{2r} may be chosen in the following way. If N_1^r and N_2^r are finite dimensional subspaces of X_1 and X_2 respectively, such that $R(d_{11}^r) \oplus N_1^r = X_1$ and $R(d_{21}^r) \oplus N_2^r = X_2$ and L_1^r and L_2^r are closed linear subspaces of $X_1 \otimes \wedge^1\mathbb{C}^n$ and $X_2 \otimes \wedge^1\mathbb{C}^m$ respectively, such that $N(d_{11}^r) \oplus L_1^r = X_1 \otimes \wedge^1\mathbb{C}^n$, and $N(d_{21}^r) \oplus L_2^r = X_2 \otimes \wedge^1\mathbb{C}^m$, then, k_{1r}, respectively k_{2r}, may be chosen as the projector onto N_1^r, respectively N_2^r, whose null space coincide with $R(d_{11}^r)$, respectively $R(d_{21}^r)$, and the map h_r, respectively g_r, may be chosen such that $h_r \circ d_{11}^r = I|L_1^r$, respectively $g_r \circ d_{21}^r = I|L_2^r$, $h_r|N_1^r = 0$, respectively $g_r|N_2^r = 0$. In particular, $R(k_{1r}) \cong H_0(K_1^r)$ and $R(k_{2r}) \cong H_0(K_2^r)$.

Now, as in Proposition 4.2, for all $r \in \mathbb{N}$ we have a well defined map $H_r \colon X_1 \widetilde{\otimes} X_2 \to (K_1^r \widetilde{\otimes} K_2^r)_1$ such that

$$d_1^r \circ H_r = I - k_{1r} \otimes k_{2r},$$

where d_1^r is the boundary map of the complex $K_1^r\widetilde{\otimes}K_2^r$ at level $p=1$. Then, since for all $r\in\mathbb{N}$, $R(k_{1r}\otimes k_{2r})=R(k_{1r})\otimes R(k_{2r})$ (see Proposition 4.2) for all $r\geqslant l$ we have that

$$\begin{aligned}\dim H_0(K^r) &= \dim H_0(K_1^r\widetilde{\otimes}K_2^r)\leqslant \dim R(k_{1r}\otimes k_{2r})\\ &= \dim R(k_{1r})\times\dim R(k_{2r}) = \dim H_0(K_1^r)\times\dim H_0(K_2^r)\\ &= \dim H_0(K_1^l)\times\dim H_0(K_2^l),\end{aligned}$$

which is impossible for $(\lambda,\mu)\in\widetilde{\mathcal{A}}(S\otimes I, I\otimes T)$ and $\dim H_0(K^r)=\operatorname{codim} M_r((S-\lambda)\otimes I, I\otimes(T-\mu))$.

By means of a similar argument it is possible to prove the second statement. ∎

6. OPERATOR IDEALS BETWEEN BANACH SPACES

In this section we extend our descriptions of the semi-Fredholm joint spectra and the semi-Browder joint spectra for tuples of left and right multiplications defined on an operator ideal between Banach spaces in the sense of [5]. We first recall the definition of such an ideal and then we introduce the tuples with which we shall work. For a complete exposition see [5].

An operator ideal J between Banach spaces X_2 and X_1 will be a *linear subspace* of $\mathrm{L}(X_2,X_1)$, equiped with a space norm α such that:

(i) $x_1\otimes x_2'\in J$ and $\alpha(x_1\otimes x_2')=\|x_1\|\,\|x_2'\|$;
(ii) $SAT\in J$ and $\alpha(SAT)\leqslant\|S\|\alpha(A)\|T\|$;

where $x_1\in X_1$, $x_2'\in X_2'$, $A\in J$, $S\in\mathrm{L}(X_1)$, $T\in\mathrm{L}(X_2)$, and $x_1\otimes x_2'$ is the usual rank one operator $X_2\to X_1$, $x_2\to\langle x_2,x_2'\rangle x_1$.

Examples of this kind of ideals are given in 1 from [5].

Let us recall that such an operator ideal J is naturally a tensor product relative to $\langle X_1,X_1'\rangle$ and $\langle X_2',X_2\rangle$, with the bilinear mappings:

$$\begin{aligned}&X_1\times X_2'\to J,\quad (x_1,x_2')\mapsto x_1\otimes x_2',\\ &\mathcal{L}(X_1)\times\mathcal{L}(X_2')\to\mathrm{L}(J),\quad (S,T')\mapsto S\otimes T',\end{aligned}$$

where $S\otimes T'(A)=SAT$.

On the other hand, if X is a Banach space and $U\in\mathrm{L}(X)$, we denote by L_U and R_U the operators of left and right multiplication in $\mathrm{L}(X)$, respectively, i.e., if $V\in\mathrm{L}(X)$, then $L_U(V)=UV$ and $R_U(V)=VU$.

Now, if $S=(S_1,\ldots,S_n)$ and $T=(T_1,\ldots,T_m)$ are tuples of commuting operators defined on X_1 and X_2 respectively, if J is seen as a tensor product of X_1 and X_2 relative to $\langle X_1,X_1'\rangle$ and $\langle X_2',X_2\rangle$, then the tuple of left and right multiplications (L_S,R_T) defined on $L(J)$, $(L_S,R_T)=(L_{S_1},\ldots,L_{S_n},R_{T_1},\ldots,R_{T_m})$, may be identified with the $(n+m)$-tuple $(S\otimes I,I\otimes T')$ defined on $X_1\widetilde{\otimes}X_2'$, where $T'=({T_1}',\ldots,{T_m}')$ and for all $i=1,\ldots,m$, ${T_i}'$ is the adjoint map associated to T_i (see 3.1 from [5]).

In addition, if $\lambda\in\mathbb{C}^n$ and $\mu\in\mathbb{C}^m$, and if we denote by K_1 and K_2' the Koszul complexes associated to S and λ and T' and μ respectively, then the total complex of the double complex obtained from the tensor product of K_1 and K_2', $K_1\widetilde{\otimes}K_2'$ is isomorphic to $\widetilde{K}$, the Koszul complex associated to $(S\otimes I,I\otimes T')$ and

(λ,μ) on $X_1\widetilde{\otimes}X_2$, which is naturally isomorphic to the Koszul complex of (L_S,R_T) and (λ,μ) on $\mathrm{L}(J)$ (see 3 from [5]).

In order to state our description of the semi-Fredholm and the semi-Browder joint spectra of the tuple (L_S,R_T), as we have done in Section 4, we first describe the defect and the approximate point spectra of the mentioned tuple.

PROPOSITION 6.1. *Let X_1 and X_2 be two complex Banach spaces, and J an operator ideal between X_2 and X_1 in the sense of* [5]. *Let us consider two tuples of commuting operators defined on X_1 and X_2, S and T respectively. Then, if (L_S,R_T) is the tuple of left and right multiplications defined on $L(J)$, we have that:*

(i) $\sigma_\delta(S)\times\sigma_\pi(T)\subseteq\sigma_\delta(L_S,R_T)\subseteq\mathrm{sp}_\delta(L_S,R_T)\subseteq\mathrm{sp}_\delta(S)\times\mathrm{sp}_\pi(T)$;

(ii) $\sigma_\pi(S)\times\sigma_\delta(T)\subseteq\sigma_\pi(L_S,R_T)\subseteq\mathrm{sp}_\pi(L_S,R_T)\subseteq\mathrm{sp}_\pi(S)\times\mathrm{sp}_\delta(T)$.

In addition, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. As we have said, J may be seen as the tensor product of X_1 and X_2', $X_1\widetilde{\otimes}X_2'$, relative to $\langle X_1,X_1'\rangle$ and $\langle X_2,X_2'\rangle$, and (L_S,R_T) may be identified with the tuple $(S\otimes I,I\otimes T')$. Moreover, if $\mathcal{K}_1$ and $\mathcal{K}_2{}'$ denote the differential space associated to K_1 and K_2' respectively, then $\widetilde{\mathcal{K}}$, the differentiable space associated to $\widetilde{K}$, is isomorphic to $\mathcal{K}_1\widetilde{\otimes}\mathcal{K}_2{}'$ (see 3 from [5]).

In addition, since for all $i=1,\dots,n$ $S_i\in\mathrm{L}(X_1)$ and for all $j=1,\dots,m$ $T_j\in\mathrm{L}(X_2)$, the differential spaces $\mathcal{K}_1$ and $\mathcal{K}_2{}'$ satisfy the conditions of 2.2 from [5], and by means of an argument similar to the one of Proposition 4.1 we have that

$$\sigma_\delta(S)\times\sigma_\delta(T')\subseteq\sigma_\delta(S\otimes I,I\otimes T')=\sigma_\delta(L_S,R_T).$$

However, by 2.0 from [8], $\sigma_\pi(T)=\sigma_\delta(T')$. Thus, we have proved the left hand side inclusion of the first statement.

The middle inclusion is clear.

In order to see the right hand inclusion, let us first observe that if $\mu\notin\mathrm{sp}_\pi(T)$, then $\mu\notin\mathrm{sp}_\delta(T')$.

In fact, if K_2 is split at level $p=m$, then by 2.2 from [8] K_2' is split at level $p=0$.

Now, by the isomorphism of 2.2 from [8], if we think the homotopy operator which gives the splitting for the complex K_2' at level $p=0$ as a matrix, then each component of the matrix is an adjoint operator. In particular, by means of the properties of the tensor product of [5], it is possible to adapt the proof of the corresponding inclusion of Proposition 4.1 in order to see that if $(\lambda,\mu)\notin\mathrm{sp}_\delta(S)\times\mathrm{sp}_\pi(T)$, then $(\lambda,\mu)\notin\mathrm{sp}_\delta(S\otimes I,I\otimes T')=\mathrm{sp}_\delta(L_S,R_T)$.

The second statement may be proved by means of a similar argument. ∎

In the following proposition we give our description of the semi-Fredholm joint spectra of the tuple (L_S,R_T).

PROPOSITION 6.2. *Let X_1 and X_2 be two complex Banach spaces, and J and operator ideal between X_2 and X_1 in the sense of* [5]. *Let us consider two tuples of commuting operators defined on X_1 and X_2, S and T respectively. Then, if (L_S, R_T) is the tuple of left and right multiplications defined on $\mathrm{L}(J)$, we have that:*

(i) $\sigma_{\Phi_-}(S) \times \sigma_\pi(T) \cup \sigma_\delta(S) \times \sigma_{\Phi_+}(T) \subseteq \sigma_{\Phi_-}(L_S, R_T) \subseteq \mathrm{sp}_{\delta\mathrm{e}}(L_S, R_T) \subseteq \mathrm{sp}_{\delta\mathrm{e}}(S) \times \mathrm{sp}_\pi(T) \cup \mathrm{sp}_\delta(S) \times \mathrm{sp}_{\pi\mathrm{e}}(T)$;

(ii) $\sigma_{\Phi_+}(S) \times \sigma_\delta(T) \cup \sigma_\pi(S) \times \sigma_{\Phi_-}(T) \subseteq \sigma_{\Phi_+}(L_S, R_T) \subseteq \mathrm{sp}_{\pi\mathrm{e}}(L_S, R_T) \subseteq \mathrm{sp}_{\pi e}(S) \times \mathrm{sp}_\delta(T) \cup \mathrm{sp}_\pi(S) \times \mathrm{sp}_{\delta\mathrm{e}}(T)$.

In addition, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. By means of an argument similar to the one of Proposition 4.2, adapted as we have done in Poposition 6.1, it is possible to see that

$$\sigma_{\Phi_-}(S) \times \sigma_\delta(T') \cup \sigma_\delta(S) \times \sigma_{\Phi_-}(T') \subseteq \sigma_{\Phi_-}(S \otimes I, I \otimes T') = \sigma_{\Phi_-}(L_S, R_T).$$

However, by 2.0 from [8] $\sigma_\delta(T') = \sigma_\pi(T)$, and by elementary properties of the adjoint of an operator it is easy to see that $\sigma_{\Phi_+}(T) \subseteq \sigma_{\Phi_-}(T')$. Thus, we have seen the left hand side inclusion of the first statement.

The middle inclusion is clear.

Let us consider $(\lambda, \mu) \in \mathrm{sp}_{\delta\mathrm{e}}(L_S, R_T) \setminus (\mathrm{sp}_{\delta\mathrm{e}}(S) \times \mathrm{sp}_\pi(T) \cup \mathrm{sp}_\delta(S) \times \mathrm{sp}_{\pi\mathrm{e}}(T))$. By Proposition 6.1 we have that $\lambda \in \mathrm{sp}_\delta(S) \setminus \mathrm{sp}_{\delta\mathrm{e}}(S)$ and $\mu \in \mathrm{sp}_\pi(T) \setminus \mathrm{sp}_{\pi\mathrm{e}}(T)$. However, by 2.2 from [8] and elementary properties of the adjoint of an operator we have that $\mu \in \mathrm{sp}_\delta(T') \setminus \mathrm{sp}_{\delta\mathrm{e}}(T')$. Then, as in Proposition 4.2, there are two linear bounded maps $h : X_1 \to X_1 \otimes \wedge^1\mathbb{C}^n$, $g' : X_2' \to X_2' \otimes \wedge^1\mathbb{C}^m$, and two finite rank projectors $k_1 : X_1 \to X_1$ and $K_2' : X_2 \to X_2$ such that

$$d_{11} \circ h = I - k_1, \quad d_{21}' \circ g' = I - K_2',$$

where d_{21}' is the boundary map of the complex K_2' at level $p = 1$.

Now, by the isomorphism of 2.2 from [8], if we think the map g' as a matrix, then each component of the matrix is an adjoint operator. Then, by the properties of the tensor product introduced in [5], we may consider the well defined map

$$H : X_1 \widetilde{\otimes} X_2' \to (K_1 \widetilde{\otimes} K_2')_1, \quad H = (h \otimes I, I \otimes g').$$

Now, by an argument similar to the one of Proposition 4.2, it is easy to see that $(\lambda, \mu) \notin \mathrm{sp}_{\delta\mathrm{e}}(S \otimes I, I \otimes T') = \mathrm{sp}_{\delta\mathrm{e}}(L_S, R_T)$, which is impossible by our assumptions.

By means of a similar argument it is possible to prove the second statement. ∎

We now give our description of the semi-Browder joint spectra of the tuple of left and right multiplications (L_S, R_T) defined on $\mathrm{L}(J)$.

THEOREM 6.3. *Let X_1 and X_2 be two complex Banach spaces, and J an operator ideal between X_2 and X_1 in the sense of* [5]. *Let us consider two tuples of commuting operators defined on X_1 and X_2, S and T respectively. Then, if (L_S, R_T) is the tuple of left and right multiplications defined on $\mathrm{L}(J)$, we have that:*

(i) $\sigma_{\mathcal{B}_-}(S)\times\sigma_\pi(T)\cup\sigma_\delta(S)\times\sigma_{\mathcal{B}_+}(T)\subseteq\sigma_{\mathcal{B}_-}(L_S,R_T)\subseteq\mathrm{sp}_{\mathcal{B}_-}(L_S,R_T)\subseteq\mathrm{sp}_{\mathcal{B}_-}(S)\times\mathrm{sp}_\pi(T)\cup\mathrm{sp}_\delta(S)\times\mathrm{sp}_{\mathcal{B}_+}(T)$;

(ii) $\sigma_{\mathcal{B}_+}(S)\times\sigma_\delta(T)\cup\sigma_\pi(S)\times\sigma_{\mathcal{B}_-}(T)\subseteq\sigma_{\mathcal{B}_+}(L_S,R_T)\subseteq\mathrm{sp}_{\mathcal{B}_+}(L_S,R_T)\subseteq\mathrm{sp}_{\mathcal{B}_+}(S)\times\mathrm{sp}_\delta(T)\cup\mathrm{sp}_\pi(S)\times\mathrm{sp}_{\mathcal{B}_-}(T)$.

In addition, if X_1 and X_2 are Hilbert spaces, the above inclusions are equalities.

Proof. In order to see the first statement, let us observe that if K_1^r is the Koszul complex associated to the tuple $(S-\lambda)^r=((S_1-\lambda_1)^r,\dots,(S_n-\lambda_n)^r)$, and if ${K'}_2^r$ is the Koszul complex associated to the tuple $(T'-\mu)^r=(({T'}_1-\mu_1)^r,\dots,(T_m-\mu_m)^r)$, then $\widetilde{K}^r$, the Koszul complex associated to the tuple $((S-\lambda)^r\otimes T, I\otimes(T'-\mu)^r)$, is isomorphic to the total complex obtained from the double complex of the tensor product of K_1^r and ${K'}_2^r$, i.e., $\widetilde{K}^r\cong K_1^r\widetilde{\otimes}{K'}_2^r$ (see 3 from [5]).

Now, we may adapt the proof of the left hand inclusion of Theorem 5.1, as we have done in Proposition 6.1, using in particular Proposition 6.2 instead of Proposition 4.2, in order to see that $\sigma_{\mathcal{B}_-}(S)\times\sigma_\delta(T')\subseteq\sigma_{\mathcal{B}_-}(S\otimes I, I\otimes T')=\sigma_{\mathcal{B}_-}(L_S,R_T)$. However, by 2.0 from [8], $\sigma_\pi(T)=\sigma_\delta(T')$. Thus, $\sigma_{\mathcal{B}_-}(S)\times\sigma_\pi(T)\subseteq\sigma_{\mathcal{B}_-}(L_S,R_T)$.

A similar argument, using in particular that $\sigma_{\mathcal{B}_-}(T')=\sigma_{\mathcal{B}_+}(T)$ (see 11 from [7]) gives us that $\sigma_\delta(S)\times\sigma_{\mathcal{B}_+}(T)\subseteq\sigma_{\mathcal{B}_-}(L_S,R_T)$.

The middle inclusion is clear.

In order to see the right hand inclusion, it is possible to adapt the proof of the corresponding part of Theorem 5.1.

Indeed, if we use Proposition 6.2 instead of Proposition 4.2, we have that $\mathrm{sp}_{\delta\mathrm{e}}(L_S,R_T)\subseteq\mathrm{sp}_{\mathcal{B}_-}(S)\times\mathrm{sp}_\pi(T)\cup\mathrm{sp}_\delta(S)\times\mathrm{sp}_{\mathcal{B}_+}(T)$. On the other hand, if we suppose that $(\lambda,\mu)\in\widetilde{\mathcal{A}}(L_S,R_T)\setminus(\mathrm{sp}_{\mathcal{B}_-}(S)\times\mathrm{sp}_\pi(T)\cup\mathrm{sp}_\delta(S)\times\mathrm{sp}_{\mathcal{B}_+}(T))$, then it is possible to adapt the argument of Theorem 5.1 in order to get a contradiction. However, in order to adapt this part of the proof, we have to observe the following facts.

First, by 2.2 from [8], if $\mu\notin\mathrm{sp}_{\pi\mathrm{e}}(T)$, then $\mu\notin\mathrm{sp}_{\delta\mathrm{e}}(T')$. Moreover, if there exists $l\in\mathbb{N}$ such that for all $r\geqslant l$ $\dim H_m(K_2^r)=\dim H_m(K_2^l)$, then by 11 from [7] it is easy to see that $\dim H_0({K'}_2^r)=\dim H_0({K'}_2^l)$, for all $r\geqslant l$. In addition, if $\mu\notin\mathrm{sp}_{\delta\mathrm{e}}(T')$, by the analytic spectral mapping theorem for the essential split defect spectrum, the complex ${K'}_2^r$ are Fredholm split for all $r\in\mathbb{N}$, i.e., there are operators $g'_r\colon X'_2\to X'_2\otimes\wedge^1\mathbb{C}^m$ and finite rank projectors $k'_{2r}\colon X'_2\to X'_2$ such that $d_{21}^{r'}\circ g'_r=I-k'_{2r}$, where $d_{21}^{r'}$ denotes the chain map of the complex ${K'}_2^r$ at level $p=1$. Furthermore, by 2.2 from [8], if for $r\in\mathbb{N}$ we think the map g'_r as a matrix, then each component of the a matrix is an adjoint operator, and by elementary properties of the adjoint of an operator, the maps g'_r and k'_{2r} may be chosen with the same properties of the maps g_r and k_{2r} of Theorem 5.1. With all these observations it is possible to conclude the proof of the right hand side inclusion of the first statement.

The second statement may be proved by means of a similar argument. ∎

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ENRICO BOASSO
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria Pab. I
(1428) Buenos Aires
REPÚBLICA ARGENTINA
E-mail: eboasso@dm.uba.ar

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