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REAL RANK AND EXPONENTIAL LENGTH OF TENSOR PRODUCTS WITH $\mathcal{O}_\infty$

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Abstract. Let D be any C^*-algebra. We prove that $\mathcal{O}_\infty \otimes D$ has real rank at most 1, exponential length at most 2π, exponential rank at most $2+\varepsilon$, and C^* projective length at most π. The algebra $\mathcal{O}_\infty$ can be replaced with any separable nuclear purely infinite simple C^*-algebra.

Keywords: *Real rank, exponential rank, exponential length, projective length, infinite C^*-algebras.*

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Let $\mathcal{O}_\infty$ be the Cuntz algebra ([4]) generated by infinitely many isometries with orthogonal ranges. The purpose of this paper is to compute several ranks of C^*-algebras of the form $\mathcal{O}_\infty \otimes D$, with D an arbitrary C^*-algebra. For these algebras, the original rank, Rieffel's (topological) stable rank, is not very interesting, at least if D is unital, since then $\mathrm{tsr}(\mathcal{O}_\infty \otimes D) = \infty$ by Proposition 6.5 of [21]. Therefore we will study the real rank ([3]) and the exponential rank and length ([19] and [22]). It is not a priori evident that these problems are related, or even should be considered together. However, the results turn out to be similar: both ranks can be at most 1 more than the corresponding rank of a purely infinite simple C^*-algebra. Moreover, the proofs use the same key property of $\mathcal{O}_\infty$ and are similar in spirit.

Unfortunately, there are very few results giving explicit relations between the real and exponential ranks.

The exponential length result $\mathrm{cel}(\mathcal{O}_\infty \otimes D) \leqslant 2\pi$ was promised in [20]. For the approach used there for the classification of nuclear purely infinite simple C^*-algebras, it is important to know that there is a finite upper bound L, not depending on D, for $\mathrm{cel}(\mathcal{O}_\infty \otimes D)$ for a number of C^*-algebras D. This kind of condition first appeared in [23], but without the name; Rørdam was evidently unaware of [22]. The upper bound $L = 3\pi$ is much easier to prove than the result here and suffices for the purposes of [20]. But results of Zhang ([24]) certainly suggested that the correct answer might be 2π, and we prove that here.

The key property of $\mathcal{O}_\infty$ is essentially that it has an asymptotically central embedding in itself. This is a consequence of results of Kirchberg ([9]; proofs in [10] and in [11]). The following form of this result is convenient for our purposes:

Theorem 0.1. (Kirchberg) *There is an isomorphism $\varphi : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$ such that the homomorphism $a \mapsto \varphi(1 \otimes a)$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_\infty}$. That is, there is a sequence (z_n) of unitaries in $\mathcal{O}_\infty$ such that we have $\lim\limits_{n\to\infty} z_n \varphi(1 \otimes a) z_n^* = a$ for all $a \in \mathcal{O}_\infty$.*

Proof. This is contained in the proof of Theorem 3.15 of [11]. But it can easily be derived from that result, as follows: Use it to choose any isomorphism $\varphi : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$, and observe that $a \mapsto \varphi(1 \otimes a)$ and $\mathrm{id}_{\mathcal{O}_\infty}$ are two unital homomorphisms from $\mathcal{O}_\infty$ to $\mathcal{O}_\infty$, hence approximately unitarily equivalent by Theorem 3.3 of [13]. ∎

In fact, there is an asymptotically central embedding of $\mathcal{O}_\infty$ in any separable nuclear purely infinite simple C^*-algebra A, but we extend our results to tensor products with such algebras A by simply observing (Theorem 3.15 of [11]) that $\mathcal{O}_\infty \otimes A \cong A$.

There are results parallel to those of this paper for C^*-algebras of the form $A \otimes C(X)$, where A is an arbitrary purely infinite simple C^*-algebra, not necessarily nuclear. Results on exponential rank and exponential length of such algebras appear in Zhang's paper ([24]). One of Zhang's results has recently been generalized as follows ([8]): If A is a unital purely infinite simple C^*-algebra, and B is any unital C^*-algebra, then for any C^* tensor product we have $\mathrm{cel}(A \otimes B) \leqslant 5\pi/2$ and $\mathrm{cer}(A \otimes B) \leqslant 3$. Theorem 3.7 of [16] shows that if A is purely infinite and simple, with $K_0(A) = 0$, then the related invariant C^* projective length satisfies $\mathrm{cpl}(A \otimes C(X)) \leqslant 3\pi/2$; this invariant is also treated here. Similar results on the real rank of $A \otimes C(X)$ appear in [14].

As noted above, in our results $\mathcal{O}_\infty$ may be replaced by any separable nuclear purely infinite simple C^*-algebra. This, and the other results described above, suggest the possibility that our estimates hold for $A \otimes B$ for any purely infinite simple C^*-algebra A, any C^*-algebra B, and any choice of the C^* tensor product. Proving this is expected to be more difficult. Indeed, by Theorem 1.4 of [7], a nonnuclear purely infinite simple C^*-algebra need not even be approximately divisible in the sense of [2].

The exponential length $\mathrm{cer}(A)$, and the related invariants $\mathrm{cel}(A)$, $\mathrm{cpr}(A)$, and $\mathrm{cpl}(A)$ have been computed or estimated for many C^*-algebras A, and we refer to [19] for a recent compilation of the known results (not including that of [8], which appeared later).

We use the following notation throughout this paper. We let A^+ denote the unitization of A. We use this notation for the C^*-algebra obtained by adding a new unit even if A is already unital. We let A_{sa} denote the set of selfadjoint elements of A, and we let K denote the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space. Moreover, we use without comment the fact (Theorem 1.4 and Proposition 1.5 of [5]) that if A is a purely infinite simple C^*-algebra, then $K_0(A)$ is the set of Murray-von Neumann equivalence classes of nonzero projections in A.

This paper consists of three sections. The first treats real rank, the second contains various lemmas needed for the result on exponential length, and the third proves the result on exponential length and uses it to treat exponential rank and projective length and rank.

1. REAL RANK

In this section, we prove that $\mathrm{RR}(\mathcal{O}_\infty \otimes D) \leqslant 1$ for any C^*-algebra D. This is the best possible general result, since it is easy to see that $\mathrm{RR}(\mathcal{O}_\infty \otimes C([0,1])) \geqslant 1$. Our proof uses two lemmas.

Lemma 1.1. *Let A be a unital purely infinite simple C^*-algebra, let $p \in A$ be a projection with $p \neq 0$, 1, let $\varepsilon > 0$, and let $\mu \in [-1,1]$. Then there is a continuous function $a : [-1,1] \to A_{\mathrm{sa}}$ with the following properties:*

(i) $\|a(\lambda) - \lambda \cdot 1\| < \varepsilon$ *for all* $\lambda \in [-1,1]$;

(ii) $a(\lambda)^2$ *commutes with* p *for all* $\lambda \in [-1,1]$;

(iii) $pa(\lambda)^2 p$ *is invertible in* pAp *for all* $\lambda \in [-1,1]$;

(iv) $a(\mu) \in \mathbb{C} \cdot 1$.

Proof. We start by constructing, for given $\delta > 0$, a continuous function $a_\delta : \mathbb{R} \to A_{\mathrm{sa}}$ satisfying (ii), (iii), and $\|a_\delta(\lambda) - \lambda \cdot 1\| \leqslant 4\delta$ for all $\lambda \in \mathbb{R}$, and also $a_\delta(\lambda) = \lambda \cdot 1$ for $|\lambda| \geqslant 2\delta$.

Define a piecewise continuous linear function $f_- : \mathbb{R} \to \mathbb{R}$ by

$$f_-(\lambda) = \begin{cases} \lambda, & |\lambda| \geqslant 2\delta; \\ -2\delta, & -2\delta \leqslant \lambda \leqslant \delta; \\ 4(\lambda - \delta) - 2\delta, & \delta \leqslant \lambda \leqslant 2\delta. \end{cases}$$

Then define $f_+(\lambda) = -f_-(-\lambda)$.

Choose a projection $q_0 \in A$ such that $q_0 \leqslant 1 - p$ and q_0 is homotopic to p. Set $q = 1 - q_0$, which is a projection satisfying $q \geqslant p$ and homotopic to $1 - p$. Then we can choose a continuous unitary path $\lambda \mapsto u(\lambda)$ in A, defined for $\lambda \in [-\delta, \delta]$, such that $u(-\delta) = 1$ and $u(\delta)(1-p)u(\delta)^* = q$. Extend the definition of u over $\mathbb{R}$ by setting $u(\lambda) = 1$ for $\lambda < -\delta$ and $u(\lambda) = u(\delta)$ for $\lambda > \delta$.

Now define

$$a_\delta(\lambda) = u(\lambda)[f_-(\lambda)p + f_+(\lambda)(1-p)]u(\lambda)^*$$

for all λ. Clearly a_δ is selfadjoint and continuous. Furthermore, from the definitions of f_+ and f_-, we get $a_\delta(\lambda) = \lambda \cdot 1$ for $|\lambda| \geqslant 2\delta$. This equation immediately implies all the other required properties of $a_\delta(\lambda)$ for these values of λ. Furthermore, the estimate $\|a_\delta(\lambda) - \lambda \cdot 1\| \leqslant 4\delta$, for all $\lambda \in \mathbb{R}$, is immediate from the easily checked inequalities $|f_-(\lambda) - \lambda| \leqslant 4\delta$ and $|f_+(\lambda) - \lambda| \leqslant 4\delta$ for all $\lambda \in \mathbb{R}$. It therefore only remains to prove (ii) and (iii) for $-2\delta \leqslant \lambda \leqslant 2\delta$.

For $-2\delta \leqslant \lambda \leqslant -\delta$, we have $u(\lambda) = 1$, from which it is clear that already $a_\delta(\lambda)$ commutes with p. Moreover, $pa_\delta(\lambda)p = f_-(\lambda)p = -2\delta p$, from which it is clear that $pa_\delta(\lambda)^2 p$ is invertible in pAp. For $\delta \leqslant \lambda \leqslant 2\delta$, we have $a_\delta(\lambda) = 2\delta q + f_-(\lambda)(1-q)$. Since $q \geqslant p$, clearly $a_\delta(\lambda)$ commutes with p, and $pa_\delta(\lambda)^2 p = 4\delta^2 p$, which is invertible in pAp. Finally, for $-\delta \leqslant \lambda \leqslant \delta$, we have $f_-(\lambda) = -2\delta$ and

$f_+(\lambda) = 2\delta$. Therefore $a_\delta(\lambda)^2 = 4\delta^2 \cdot 1$, from which it is clear that $a_\delta(\lambda)^2$ commutes with p and $pa_\delta(\lambda)^2 p$ is invertible in pAp. This proves the properties required of a_δ.

Now let $\varepsilon > 0$. Set $\delta = \frac{1}{8}\varepsilon$ and $\lambda_0 = 3\delta$. If $\mu \geqslant 0$, define $a(\lambda) = a_\delta(\lambda + \lambda_0)$. For (i), note that

$$\|a(\lambda) - \lambda \cdot 1\| \leqslant \|a_\delta(\lambda + \lambda_0) - (\lambda + \lambda_0) \cdot 1\| + |\lambda_0| \leqslant 4 \cdot \frac{\varepsilon}{8} + \frac{3\varepsilon}{8} < \varepsilon.$$

Properties (ii) and (iii) follow from the corresponding properties of a_δ. For (iv), note that $a(\mu) = a_\delta(\mu + \lambda_0) = (\mu + \lambda_0) \cdot 1 \in \mathbb{C} \cdot 1$ because $\mu + \lambda_0 > 2\delta$. If instead $\mu \leqslant 0$, define $a(\lambda) = a_\delta(\lambda - \lambda_0)$ to get the same conclusions. ∎

Lemma 1.2. *Let D be any C^*-algebra, let $a_0, a_1 \in (D^+)_{\mathrm{sa}}$, and let $\varepsilon > 0$. Then there exist $b_0, b_1 \in [(\mathcal{O}_\infty \otimes D)^+]_{\mathrm{sa}}$ such that $\|b_0 - 1 \otimes a_0\|$, $\|b_1 - 1 \otimes a_1\| < \varepsilon$ and $b_0^2 + b_1^2$ is invertible.*

Proof. By scaling, we may assume without loss of generality that $\|a_0\|$, $\|a_1\| \leqslant 1$. Then there are homomorphisms $\varphi_0, \varphi_1 : C([-1,1]) \to D^+$ such that, with $f(\lambda) = \lambda$ for all λ, we have $\varphi_0(f) = a_0$ and $\varphi_1(f) = a_1$. Choose $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $a_0 - \lambda_0 \cdot 1$, $a_1 - \lambda_1 \cdot 1 \in D_{\mathrm{sa}}$. It follows that $\varphi_j(g) \in D$ exactly when $g(\lambda_j) = 0$.

Choose nonzero projections $p_0, p_1 \in \mathcal{O}_\infty$ such that $p_0 + p_1 = 1$. Use Lemma 1.1 to choose $d_0, d_1 \in [\mathcal{O}_\infty \otimes C([-1,1])]_{\mathrm{sa}}$, taking $A = \mathcal{O}_\infty$, taking ε as above, taking p to be p_0 and p_1 respectively, and taking μ to be λ_0 and λ_1 respectively. Condition (iv) of Lemma 1.1 implies that $d_j \in [\mathcal{O}_\infty \otimes C_0([-1,1] \setminus \{\lambda_j\})]^+$. Therefore

$$b_j = (\mathrm{id}_{\mathcal{O}_\infty} \otimes \varphi_j)(d_j) \in [(\mathcal{O}_\infty \otimes D)^+]_{\mathrm{sa}}.$$

Moreover, we can transfer the properties of the d_j to $(\mathcal{O}_\infty \otimes D)^+$ via $\mathrm{id}_{\mathcal{O}_\infty} \otimes \varphi_j$. So $\|b_j - 1 \otimes a_j\| < \varepsilon$, and also

$$\begin{aligned} b_0^2 + b_1^2 &= (p_0 \otimes 1)b_0^2(p_0 \otimes 1) + ((1 - p_0) \otimes 1)b_0^2((1 - p_0) \otimes 1) \\ &\quad + (p_1 \otimes 1)b_1^2(p_1 \otimes 1) + ((1 - p_1) \otimes 1)b_1^2((1 - p_1) \otimes 1) \\ &\geqslant (p_0 \otimes 1)b_0^2(p_0 \otimes 1) + (p_1 \otimes 1)b_1^2(p_1 \otimes 1). \end{aligned}$$

Each term in the last expression is invertible in its corner, and $p_0 + p_1 = 1$, so it follows that $b_0^2 + b_1^2$ is invertible, as desired. ∎

Theorem 1.3. *Let D be any C^*-algebra. Then $\mathrm{RR}(\mathcal{O}_\infty \otimes D) \leqslant 1$.*

Proof. Let $\varphi : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$ and $(z_n) \in \mathcal{O}_\infty$ be as in Theorem 0.1, that is, $\lim_{n \to \infty} z_n \varphi(1 \otimes a) z_n^* = a$ for all $a \in \mathcal{O}_\infty$. Then for any $a \in (\mathcal{O}_\infty \otimes D)^+$ we have

$$\lim_{n \to \infty} (z_n \otimes 1) \left[(\varphi \otimes \mathrm{id}_{\mathcal{O}_\infty})(1 \otimes a)\right] (z_n \otimes 1)^* = a$$

in $\mathcal{O}_\infty \otimes D^+$. One furthermore checks that

$$(z_n \otimes 1) \left[(\varphi \otimes \mathrm{id}_{\mathcal{O}_\infty})([\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D]^+)\right] (z_n \otimes 1)^* \subset (\mathcal{O}_\infty \otimes D)^+.$$

Let $a_0, a_1 \in (\mathcal{O}_\infty \otimes D)^+$ be selfadjoint, and let $\varepsilon > 0$. Choose n such that

$$\|(z_n \otimes 1) \left[(\varphi \otimes \mathrm{id}_{\mathcal{O}_\infty})(1 \otimes a_j)\right] (z_n \otimes 1)^* - a_j\| < \frac{1}{2}\varepsilon$$

for $j = 0, 1$. Use Lemma 1.2 to choose selfadjoint elements $b_0, b_1 \in (\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D)^+$ such that $\|b_j - 1 \otimes a_j\| < \frac{1}{2}\varepsilon$ and $b_0^2 + b_1^2$ is invertible. Set

$$c_j = (z_n \otimes 1)\,[(\varphi \otimes \mathrm{id}_{\mathcal{O}_\infty})(1 \otimes b_j)]\,(z_n \otimes 1)^*.$$

Then $c_j \in [(\mathcal{O}_\infty \otimes D)^+]_{\mathrm{sa}}$, $c_0^2 + c_1^2$ is invertible, and $\|c_j - a_j\| < \varepsilon$ for $j = 0, 1$. ∎

COROLLARY 1.4. *Let A be any separable nuclear purely infinite simple C^*-algebra, and let D be any C^*-algebra. Then* $\mathrm{RR}(A \otimes D) \leqslant 1$.

Proof. This follows from the theorem and the isomorphism $\mathcal{O}_\infty \otimes A \cong A$ ([10]; Theorem 3.15 of [11]). ∎

2. APPROXIMATE ABSORPTION

This section contains various lemmas needed for the proof that $\mathrm{cel}(\mathcal{O}_\infty \otimes D) \leqslant 2\pi$. The last one is an approximate absorption result, in a sense related to that of [12]. We begin by recalling the definition of exponential length ([22]) in a form suitable for our purposes, and extending it to nonunital C^*-algebras.

DEFINITION 2.1. Let A be a C^*-algebra. Define $U(A) = \{u \in U(A^+) : u - 1 \in A\}$. When A is unital, we identify this group with the usual unitary group of A by sending a unitary $u \in A$ to $(u, 1) \in A^+ \cong A \oplus \mathbb{C}$. Define $U_0(A)$ to be the connected component of $U(A)$ which contains 1.

If $u : [0, 1] \to U(A)$ is a continuous path, then we define its *length* by

$$l(u) = \lim_P \sum_{j=1}^{n} \|u(t_j) - u(t_{j-1})\|,$$

where the limit is taken over all partitions $P = \{t_0, t_1, \ldots, t_n\}$ of $[0, 1]$ (that is, $0 = t_0 < t_1 < \cdots < t_n = 1$), ordered by refinement (inclusion). (Compare with Definition 1.1 of [16].) Now for $v \in U_0(A)$, define

$$\mathrm{cel}(v) = \inf\{l(u) : u \text{ is a continuous unitary path with } u(0) = 1 \text{ and } u(1) = v\},$$

and set

$$\mathrm{cel}(A) = \sup\{l(v) : v \in U_0(A)\}.$$

Proposition 2.9 of [22] shows that this definition of $\mathrm{cel}(v)$ agrees with that of [22]. Therefore our definition of $\mathrm{cel}(A)$ agrees with that of [22] for unital A. For nonunital A, this definition differs from that suggested in Section 6 of [19] and used in [8] ($\mathrm{cel}(A) = \mathrm{cel}(A^+)$), but seems more natural.

Lemma 2.2. *Let u and v be continuous unitary paths in a unital C^*-algebra A, and let $\varphi : A \to B$ be a unital homomorphism. Then:*

(i) *the path $(uv)(t) = u(t)v(t)$ satisfies $l(uv) \leqslant l(u) + l(v)$;*
(ii) *the path $u^*(t) = u(t)^*$ satisfies $l(u^*) = l(u)$;*
(iii) *the path $\varphi(u)(t) = \varphi(u(t))$ satisfies $l(\varphi(u)) \leqslant l(u)$.*

Proof. These are immediate from the following three estimates:

$$\begin{aligned}\|u(s)v(s) - u(t)v(t)\| &\leqslant \|u(s) - u(t)\| \, \|v(s)\| + \|u(t)\| \, \|v(s) - c(t)\| \\ &= \|u(s) - u(t)\| + \|v(s) - v(t)\|,\end{aligned}$$

and

$$\|u(s)^* - u(t)^*\| = \|u(s) - u(t)\| \quad \text{and} \quad \|\varphi(u(s)) - \varphi(u(t))\| \leqslant \|u(s) - u(t)\|. \quad ∎$$

Lemma 2.3. *Let A be a purely infinite simple separable C^*-algebra, and let B be a hereditary subalgebra of the function algebra $C([0,1], A)$. Let $\mathrm{ev}_t : C([0,1], A) \to A$ be evaluation at $t \in [0,1]$, and assume that $\mathrm{ev}_t(B) \subset A$ is nonzero and nonunital for all t. Then there is an isomorphism $\varphi : B \to C([0,1], A \otimes K)$ which has the form $\varphi(b)(t) = \varphi_t(b)$ for isomorphisms $\varphi_t : \mathrm{ev}_t(B) \to A \otimes K$.*

Proof. By Proposition 2.6 of [12], B has an increasing approximate identity (p_n) consisting of projections. We may assume strict inequality at zero: $0 < p_1(0) < p_2(0) < \cdots$. We now claim that there exist projections $q_n \in B$ such that

$$0 = q_0 < p_1 < q_1 < p_2 < q_2 < \cdots$$

and $[q_{n+1} - q_n] = 0$ in $K_0(C([0,1], A))$. The q_n are constructed by induction, starting with $q_0 = 0$. Assume q_{n-1} has been chosen. Standard methods provide a unitary $u \in C([0,1], A^+)$ such that $u(0) = 1$ and up_nu^* and $up_{n+1}u^*$ are constant projections in $C([0,1], A)$, with values $p_n(0)$ and $p_{n+1}(0)$. Since A is purely infinite and simple, there exists a projection $f \in A$ with $p_n(0) < f < p_{n+1}(0)$ and $[f] = [q_{n-1}(0)]$ in $K_0(A)$. (We actually have $[q_{n-1}(0)] = 0$.) Regard f as a constant projection in $C([0,1], A)$ and define $q_n = u^*fu$. Note that $q_n \in B$ because $q_n \leqslant p_{n+1}$ and $p_{n+1} \in B$. This completes the induction step and proves the claim.

Clearly (q_n) is again an increasing approximate identity of projections for B. Since A is purely infinite and simple, there are partial isometries $v_n \in A$ such that

$$v_nv_n^* = q_n(0) - q_{n-1}(0) \quad \text{and} \quad v_n^*v_n = q_1(0).$$

Standard methods now give partial isometries $w_n \in C([0,1], A)$ such that

$$w_nw_n^* = q_n - q_{n-1} \quad \text{and} \quad w_n^*w_n = q_1.$$

Since B is hereditary, we actually have $w_n \in B$.

Let (e_{jk}) be a system of matrix units for K. Then the formula $\psi(a \otimes e_{jk}) = w_jaw_k^*$ defines an isometric isomorphism

$$\bigcup_{n=1}^{\infty} q_1C([0,1], A)q_1 \otimes M_n \to \bigcup_{n=1}^{\infty} q_nC([0,1], A)q_n$$

which extends by continuity to an isomorphism $\psi : q_1C([0,1], A)q_1 \otimes K \to B$. (Recall that (q_n) is an approximate identity for B.) Now

$$q_1C([0,1], A)q_1 \cong C([0,1], q_1(0)Aq_1(0))$$

via a unitary equivalence of q_1 with a constant projection, and $q_1(0)Aq_1(0) \otimes K \cong A \otimes K$. We therefore obtain an isomorphism $\sigma : q_1C([0,1], A)q_1 \to C([0,1], A \otimes K)$, and we can take $\varphi = \sigma \circ \psi^{-1}$. ∎

Lemma 2.4. *Let $p_1, \ldots, p_n$ be mutually orthogonal projections in $\mathcal{O}_\infty \otimes C(S^1)$. Then there exist mutually orthogonal projections $q_1, \ldots, q_n \in \mathcal{O}_\infty$ and a unitary $z \in \mathcal{O}_\infty \otimes C(S^1)$ such that $z(q_j \otimes 1)z^* = p_j$ for $j = 1, \ldots, n$.*

Proof. By considering $p_1, \ldots, p_n, 1 - \sum_{j=1}^{n} p_j$, we reduce to the case $\sum_{j=1}^{n} p_j = 1$. Now identify $\mathcal{O}_\infty \otimes C(S^1)$ with the algebra of continuous functions from S^1 to $\mathcal{O}_\infty$, and set $q_j = p_j(1)$. If $z_j \in \mathcal{O}_\infty \otimes C(S^1)$ satisfies $z_j z_j^* = p_j$ and $z_j^* z_j = q_j \otimes 1$, then the unitary $z = \sum_{j=1}^{n} z_j$ will satisfy the conclusion of the lemma. It therefore suffices to prove, for a single projection $p \in \mathcal{O}_\infty \otimes C(S^1)$, that p is Murray-von Neumann equivalent to $p(1) \otimes 1$.

Standard methods give a continuous path $t \mapsto w_t$ of unitaries in $\mathcal{O}_\infty$, defined for $t \in [0, 1]$, such that $w_0 = 1$ and $w_t p(1) w_t^* = p(\exp(2\pi i t))$ for all t. Then w_1 commutes with $p(1)$, so $p(1)w_1p(1)$ is a unitary in $p(1)\mathcal{O}_\infty p(1)$. The unitary group of this algebra is connected, so there is a continuous path $t \mapsto c_t$ of unitaries in $p(1)\mathcal{O}_\infty p(1)$, defined for $t \in [0, 1]$, such that $c_0 = p(1)$ and $c_1 = p(1)w_1p(1)$. Set $z(\exp(2\pi i t)) = w_t p(1) c_t^*$ for $t \in [0, 1]$. Our choices ensure that z is well defined and continuous, and defines an element $z \in \mathcal{O}_\infty \otimes C(S^1)$ such that $zz^* = p$ and $z^* z = p(1) \otimes 1$. ∎

The following lemma is the analog in our situation of Lemma 6 of [17], itself a generalization of Lemma 1.7 of [5].

Lemma 2.5. *Let D be a unital C^*-algebra, let $v \in D$ be unitary, let $\lambda_1, \ldots, \lambda_n \in S^1$ be distinct, and let $\varepsilon > 0$. Let $u_0 \in \mathcal{O}_\infty$ be any unitary with $\mathrm{sp}(u_0) = S^1$. Then there exist nonzero mutually orthogonal projections $q_1, \ldots, q_n \in \mathcal{O}_\infty$ and unitaries $w, z \in \mathcal{O}_\infty \otimes D$ such that $\|u_0 \otimes v - w\| < \varepsilon$ and the projections $p_j = z(q_j \otimes 1)z^*$ satisfy $p_j w = w p_j = \lambda_j p_j$.*

Proof. It suffices to prove this with $D = C(S^1)$ and v given by $v(\zeta) = \zeta$ for $\zeta \in S^1$. (For D and v arbitrary, consider the image of everything under the homomorphism $a \otimes f \mapsto a \otimes f(v)$ from $\mathcal{O}_\infty \otimes C(S^1)$ to $\mathcal{O}_\infty \otimes D$.) By the previous lemma, in this case we need only find w with $\|u_0 \otimes v - w\| < \varepsilon$ and nonzero orthogonal projections $p_1, \ldots, p_n \in \mathcal{O}_\infty \otimes C(S^1)$ such that $p_j w = w p_j = \lambda_j p_j$; the existence of the q_j and of z will then be automatic.

Choose nonzero positive continuous functions $h_1, \ldots, h_n \in C(S^1)$ whose supports are disjoint and contained in the sets $\{\zeta \in S^1 : |\zeta - \lambda_j| < \varepsilon_0\}$. We show below that there is a nonzero projection p_j in the hereditary subalgebra B_j of $\mathcal{O}_\infty \otimes C(S^1)$ generated by $h_j(u_0 \otimes v)$. As in [17], the proof of Lemma 1.7 of [5] will then show that, if ε_0 is chosen sufficiently small, then the unitary part of the polar decomposition of

$$\left(1 - \sum_{j=1}^{n} p_j\right)(u_0 \otimes v)\left(1 - \sum_{j=1}^{n} p_j\right) + \sum_{j=1}^{n} \lambda_j p_j$$

is the required w.

We now find p_j. Identify $\mathcal{O}_\infty \otimes C(S^1)$ with the algebra of continuous functions from $[0, 1]$ to $\mathcal{O}_\infty$ which take the same value at 0 and 1, in such a way that v

becomes $v(t) = \exp(2\pi i t)$. Let $E_j \subset \mathcal{O}_\infty \otimes C([0,1])$ be the hereditary subalgebra of $\mathcal{O}_\infty \otimes C([0,1])$ generated by $h_j(u_0 \otimes v)$, so that $B_j = E_j \cap [\mathcal{O}_\infty \otimes C(S^1)]$. Let $\mathrm{ev}_t : \mathcal{O}_\infty \otimes C([0,1]) \to \mathcal{O}_\infty$ be evaluation at t. Then $\mathrm{ev}_t(E_j)$ is the hereditary subalgebra of $\mathcal{O}_\infty$ generated by $\mathrm{ev}_t(h_j(u_0 \otimes v)) = h_j(\exp(2\pi i t)u_0)$. Note that for every t this subalgebra is nonunital and nonzero. Lemma 2.3 therefore implies that $E_j \cong \mathcal{O}_\infty \otimes C([0,1]) \otimes K$. Moreover, the special form of the isomorphism in that lemma shows that there is an automorphism α_j of $\mathcal{O}_\infty \otimes C([0,1]) \otimes K$ such that the image of B_j is

$$C_j = \{b \in \mathcal{O}_\infty \otimes C([0,1]) \otimes K : b(1) = \alpha_j(b(0))\}.$$

It suffices to find a nonzero projection in C_j. Let $f \in \mathcal{O}_\infty \otimes K$ be any nonzero projection with $[f] = 0$ in $K_0(\mathcal{O}_\infty \otimes K)$. Then also $\alpha_j(f) \neq 0$ and $[\alpha_j(f)] = 0$ in $K_0(\mathcal{O}_\infty \otimes K)$. So f and $\alpha_j(f)$ are homotopic and this homotopy is the desired nonzero projection in C_j. ∎

Lemma 2.6. *Let A be a unital C^*-algebra, let $a \in A$ be selfadjoint with $\mathrm{sp}(a) \subset [\alpha, \beta]$, and let $n \in \mathbb{N}$. Let $b \in M_{n+1}(A)$ be the diagonal matrix*

$$b = \mathrm{diag}\left(\alpha, \alpha + \frac{\beta - \alpha}{n}, \alpha + \frac{2(\beta - \alpha)}{n}, \ldots, \beta\right),$$

and let $a_0 = \mathrm{diag}(a, 0, \ldots, 0)$. Then there exist a projection p and a unitary u in $M_{n+1}(A)$ such that

$$upu^* = \mathrm{diag}(1, 0, \ldots, 0), \quad \|pb - bp\| \leqslant \frac{\beta - \alpha}{2n}, \quad upbpu^* = a_0,$$

and

$$\left\| u(1-p)b(1-p)u^* - \mathrm{diag}\left(0, \alpha + \frac{\beta - \alpha}{n}, \alpha + \frac{2(\beta - \alpha)}{n}, \ldots, \beta\right) \right\| \leqslant \frac{\beta - \alpha}{n}.$$

Much more general results are now known, involving homomorphisms from commutative C^*-algebras in place of selfadjoint elements. (See for example [6].) But this lemma has the advantage of being explicit (which will make the proof of the next lemma easier), and most of the work has already been done.

Proof of Lemma 2.6. All but the last of the four conclusions is in Lemma 2.4 of [18]. For the last one, we reduce as there to the case handled in the proof of that lemma ($\alpha = 0$, $\beta = 1$, $A = C(T)$ with $T \subset [0,1]$, and $a(t) = t$). With $p(t)$ as there, for

$$t \in \left[\frac{l-1}{n}, \frac{l}{n}\right] \quad \text{and} \quad \lambda = n\left(t - \frac{l-1}{n}\right) \in [0,1],$$

and with e being the rank one projection

$$e(t) = \begin{pmatrix} \lambda & -\sqrt{\lambda(1-\lambda)} \\ -\sqrt{\lambda(1-\lambda)} & 1-\lambda \end{pmatrix} \in M_2,$$

a computation shows that

$$[1-p(t)]b(t)[1-p(t)] = \mathrm{diag}\left(0, \frac{1}{n}, \ldots, \frac{l-2}{n}\right) \oplus \left(\frac{l-\lambda}{n}\right)e(t) \oplus \mathrm{diag}\left(\frac{l+1}{n}, \ldots, 1\right).$$

It is easy to choose $u(t)$ to conjugate this element to

$$\operatorname{diag}\Big(0, 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{l-2}{n}, \frac{l-\lambda}{n}, \frac{l+1}{n}, \dots, 1\Big),$$

which differs by at most $\frac{1}{n}$ from

$$\operatorname{diag}\left(0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{l-1}{n}, \frac{l}{n}, \frac{l+1}{n}, \dots, 1\right). \quad \blacksquare$$

Definition 2.7. Let A be a unital C^*-algebra, $x, y \in A$. Then we say x is *approximately unitarily equivalent* to y if for all $\varepsilon > 0$ there is a unitary $c \in A$ such that $\|cxc^* - y\| < \varepsilon$. If x and y are normal with spectrum contained in S, this is the same as saying that the two homomorphisms from $C(S)$ to A, given by applying functional calculus to x and y respectively, are approximately unitarily equivalent in the usual sense. (See for example Definition 1.1 of [12].)

Lemma 2.8. *Let D be a unital C^*-algebra, let $v \in D$ be unitary, and let $u_0 \in \mathcal{O}_\infty$ be a unitary with $\operatorname{sp}(u_0) = S^1$. Then the unitary $u_0 \otimes v \in \mathcal{O}_\infty \otimes D$ is approximately absorbing in the following sense. Let $e \in \mathcal{O}_\infty$ be a nonzero projection with $[e] = 0$ in $K_0(\mathcal{O}_\infty)$, let $a \in (e \otimes 1)(\mathcal{O}_\infty \otimes D)(e \otimes 1)$ be selfadjoint, and let $s \in \mathcal{O}_\infty$ satisfy $ss^* = 1 - e$ and $s^*s = 1$. Then $(s \otimes 1)(u_0 \otimes v)(s \otimes 1)^* + \exp(\mathrm{i}a)$ is approximately unitarily equivalent to $u_0 \otimes v$.*

Proof. Let e, a and s as in the statement be given, and let $\varepsilon > 0$. First, note that we may replace $u_0 \otimes v$ by any other unitary w with $\|u_0 \otimes v - w\| < \frac{1}{3}\varepsilon$, provided we prove that there is a unitary y with

$$\|y\,[(s \otimes 1)w(s \otimes 1)^* + \exp(\mathrm{i}a)]\,y^* - w\| < \frac{1}{3}\varepsilon. \tag{2.1}$$

Choose $\alpha \geqslant \|a\|$ with $\alpha/(2\pi)$ irrational, choose an integer $N > 6\alpha\exp(\alpha)/\varepsilon$, and define $\lambda_k = \exp(\mathrm{i}\alpha k/N)$ for $-N \leqslant k \leqslant N$. Then choose w, z, and q_k for $-N \leqslant k \leqslant N$ following Lemma 2.5, except using $\frac{1}{3}\varepsilon$ for ε. Note that the conclusion of this lemma will still be satisfied if q_k is replaced by any smaller projection. We may therefore reduce the size of q_k so as to have $[q_k] = 0$ in $K_0(\mathcal{O}_\infty)$ for all k.

Replacing w by z^*wz changes the unitary equivalence classes of neither w nor $(s \otimes 1)w(s \otimes 1)^* + \exp(\mathrm{i}a)$. Therefore we may in fact assume

$$(q_k \otimes 1)w = w(q_k \otimes 1) = \lambda_k q_k \otimes 1 \tag{2.2}$$

for $-N \leqslant k \leqslant N$.

Define $e_k = q_k$ for $k \neq -N$, and write $q_{-N} = e_{-N} + f$ with e_{-N}, f nonzero projections satisfying $[e_{-N}] = [f] = 0$ in $K_0(\mathcal{O}_\infty)$. Observe that the unitary equivalence class of $(s \otimes 1)w(s \otimes 1)^* + \exp(\mathrm{i}a)$ is unchanged if, for some unitary $z \in \mathcal{O}_\infty$, we replace e by zez^*, a by $(z \otimes 1)a(z \otimes 1)^*$, and s by zs. We may therefore assume that $e = e_{-N}$. The unitary equivalence class is also unchanged if we replace s by any other isometry with the same range projection. We may therefore specifically choose $s = 1 - e_{-N} - f + t$, for some t satisfying $tt^* = f$ and $t^*t = e_{-N} + f$. (Such a t exists because $[f] = [e_{-N} + f] = 0$ in $K_0(\mathcal{O}_\infty)$.)

Define $\overline{e} = \sum_{k=-N}^{N} e_k$. Using (2.2), we may write

$$w = \sum_{k=-N}^{N} \lambda_k (e_k \otimes 1) + \lambda_{-N}(f \otimes 1) + w_0, \tag{2.3}$$

with w_0 a unitary in $(1 - \overline{e} - f)\mathcal{O}_\infty(1 - \overline{e} - f) \otimes D$. Also,

$$(s \otimes 1)w(s \otimes 1)^* = \sum_{k=-N+1}^{N} \lambda_k (e_k \otimes 1) + \lambda_{-N}(f \otimes 1) + w_0. \tag{2.4}$$

(The difference from w is that the term $\lambda_{-N}(e_{-N} \otimes 1)$ is missing.)

Define

$$b = \sum_{k=-N}^{N} \left(\frac{\alpha k}{N} \right) e_k \otimes 1 \in (\overline{e} \otimes 1)(\mathcal{O}_\infty \otimes D)(\overline{e} \otimes 1).$$

Lemma 2.6 yields a projection p, a unitary u, and a selfadjoint element b_0, all in $(\overline{e} \otimes 1)(\mathcal{O}_\infty \otimes D)(\overline{e} \otimes 1)$, such that

$$upu^* = e_{-N}, \quad \|pb - bp\| \leqslant \frac{\alpha}{2N}, \quad upbpu^* = a,$$

$$\|b_0 - (1-p)b(1-p)\| \leqslant \frac{\alpha}{N}, \quad \text{and} \quad ub_0u^* = \sum_{k=-N+1}^{N} \frac{\alpha k}{N} e_k \otimes 1.$$

Since $\lambda_k = \exp(\mathrm{i}\alpha k/N)$, we have

$$\exp(\mathrm{i}[upbpu^* + ub_0u^*]) + \lambda_{-N}(f \otimes 1) + w_0 = (s \otimes 1)w(s \otimes 1)^* + \exp(\mathrm{i}a)$$

by (2.4). The expression on the left is unitarily equivalent to

$$\exp(\mathrm{i}[pbp + b_0]) + \lambda_{-N}(f \otimes 1) + w_0.$$

Furthermore, from (2.2) and the definition of b, we have

$$w = \exp(\mathrm{i}b) + \lambda_{-N}(f \otimes 1) + w_0.$$

We want to prove that there is a unitary y satisfying (2.1), and it is clearly enough to prove

$$\|\exp(\mathrm{i}b) - \exp(\mathrm{i}[pbp + b_0])\| < \frac{1}{3}\varepsilon.$$

From the choices in the previous paragraph, we have

$$\begin{aligned} \|b - (pbp + b_0)\| &\leqslant \|b - [pbp + (1-p)b(1-p)]\| + \|(1-p)b(1-p) - b_0\| \\ &\leqslant \|pb(1-p)\| + \|(1-p)bp\| + \frac{\alpha}{N} \leqslant 2\|pb - bp\| + \frac{\alpha}{N} \leqslant \frac{2\alpha}{N}. \end{aligned}$$

Also clearly $\|b\|$, $\|pbp + b_0\| \leqslant \|a\| < \alpha$. So Lemma 2.1 of [18] yields

$$\|\exp(\mathrm{i}b) - \exp(\mathrm{i}[pbp + b_0])\| \leqslant \frac{2\alpha \exp(\alpha)}{N} < \frac{1}{3}\varepsilon,$$

as desired. (Note that there are misprints in the proof in [18]: the factor $\|a - b\|$ was accidentally dropped halfway through the estimates.) ∎

3. EXPONENTIAL LENGTH AND RELATED INVARIANTS

In this section, we use the lemmas of the previous section to show that the exponential length of a tensor product with $\mathcal{O}_\infty$ can be at most 2π. It follows that the exponential rank can be at most $2+\varepsilon$. A similar argument shows that the C^* projective length and rank [16] of a tensor product with $\mathcal{O}_\infty$ can be at most π and $2+\varepsilon$ respectively, the same as for a purely infinite simple C^*-algebra. As noted after the theorems, these results are essentially the best possible.

THEOREM 3.1. *Let D be any C^*-algebra. Then* $\mathrm{cel}(\mathcal{O}_\infty \otimes D) \leqslant 2\pi$.

Proof. We prove that if $u \in U_0((\mathcal{O}_\infty \otimes D)^+)$ with $u - 1 \in \mathcal{O}_\infty \otimes D$, and $\varepsilon > 0$, then there is a unitary path $t \mapsto c(t)$ from 1 to u of length less than $2\pi + \varepsilon$ and satisfying $c(t) - 1 \in \mathcal{O}_\infty \otimes D$ for all t.

We claim that it suffices to prove this for unitaries of the form $1 \otimes v$ with $v \in U_0(D^+)$ satisfying $v - 1 \in D$. To see this, let $\varphi : \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$ and $z_n \in U(\mathcal{O}_\infty)$ be as in Theorem 0.1. For $d \in (\mathcal{O}_\infty \otimes D)^+$ we then have

$$\lim_{n\to\infty} (z_n \otimes 1)(\varphi \otimes \mathrm{id}_D)^+(1 \otimes d)(z_n \otimes 1)^* = d.$$

Putting $d = u$, we see that if $1 \otimes u \in (\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes D)^+$ can be connected to 1 by a path of the required form, then $u \in (\mathcal{O}_\infty \otimes D)^+$ is a limit of unitaries (namely $(z_n \otimes 1)(\varphi \otimes \mathrm{id}_D)^+(1 \otimes u)(z_n \otimes 1)^*$) which can be connected to 1 by such paths, and therefore can itself be connected to 1 by such a path. This proves the claim (with $\mathcal{O}_\infty \otimes D$ replacing D).

So let $v \in U_0(D^+)$ satisfy $v - 1 \in D$. If $\mathrm{sp}(v) \neq S^1$ then functional calculus yields a selfadjoint $h \in D$ with $\|h\| \leqslant 2\pi$ and $\exp(\mathrm{i}h) = v$. So certainly $\mathrm{cel}(1 \otimes v) \leqslant 2\pi$. Thus, we may assume $\mathrm{sp}(v) = S^1$. Choose a unitary $u_0 \in U_0(\mathcal{O}_\infty)$ with $\mathrm{sp}(u_0) = S^1$. (It follows from a result on page 61 of [1] that there is a selfadjoint $a \in \mathcal{O}_\infty$ with $\mathrm{sp}(a) = [0, 1]$. Take $u_0 = \exp(2\pi \mathrm{i}a)$.) Then $u_0 \otimes v \in U_0(\mathcal{O}_\infty \otimes D^+)$. We claim that $\mathrm{cel}(u_0 \otimes v) \leqslant \pi$, in the algebra $\mathcal{O}_\infty \otimes D^+$.

Given this claim, we connect $1 \otimes v$ to 1 by a path $t \mapsto c(t)$ of length at most $2\pi + \varepsilon$ and with $c(t) - 1 \in \mathcal{O}_\infty \otimes D$, as follows. Let $\pi : \mathcal{O}_\infty \otimes D^+ \to \mathcal{O}_\infty$ be the homomorphism induced by the standard map $D^+ \to \mathbb{C}$, and let $\iota : \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes D^+$ be the homomorphism $\iota(a) = a \otimes 1$ induced by the standard map $\mathbb{C} \to D^+$. Since $\mathrm{cel}(u_0 \otimes v) \leqslant \pi$, there is a unitary path $t \mapsto w(t)$ in $\mathcal{O}_\infty \otimes D^+$ with length $l(w) < \pi + \frac{1}{2}\varepsilon$ and with $w(0) = 1$ and $w(1) = u_0 \otimes v$. Now define

$$c(t) = w(t)(\iota \circ \pi)(w(t)^*).$$

Lemma 2.2 implies that $l(c) < 2\pi + \varepsilon$. It is immediate to check that $\pi(c(t)) = 1$ for all t, which implies that $c(t) \in (\mathcal{O}_\infty \otimes D)^+$. Obviously $c(0) = 1$. Since $v - 1 \in D$, we have $\pi(1 \otimes v) = 1$, and thus $\pi(u_0 \otimes v) = u_0$. This implies that $c(1) = 1 \otimes v$. So c is the required path.

We now prove the claim. We do this by showing that for all $\varepsilon > 0$ there is a selfadjoint $h \in \mathcal{O}_\infty \otimes D^+$ such that $\|h\| \leqslant \pi$ and $\|\exp(\mathrm{i}h) - u_0 \otimes v\| < \varepsilon$. Let $\alpha \mapsto y(\alpha)$ be any continuous unitary path in $\mathcal{O}_\infty \otimes D^+$ with $y(0) = 1$ and $y(1) = u_0 \otimes v$. Choose $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ such that $\|y(\alpha_j) - y(\alpha_{j-1})\| < \frac{1}{3}\varepsilon$, and set $y_j = y(\alpha_j)$. Thus $y_0 = 1$ and $y_n = u_0 \otimes v$. Choose a nonzero projection $e \in \mathcal{O}_\infty$ with $[e] = 0$ in $K_0(\mathcal{O}_\infty)$, and an isometry s with $s^*s = 1 - e$. Choose a projection

$f_0 \in \mathcal{O}_\infty$ with $[f_0] = -2n[1]$ in $K_0(\mathcal{O}_\infty)$, and define $f = f_0 \oplus 1_{2n} \in M_{2n+1}(\mathcal{O}_\infty)$. Then there is a partial isometry $t \in M_{2n+1}(\mathcal{O}_\infty)$ with $t^*t = f$ and $tt^* = e \oplus 0$. Using s and t, we can construct an isomorphism

$$\psi : (f \oplus 1)M_{2n+2}(\mathcal{O}_\infty)(f \oplus 1) \to \mathcal{O}_\infty$$

such that $\psi(f) = e$ and $\psi(0 \oplus b) = sbs^*$ for $b \in \mathcal{O}_\infty$. Define

$$\overline{y} = (f_0 \otimes 1) \oplus y_0 \oplus y_0^* \oplus y_1 \oplus y_1^* \oplus \cdots \oplus y_{n-1} \oplus y_{n-1}^* \in fM_{2n+1}(\mathcal{O}_\infty)f \otimes D.$$

Then $\overline{y}$ is unitary, and, by Corollary 5 of [17], $\overline{y}$ is a limit of elements $\exp(\mathrm{i}a)$ with a selfadjoint. Lemma 2.8 therefore shows that $y_n = u_0 \otimes v$ is approximately unitarily equivalent to $\psi(\overline{y} \oplus y_n)$. Choose a unitary z such that $\|z\psi(\overline{y} \oplus y_n)z^* - u_0 \otimes v\| < \frac{1}{3}\varepsilon$.

Since $\|y_j - y_{j-1}\| < \frac{1}{3}\varepsilon$ we have

$$\|\overline{y} \oplus y_n - [(f_0 \otimes 1) \oplus y_0 \oplus y_1^* \oplus y_1 \oplus y_2^* \oplus \cdots \oplus y_{n-1} \oplus y_n^* \oplus y_n]\| < \frac{1}{3}\varepsilon.$$

Since $y_0 = 1$, we obtain, again from Corollary 5 of [17], a selfadjoint element $h_0 \in (f \oplus 1)M_{2n+2}(\mathcal{O}_\infty)(f \oplus 1) \otimes D$ such that

$$\| \exp(\mathrm{i}h_0) - [(f_0 \otimes 1) \oplus y_0 \oplus y_1^* \oplus y_2^* \oplus \cdots \oplus y_{n-1} \oplus y_n^* \oplus y_n]\| < \frac{1}{3}\varepsilon.$$

From the proof it is clear that we may require $\|h_0\| \leqslant \pi$. Then $\| \exp(\mathrm{i}h_0) - \overline{y} \oplus y_0\| < \frac{2}{3}\varepsilon$. Setting $h = z\psi(h_0)z^*$, we get $\| \exp(\mathrm{i}h) - u_0 \otimes v\| < \varepsilon$. ∎

Corollary 3.2. *Let A be any separable nuclear purely infinite simple C^*-algebra, and let D be any C^*-algebra. Then*

$$\mathrm{cel}(A \otimes D) \leqslant 2\pi \quad \textit{and} \quad \mathrm{cer}(A \otimes D) \leqslant 2 + \varepsilon.$$

Proof. We have $\mathcal{O}_\infty \otimes A \cong A$ ([10]; Theorem 3.15 of [11]). So $\mathrm{cel}(A \otimes D) \leqslant 2\pi$ follows from the theorem. Given this, in the unital case $\mathrm{cer}(A \otimes D) \leqslant 2 + \varepsilon$ follows from Corollary 2.7 of [22]. Since scalars have logarithms in the center of the algebra, the same argument also proves the nonunital case. ∎

The upper bounds in this result can't be improved, except possibly by dropping the ε in the estimate on $\mathrm{cer}(A \otimes D)$. It was shown in Proposition 10 of [17] that $\mathrm{cer}(C(S^1) \otimes \mathcal{O}_2) \geqslant 2$, whence also $\mathrm{cel}(C(S^1) \otimes \mathcal{O}_2) \geqslant 2\pi$.

Essentially the same methods enable us to estimate the C^* projective length.

Theorem 3.3. *Let D be any C^*-algebra. Then* $\mathrm{cpl}(\mathcal{O}_\infty \otimes D) \leqslant \pi$.

Proof. Using the reasoning of the second paragraph of the proof of Theorem 3.1, we see that it suffices to show that if $p, q \in D$ are homotopic projections and $\varepsilon > 0$, then the projections $1 \otimes p$, $1 \otimes q \in \mathcal{O}_\infty \otimes D$ can be connected by a continuous path of projections of length less $\pi + \varepsilon$.

Since p is homotopic to q, there is a unitary $v \in U_0(D^+)$ such that $vpv^* = q$. If $\mathrm{sp}(v) \neq S^1$, then we may replace v by λv for a suitable $\lambda \in S^1$ so as to have $-1 \notin \mathrm{sp}(v)$. Then $\mathrm{cel}(v) \leqslant \pi$, so by Theorem 1.9 of [16] the rectifiable distance from p to q in D is at most π. Therefore the same is true of $1 \otimes p$ and $1 \otimes q$.

If $\mathrm{sp}(v) = S^1$, choose $u \in U_0(\mathcal{O}_\infty)$ with $\mathrm{sp}(u_0) = S^1$ as in the proof of Theorem 3.1. The claim proved in the last three paragraphs of that proof shows that $\mathrm{cel}(u_0 \otimes v) \leqslant \pi$. Since $(u_0 \otimes v)(1 \otimes p)(u_0 \otimes v)^* = 1 \otimes q$, it again follows from Theorem 1.9 of [16] that the rectifiable distance from $1 \otimes p$ to $1 \otimes q$ is at most π. ∎

Corollary 3.4. *Let A be any separable nuclear purely infinite simple C^*-algebra, and let D be any C^*-algebra. Then*

$$\mathrm{cpl}(A \otimes D) \leqslant \pi \quad \textit{and} \quad \mathrm{cpr}(A \otimes D) \leqslant 2 + \varepsilon.$$

Proof. Use Theorem 2.4 (1) of [16] in place of [22] in the proof of Corollary 3.2. ∎

Again, the upper bounds in this result can't be improved, except possibly by dropping the ε in the estimate on $\mathrm{cpr}(A \otimes D)$. Even for a purely infinite simple C^*-algebra A, we have $\mathrm{cpl}(A) = \pi$ and $\mathrm{cpr}(A) = 2$. (See Theorem 3.3 of [16].)

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